

Part III

Spherical functions on p -adic homogeneous spaces

Yumiko Hironaka

Introduction

Let \mathbb{G} be a reductive linear algebraic group defined over k , and \mathbb{X} be an affine algebraic variety defined over k which is \mathbb{G} -homogeneous, where and henceforth k stands for a non-archimedean local field of characteristic 0. The Hecke algebra $\mathcal{H}(G, K)$ of G with respect to K acts by convolution product on the space of $\mathcal{C}^\infty(K \backslash X)$ of K -invariant \mathbb{C} -valued functions on X , where K is a maximal compact open subgroup of $G = \mathbb{G}(k)$ and $X = \mathbb{X}(k)$.

A nonzero function in $\mathcal{C}^\infty(K \backslash X)$ is called a *spherical function on X* if it is a common $\mathcal{H}(G, K)$ -eigen function.

Spherical functions on homogeneous spaces comprise an interesting topic to investigate and a basic tool to study harmonic analysis on G -space X . They have been studied also as spherical vectors of distinguished models, Shalika functions and Whittaker-Shintani functions, and are closely related to theory of automorphic forms and representation theory. When \mathbb{G} and \mathbb{X} are defined over \mathbb{Q} , spherical functions appear in local factors of global objects, e.g. Rankin-Selberg convolutions and Eisenstein series (e.g. [CS], [F1], [HS3], [Jac], [KMS], [Sf2]).

The theory of spherical functions also has applications to classical number theory. For example when X is the space of symmetric forms, alternating forms or hermitian forms, spherical functions can be considered as generating functions of local densities, and have been applied to obtain their explicit formulas (cf. [HS1], [HS2], [H1]-[H4]).

To obtain explicit expressions of spherical functions is one of basic problems. For the group cases, it has been done by I. G. Macdonald and afterwards by W. Casselman by a representation theoretical method (cf. [Ma], [Cas]). There are some results on homogeneous space cases mainly for the case that the space of spherical functions attached to each Satake parameter is of dimension one (e.g. [CS], [KMS], [Of]).

In this paper, following the preliminaries in §1, we give a general expression of spherical functions on X of dimension not necessary one based on the data of the group G and functional equations of spherical functions in §2. Then we show a unified method to obtain functional equations of spherical functions on X , and explain that functional equations are reduced to those of p -adic local zeta functions of *small* prehomogeneous vector spaces in §3. These are improvements of some results in [H3] and [H6]. We devote §4 to examples. For general references for algebraic groups, one may refer to [Bo] and [PR].

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§1

1.1. Let \mathbb{H} be a connected linear algebraic group and \mathbb{Y} an affine algebraic variety on which \mathbb{H} acts, where everything is assumed to be defined over k . We denote by $\mathfrak{X}(\mathbb{H})$ the group of k -rational characters of \mathbb{H} , which is a free abelian group of finite rank. We set $\mathfrak{X}_0(\mathbb{H})$ for the subgroup consisting of characters corresponding to some relative \mathbb{H} -invariants on \mathbb{Y} , where a rational function f on \mathbb{Y} defined over k is called *relative \mathbb{H} -invariant* if it satisfies, for some $\psi \in \mathfrak{X}(\mathbb{H})$,

$$f(g \cdot y) = \psi(g)f(y), \quad g \in \mathbb{H}.$$

We say a set $\{f_i(y) \mid 1 \leq i \leq n\}$ is *basic*, if the corresponding characters form a basis for $\mathfrak{X}_0(\mathbb{H})$; then every relative \mathbb{H} -invariant on \mathbb{Y} has a following form

$$c \cdot \prod_{i=1}^n f_i(y)^{e_i}, \quad c \in k^\times, \quad e_i \in \mathbb{Z}.$$

We consider the following conditions for (\mathbb{H}, \mathbb{Y}) .

(A1') \mathbb{Y} has a Zariski open \mathbb{H} -orbit.

(A1) \mathbb{Y} has only a finite number of \mathbb{H} -orbits.

(A2) A basic set of relative \mathbb{H} -invariants on \mathbb{Y} can be taken by regular functions on \mathbb{Y} .

(A3) For $y \in \mathbb{Y}$ not contained in open orbits, there exists some ψ in $\mathfrak{X}_0(\mathbb{H})$ whose restriction to the identity component of the stabilizer \mathbb{H}_y is not trivial.

(A4) The rank of $\mathfrak{X}_0(\mathbb{H})$ coincides with that of $\mathfrak{X}(\mathbb{H})$.

Remark 1.1. Assume that \mathbb{Y} is a homogeneous space of a connected reductive linear algebraic group \mathbb{G} (like as § 1. 2). Then \mathbb{Y} is irreducible and there is at most one Zarisky open orbit, and (A1) implies (A1'). The condition (A1) is satisfied if \mathbb{Y} is a spherical homogeneous space of \mathbb{G} and \mathbb{H} is a minimal parabolic subgroup of \mathbb{G} , and symmetric spaces are spherical, especially the spaces of type $\mathbb{G}/\mathbb{G}^\theta$ (\mathbb{G}^θ is the set of fixed points of an involution θ on \mathbb{G}) are spherical(cf. [Sf4]). As for (A2), we note here that for the case of prehomogeneous vector spaces, basic relative invariants can be chosen as polynomial functions ([Sf1]-Lemma 1.3). The condition (A3) assures us a good condition for distributions on $\mathbb{Y}(k)$, which we need when we consider functional equations of spherical functions.

1.2. Hereafter, let \mathbb{G} be a connected reductive linear algebraic group and \mathbb{X} be an affine algebraic variety which is \mathbb{G} -homogeneous, where everything is assumed to be defined over k . For an algebraic set, we use the same ordinary letter for the set of k -rational points, e.g. $G = \mathbb{G}(k)$, $X = \mathbb{X}(k)$. Let K be a maximal compact open subgroup of G , and \mathbb{B} a minimal parabolic subgroup of \mathbb{G} defined over k satisfying $G = KB = BK$. The group \mathbb{B} is not necessarily a Borel subgroup. We denote by $|\cdot|$ the absolute value on k normalized by $|\pi| = q^{-1}$, where π is a prime element of k and q is the cardinal number of the residue class field of k , we understand $|0| = 0$ for simplicity.

Assume that (\mathbb{B}, \mathbb{X}) satisfies (A1') and (A2), and let $\{f_i(x) \mid 1 \leq i \leq n\}$ be a basic set of regular relative \mathbb{B} -invariants, and $\psi_i \in \mathfrak{X}_0(\mathbb{B})$ the corresponding character to $f_i(x)$, and $n = \text{rank}(\mathfrak{X}_0(\mathbb{B}))$. The open \mathbb{B} -orbit \mathbb{X}^{op} is decomposed into a finite number of open B -orbits over k (cf. [Sr]-III-4.4), which we write

$$\mathbb{X}^{op}(k) = \bigsqcup_{u \in J(X)} X_u.$$

For $x \in X$, $s \in \mathbb{C}^n$ and $u \in J(X)$, we define

$$\omega(x; s) = \int_K |f(k \cdot x)|^s dk, \quad \omega_u(x; s) = \int_K |f(k \cdot x)|_u^s dk, \quad (1.1)$$

where dk is the Haar measure on K normalized by $\int_K dk = 1$, and

$$|f(x)|^s = \prod_{i=1}^n |f_i(x)|^{s_i}, \quad |f(x)|_u^s = \begin{cases} |f(x)|^s & \text{if } x \in X_u, \\ 0 & \text{otherwise.} \end{cases}$$

We set

$$|\psi(p)|^s = \prod_{i=1}^n |\psi_i(p)|^{s_i}.$$

By the following proposition, we see $\omega(x; s)$ and $\omega_u(x; s)$ are spherical functions on X , where we give also the ‘eigenvalues’ for them.

Proposition 1.2. *The integrals in (1.1) are absolutely convergent if $\operatorname{Re}(s_i) \geq 0$, $1 \leq i \leq n$, analytically continued to rational functions of q^{s_1}, \dots, q^{s_n} , and become $\mathcal{H}(G, K)$ -common eigen functions. In particular $\omega_u(x; s)$, $u \in J(X)$, are spherical functions on X and linearly independent for generic s . More precisely, for each $\phi \in \mathcal{H}(G, K)$, one has $(\phi * \omega(\ ; s))(x) = \lambda_s(\phi)\omega(x; s)$ and $(\phi * \omega_u(\ ; s))(x) = \lambda_s(\phi)\omega_u(x; s)$ with*

$$\lambda_s(\phi) = \int_B \phi(p) |\psi(p)|^{-s} \delta(p) dp = \int_G \phi(g) |\psi(p(g))|^{-s} \delta(p(g)) dg,$$

where dp is the left invariant Haar measure on B normalized by $\int_{K \cap B} dp = 1$ and $p(g) \in B$ for which $p(g)^{-1}g \in K$.

For a proof we refer to [H3]-Proposition 1.1. To make it sure we note here the action of $\mathcal{H}(G, K)$ on $\omega_u(x; s)$: for $\phi \in \mathcal{H}(G, K)$ and $x \in X$,

$$\begin{aligned} (\phi * \omega_u(\ ; s))(x) &= \int_G \phi(g) \int_K |f(kg^{-1} \cdot x)|_u^s dk dg \\ &= \int_K \int_G \phi(gk) |f(g^{-1} \cdot x)|_u^s dg dk = \int_G \phi(g) |f(g^{-1} \cdot x)|_u^s dg \\ &= \int_K \int_B \phi(kp) |f(p^{-1}k^{-1} \cdot x)|_u^s d_r p dk \\ &= \int_K \int_B \phi(p) |\psi(p)|^{-s} |f(k^{-1} \cdot x)|_u^s d_r p dk \\ &= \int_B \phi(p) |\psi(p)|^{-s} \delta(p) dp \cdot \omega_u(x; s). \end{aligned}$$

□

Remark 1.3. When we assume also (A4), we can determine $\varepsilon_0 \in \mathbb{Q}^n$ by

$$|\psi(p)|^{\varepsilon_0} = \delta^{\frac{1}{2}}(p), \quad p \in B,$$

and it is better to modify the definition of spherical functions as follows:

$$\tilde{\omega}_u(x; s) = \int_K |f(k \cdot x)|_u^{s+\varepsilon_0} dk. \quad (1.2)$$

Then, instead of Proposition 1.2, we have

$$(\phi * \tilde{\omega}(\ ; s))(x) = \tilde{\lambda}_s(\phi) \tilde{\omega}(x; s), \quad (\phi * \tilde{\omega}_u(\ ; s))(x) = \tilde{\lambda}_s(\phi) \tilde{\omega}_u(x; s),$$

where

$$\tilde{\lambda}_s(\phi) = \int_B \phi(p) |\psi(p)|^{-s+\varepsilon_0} dp = \int_G \phi(g) |\psi(p(g))|^{-s+\varepsilon_0} dg, \quad \phi \in \mathcal{H}(G, K).$$

Remark 1.4. The value $f_i(x) \bmod \psi_i(B)$ is constant in $k^\times / \psi_i(B)$ on each open B -orbit, which we call *the signature of f_i* . If we can parametrize open B -orbits in X^{op} by the signatures of $\{f_i(x) \mid 1 \leq i \leq n\}$, then $J(X)$ can be naturally identified with a subset of the finite abelian group

$$(k^\times)^n / \prod_{i=1}^n \psi_i(B). \quad (1.3)$$

Such cases often occur, and then, it is natural to consider spherical functions with character as follows. Assume \mathcal{U} is $J(X)$ or its suitable subset which is canonically identified with a subgroup of (1.3). Taking a character χ of \mathcal{U} , we set

$$\omega(x; \chi; s) = \int_K \chi(f(k \cdot x)) |f(k \cdot x)|^s dk = \sum_{u \in \mathcal{U}} \chi(u) \omega_u(x; s), \quad (1.4)$$

$$\tilde{\omega}(x; \chi; s) = \int_K \chi(f(k \cdot x)) |f(k \cdot x)|^{s+\varepsilon_0} dk = \sum_{u \in \mathcal{U}} \chi(u) \tilde{\omega}_u(x; s), \quad (1.5)$$

the latter can be considered only when (A4) is satisfied.

Let W be the relative Weyl group of \mathbb{G} with respect to \mathbb{T} , where \mathbb{T} is a maximal k -split torus contained in \mathbb{B} . The group W acts on $\mathfrak{X}(\mathbb{B})$ as $(\sigma\xi)(b) = \xi(n_\sigma^{-1}bn_\sigma)$ by taking a representative $n_\sigma \in N_G(T)$ of $\sigma \in W$, hence it acts on $s \in \mathbb{C}^n$ through the identification $\mathbb{C}^n \cong \mathfrak{X}_0(\mathbb{B}) \otimes_{\mathbb{Z}} \mathbb{C} \subset \mathfrak{X}(\mathbb{B}) \otimes_{\mathbb{Z}} \mathbb{C}$.

§2

In this section, we will give a general expression for spherical functions based on the data of the group G and functional equations of spherical functions.

We follow the notation in §1.2, and take K as a special, good, maximal compact subgroup in the sense of Bruhat and Tits (cf. [Cas]-§3.5), and Iwahori subgroup U of K compatible with B .

2.1. In this subsection, we prepare some results from representation theory (cf. [H3]-§1, [Cas]).

We denote by $\mathcal{S}(G)$ the Schwartz-Bruhat space on G , namely the space of locally constant compactly supported functions on G , and set $\mathcal{D}(G) = \text{Hom}_{\mathbb{C}}(\mathcal{S}(G), \mathbb{C})$, the space of distributions on G , and the pairing on $\mathcal{D}(G) \times \mathcal{S}(G)$

$$\langle T, \phi \rangle = \langle T, \phi \rangle_{\mathcal{D} \times \mathcal{S}} = T(\phi), \quad (T \in \mathcal{D}(G), \phi \in \mathcal{S}(G)).$$

Then the space $\mathcal{C}^\infty(G)$ of locally constant functions can be regarded as a subspace of $\mathcal{D}(G)$ by

$$\langle \psi, \phi \rangle = \int_G \psi(g)\phi(g)dg, \quad (\psi \in \mathcal{C}^\infty(G), \phi \in \mathcal{S}(G)).$$

We regard $\mathcal{C}^\infty(G)$ as a two-sided G -module and $\mathcal{S}(G)$ as a submodule by

$$g \cdot \psi(x) = \psi(xg), \quad \psi^g(x) = \psi(gx), \quad (\psi \in \mathcal{C}^\infty(G), g, x \in G).$$

Then, $\mathcal{D}(G)$ becomes also a two-sided G -module by the dual action:

$$\langle g \cdot T, \phi \rangle = \langle T, g^{-1} \cdot \phi \rangle, \quad \langle T^g, \phi \rangle = \langle T, \phi^{g^{-1}} \rangle, \quad (T \in \mathcal{D}(G), \phi \in \mathcal{S}(G), g \in G).$$

For a subspace Γ of $\mathcal{D}(G)$ and a subgroup H of G , we denote by Γ^H the set of left H -invariant elements in Γ .

Let χ be an unramified regular character of the centralizer $Z_G(T)$, i.e. $\chi|_{Z_G(T) \cap K} \equiv 1$, and $\sigma\chi = \chi$ implies $\sigma = 1$ for $\sigma \in W$, which is canonically extended to be a character of B . We recall the induced representation (principal series representation) of G :

$$\begin{aligned} I(\chi) &= \text{Ind}_B^G(\chi) = \left\{ \phi \in \mathcal{C}^\infty(G) \mid \phi(pg) = \chi\delta^{\frac{1}{2}}(p)\phi(g) \quad (p \in B, g \in G) \right\} \\ &= \left\{ \phi \in \mathcal{C}^\infty(G) \mid \phi^p = \chi\delta^{\frac{1}{2}}(p)\phi, \quad (p \in B) \right\}, \end{aligned} \quad (2.1)$$

which is a left G -submodule of $\mathcal{C}^\infty(G)$. Then we have a left G -equivariant surjection

$$\begin{aligned} \mathcal{P}_\chi : \mathcal{S}(G) &\longrightarrow I(\chi), \\ \mathcal{P}_\chi(\phi)(x) &= \int_B \chi^{-1}\delta^{\frac{1}{2}}(p)\phi(px)dp, \quad (\phi \in \mathcal{S}(G), x \in G). \end{aligned}$$

We set $\varphi_{K,\chi} = \mathcal{P}_\chi(ch_K)$, where ch_K is the characteristic function of K .

The map $\mathcal{P}_{\chi^{-1}}$ induces a left G -equivariant injection $\mathcal{P}_{\chi^{-1}}^*$ from $I(\chi^{-1})^* = \text{Hom}_{\mathbb{C}}(I(\chi^{-1}), \mathbb{C})$ to $\mathcal{D}(G)$ determined by

$$\langle \mathcal{P}_{\chi^{-1}}^*(T), \phi \rangle = T(\mathcal{P}_{\chi^{-1}}(\phi)), \quad (T \in I(\chi^{-1})^*, \phi \in \mathcal{S}(G)),$$

and we obtain the following ([H3]-Lemma 1.2, Corollary 1.3).

Proposition 2.1. *By the dual map $\mathcal{P}_{\chi^{-1}}^*$ of $\mathcal{P}_{\chi^{-1}}$, one has a left G -isomorphism*

$$I(\chi^{-1})^* \cong \mathcal{D}(G)_\chi := \left\{ T \in \mathcal{D}(G) \mid T^p = \chi\delta^{\frac{1}{2}}(p)T \quad (p \in B) \right\}.$$

Further, by this isomorphism, $I(\chi)$ and $I(\chi^{-1})$ can be understood as the smooth dual of the each other with pairing

$$\langle f_1, f_2 \rangle = \int_K f_1(k)f_2(k)dk, \quad (f_1 \in I(\chi), f_2 \in I(\chi^{-1})).$$

Indeed we calculate the pairing on $I(\chi) \times I(\chi^{-1})$ in the following: for $(f_1, f_2) \in I(\chi) \times I(\chi^{-1})$ and $\phi \in \mathcal{S}(G)$ such that $\mathcal{P}_{\chi^{-1}}(\phi) = f_2$,

$$\langle f_1, f_2 \rangle = \langle (\mathcal{P}_{\chi^{-1}}^*)^{-1}(f_1), \mathcal{P}_{\chi^{-1}}(\phi) \rangle_{I(\chi^{-1})^* \times I(\chi^{-1})}$$

$$\begin{aligned}
&= \langle f_1, \phi \rangle_{\mathcal{D} \times \mathcal{S}} = \int_G f_1(g) \phi(g) dg = \int_K \int_B f_1(pk) \phi(pk) dp dk \\
&= \int_K f_1(k) \int_B \chi^{\frac{1}{2}}(p) \phi(pk) dp dk = \int_K f_1(k) \mathcal{P}_{\chi^{-1}}(\phi)(k) dk \\
&= \int_K f_1(k) f_2(k) dk.
\end{aligned}$$

For $\sigma \in W$, there is a unique left G -equivariant map satisfying

$$T_\sigma^\chi : I(\chi) \longrightarrow I(\sigma\chi), \quad T_\sigma^\chi(\varphi_{K,\chi}) = c_\sigma(\chi) \varphi_{K,\sigma\chi}, \quad (2.2)$$

where

$$\begin{aligned}
c_\sigma(\chi) &= \prod_{\alpha \in \Sigma^+, \sigma(\alpha) < 0} c_\alpha(\chi), \\
c_\alpha(\chi) &= \frac{(1 - q^{-\frac{1}{2}} q_\alpha^{-1} \chi(a_\alpha))(1 + q^{\frac{1}{2}} \chi(a_\alpha))}{1 - \chi(a_\alpha)^2} \\
&= \frac{1 - q^{-1} \chi(a_\alpha)}{1 - \chi(a_\alpha)} \quad \text{if } G \text{ is split}.
\end{aligned}$$

Here Σ^+ is the set of positive roots \mathbb{G} with respect to \mathbb{T} and \mathbb{B} , and for the definition of $a_\alpha \in T$ and numbers $q_\alpha, q_{\frac{1}{2}\alpha}$ ($\alpha \in \Sigma$), see [Cas]. It is known that

$$T_\sigma^\chi \text{ is an isomorphism if and only if } c_\sigma(\chi) c_{\sigma^{-1}}(\sigma(\chi)) \neq 0. \quad (2.3)$$

For a compact open subgroup V of G , we define an operator \mathcal{P}_V on $\mathcal{D}(G)$ by

$$\langle \mathcal{P}_V(T), \phi \rangle = \int_V \langle u \cdot T, \phi \rangle du = \int_V \langle T, u^{-1} \cdot \phi \rangle du, \quad (T \in \mathcal{D}(G), \phi \in \mathcal{S}(G)),$$

where du is the Haar measure on V normalized by $\int_V du = 1$.

As the adjoint G -morphism of $T_{\sigma^{-1}}^{\sigma\chi^{-1}}$, we have (under the identification through $\mathcal{P}_{\chi^{-1}*}$ and $\mathcal{P}_{\sigma\chi^{-1}*}$ by Proposition 2.1)

$$\left(T_{\sigma^{-1}}^{\sigma\chi^{-1}}\right)^* : \mathcal{D}(G)_\chi = I(\chi^{-1})^* \longrightarrow \mathcal{D}(G)_{\sigma\chi} = I(\sigma\chi^{-1})^*.$$

Then we see the following(cf. [H3]-Proposition 1.6, Proposition 1.7).

Proposition 2.2. *Assume $c_\sigma(\chi) c_\sigma(\chi^{-1}) c_{\sigma^{-1}}(\sigma\chi) c_{\sigma^{-1}}(\sigma\chi^{-1}) \neq 0$. Then*

$$\tilde{T}_\sigma^\chi = \frac{c_\sigma(\chi)}{c_{\sigma^{-1}}(\sigma\chi^{-1})} \left(T_{\sigma^{-1}}^{\sigma\chi^{-1}}\right)^* : \mathcal{D}(G)_\chi \longrightarrow \mathcal{D}(G)_{\sigma\chi}$$

is an extension of the G -isomorphism $T_\sigma^\chi : I(\chi) \longrightarrow I(\sigma\chi)$. Further, for a compact open subgroup V of G , one has

$$\mathcal{P}_V \circ \tilde{T}_\sigma^\chi = T_\sigma^\chi \circ \mathcal{P}_V.$$

We recall Casselman basis $\{f_{\sigma,\chi} \mid \sigma \in W\}$ for $I(\chi)^U$, which satisfies the following(cf. [Cas])

$$\begin{aligned} T_\sigma^X(f_{\tau,s})(1) &= \delta_{\sigma,\tau}, \\ \mathcal{P}_K(f_{\sigma,\chi})(1) &= \frac{\gamma(\sigma(\chi))}{Q \cdot c_\sigma(\chi)}, \end{aligned}$$

where $\delta_{\sigma,\tau}$ is the Kronecker delta, and

$$\gamma(\chi) = \prod_{\alpha \in \Sigma^+} c_\alpha(\chi), \quad Q = \sum_{\sigma \in W} [U\sigma U : U]^{-1}.$$

Let us recall our situation in §1. Since \mathbb{G} is reductive and \mathbb{X} is affine, $\mathbb{H} = \mathbb{G}_{x_0}$, ($x_0 \in X^{op}$) is reductive also(cf. [Sp]-Satz 3.3). Then there is a G -invariant measure on G/H , since G and H are unimodular. Since BH/H is open in G/H and isomorphic to B/B_0 with $B_0 = B \cap H$, there is an invariant measure on B/B_0 , so the modulus character of B_0 coincides with $\delta|_{B_0}$.

Now we set

$$I(\chi, BH) = \{\phi \in I(\chi) \mid \text{Supp}(\phi) \subset BH\}.$$

The next lemma is based on an idea of O. Offen used in [Of]. It will play a key role to restrict the summation with respect W to a certain subgroup W_0 in §2.2.

Lemma 2.3. *If there is a nonzero left H -invariant distribution in $I(\chi)^*$ which is not identically zero on $I(\chi, BH)$, then $\chi = \delta^{\frac{1}{2}}$ on B_0 .*

Proof. Assume a distribution $\Lambda \in I(\chi)^*$ satisfies the condition as above. The space $I(\chi, BH)$ can be identified (by the restriction) with

$$\text{Ind}_{B_0}^H(\chi|_{B_0}) = \left\{ f \in C^\infty(H) \mid f(p_0h) = \chi\delta^{\frac{1}{2}}(p_0)f(h) \quad (p_0 \in B_0, h \in H) \right\},$$

on which there is a left H -invariant surjection $\mathcal{P}_{H,\chi}$ from $\mathcal{S}(H)$ given by

$$\mathcal{P}_{H,\chi}(\varphi)(h) = \int_{B_0} \chi^{-1}\delta^{\frac{1}{2}}(p_0)\varphi(p_0h)dp_0,$$

where dp_0 is a left invariant Haar measure on B_0 . Then we have a nonzero H -invariant distribution T on H determined by

$$\langle T, \varphi \rangle_{\mathcal{D} \times \mathcal{S}} = \langle \Lambda, \mathcal{P}_{H,\chi}(\varphi) \rangle,$$

thus we have a left invariant measure on H , which becomes also right invariant since H is unimodular. On the other hand, since we have for $p \in B_0$

$$\begin{aligned} \langle T^{p^{-1}}, \varphi \rangle_{\mathcal{D} \times \mathcal{S}} &= \langle T, \varphi^p \rangle_{\mathcal{D} \times \mathcal{S}} = \langle \Lambda, \mathcal{P}_{H,\chi}(\varphi^p) \rangle = \langle \Lambda, \chi\delta^{-\frac{1}{2}}(p)\mathcal{P}_{H,\chi}(\varphi) \rangle \\ &= \chi\delta^{-\frac{1}{2}}(p)\langle T, \varphi \rangle_{\mathcal{D} \times \mathcal{S}}, \end{aligned}$$

we obtain $\chi = \delta^{\frac{1}{2}}$ on B_0 . □

2.2. Take $x_0 \in X^{op}$ and set $\mathbb{H} = \mathbb{G}_{x_0}$ and $\mathcal{U} = \{\nu \in J(X) \mid G \cdot x_0 \cap X_\nu \neq \emptyset\}$. We define the subgroup W_0 of W by

$$W_0 = \{\sigma \in W \mid \sigma(|\psi|^s) \equiv 1 \text{ and } \sigma(\delta) = \delta \text{ on } B \cap H\},$$

where $s \in \mathbb{C}^n$ is considered as a variable. Though we do not assume the condition (A4), $\sigma(|\psi|^s)$ is contained in $\mathfrak{X}_0(\mathbb{B}) \otimes_{\mathbb{Z}} \mathbb{C}$ if $\sigma \in W_0$, and in this case $|f(x)|_u^{\sigma(s)}$ and $|\psi|^{\sigma(s)} = \sigma(|\psi|^s)$ are well defined. For $\sigma \in W_0$ we define $\varepsilon_\sigma \in \mathbb{Q}^n$ by

$$|\psi|^{2\varepsilon_\sigma} = \delta\sigma(\delta^{-1}). \quad (2.4)$$

For $s \in \mathbb{C}^n$, let $\chi = \chi_s$ be the character of B given by

$$\chi = |\psi|^s \delta^{-\frac{1}{2}}, \quad \text{i.e.,} \quad \chi(p) = |\psi(p)|^s \delta^{-\frac{1}{2}}(p), \quad p \in B. \quad (2.5)$$

Then we have

$$\sigma(\chi) = |\psi|^{\sigma(s) + \varepsilon_\sigma} \delta^{-\frac{1}{2}}, \quad \sigma \in W_0.$$

For each $\nu \in \mathcal{U}$, we set (through the analytic continuation for general $s \in \mathbb{C}^n$)

$$\begin{aligned} \Psi_\nu(x, s; g) &= |f(g \cdot x)|_\nu^s \in \mathcal{D}(G)_\chi \cong I(\chi^{-1})^*, \\ \tilde{\Psi}_\nu(x, s; g) &= \mathcal{P}_U(\Psi_\nu(x, s; \cdot))(g) = \int_U |f(gu \cdot x)|_\nu^s du \in I(\chi)^U, \end{aligned} \quad (2.6)$$

where we note that

$$\Psi_\nu(g_1 \cdot x, s; g) = g_1 \cdot \Psi_\nu(x, s; \cdot)(g) = \Psi_\nu(x, s; gg_1), \quad (g, g_1 \in G). \quad (2.7)$$

The condition (A3) is crucial for the next lemma (cf. [H3]-Lemma 1.8).

Lemma 2.4. *Assume (A1), (A2) and (A3) for (\mathbb{B}, \mathbb{X}) . Then for each $x \in G \cdot x_0$ and generic s , the set $\{\Psi_u(x, \sigma(s) + \varepsilon_\sigma; g) \mid u \in \mathcal{U}\}$ forms a basis for $\mathcal{D}(G)_{\sigma\chi}^{G \cdot x}$ for any $\sigma \in W_0$. Here, ‘generic’ means to avoid a finite number of linear relations of type $\sum_{i=1}^n m_i s_i - \alpha \in (\frac{2\pi\sqrt{-1}}{\log q})$ with $m_i \in \mathbb{Z}$, $\alpha \in \mathbb{C}$.*

In the following, we say s is *generic* if s is generic in the sense of Lemma 2.4, s is neither a pole nor a zero of $\omega_u(x; s)$ ($u \in \mathcal{U}$), $\chi = \chi_s$ is regular, and $c_\sigma(\chi)c_\sigma(\chi^{-1})c_\sigma(\sigma^{-1}\chi)c_\sigma(\sigma^{-1}\chi^{-1}) \neq 0$ for every $\sigma \in W_0$ (cf. Proposition 2.2).

We set

$$\begin{aligned} \mathcal{R} &= \{x \in G \cdot x_0 \cap X^{op} \mid U \cdot x \subset B \cdot x_0\}, \\ \mathcal{R}^+ &= \{x \in \mathcal{R} \mid |f(u \cdot x)|^s = |f(x)|^s, (u \in U)\}. \end{aligned} \quad (2.8)$$

Our main theorem in this section is the following, which is a refinement of [H3]-Proposition 1.9, where we assumed the condition (A4).

Theorem 2.5. *Assume (A1), (A2) and (A3) for (\mathbb{B}, \mathbb{X}) and s is generic. For $x \in \mathcal{R}$, one has*

$$(\omega_\nu(x; s))_{\nu \in \mathcal{U}} = \frac{1}{Q} \sum_{\sigma \in W_0} \gamma(\sigma(\chi)) \cdot B_\sigma(\chi) \cdot \left(\int_U |f(u \cdot x)|_\nu^{\sigma(s) + \varepsilon_\sigma} du \right)_{\nu \in \mathcal{U}}.$$

Moreover, if $x \in \mathcal{R}^+$, one has

$$(\omega_\nu(x; s))_{\nu \in \mathcal{U}} = \frac{1}{Q} \sum_{\sigma \in W_0} \gamma(\sigma(\chi)) \cdot B_\sigma(\chi) \cdot \left(|f(x)|_\nu^{\sigma(s) + \varepsilon_\sigma} \right)_{\nu \in \mathcal{U}}.$$

Here the constant Q and the rational function $\gamma(\chi)$ of q^{s_1}, \dots, q^{s_n} are determined by the group G as in §2.1, and the matrix $B_\sigma(\chi)$ is determined by the functional equation

$$(\omega_\nu(x; s))_{\nu \in \mathcal{U}} = B_\sigma(\chi) (\omega_\nu(x; \sigma(s) + \varepsilon_\sigma))_{\nu \in \mathcal{U}}.$$

We give here an outline of a proof. By definition of Ψ_ν and $\tilde{\Psi}_\nu$, we have

$$\omega_\nu(x; s) = \int_K \Psi_\nu(x, s; k) dk = \int_K \tilde{\Psi}_\nu(x, s; k) dk = \mathcal{P}_K(\tilde{\Psi}_\nu(x, s; \cdot))(1), \quad (2.9)$$

and we may write by using Casselman basis

$$\tilde{\Psi}_\nu(x; s; \cdot) = \sum_{\sigma \in W} a_{\nu, \sigma}(x; s) f_{\sigma, \chi},$$

where

$$a_{\nu, \sigma}(x; s) = T_\sigma^\chi(\tilde{\Psi}_\nu(x, s; \cdot))(1) = (\mathcal{P}_U \circ \tilde{T}_\sigma^\chi)(\Psi_\nu(x, s; \cdot))(1).$$

Now we set $\Lambda = \tilde{T}_\sigma^\chi(\Psi_\nu(x_0, s; \cdot)) \in I(\sigma\chi^{-1})^*$, which is left H -invariant, and $x = g_1 \cdot x_0$. Then we have (cf. (2.7))

$$\begin{aligned} a_{\nu, \sigma}(x; s) &= (\mathcal{P}_U \circ \tilde{T}_\sigma^\chi)(g_1 \cdot \Psi_\nu(x_0, s; \cdot))(1) \\ &= (\mathcal{P}_U \circ (g_1 \cdot \Lambda))(1). \end{aligned}$$

Since $\mathcal{P}_U(g_1 \cdot \Lambda)$ is regarded as an element of $I(\sigma\chi)^U$ by Proposition 2.1, taking $\varphi_U \in I(\sigma\chi^{-1})$ as supported by BU and $\varphi_U(u) = 1$ for $u \in U$, we can continue

$$\begin{aligned} a_{\nu, \sigma}(x; s) &= \int_K \mathcal{P}_U \circ (g_1 \cdot \Lambda)(k) \varphi_U(k) dk \\ &= \langle \mathcal{P}_U \circ (g_1 \cdot \Lambda), \varphi_U \rangle = \int_U \langle g_1 \cdot \Lambda, u \cdot \varphi_U \rangle du \\ &= \langle g_1 \cdot \Lambda, \varphi_U \rangle = \langle \Lambda, g_1^{-1} \cdot \varphi_U \rangle, \end{aligned}$$

where $\text{Supp}(g_1^{-1} \cdot \varphi_U) = BUg_1$.

If $x \in \mathcal{R}$, then $\text{Supp}(g_1^{-1} \cdot \varphi_U) \subset BH$, and $a_{\nu, \sigma}(x; s) = 0$ unless $\sigma\chi^{-1} = \delta^{\frac{1}{2}}$ on B_0 by Lemma 2.3, i.e., $a_{\nu, \sigma}(x; s) = 0$ unless $\sigma \in W_0$, by our choice of χ and W_0 . Thus we have, for $x \in \mathcal{R}$

$$\omega_\nu(x; s) = \frac{\gamma(\sigma(\chi))}{Q \cdot c_\sigma(\chi)} \cdot \sum_{\sigma \in W_0} \mathcal{P}_U(\tilde{T}_\sigma^\chi(\Psi_\nu(x, s; \cdot)))(1). \quad (2.10)$$

On the other hand, by Lemma 2.4, there exists an invertible matrix $A_\sigma(\chi)$ for $\sigma \in W_0$ satisfying

$$\left(\tilde{T}_\sigma^\chi(\Psi_\nu(x, s; \cdot)) \right)_{\nu \in \mathcal{U}} = A_\sigma(\chi) \left(\Psi_\nu(x, \sigma(s) + \varepsilon_\sigma; \cdot) \right)_{\nu \in \mathcal{U}}, \quad (2.11)$$

where $A_\sigma(\chi)$ depends only on the G -orbit containing x , since \tilde{T}_σ^χ is G -equivariant and (2.7).

For $x \in \mathcal{R}$, we obtain by (2.10) and (2.11)

$$\begin{aligned} (\omega_\nu(x; s))_{\nu \in \mathcal{U}} &= \frac{1}{Q} \sum_{\sigma \in W_0} \frac{\gamma(\sigma(\chi))}{c_\sigma(\chi)} \cdot A_\sigma(\chi) \left(\mathcal{P}_U(\Psi_\nu(x, \sigma(s) + \varepsilon_\sigma; \cdot)) \right)_{\nu} \\ &= \frac{1}{Q} \sum_{\sigma \in W_0} \gamma(\sigma(\chi)) \cdot B_\sigma(\chi) \left(\tilde{\Psi}_\nu(x, \sigma(s) + \varepsilon_\sigma; 1) \right)_{\nu}, \end{aligned}$$

where we set $B_\sigma(\chi) = c_\sigma(\chi)^{-1} A_\sigma(\chi)$. By (2.3) and Proposition 2.2, we see the invertible matrix $B_\sigma(\chi)$ satisfies the cocycle relation

$$B_{\sigma\tau}(\chi) = B_\tau(\chi) B_\sigma(\tau(\chi)), \quad \sigma, \tau \in W_0.$$

Hence

$$\begin{aligned} (\omega_\nu(x; \tau(s) + \varepsilon_\tau))_\nu &= \frac{1}{Q} \sum_{\sigma \in W_0} \gamma(\sigma\tau(\chi)) B_\sigma(\tau(\chi)) \left(\tilde{\Psi}_\nu(x, \sigma\tau(s) + \varepsilon_{\sigma\tau}; 1) \right)_\nu \\ &= B_\tau(\chi)^{-1} (\omega_\nu(x; s))_\nu, \end{aligned} \quad (2.12)$$

and the above relation determines $B_\sigma(\chi)$, since $\{\omega_\nu(x; \sigma(s) + \varepsilon_\sigma) \mid \nu \in \mathcal{U}\}$ is linearly independent for each $\sigma \in W_0$.

Finally, if $x \in \mathcal{R}^+$, we have $\tilde{\Psi}_\nu(x, s; 1) = |f(x)|^s$. \square

Now we assume the condition (A.4) for (\mathbb{B}, \mathbb{X}) and recall $\tilde{\omega}_u(x; s)$, then we do not need to consider the subgroup W_0 , i.e. $W_0 = W$. We have $\chi = \chi_s = |\psi|^s$ for $\tilde{\omega}_u(x; s)$ (cf. (2.5), (2.6)), and instead of Theorem 2.5, we have the following.

Theorem 2.6. *Assume (A1), (A2), (A3) and (A4) for (\mathbb{B}, \mathbb{X}) and s is generic. For $x \in \mathcal{R}$, one has*

$$(\tilde{\omega}_\nu(x; s))_{\nu \in \mathcal{U}} = \frac{1}{Q} \sum_{\sigma \in W} \gamma(\sigma(s)) \cdot B_\sigma(s) \cdot \left(\int_U |f(u \cdot x)|_\nu^{\sigma(s)} du \right)_{\nu \in \mathcal{U}}.$$

Moreover, if $x \in \mathcal{R}^+$, one has

$$(\tilde{\omega}_\nu(x; s))_{\nu \in \mathcal{U}} = \frac{1}{Q} \sum_{\sigma \in W} \gamma(\sigma(s)) \cdot B_\sigma(s) \cdot \left(|f(x)|_\nu^{\sigma(s)} \right)_{\nu \in \mathcal{U}}.$$

Here the constant Q and the rational function $\gamma(s) = \gamma(|\psi|^s)$ of q^{s_1}, \dots, q^{s_n} are determined by the group G as in §2.1, and the matrix $\tilde{B}_\sigma(s)$ is determined by the functional equation

$$(\tilde{\omega}_\nu(x; s))_{\nu \in \mathcal{U}} = \tilde{B}_\sigma(s) (\tilde{\omega}_\nu(x; \sigma(s)))_{\nu \in \mathcal{U}}.$$

§3

We follow the previous notations, and assume that (\mathbb{B}, \mathbb{X}) satisfies (A1), (A2) and (A3). In this section we give a condition to assure the existence of functional equations for $\sigma \in W$ attached to a simple root α , and explain how the functional equations are reduced to those of p -adic local zeta functions of small prehomogeneous vector spaces of limited type. A basic reference is [H6], where we assumed (A4) also.

3.1. For a simple root α whose associated reflection $\sigma = \sigma_\alpha$ belongs to W_0 , denote by \mathbb{P} the standard parabolic subgroup $\mathbb{P}_{\{\alpha\}}$ in the sense of [Bo]-21.11, and consider the following condition.

(A5) There exists a k -rational representation $\rho: \mathbb{P} \rightarrow R_{k'/k}(GL_2)$ satisfying

$$\begin{aligned} \rho(\mathbb{P}) &= R_{k'/k}(GL_2) \text{ or } R_{k'/k}(SL_2), \quad \rho(\sigma) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (= \mathbf{j}, \text{ say}), \\ \rho^{-1}(\mathbb{B}_2) &\subset \mathbb{B}, \quad \rho(K \cap \mathbb{P}) \supset R_{k'/k}(SL_2)(\mathcal{O}), \end{aligned} \quad (3.1)$$

where k' is a finite unramified extension of k , $R_{k'/k}$ is the restriction functor of base field, and \mathbb{B}_2 is the Borel subgroup of $\rho(\mathbb{P})$ consisting of upper triangular matrices.

Chevalley groups are typical examples which have ρ as above for $k = k'$ for each simple root (cf. [Sf2]-§4.1.). As for $R_{k'/k}$, we note an example in §4.3.

For each $u \in J(X)$, we set $J_u = \{\nu \in J(X) \mid P \cdot X_\nu = P \cdot X_u\}$. Denote by e the group index $[\mathfrak{X}(\mathbb{B}) \cap (\mathfrak{X}_0(\mathbb{B}) \otimes_{\mathbb{Z}} \mathbb{Q}) : \mathfrak{X}_0(\mathbb{B})]$, by d the extension degree of k'/k , and let ε_σ be the same as in (2.4). Our first main result of this section is the following (cf. [H6]-Theorem 2.6).

Theorem 3.1. *We assume (A1), (A2), (A3) for (\mathbb{B}, \mathbb{X}) . For a simple root whose associated reflection σ belongs to W_0 , we assume (A5), and keep the notations above. Then, there exists a functional equation*

$$\omega_u(x; s) = \frac{1 - q^{-2d - \sum_i e_i s_i}}{1 - q^{-2d - \sum_i e_i (\sigma(s)_i + \varepsilon_i)}} \times \sum_{\nu \in J_u} \gamma_{u\nu}(s) \cdot \omega_\nu(x; \sigma(s) + \varepsilon_\sigma),$$

where ε_i is the i -th component of ε_σ , $\gamma_{u\nu}(s)$'s are rational functions of $q^{\frac{s_i}{e}}$, and e_i is defined in (3.5).

Hereafter we assume (\mathbb{B}, \mathbb{X}) satisfies (A1), (A2) and (A3) and \mathbb{P} satisfies (A5). In order to prove the above theorem and explain about gamma-factors, we introduce the following space.

Set $\tilde{\mathbb{X}} = \mathbb{X} \times \mathbb{V}$ with $\mathbb{V} = R_{k'/k}(M_{21})$ and $\tilde{\mathbb{P}} = \mathbb{P} \times R_{k'/k}(GL_1)$, and define the action

$$(p, t) \cdot (x, v) = (p \cdot x, \rho(p)vt^{-1}), \quad (p, t) \in \tilde{\mathbb{P}}, (x, v) \in \tilde{\mathbb{X}}. \quad (3.2)$$

Here we identify k' with its image by the regular representation in $M_d(k)$ (with respect to a fixed basis for k'/k) and realize $R_{k'/k}(GL_2)$ (resp. \mathbb{V}) in $GL_{2d}(\bar{k})$ (resp. $M_{2d,d}(\bar{k})$), where \bar{k} is the algebraic closure of k . Then we may identify as $\tilde{P} = P \times GL_1(k')$ and $V = k'^2$. Further we regard \mathbb{B} as a subgroup of $\tilde{\mathbb{P}}$ by the embedding

$$\mathbb{B} \longrightarrow \tilde{\mathbb{P}}, \quad b \longrightarrow (b, \rho(b)_1), \quad (3.3)$$

where $\rho(b)_1$ is the upper left d by d block of $\rho(b) \in R_{k'/k}(GL_2)$. Then one can identify \mathbb{B} as the stabilizer subgroup of $\tilde{\mathbb{P}}$ at $v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in V$, i.e.

$$\mathbb{B} \cong \tilde{\mathbb{P}}_{v_0} = \left\{ (p, t) \in \tilde{\mathbb{P}} \mid \rho(p)v_0t^{-1} = v_0 \right\}. \quad (3.4)$$

Then we have the following (cf. [H6]-Lemma 1.1, Proposition 1.2).

Proposition 3.2. (i) *One has the following isomorphism:*

$$\begin{aligned} \mathfrak{X}(\tilde{\mathbb{P}}) \cong \mathfrak{X}(\mathbb{P}) \times \mathfrak{X}(R_{k'/k}(GL_1)) &\longrightarrow \mathfrak{X}(\mathbb{B}) \\ (\psi_1, \psi_2) &\longmapsto [p \mapsto \psi_1(p)\psi_2(\rho(p)_1)]. \end{aligned}$$

(ii) *The space $(\tilde{\mathbb{P}}, \tilde{\mathbb{X}})$ satisfies (A1), (A2) and (A3). Further, if (\mathbb{B}, \mathbb{X}) satisfies also (A4), then so does $(\tilde{\mathbb{P}}, \tilde{\mathbb{X}})$, and $[\mathfrak{X}(\mathbb{B}) : \mathfrak{X}_0(\mathbb{B})] = [\mathfrak{X}(\tilde{\mathbb{P}}) : \mathfrak{X}_0(\tilde{\mathbb{P}})]$.*

(iii) *The set of open B -orbits in X corresponds bijectively to the set of open \tilde{P} -orbits in \tilde{X} by the map $B \cdot x \mapsto \tilde{P} \cdot (x, v_0)$.*

Let $\left\{ \tilde{f}_i(x, v) \mid 1 \leq i \leq n \right\}$ be the basic set of relative $\tilde{\mathbb{P}}$ -invariants, which are regular on $\tilde{\mathbb{X}}$ and satisfy $f_i(x) = \tilde{f}_i(x, v_0)$. Since $\tilde{f}_i(x, v)$ is a relative $R_{k'/k}(GL_1)$ -invariant with respect to the action on v , it is homogeneous in the coordinates of v over k , and we set

$$e_i = \deg_v \tilde{f}_i(x, v), \quad 1 \leq i \leq n. \quad (3.5)$$

We denote by $\tilde{\psi}_i$ the character corresponding to $\tilde{f}_i(x, v)$, then $\psi_i = \tilde{\psi}_i|_{\mathbb{B}}$ for each i . For each $u \in J(X)$ denote by \tilde{X}_u the \tilde{P} -orbit corresponding to X_u by Proposition 3.2, then we have

$$X^{op} = \bigsqcup_{u \in J(X)} X_u, \quad \tilde{X}^{op} = \bigsqcup_{u \in J(X)} \tilde{X}_u.$$

Further we see

$$\tilde{X}_u = \bigsqcup_{\ell \in \mathbb{Z}} (1, \pi^\ell) \cdot \widetilde{X_{u,0}}, \quad \widetilde{X_{u,0}} = \bigsqcup_{h \in SL_2(\mathcal{O}')/\Gamma} (\tilde{h}, 1) \cdot (X_u \times \{v_0\}), \quad (3.6)$$

where \mathcal{O}' is the ring of integers in k' , $\tilde{h} \in K \cap \mathbb{P}$ satisfying $\rho(\tilde{h}) = h$ for each $h \in SL_2$ and $\Gamma = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in SL_2(\mathcal{O}') \mid a \in \mathcal{O}' \right\}$.

Denote by $\mathcal{S}(X)$ and $\mathcal{S}(\tilde{X})$ the spaces of Schwartz-Bruhat functions on X and \tilde{X} , respectively. For $s \in \mathbb{C}^n$ and $u \in J(X)$, we consider the following integrals, which we call zeta integrals,

$$\begin{aligned} \Omega_u(\phi; s) &= \int_X \phi(x) \cdot |f(x)|_u^s dx, & \phi &\in \mathcal{S}(X), \\ \tilde{\Omega}_u(\tilde{\phi}; s) &= \int_{\tilde{X}} \tilde{\phi}(x, v) \cdot \left| \tilde{f}(x, v) \right|_u^s dx dv, & \tilde{\phi} &\in \mathcal{S}(\tilde{X}), \end{aligned}$$

where dx is a G -invariant measure on X , dv is a Haar measure on V , $|f(x)|_u^s$ is defined in (1.1), and

$$\left| \tilde{f}(x, v) \right|_u^s = \prod_{i=1}^n \left| \tilde{f}_i(x, v) \right|^{s_i}, \quad \left| \tilde{f}(x, v) \right|_u^s = \begin{cases} \left| \tilde{f}(x, v) \right|^s & \text{if } (x, v) \in \tilde{X}_u \\ 0 & \text{otherwise.} \end{cases}$$

The above integrals are absolutely convergent for $\text{Re}(s_i) \geq 0$, $1 \leq i \leq n$, and analytically continued to rational functions of q^{s_i} , $1 \leq i \leq n$. We have the following, where the assertion (i) is clear, and the assertion (ii) follows from (3.6).

Lemma 3.3. (i) Let ch_x be the characteristic function of $K \cdot x$ in $\mathcal{S}(X)$, then

$$\omega_u(x; s) = v(K \cdot x)^{-1} \cdot \Omega_u(ch_x; s), \quad x \in X, u \in J(X), \quad (3.7)$$

where $v(K \cdot x)$ is the volume of $K \cdot x$ by the above measure dx .

(ii) Let $\tilde{\phi} = \phi \otimes ch_{V(m)}$, where $\phi \in \mathcal{S}(K \backslash X)$ and $ch_{V(m)}$ is the characteristic function of $V(\pi^m \mathcal{O}_{k'})$ in $\mathcal{S}(V)$. Then

$$\tilde{\Omega}_u(\tilde{\phi}; s) = c \cdot \frac{q^{-m(2d + \sum_i e_i s_i)}}{1 - q^{-2d - \sum_i e_i s_i}} \times \Omega_u(\phi; s), \quad u \in J(X), \quad (3.8)$$

where c is a constant depending only on the normalization of measures, and independent of u .

In order to study the action of σ on s for $\tilde{\Omega}_u(\tilde{\phi}; s)$, we define the partial Fourier transform \mathcal{F} on $\mathcal{S}(\tilde{X})$ by

$$\mathcal{F}(\tilde{\phi})(x, v) = \int_V \eta({}^t v \mathbf{j} w) \tilde{\phi}(x, w) dw,$$

where η is an additive character on k' of conductor ℓ , and consider the following distribution on $\mathcal{S}(\tilde{X})$

$$T_{u,s}(\tilde{\phi}) = \tilde{\Omega}_u(\tilde{\phi}; s), \quad T_{u,s}^*(\tilde{\phi}) = T_{u,s}(\mathcal{F}(\tilde{\phi})).$$

We examine the relative invariancy of these distribution concerning the action of \tilde{P} on $\mathcal{S}(\tilde{X})$, where $\tilde{p}\tilde{\phi}(x, v) = \tilde{\phi}(\tilde{p}^{-1} \cdot (x, v))$, $\tilde{p} \in \tilde{P}$. We note the action of σ on characters in the following (cf. [H6]-Lemma 2.1).

Lemma 3.4. (i) For a character $\xi \in \mathfrak{X}(\tilde{\mathbb{P}})$,

$$\sigma(\xi)(p, t) = \xi(p, \frac{\det \rho(p)}{t}), \quad (p, t) \in \tilde{P}.$$

$$(ii) \left| \tilde{\psi}(p, t) \right|^{\varepsilon_\sigma} = |N_{k'/k}(t^2 \det \rho(p)^{-1})|, \quad (p, t) \in \tilde{P}.$$

Proposition 3.5. The distributions $T_{u,s}^*$ and T_{u,s^*} with $s^* = \sigma(s) + \varepsilon_\sigma$ have the same relative invariancy with respect to the action of \tilde{P} .

Proof. First we obtain

$$\begin{aligned} T_{u,s}^{(p,t)}(\tilde{\phi}) &= \int_{X \times V} \tilde{\phi}(p^{-1} \cdot x, \rho(p)^{-1}vt) \left| \tilde{f}(x, v) \right|^s dx dv \\ &= |N_{k'/k}(t^{-2} \det \rho(p))| \int_{X \times V} \tilde{\phi}(x, v) \left| \tilde{f}(p \cdot x, \rho(p)vt^{-1}) \right|^s dx dv \\ &= \left| \tilde{\psi}(p, t) \right|^{s-\varepsilon_\sigma} T_{u,s}(\tilde{\phi}), \end{aligned}$$

where we use Lemma 3.4(ii) and G -invariancy of dx . Next, since $\mathbf{j}\rho(p) = \det \rho(p)^t \rho(p)^{-1} \mathbf{j}$, we have

$$\begin{aligned} \mathcal{F}^{(p,t)}(\tilde{\phi})(x, v) &= \int_V \eta({}^t v \mathbf{j} w) \tilde{\phi}(p^{-1} \cdot x, \rho(p)^{-1}wt) dw \\ &= \left| \tilde{\psi}(p, t) \right|^{-\varepsilon_\sigma} \int_V \eta({}^t (\rho(p)^{-1} v \frac{\det \rho(p)}{t}) \mathbf{j} w) \tilde{\phi}(p^{-1} \cdot x, w) dw \\ &= \left| \tilde{\psi}(p, t) \right|^{-\varepsilon_\sigma} \left((p, \frac{\det \rho(p)}{t}) \mathcal{F}(\tilde{\phi}) \right) (x, v). \end{aligned}$$

By the above calculation together with Lemma 3.4 (i), we obtain

$$T_{u,s}^*(\tilde{\phi}) = \left| \tilde{\psi}(p, t) \right|^{-\varepsilon_\sigma + \sigma(s-\varepsilon_\sigma)} T_{u,s}(\tilde{\phi}) = \left| \tilde{\psi}(p, t) \right|^{\sigma(s)} T_{u,s}^*(\tilde{\phi}).$$

□

Because of the uniqueness of the relatively invariant distribution on homogeneous space(cf. [Ig]-Proposition 7.2.1), we have the following identity

$$T_{u,s}^*(\tilde{\phi}) = \sum_{\nu \in J(X)} \gamma_{u,\nu}^\eta(s) T_{\nu,s^*}(\tilde{\phi}), \quad \tilde{\phi} \in \mathcal{S}(\tilde{X}^{op}), \quad (3.9)$$

where $\gamma_{u,\nu}^\eta(s)$ is a constant independent of $\tilde{\phi}$. Since $T_{u,s}(\tilde{\phi})$ and $T_{u,\nu}^*(\tilde{\phi})$ are continued to rational functions of q^{s_1}, \dots, q^{s_n} and $[\mathfrak{X}(\tilde{\mathbb{P}}) \cap (\mathfrak{X}_0(\tilde{\mathbb{P}}) \otimes_{\mathbb{Z}} \mathbb{Q}) : \mathfrak{X}_0(\tilde{\mathbb{P}})] = e$, the above $\gamma_{u,\nu}^\eta(s)$ are rational functions of $q^{s_1/e}, \dots, q^{s_n/e}$.

On the other hand, under our assumption, essentially by (A1) and (A3), it is known that there is no nonzero distribution for generic s whose support is contained in $\tilde{X} \setminus \tilde{X}^{op}$ and relative invariance for \tilde{P} is $|\tilde{\psi}|^s$, where ‘generic’ means the same as in Lemma 2.4 (cf. [H6]-(F5), [SF3]-Lemma 2.3, Corollary 2.4). Hence the identity (3.9) holds for any $\tilde{\phi} \in \mathcal{S}(\tilde{X})$. Finally, if $\tilde{\phi} \in \mathcal{S}(\tilde{X})$ is zero outside of $P \cdot X_u \times V$, then so does $\mathcal{F}(\tilde{\phi})$, hence we see

$$\gamma_{u,\nu}^\eta(s) = 0 \quad \text{unless } P \cdot X_u = P \cdot X_\nu, \quad \text{i.e., } \nu \in J_u.$$

Thus we obtain the following theorem.

Theorem 3.6. *There exist rational functions $\gamma_{uv}^\eta(s)$ of $q^{\frac{s_1}{e}}, \dots, q^{\frac{s_n}{e}}$, which satisfy the following functional equation :*

$$\tilde{\Omega}_u(\mathcal{F}(\tilde{\phi}); s) = \sum_{\nu \in J_u} \gamma_{uv}^\eta(s) \cdot \tilde{\Omega}_\nu(\tilde{\phi}; \sigma(s) + \varepsilon_\sigma), \quad \tilde{\phi} \in \mathcal{S}(\tilde{X}).$$

We note here that $\gamma_{uv}^\eta(s)$ depends on the choice of the character η and the normalization of dv on V , since $\mathcal{F}(\tilde{\phi})$ does. Let normalize dv on V to be self dual with respect to the inner product $(v, w) \mapsto \eta(\ell v \mathbf{j} w)$, so $\text{vol}(\mathbb{V}(\mathcal{O})) = q^{\ell d}$.

Corollary 3.7. *For any $\phi \in \mathcal{S}(K \setminus X)$, we have*

$$\Omega_u(\phi; s) = \frac{1 - q^{-2d - \sum_i e_i s_i}}{1 - q^{-2d - \sum_i e_i (\sigma(s)_i + \varepsilon_i)}} \times \sum_{\nu \in J_u} \gamma_{uv}(s) \cdot \Omega_\nu(\phi; \sigma(s) + \varepsilon_\sigma),$$

where ε_i is the i -th component of ε_σ and

$$\gamma_{uv}(s) = q^{\ell(d + \sum_i e_i s_i)} \cdot \gamma_{uv}^\eta(s),$$

which is independent of the choice of the character η on k' .

Now Theorem 3.1 follows from Corollary 3.7 and Lemma 3.3.

3.2. In this subsection we look at $\mathbb{V} = R_{k'/k}(M_{21})$ together with the action of $\rho(\mathbb{P}_x) \times R_{k'/k}(GL_1)$ for $x \in X^{op}$. First observation is the following ([H6]-Lemma 3.1).

Lemma 3.8. *For each $x \in X^{op}$, $(\rho(\mathbb{P}_x) \times R_{k'/k}(GL_1), \mathbb{V})$ is a prehomogeneous vector space defined over k . Further, for $v \in V$, $\rho(P_x)v k'^{\times}$ is open in V if and only if $\tilde{P} \cdot (x, v)$ is open in \tilde{X} .*

For each $u \in J(X)$, fix an element $x_u \in X_u$ and denote by \mathbb{P}_u the stabilizer of x_u in \mathbb{P} . Then we obtain ([H6]-Lemma 3.2)

Lemma 3.9. (i) *For any $u, \nu \in J(X)$, prehomogeneous vector spaces $(\mathbb{P}_\nu \times R_{k'/k}(GL_1), \mathbb{V})$ and $(\mathbb{P}_u \times R_{k'/k}(GL_1), \mathbb{V})$ are isomorphic. If $\nu \in J_u$, they are isomorphic over k .*

(ii) *The set of k -rational points of the open orbit in $(\rho(\mathbb{P}_u) \times R_{k'/k}(GL_1), \mathbb{V})$ decomposes as*

$$(\rho(\mathbb{P}_u)v_0 R_{k'/k}(GL_1))(k) = \bigsqcup_{\nu \in J_u} \rho(P_u p_\nu)v_0 k'^{\times}, \quad (3.10)$$

where $p_\nu \in P$ satisfying $p_\nu^{-1} \cdot x_u \in X_\nu$.

For $\tilde{\phi} = \phi_1 \otimes \phi_2$ with $\phi_1 \in \mathcal{S}(X)$ and $\phi_2 \in \mathcal{S}(V)$, we have

$$\mathcal{F}(\tilde{\phi}) = \phi_1 \otimes \mathcal{F}_V(\phi_2),$$

where

$$\mathcal{F}_V(\phi_2)(v) = \int_V \eta({}^t v \mathbf{j} w) \phi_2(w) dw.$$

Taking these $\tilde{\phi}$ in Theorem 3.6 and pulling out the V -part, we obtain the following theorem (cf. [H6]-Theorem 3.3) which shows that the functional equations of spherical functions $\omega_u(x; s)$ are reduced to those for “small” prehomogeneous vector spaces, and oppositely gamma factors can be calculated from those of these prehomogeneous vector spaces.

Theorem 3.10. *The prehomogeneous vector space $(\mathbb{P}_u \times GL_1, \mathbb{V})$ has the following functional equation:*

$$\begin{aligned} & \int_V \mathcal{F}_V(\phi)(v) \left| \tilde{f}(x_u, v) \right|_u^s dv \\ &= \sum_{\nu \in J_u} \gamma_{u\nu}^\eta(s) \int_V \phi(v) \left| \tilde{f}(x_u, v) \right|_\nu^{\sigma(s) + \varepsilon_\sigma} dv, \quad \phi \in \mathcal{S}(V), \end{aligned}$$

where the gamma factors $\gamma_{u\nu}^\eta(s)$ are the same as those for $\tilde{\Omega}_u(\tilde{\phi}; s^*)$ in Theorem 3.6.

Because of the existence of the functional equations of the above type, we see the following (cf. [H6]-Theorem 3.6).

Theorem 3.11. *For the prehomogeneous vector space $(\rho(\mathbb{P}_u) \times R_{k'/k}(GL_1), \mathbb{V})$, the identity component of $\rho(\mathbb{P}_u) \times R_{k'/k}(GL_1)$ is isomorphic to $R_{k'/k}(GL_1 \times GL_1)$ over the algebraic closure \bar{k} of k .*

3.3. If the condition (A4) is also satisfied by (\mathbb{B}, \mathbb{X}) , we should consider $\tilde{\omega}_u(X; s)$. In this subsection, we assume (A1), (A2), (A3) and (A4) for (\mathbb{B}, \mathbb{X}) . We assume (A5) for a simple root α whose associated reflection σ , and keep the notations before. Then we have Theorem 3.12 instead of Theorem 3.6, and based on it, we obtain Theorem 3.13 and Theorem 3.14, these are the original formulation in [H6]. We do not need to modify Theorem 3.11.

Theorem 3.12. *There exist rational functions $\tilde{\gamma}_{u\nu}^\eta(s)$ of $q^{\frac{s_1}{e}}, \dots, q^{\frac{s_n}{e}}$, which satisfy the following functional equation :*

$$\tilde{\Omega}'_u(\mathcal{F}(\tilde{\phi}); s) = \sum_{\nu \in J_u} \tilde{\gamma}_{u\nu}^\eta(s) \cdot \tilde{\Omega}'_\nu(\tilde{\phi}; \sigma(s)), \quad \tilde{\phi} \in \mathcal{S}(\tilde{X}),$$

where

$$\tilde{\Omega}'_u(\tilde{\phi}; s) = \int_{\tilde{X}} \tilde{\phi}(x, v) \cdot \left| \tilde{f}(x, v) \right|_u^{s + \varepsilon_0} dx dv.$$

Theorem 3.13. *Then exists a functional equation*

$$\tilde{\omega}_u(x; s) = \frac{1 - q^{-2d - \sum_i e_i(s_i + \varepsilon_i)}}{1 - q^{-2d - \sum_i e_i(\sigma(s)_i + \varepsilon_i)}} \times \sum_{\nu \in J_u} \tilde{\gamma}_{u\nu}(s) \cdot \tilde{\omega}_\nu(x; \sigma(s)),$$

where ε_i is the i -th component of ε_0 , $\tilde{\gamma}_{u\nu}(s)$'s are rational functions of $q^{\frac{s_i}{e}}$, and e_i is defined in (3.5).

Theorem 3.14. *The prehomogeneous vector space $(\mathbb{P}_u \times GL_1, \mathbb{V})$ has the following functional equation:*

$$\begin{aligned} & \int_V \mathcal{F}_V(\phi)(v) \left| \tilde{f}(x_u, v) \right|_u^{s+\varepsilon_0} dv \\ &= \sum_{\nu \in J_u} \tilde{\gamma}_{u\nu}^\eta(s) \int_V \phi(v) \left| \tilde{f}(x_u, v) \right|_\nu^{\sigma(s)+\varepsilon_0} dv, \quad \phi \in \mathcal{S}(V), \end{aligned}$$

where the gamma factors $\tilde{\gamma}_{u\nu}^\eta(s)$ are the same as those for $\tilde{\Omega}'_u(\tilde{\phi}; s)$ in Theorem 3.12.

Remark 3.15. We recall Remark 1.4. Assume \mathcal{U} is a subset containing J_u and canonically identified with a subgroup of $k^{\times n} / \prod_i \psi_i(B)$, and denote by $\hat{\mathcal{U}}$ the character group of \mathcal{U} . We may define similarly $\Omega(\phi; \chi; s)$ and $\tilde{\Omega}(\tilde{\phi}; \chi; s)$ for $\chi \in \hat{\mathcal{U}}$. Then, instead of Theorem 3.1, we have

$$\omega(x; \chi; s) = \frac{1 - q^{-2d - \sum_i e_i(s_i)}}{1 - q^{-2d - \sum_i e_i(\sigma(s_i) + \varepsilon_i)}} \times \sum_{\xi \in \hat{\mathcal{U}}} A_{\chi\xi}(s) \omega(x; \xi; \sigma(s) + \varepsilon_\sigma), \quad (3.11)$$

where

$$\begin{aligned} A_{\chi\xi}(s) &= \frac{1}{\#\mathcal{U}} \sum_{u, \nu \in \mathcal{U}} \chi(u) \bar{\xi}(\nu) \gamma_{u\nu}(s), \quad \gamma_{u\nu}(s) = 0 \text{ unless } \nu \in J_u, \\ \gamma_{u\nu}(s) &= \frac{1}{\#\mathcal{U}^2} \sum_{\chi, \xi \in \hat{\mathcal{U}}} \bar{\chi}(u) \xi(\nu) A_{\chi\xi}(s). \end{aligned}$$

We have a similar formula for $\tilde{\omega}(x; \chi; s)$.

§4

We prepare some notations. For a matrix $x \in M_n$, we denote by $d_i(x)$ is the determinant of upper left i by i block of x , by x_{ij} the (i, j) -component of x , and $x_i = x_{ii}$. We set $J_n = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix} \in GL_n$, i.e., the matrix whose anti-diagonal components are 1 and 0 elsewhere.

Set $\Lambda_n = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \dots \geq \lambda_n\}$ and $\Lambda_n^+ = \{\lambda \in \Lambda_n \mid \lambda_n \geq 0\}$.

In the following, we may take $K = \mathbb{G}(\mathcal{O})$, where \mathcal{O} is the ring of integers in k , each \mathbb{X} is a symmetric space except the one in §4.5, which is a spherical homogeneous space, and (\mathbb{B}, \mathbb{X}) satisfies (A1) and (A2). In each case, the open orbit \mathbb{X}^{op} is given as the non-vanishing set of basic relative invariants, and one can consider spherical functions with character, since $J(X)$ has a canonical group structure (cf. Remark 1.4).

4.1. The space of symmetric forms.

$\mathbb{G} = GL_n, \mathbb{X} = \{x \in \mathbb{G} \mid {}^t x = x\}, g \cdot x = gx^t g$.

\mathbb{B} is the Borel group consisting of lower triangular matrices in \mathbb{G} .

$f_i(x) = d_i(x), \psi_i(p) = (p_1 \cdots p_i)^2, 1 \leq i \leq n, W = W_0 \cong S_n$.

$J(X) \cong (k^\times / k^{\times 2})^n (\cong (\mathbb{Z}/4\mathbb{Z})^n \text{ if } 2 \notin (\pi))$.

(\mathbb{B}, \mathbb{X}) satisfies (A1)–(A4).

We may take representatives of $K \backslash X$ in \mathcal{R}^+ , since $Diag(a_1, \dots, a_n) \in X$ with $v_\pi(a_1) \leq \dots \leq v_\pi(a_n)$ is contained in \mathcal{R}^+ .

The condition (A5) is satisfied for each simple root, indeed for the transposition $(\alpha \ \alpha + 1)$, $1 \leq \alpha \leq n - 1$, we have

$$\mathbb{P} = \mathbb{P}_\alpha = \{(p) \in \mathbb{G} \mid p_{ij} = 0 \text{ unless } i \geq j \text{ or } (i, j) = (\alpha, \alpha + 1)\},$$

$$\rho : \mathbb{P} \longrightarrow GL_2, \ p \longmapsto \begin{pmatrix} p_{\alpha+1, \alpha+1} & -p_{\alpha+1, \alpha} \\ -p_{\alpha, \alpha+1} & p_{\alpha, \alpha} \end{pmatrix}.$$

The small prehomogeneous vector spaces are of type $(O(T) \times GL_1, V)$ for some symmetric matrix T of size 2.

Functional equations with respect to the Weyl group have been known by a different method based on the explicit expressions of spherical functions of size 2 ([H1]-III), and one can apply Theorem 2.5 formally, but good expressions of spherical functions are not known for general $n \geq 3$, only partial results are known([H1]).

4.2. The space of alternating forms.

$\mathbb{G} = GL_{2n}, \mathbb{X} = \{x \in \mathbb{G} \mid {}^t x = -x\}, g \cdot x = gx^t g.$

\mathbb{B} = the Borel group consisting of lower triangular matrices in \mathbb{G} .

$f_i(x) = \text{pf}_i(x), \psi_i(p) = p_1 \cdots p_{2i}, 1 \leq i \leq n$, where $\text{pf}_i(x)$ is the phaffian of the upper left $2i$ by $2i$ block of x .

$W = S_{2n} \supset W_0 \cong S_n$, in fact for each $\sigma \in S_n$, we associate $w_\sigma \in W$ such that $w_\sigma(2i - 1) = 2\sigma(i) - 1, w_\sigma(2i) = 2\sigma(i), 1 \leq i \leq n$.

X^{op} is a single B -orbit. As a set of complete representatives of $K \backslash X$, we may take

$$\left\{ \pi^\lambda \mid \lambda \in \Lambda_n \right\}, \quad \pi^\lambda = \begin{pmatrix} 0 & \pi^{\lambda_1} \\ -\pi^{\lambda_1} & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & \pi^{\lambda_n} \\ -\pi^{\lambda_n} & 0 \end{pmatrix} \in X,$$

and $J_{2n} \cdot \pi^\lambda \in \mathcal{R}^+$.

The explicit formula of $\omega(x; s)$ was calculated by another method ([HS1]), and it can be reproduced also by Theorem 2.5. We introduce a new variable z related to s by

$$s_i = -z_i + z_{i+1} - 2 \quad (1 \leq i \leq n - 1), \quad s_n = -z_n + n - 1$$

and write $\omega(x; z) = \omega(x; s)$. For each $\lambda \in \Lambda_n$

$$\omega(\pi^\lambda; s) = c_\lambda \cdot \prod_{1 \leq i < j \leq n} \frac{1 - q^{z_i - z_j - 1}}{1 - q^{z_i - z_j + 1}} \cdot P_\lambda(q^{z_1}, \dots, q^{z_n}; q^{-2}),$$

where c_λ is an explicitly given constant in $\mathbb{Q}(q^{-1})$ and P_λ is a Hall-Littlewood symmetric polynomial (a symmetric Laurent polynomial of q^{z_1}, \dots, q^{z_n}). The Hall-Littlewood polynomial $P_\lambda(x; t)$ is defined as follows (cf. [Ma2])

$$P_\lambda(x; t) = P_\lambda(x_1, \dots, x_n; t) = \frac{(1-t)^n}{w_\lambda(t)} \cdot \sum_{\sigma \in S_n} x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(n)}^{\lambda_n} \prod_{i < j} \frac{x_{\sigma(i)} - tx_{\sigma(j)}}{x_{\sigma(i)} - x_{\sigma(j)}},$$

$$w_\lambda(t) = \prod_{j=1}^r \prod_{i=1}^{n_i} (1 - t^i), \quad \text{for } \lambda = (\ell_1^{n_1} \cdots \ell_r^{n_r}), \ell_1 > \cdots > \ell_r, n_1 + \cdots + n_r = n,$$

where the set $\{P_\lambda(x; t) \mid \lambda \in \Lambda_n^+\}$ forms a $\mathbb{Z}[t]$ -basis for $\mathbb{Z}[t][x_1, \dots, x_n]^{S_n}$, and the set $\{P_\lambda(x; t) \mid \lambda \in \Lambda_n\}$ forms a $\mathbb{Z}[t]$ -basis for $\mathbb{Z}[t][x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{S_n}$.

Setting

$$\Psi(x; z) = \omega(x; z) / \omega(\pi^{\mathbf{0}}; z), \quad \mathbf{0} = (0, \dots, 0) \in \Lambda_n,$$

we have the spherical transform which is a surjective $\mathcal{H}(G, K)$ -module homomorphism

$$\mathcal{S}(K \backslash X) \longrightarrow \mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^{S_n}, \quad \phi \longmapsto \int_X \phi(x) \Psi(x; z) dx$$

where dx is G -invariant measure on X and $\mathcal{H}(G, K)$ acts on the right hand side via $\lambda_z(\phi) = \lambda_s(\phi)$, a specialization of Satake transform $\mathcal{H}(G, K) \cong \mathbb{C}[q^{\pm t_1}, \dots, q^{\pm t_{2n}}]^{S_{2n}}$.

Each spherical function on X is associated to some $z \in \mathbb{C}^n/S_n$ through λ_z , and it is a constant multiple of $\Psi(x; z)$.

4.3. The space of hermitian forms.

For a quadratic extension k'/k with involution $*$, we consider spherical functions on the space of hermitian forms $X = \{x \in GL_n(k') \mid x^* = x\}$ with canonical action of $G = GL_n(k')$, where (i, j) -component of g^* for $g = (g_{ij}) \in G$ is g_{ji}^* . We have to realize these objects as the sets of k -rational points: taking $u \in k'$ such $k' = k(u)$ and $u^2 \in k$, we identify k' with the image of the inclusion

$$k' \longrightarrow M_2(k), \quad a + bu \longmapsto \begin{pmatrix} a & bu^2 \\ b & a \end{pmatrix}$$

and realize $\mathbb{G} = R_{k'/k}(GL_n)$ and the space \mathbb{X} of hermitian forms in the following.

$$\begin{aligned} \mathbb{G} &= \left\{ (g_{ij}) \in GL_{2n}(\bar{k}) \mid g_{ij} = \begin{pmatrix} a_{ij} & b_{ij}u^2 \\ b_{ij} & a_{ij} \end{pmatrix} \in M_2(\bar{k}) \quad (1 \leq i, j \leq n) \right\}, \\ \mathbb{X} &= \left\{ (x_{ij}) \in \mathbb{G} \mid x_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x_{ji} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1 \leq i, j \leq n) \right\}, \\ \mathbb{B} &= \{g = (g_{ij}) \in \mathbb{G} \mid g_{ij} = 0 \text{ unless } n \geq i \geq j \geq 1\}. \end{aligned}$$

Then $G = \mathbb{G}(k) = GL_n(k')$, $X = \mathbb{X}(k) = \{x \in G \mid x^* = x\}$, $g \cdot x = gxg^*$.

$B = \mathbb{B}(k)$ is the Borel subgroup of G consisting of lower triangular matrices.

$f_i(x) = d_i(x) \in k$ for $x \in X$, and $\psi_i(p) = N_{k'/k}(p_1 \cdots p_i)$ for $p \in B$, $1 \leq i \leq n$.

$W = W_0 \cong S_n$. $J(X) \cong (k^\times / N_{k'/k}(k'^\times))^n \cong (\mathbb{Z}/2\mathbb{Z})^n$.

(\mathbb{B}, \mathbb{X}) satisfies (A1) – (A4).

Functional equations with respect to the Weyl group have been known by a different method based on the explicit expressions of spherical functions of size 2 ([H1]-III), and one applies Theorem 2.5 formally.

Theorem 2.5 (of its original formulation) was used to obtain the explicit formula for the case unramified hermitian forms ([H3]-§2). We consider the spherical function of type (1.5)

$$\omega(x; z) = \omega(x; s) = \int_K |f(k \cdot x)|^{s+\varepsilon} dk,$$

where $\varepsilon = (-1, \dots, -1, -\frac{n-1}{2}) + (\frac{\pi\sqrt{-1}}{\log q}, \dots, \frac{\pi\sqrt{-1}}{\log q}) \in \mathbb{C}^n$, and z is the new variable related to s by

$$s_i = -z_i + z_{i+1} \quad (1 \leq i \leq n-1), \quad s_n = -z_n.$$

Though we are shifting the variable s here, $\omega(x; z)$ is the same as before. We have the functional equations

$$\omega(x; z) = \prod_{\substack{1 \leq i < j \leq n \\ \sigma(i) > \sigma(j)}} \frac{q^{z_{\sigma(i)}} - q^{z_{\sigma(j)}-1}}{q^{z_{\sigma(j)}} - q^{z_{\sigma(i)}-1}} \times \omega(x; \sigma(z)), \quad \sigma \in S_n. \quad (4.1)$$

A set of complete representatives of $K \backslash X$ can be taken as

$$\left\{ \pi^\lambda = \text{Diag}(\pi^{\lambda_1}, \dots, \pi^{\lambda_n}) \mid \lambda \in \Lambda_n \right\}$$

and $J_n \cdot \pi^\lambda \in \mathcal{R}^+$. The explicit formula is given as

$$\omega(\pi^\lambda; z) = c_\lambda \cdot \prod_{1 \leq i < j \leq n} \frac{1 - q^{z_i - z_j - 1}}{1 + q^{z_i - z_j}} \cdot P_\lambda(q^{z_1}, \dots, q^{z_n}; -q^{-1}),$$

where c_λ is an explicitly given constant in $\mathbb{Q}(q^{-1})$ and P_λ is a Hall-Little symmetric polynomial (cf. §4.2). Setting

$$\Psi(x; z) = \omega(x; z) / \omega(1_n; z),$$

we have the spherical transform which is an $\mathcal{H}(G, K)$ -module isomorphism

$$\mathcal{S}(K \backslash X) \cong \mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^{S_n}, \phi \longmapsto \int_X \phi(x) \Psi(x; z) dx,$$

where dx is the G -invariant measure on X and $\mathcal{H}(G, K)$ acts on the right hand side via Satake transform $\tilde{\lambda}_z = \tilde{\lambda}_s : \mathcal{H}(G, K) \cong \mathbb{C}[q^{\pm 2z_1}, \dots, q^{\pm 2z_n}]^{S_n}$. In particular $\mathcal{S}(K \backslash X)$ is a free $\mathcal{H}(G, K)$ -module of rank 2^n .

Each spherical function on X is associated to some $z \in \mathbb{C}^n / S_n$ through $\tilde{\lambda}_z$, and the space of spherical functions associated to $\tilde{\lambda}_z$ has dimension 2^n and a basis

$$\left\{ \Psi(x; z + \epsilon) \mid \epsilon \in \left\{ 0, \frac{\pi\sqrt{-1}}{\log q} \right\}^n \right\}.$$

For a simple root associated to the transposition $(\alpha \ \alpha+1)$, $1 \leq \alpha \leq n-1$, there is a representation ρ satisfying (A5), similarly given to the case of symmetric form, but we have to use $R_{k'/k}(GL_2)$ in order to define ρ over k (cf. [H6]-§4.2). The small prehomogeneous vector spaces are of type $(U(T) \times GL_1, V)$ for some hermitian matrix T of size 2.

Assume that k'/k is ramified and take a prime element π' of k' as $\pi'^2 = -\pi$. Then some representatives of $K \backslash X$ contain a matrix of type $\begin{pmatrix} 0 & \pi'^{2m+1} \\ -\pi'^{2m+1} & 0 \end{pmatrix}$ as a direct summand, and they are not in \mathcal{R}^+ , functional equations of spherical functions are much more complicated (cf. [H1]-III), and no good expressions of spherical functions are known for general $n \geq 3$.

4.4.

(i) For a nondegenerate symmetric matrix A of size n , we set $O(A) = \{g \in GL_n \mid A[g] = A\}$ and $SO(A) = O(n) \cap SL_n$ where $A[g] = {}^t g A g$. Set

$$\mathbb{G} = SO(H_n), H_n = \frac{1}{2} \begin{pmatrix} & 1_n \\ 1_n & \end{pmatrix}, \mathbb{B} = \left\{ \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \mathbb{G} \mid \begin{array}{l} b_i \in M_n, b_3 = 0, \\ b_1 \text{ is upper triangular} \end{array} \right\}.$$

In the following we consider the set of k -rational points. Fix a symmetric matrix $T \in GL_r(k)$, set

$$\mathfrak{X}_T = \{x \in M_{2n,r} \mid H_n[x] = T\}, \quad X_T = \mathfrak{X}_T / O(T) \ni \bar{x} = xO(T),$$

and consider them as \mathbb{G} -spaces by left multiplication. The stabilizer of

$$\bar{x}_T \in X_T, \quad x_T = {}^t(T \ 0_{n-r,r} \ 1_r \ 0_{n-r,r}) \in \mathfrak{X}_T$$

is isomorphic to $S(O(T) \times O(T \perp H_{n-r}))$, so the space X_T is isomorphic to $SO(2n) / S(O(r) \times O(2n-r))$ over the algebraic closure of k .

A set of basic B -relative invariants and associated characters are given as follows

$$f_{T,i}(\bar{x}) = d_i(T^{-1}[{}^t x_2]) = d_i(x_2 T^{-1} x_2), \quad \psi_i(p) = (p_1 \cdots p_i)^{-2}, \quad 1 \leq i \leq r,$$

where x_2 is the submatrix of $x \in \mathfrak{X}_T$ consisting of its $(n+i)$ -th row, $1 \leq i \leq r$, in order, and p_i is the i -th diagonal component of $p \in B$. So $\text{rank}(\mathfrak{X}_0(B)) = r$, whereas $\text{rank}(\mathfrak{X}(B)) = n$.

$W = S_n \times C_2^{n-1} \supset W_0 \cong S_r \times C_2^{r-1}$, and $J(X_T) \cong (k^\times/k^{\times 2})^{r-1}$.

Functional equations with respect to the S_r -part are reduced to the case of symmetric forms. When $r = n$, the condition (A5) is satisfied for each simple root and small prehomogeneous vector spaces are isomorphic to $(O(t) \times GL_1, V)$ for some symmetric matrix t of size 2, for details see [H6]-§4-3. The spherical functions on this space with respect to the Siegel parabolic subgroup have a close relation to Siegel singular series (cf. [HS4]).

For odd $2n+1$, we may start with

$$\mathbb{G} = SO(H_n), \quad H_n = \begin{pmatrix} & & & 1_n \\ & & & \\ & & 1 & \\ & & & \\ 1_n & & & \end{pmatrix} \in GL_{2n+1},$$

and consider the space for symmetric $T \in GL_r(k)$

$$X_T = \mathfrak{X}_T/O(T), \quad \mathfrak{X}_T = \{x \in M_{2n+1,r} \mid H_n[x] = T\}.$$

Relative invariants are given similarly to the even case, and we have $W = S_n \times C_2^n \supset W_0 \cong S_r \times C_2^r$.

(ii) For a nondegenerate hermitian matrix A of size n , we set $U(A) = \{g \in GL_n \mid A[g] = g\}$, where $A[g] = g^*Ag$ and g^* is the same as in §4.3. In the following we write by the set of k -rational points for simplicity of notations (cf. §4.3). Setting

$$G = U(H_n), \quad H_n = \begin{pmatrix} & 1_n \\ 1_n & \end{pmatrix}, \quad B = \left\{ \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in G \mid \begin{array}{l} b_i \in M_n, ; b_3 = 0 \\ b_1 \text{ is upper triangular} \end{array} \right\},$$

define the G -spaces \mathfrak{X}_T and $X_T = \mathfrak{X}_T/U(T)$ similarly to the case (i) for hermitian $T \in GL_r(k')$. Then,

$F_{T,i}(\bar{x}) = d_i(T^{-1}[x_2^*])$, $\psi_i(p) = N_{k'/k}(p_1 \cdots p_i)^{-1}$, $1 \leq i \leq r$, $J(X_T) \cong (k^\times/N_{k'/k}(k'^{\times}))^{r-1}$.
 $\text{rank}(\mathfrak{X}(\mathbb{B})) = n$, $\text{rank}(\mathfrak{X}_0(\mathbb{B})) = r$, and $W = S_n \times C_2^n \supset W_0 = S_r \times C_2^r$.

Functional equations with respect to the S_r -part are reduced to the case of hermitian forms, so we know well if k'/k is unramified (cf. §4.3).

Let us assume $r = n$ and k'/k is unramified, and consider the spherical function on X_T of type (1.5)

$$\omega_T(\bar{x}; z) = \omega_T(\bar{x}; s) = \int_K |f_T(kx)|^{s+\varepsilon} dk,$$

where $\varepsilon = (-1, \dots, -1, -\frac{1}{2}) + (\frac{\pi\sqrt{-1}}{\log q}, \dots, \frac{\pi\sqrt{-1}}{\log q}) \in \mathbb{C}^n$, and z is the new variable related to s by

$$s_i = -z_i + z_{i+1} \quad (1 \leq i \leq n-1), \quad s_n = -z_n.$$

This $\omega_T(\bar{x}; z)$ satisfies the same functional equation as in (4.1) with respect to S_n , independent of the choice of T . Further we obtain

$$\omega_T(\bar{x}; z) = |2|_k^{2z_n} \omega_T(\bar{x}; \tau(z)), \quad \tau(z) = (z_1, \dots, z_{n-1}, -z_n), \quad (4.2)$$

and we have functional equations with respect to whole W by cocycle relations. The parabolic subgroup attached to τ does not have the representation satisfying (A5) and the above functional equation (4.2) does not come from prehomogeneous vector space. Explicit formulas of $\omega_T(\bar{x})$ for some particular points are obtained by using these functional equations and Theorem 2.6.

Similar to the case (i), spherical functions on this space with respect to the Siegel parabolic subgroup have a close relation to hermitian Siegel series, for details see [H7].

4.5. $Sp_2 \times (Sp_1)^2$ -space Sp_2 (cf. [H5]).

We assume that k has odd residual characteristic.

Setting

$$Sp_n = \{x \in GL_{2n} \mid H_n[x] = H_n\}, \quad H_n = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}, \quad A[x] = {}^t xAx,$$

we embed $(Sp_1)^2 = (SL_2)^2$ into Sp_2 by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right) \mapsto \left(\begin{array}{c|c} a & b \\ \hline e & f \\ c & d \\ \hline g & h \end{array} \right),$$

and define the action of $\mathbb{G} = Sp_2 \times (Sp_1)^2$ on $\mathbb{X} = Sp_2$

$$\tilde{g} \cdot x = g_1 x^t g_2, \quad \tilde{g} = (g_1, g_2) \in \mathbb{G}, \quad x \in \mathbb{X}.$$

Then this space \mathbb{X} is not a symmetric space, but a spherical homogeneous space, whereas $Sp_{2n}/(Sp_n \times Sp_n)$ -space Sp_{2n} is no longer spherical for $n \geq 2$ (and hence there is no open Borel orbit in it).

We take the Borel subgroup of \mathbb{G} consisting of matrices of type

$$b = \left(\begin{array}{cc|c} * & * & * \\ 0 & * & \\ \hline & & b_1 \ 0 \\ 0 & & * \ b_2 \end{array} \right), \left(\begin{array}{cc|c} b_3 & 0 & 0 \\ 0 & b_4 & \\ \hline * & 0 & * \ 0 \\ 0 & * & 0 \ * \end{array} \right) \in \mathbb{G},$$

then a set of basic relative \mathbb{B} -invariants on \mathbb{X} and associated characters of \mathbb{B} are given as

$$\begin{aligned} f_1(x) &= x_{31}, & \psi_1(b) &= b_1 b_3, \\ f_2(x) &= x_{32}, & \psi_2(b) &= b_1 b_4, \\ f_3(x) &= x_{31} x_{42} - x_{32} x_{41}, & \psi_3(b) &= b_1 b_2 b_3 b_4, \\ f_4(x) &= x_{31} x_{43} - x_{41} x_{33}, & \psi_4(b) &= b_1 b_2, \end{aligned}$$

where x_{ij} is the (i, j) -component of x and b_i comes from the above expression of $b \in \mathbb{B}$. $J(X) \cong k^\times/k^{\times 2}$, and (\mathbb{B}, \mathbb{X}) satisfies (A1) – (A4).

We consider the spherical function of type (1.5)

$$\tilde{\omega}(x; \chi; z) = \tilde{\omega}(x; \chi; s) = \int_K \chi(f(k \cdot x)) |f(k \cdot x)|^{s+\varepsilon} dk,$$

where χ is a character of $k^\times/k^{\times 2}$, $\varepsilon = (-\frac{1}{2}, \dots, -\frac{1}{2})$, and z is related to s by

$$\begin{aligned} z_1 &= s_1 + s_2 + s_3 + s_4, & z_2 &= s_3 + s_4, \\ z_3 &= s_1 + s_3, & z_4 &= s_2 + s_3. \end{aligned}$$

$W = W_0 \cong (S_2 \times (C_2)^2) \times (S_2)^2$, and (A5) is satisfied for each simple root. Calculating functional equations by using zeta integrals of type $\tilde{\Omega}(\tilde{\phi}, \chi; s)$ (in Remark 3.15), we have explicit formulas of spherical functions on X . By spherical transform we see $\mathcal{S}(K \backslash X)$ is a free $\mathcal{H}(G, K)$ -module of rank 4. Each spherical function is associated to some $z \in \mathbb{C}^4/W$ through $\tilde{\lambda}_z$, and the space of spherical functions associated to $\tilde{\lambda}_z$ has dimension 4 and there is a basis explicitly given by terms of $\omega(x; \chi; z)$. The small prehomogeneous vector spaces are isomorphic to $(GL_1 \times GL_1, V)$ over k (cf. [H6]-§4.1), and those functional equations are reduced to Tate's formula ([Ta]-§2).

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Yumiko HIRONAKA

Department of Mathematics
Faculty of Education and Integrated Sciences
Waseda University
Nishi-Waseda, Tokyo 169-8050, JAPAN

e-mail : hironaka@waseda.jp