CHAPTER 2

Stable commutator length

Many natural problems in topology and geometric group theory can be formulated as a kind of *genus problem.* In the absolute version of this problem, one is given a space X and tries to find a surface in X with prescribed properties, of least genus. Examples of the kind of properties one wants for the surface are that it represent a given class in $H_2(X)$, that it is a Heegaard surface (in a 3-manifold), that $\pi_1(X)$ splits nontrivially over its image, that it is pseudoholomorphic, etc. In the relative version one is given X and a loop γ in X and tries to find a surface (again with prescribed properties) of least genus with boundary γ. In its purest form, the analogue of this second problem in group theory asks to determine the *commutator length* of an element in the commutator subgroup of a group, and it is this problem (or rather its stabilization) with which we are preoccupied in this chapter (we give precise definitions in § 2.1). We will use the algebraic and geometric language interchangeably in what follows; however our *methods* and *arguments* are mostly geometric.

There is a dual formulation of these problems, in terms of (bounded) cohomology and *quasimorphisms* — real-valued functions on a group which are additive, up to bounded error. This duality is expressed in the fundamental *Bavard Duality theorem* from [**8**], which gives a precise relationship between (stable) commutator length and bounded cohomology, and reconciles the homotopy theoretic and the (co)-homological points of view of surfaces and the genus problem. The main goal of this chapter is to give a self-contained exposition of this fundamental result and some generalizations, including all the necessary background and details. Our aim is to keep the presentation elementary wherever possible, although certain arguments are streamlined by using the language of abstract functional analysis.

In many places we follow Bavard's original paper [**8**], though occasionally our emphasis is different. We also enumerate and prove some useful properties of scl and bounded cohomology which are used in subsequent chapters.

2.1. Commutator length and stable commutator length

Definition 2.1. Let G be a group, and $a \in [G, G]$. The *commutator length* of a, denoted $\mathrm{cl}(a)$, is the least number of commutators in G whose product is equal to a.

By convention we define $\mathrm{cl}(a) = \infty$ for a not in $[G, G]$.

Definition 2.2. For $a \in [G, G]$, the *stable commutator length*, denoted $\mathrm{scl}(a)$, is the following limit:

$$\mathrm{scl}(a) = \lim_{n \to \infty} \frac{\mathrm{cl}(a^n)}{n}$$

For each fixed a, the function $n \to \mathrm{cl}(a^n)$ is non-negative and subadditive; hence this limit exists. If a is not in $[G,G]$ but has a power a^n which is, define $\mathrm{scl}(a) = \mathrm{scl}(a^n)/n$, and by convention define $\mathrm{scl}(a) = \infty$ if and only if a represents a nontrivial element in $H_1(G;\mathbb{Q})$.

Remark 2.3. Computing commutator length is almost always difficult, even in finite groups. Ore [**164**] famously conjectured in 1951 that every element of a finite non-cyclic simple group is a commutator, and proved his conjecture for alternating groups A_n where $n \geq 5$. After receiving considerable attention (see e.g. [**72, 121**]), Ore's conjecture was finally proved in 2008 by Liebeck–O'Brien–Shalev–Tiep [**135**].

Commutator length in free groups has been studied by many people, with effective (though inefficient) procedures for calculating commutator length first obtained by Edmunds [**68, 69**]. The use of geometric methods to study genus was pioneered by Culler [**59**]. Several authors ([**98, 99, 178**]) used minimal surface techniques to obtain estimates of commutator length under geometric hypotheses.

Thurston [**196**], studied the absolute genus problem in the context of *embedded* surfaces in 3-manifolds, and showed how a stabilization of this problem gives rise to a *norm* on homology with several remarkable properties. Gromov [**99**] also emphasized the importance of stabilization, and posed a number of very general problems about genus and stable genus, especially their interaction with negative curvature. Gromov further stressed the relationship between the stable genus problem and bounded cohomology, which he systematically introduced and studied in [**97**]. This connection was also studied by Matsumoto and Morita; the paper [**150**] describes a fundamental relationship between homological "filling" norms and the kernel of the natural map from bounded to ordinary cohomology.

The most important property of cl and scl is their *monotonicity* under homomorphisms:

Lemma 2.4 (monotonicity). *Let* $\varphi : G \to H$ *be a homomorphism of groups. Then* $\mathrm{scl}_H(\varphi(a)) \leq \mathrm{scl}_G(a)$ *for all* $a \in G$ *and similarly for* cl.

Proof. The image of a commutator under a homomorphism is a commutator. It follows that both cl and scl are monotone decreasing. □

The following corollaries are immediate:

Corollary 2.5 (retraction). *Let* $\varphi : G \to H$ *be a monomorphism with a left inverse; i.e. there is* $\psi : H \to G$ *with* $\psi \circ \varphi : G \to G$ *the identity. Then*

$$\mathrm{scl}(\varphi(a)) = \mathrm{scl}(a)$$

for all $a \in G$.

Corollary 2.6 (characteristic). *The functions* cl *and* scl *are constant on orbits of* $\mathrm{Aut}(G)$.

Remark 2.7. Corollary 2.6 is especially interesting when $\mathrm{Out}(G)$ is large.

For most interesting phenomena concerning scl, it suffices to restrict attention to *countable* groups, as the following Lemma shows.

Lemma 2.8 (countable). *Let* G *be a group, and* $a \in G$ *an element. Then there is a countable subgroup* $H < G$ *containing* a, *such that* $\mathrm{scl}_H(a) = \mathrm{scl}_G(a)$.

Proof. For each n, exhibit a^n as a product of $\mathrm{cl}(a^n)$ commutators in G, and let H_n be the subgroup generated by the elements appearing in these commutators. Then let H be the subgroup generated by $\cup_n H_n$. □

The algebraic definitions of cl and scl are almost useless for the purposes of computation. Products and powers of commutators satisfy many identities which at first glance might appear quite mysterious.

EXAMPLE 2.9 (Culler [**59**]). For any elements a, b in any group, there is an identity

$$[a,b]^3 = [aba^{-1}, b^{-1}aba^{-1}][b^{-1}ab, b^2]$$

These properties are often more clear from a geometric perspective (for instance, Example 2.9 is really just Remark 1.13 in disguise). Given a group G, one can construct a space X (for example, a CW complex) with $\pi_1(X) = G$. A conjugacy class $a \in G$ corresponds to a free homotopy class of loop γ in X. From the definitions and the discussion in § 1.1.5 it follows that the commutator length of a is the least genus of a surface with one boundary component mapping to X in such a way that the boundary represents the free homotopy class of γ, and the stable commutator length of a may be obtained by estimating the genus of surfaces whose boundary wraps multiple times around γ.

Once we have recast this problem in geometric terms, a number of facts become immediately apparent:

1. genus is not multiplicative under coverings whereas Euler characteristic is
2. there is no good reason to restrict attention to surfaces with exactly one boundary component

As in § 1.2.5, given a (not necessarily connected) compact oriented surface S, let $-\chi^-(S)$ denote the sum of $\max(-\chi(\cdot), 0)$ over the components of S. Given a space X and a loop $\gamma : S^1 \to X$ we say that a map $f : S \to X$ is *admissible* if there is a commutative diagram:

$$\begin{tikzcd}
\partial S \arrow[r, "i"] \arrow[d, "\partial f"'] & S \arrow[d, "f"'] \\
S^1 \arrow[r, "\gamma"'] & X
\end{tikzcd}$$

Since S is oriented, the boundary of S inherits an orientation, and it makes sense to define the fundamental class $[\partial S]$ in $H_1(\partial S)$. Similarly, one has a fundamental class $[S^1] \in H_1(S^1)$. Define $n(S)$ by the formula

$$\partial f_*[\partial S] = n(S)[S^1]$$

Note that by orienting S appropriately, we can ensure that $n(S) \geq 0$. The number $n(S)$ is just the (total algebraic) *degree* of the map $\partial S \to S^1$ between oriented closed manifolds.

With this notation, one can give an intrinsically geometric definition of scl, which is contained in the following proposition.

PROPOSITION 2.10. *Let $\pi_1(X) = G$, and let $\gamma : S^1 \to X$ be a loop representing the conjugacy class of $a \in G$. Then*

$$\mathrm{scl}(a) = \inf_S \frac{-\chi^-(S)}{2n(S)}$$

where the infimum is taken over all admissible maps as above.

Proof. An inequality in one direction is obvious: $\mathrm{cl}(a^n) \le g$ if and only if there is an admissible map $f : S \to X$, where S has exactly one boundary component and satisfies $n(S) = n$ and $2g - 1 = -\chi^-(S)$. Hence $\lim_n \mathrm{cl}(a^n)/n \ge \inf_S -\chi^-(S)/2n(S)$.

Conversely, suppose $f : S \to X$ is admissible. If S has multiple components, at least one of them S_i satisfies $-\chi^-(S_i)/2n(S_i) \le -\chi^-(S)/2n(S)$, so without loss of generality we can assume S is connected. Since $-\chi^-(\cdot)$ and $2n(\cdot)$ are both multiplicative under covers, we can replace S with any finite cover without changing their ratio, so we may additionally assume that S has $p \ge 2$ boundary components.

As in Lemma 1.12, we can find a finite cover $S' \to S$ of degree $N \gg 1$ such that S' also has p boundary components. Observe that $-\chi^-(S') = -N\chi^-(S)$ and $n(S') = Nn(S)$. We may modify S' by attaching 1-handles to connect up the different boundary components, and extend $\partial f'$ over these 1-handles by a trivial map to a basepoint of S^1. Adding a 1-handle increases genus by 1 and reduces the number of boundary components by 1, so it increases $-\chi^-$ by 1. The result of this is that we can find a new surface S'' with *exactly one* boundary component and a map f'' satisfying $-\chi^-(S'') = -\chi^-(S') + p - 1$ and $n(S'') = n(S')$. We estimate

$$\frac{-\chi^-(S'')}{2n(S'')} = \frac{p - 1 - N\chi^-(S)}{2Nn(S)}$$

Since S is arbitrary, and given S the number p is fixed but N may be taken to be as large as desired, the right hand side may be taken to be arbitrarily close to $\inf_S -\chi^-(S)/2n(S)$. On the other hand, since the genus of S'' may be chosen to be as large as desired, and since S'' has exactly one boundary component, we have $\mathrm{cl}(a^{n(S'')}) \le -2\chi^-(S'') + 1$. The proof follows. □

Notice that for any element a of infinite order, we have an inequality $\mathrm{scl}(a) \le \mathrm{cl}(a^n)/n - 1/2n$. It follows that *no* surface can realize the infimum of $\mathrm{cl}(a^n)/n$. On the other hand, it is entirely possible for a surface to realize the infimum of $-\chi^-(S)/2n(S)$. Such surfaces are sufficiently useful and important that they deserve to be given a name.

Definition 2.11. A surface $f : S \to X$ realizing the infimum of $-\chi^-(S)/2n(S)$ is said to be *extremal*.

We will return to extremal surfaces in § 4.1.10.

At this point it is convenient to state and prove another proposition about the kinds of admissible surfaces we need to consider.

Definition 2.12. An admissible map $f : S, \partial S \to X, \gamma$ is *monotone* if for every boundary component ∂_i of ∂S, the degree of $\partial f : \partial_i \to S^1$ has the same sign.

Proposition 2.13. *Let S be connected with $\chi(S) < 0$, and let $f : S, \partial S \to X, \gamma$ be admissible. Then there is a* monotone *admissible map $f' : S', \partial S' \to X, \gamma$ with $-\chi^-(S')/2n(S') \le -\chi^-(S)/2n(S)$.*

Proof. Each boundary component ∂_i of ∂S maps to S^1 with degree n_i (which may be positive, negative or zero), where $\sum_i n_i = n(S)$. If some n_i is zero, the image $f(\partial_i)$ is homotopically trivial in X, so we may reduce $-\chi^-$ by compressing ∂_i. Hence we may assume every n_i is nonzero.

If S is a planar surface, then since $\chi(S) < 0$, there is a finite cover of S with positive genus. If S is a surface with positive genus and negative Euler characteristic, there is a degree 2 cover $S' \to S$ such that each boundary component in S has exactly two preimages. Hence, after passing to a finite cover if necessary, we can assume that the boundary components ∂_i come in *pairs* with equal degrees n_i.

Now let N be the least common multiple of the $|n_i|$. Define ϕ as a function on the set of boundary components with values in $\mathbb{Z}/N\mathbb{Z}$ as follows. For each pair of boundary components ∂_i, ∂_j with $n_i = n_j$, define $\phi(\partial_i) = n_i$ and $\phi(\partial_j) = -n_i$. Then $\sum_i \phi(\partial_i) = 0$, so ϕ extends to a surjective homomorphism from $\pi_1(S)$ to $\mathbb{Z}/N\mathbb{Z}$. If S' is the cover associated to the kernel, then each component of $\partial S'$ has degree $\pm N$. Pairs of components for which the sign of the degree is opposite can be glued up (which does not affect χ or $n(\cdot)$) until all remaining components have degrees with the same signs. ☐

Consequently it suffices to take the infimum of $-\chi^-/2n$ over monotone surfaces to determine scl.

Remark 2.14. Note that the surface constructed in Proposition 2.13 is not merely monotone, but has the property that all boundary components map with the *same* degree.

2.2. Quasimorphisms

We now have two different definitions of stable commutator length: an algebraic definition and a (closely related) topological definition. It turns out that one can also give a *functional analysis* definition, couched not directly in terms of groups and elements, but dually in terms of certain kinds of functions on groups, namely *quasimorphisms*. This particular form of duality is known as *Bavard duality*; the precise statement of this duality is Theorem 2.70.

2.2.1. Definition.

Definition 2.15. Let G be a group. A *quasimorphism* is a function

$$\phi : G \to \mathbb{R}$$

for which there is a least constant $D(\phi) \ge 0$ such that

$$|\phi(ab) - \phi(a) - \phi(b)| \le D(\phi)$$

for all $a, b \in G$. In words, *a quasimorphism is a real-valued function which is additive up to bounded error.* The constant $D(\phi)$ is called the *defect* of ϕ.

Example 2.16. Any bounded function is a quasimorphism. A quasimorphism has defect 0 if and only if it is a homomorphism.

Lemma 2.17. *Let S be a (possibly infinite) generating set for G. Let w be a word in the generators, representing an element of G. Let $|w|$ denote the length of w, and let w_i denote the ith letter. Then*

$$\Big|\phi(w) - \sum_{i=1}^{|w|} \phi(w_i)\Big| \le (|w| - 1)D(\phi)$$

Proof. This follows from the defining property of a quasimorphism, the triangle inequality, and induction. ☐

The set of all quasimorphisms on a fixed group G is easily seen to be a (real) vector space; we denote this vector space by $\widehat{Q}(G)$. In anticipation of what is to come, we denote the space of (real-valued) bounded functions on G by $C_b^1(G)$, and observe that C_b^1 is a vector subspace of $\widehat{Q}$.

2.2.2. Antisymmetric and homogeneous quasimorphisms. Some quasimorphisms are better behaved than others.

Definition 2.18. A quasimorphism ϕ is *antisymmetric* if

$$\phi(a^{-1}) = -\phi(a)$$

for all a. Any quasimorphism ϕ can be *antisymmetrized* $\phi \to \phi'$ by the formula

$$\phi'(a) = \frac{1}{2}(\phi(a) - \phi(a^{-1}))$$

Lemma 2.19. *For any quasimorphism ϕ, the antisymmetrization ϕ' satisfies*

$$D(\phi') \le D(\phi)$$

Proof. We calculate

$$D(\phi') = \sup_{a,b} |\phi'(ab) - \phi'(a) - \phi'(b)|$$

$$= \sup_{a,b} \frac{1}{2}|\phi(ab) - \phi(a) - \phi(b) - \phi(b^{-1}a^{-1}) + \phi(a^{-1}) + \phi(b^{-1})| \le D(\phi)$$

□

Observe that for any antisymmetric quasimorphism ϕ there is an inequality

$$|\phi([a,b])| = |\phi(aba^{-1}b^{-1}) - \phi(a) - \phi(b) - \phi(a^{-1}) - \phi(b^{-1})| \le 3D(\phi)$$

and in general (by Lemma 2.17), $|\phi(\prod_{i=1}^n [a_i,b_i])| \le (4n-1)D(\phi)$.

Definition 2.20. A quasimorphism is *homogeneous* if it satisfies the additional property

$$\phi(a^n) = n\phi(a)$$

for all $a \in G$ and $n \in \mathbb{Z}$. Denote the vector space of homogeneous quasimorphisms on G by $Q(G)$.

Lemma 2.21. *Let ϕ be a quasimorphism on G. For each $a \in G$, define*

$$\overline{\phi}(a) := \lim_{n\to\infty} \frac{\phi(a^n)}{n}$$

The limit exists, and defines a homogeneous quasimorphism. Moreover, for any $a \in G$ there is an estimate $|\overline{\phi}(a) - \phi(a)| \le D(\phi)$

Proof. For each positive integer i, there is an inequality

$$|\phi(a^{2^i}) - 2\phi(a^{2^{i-1}})| \le D(\phi)$$

dividing by 2^i and applying the triangle inequality and induction, we see that for any $j < i$,

$$|\phi(a^{2^i})2^j/2^i - \phi(a^{2^j})| \le D(\phi)$$

so $\phi(a^{2^i})2^{-i}$ is a Cauchy sequence. Define $\overline{\phi}(a)$ to be the limit $\lim_{i\to\infty}\phi(a^{2^i})2^{-i}$ and observe that $|\overline{\phi}(a) - \phi(a)| \le D(\phi)$ for all a.

Sine $\overline{\phi} - \phi$ is in C_b^1, we conclude that $\overline{\phi}$ is a quasimorphism. It remains to show that $\overline{\phi}$ is homogeneous. For any j, by the definition of $\overline{\phi}$ we have

$$|\overline{\phi}(a^j) - j\overline{\phi}(a)| = \lim_{i\to\infty} 2^{-i}|\phi(a^{j2^i}) - j\phi(a^{2^i})| \le \lim_{i\to\infty}(j-1)D(\phi)\cdot 2^{-i} = 0$$

where the last inequality follows from Lemma 2.17. □

Remark 2.22. Since $|\overline{\phi}(a)-\phi(a)| \leq D(\phi)$ for any element a, the triangle inequality implies that $D(\overline{\phi}) \leq 4 \cdot D(\phi)$. In fact, a more involved argument (Lemma 2.58) will give a better estimate of the defect.

Homogeneous quasimorphisms are often easier to work with than ordinary quasimorphisms, but ordinary quasimorphisms are easier to construct. We use this averaging procedure to move back and forth between the two concepts. Note that a homogeneous quasimorphism is already antisymmetric, and that homogenization commutes with antisymmetrization.

Remark 2.23. If ϕ takes values in some additive subgroup $R \subset \mathbb{R}$ then the antisymmetrization may take values in $\frac{1}{2}R$, and the homogenization may take arbitrary values in $\mathbb{R}$.

2.2.3. Commutator estimates.

If ϕ is homogeneous, then

$$|\phi(aba^{-1}) - \phi(b)| = \frac{1}{n}|(\phi(ab^na^{-1}) - \phi(b^n))| \leq \frac{2D(\phi)}{n}$$

Hence ϕ is constant on conjugacy classes; i.e. *homogeneous quasimorphisms are class functions.* It follows that for any commutator $[a,b] \in G$ and any homogeneous quasimorphism ϕ we have an inequality

$$|\phi([a,b])| \leq D(\phi)$$

In fact, this inequality is always *sharp*:

Lemma 2.24 (Bavard, Lemma 3.6. [**8**]). *Let ϕ be a homogeneous quasimorphism on G. Then there is an equality*

$$\sup_{a,b} |\phi([a,b])| = D(\phi)$$

Proof. First we show that we can write $a^{2n}b^{2n}(ab)^{-2n}$ as a product of n commutators. If $n=1$ this is just the identity

$$a^2ba^{-1}b^{-1}a^{-1} = a[a,b]a^{-1} = [a, aba^{-1}]$$

Also,

$$a^{2n}b^{2n}(ab)^{-2n} = a(a^{2n-1}b^{2n-1}(ba)^{-2n+1})a^{-1}$$

so it suffices to show that $a^{2n-1}b^{2n-1}(ba)^{-2n+1}$ can be written as a product of n commutators.

We proceed by induction, and assume we have proved this for $n \leq m$. Then

$$\begin{aligned}[][a^{-2m+1}b^{-2m}a^{-2}, ab^{-1}a^{2m-1}] &= a^{-2m+1}b^{-2m}a^{-1}b^{-1}a^{2m+1}b^{2m+1}a^{-1} \\ &= a(a^{-2m}b^{-2m}a^{-1}b^{-1}a^{2m+1}b^{2m+1})a^{-1}\end{aligned}$$

By induction, and after interchanging a and b for a^{-1} and b^{-1}, the expression $a^{-2m}b^{-2m}$ can be written as a product of m commutators times $(a^{-1}b^{-1})^{2m}$. It follows that $(a^{-1}b^{-1})^{2m+1}a^{2m+1}b^{2m+1}$ can be written as a product of $m+1$ commutators, and the induction step is complete, proving the claim.

Now let a, b be chosen so that $|\phi(ab) - \phi(a) - \phi(b)| \geq D(\phi) - \epsilon$ for some small ϵ (to be chosen later). Since ϕ is homogeneous, for any n we have

$$|\phi((ab)^{2n}) - \phi(a^{2n}) - \phi(b^{2n})| \geq 2n(D(\phi) - \epsilon)$$

On the other hand, we have shown that $(ab)^{2n}$ can be expressed as a product of n commutators c_i (which depend on a and b) times $a^{2n}b^{2n}$. Hence by Lemma 2.17,

$$|\phi((ab)^{2n}) - \phi(a^{2n}) - \phi(b^{2n}) - \sum_{i=1}^{n} \phi(c_i)| \le (n+1)D(\phi)$$

By the triangle inequality,

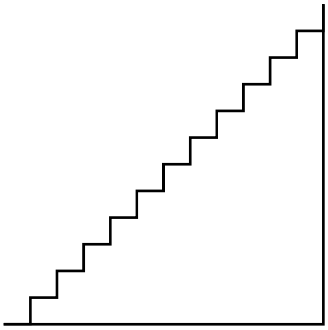
$$|\sum_{i=1}^{n} \phi(c_i)| \ge (n-1)D(\phi) - 2n\epsilon$$

Since $\phi(c_i) \le D(\phi)$ for every commutator, taking n to be big, and then ϵ small compared to $1/n$, we see that some commutator c_i has $\phi(c_i)$ as close to $D(\phi)$ as we like. $\square$

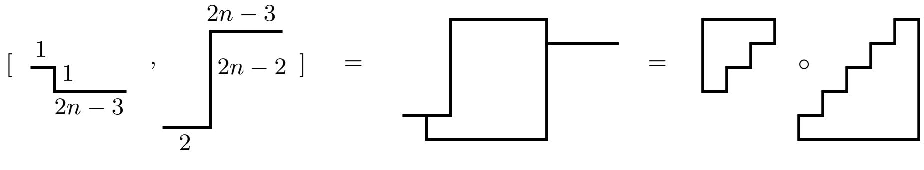
2.2.4. Graphical calculus. The argument that $a^{2n}b^{2n}(ab)^{-2n}$ can be written as a product of n commutators can be expressed more simply in the form of a graphical calculus.

A word w in F_2 determines a path in the square lattice $\mathbb{Z}^2$. Such a path corresponds to a reduced word if and only if it has no backtracking. It represents a commutator in F_2 if and only if it closes up to a loop. If one disregards basepoints, loops correspond to cyclic conjugacy classes of elements in $[F_2, F_2]$.

In this calculus, the word $a^{2n}b^{2n}(ab)^{-2n}$ is represented by the loop indicated in the figure. Note that this word is unreduced: there are two spurious backtracks, each of length 1. After removing these backtracks, one obtains a loop representing the word $a^{2n-1}b^{2n-1}(ba)^{-2n+1}$



Informally, the word $a^{2n-1}b^{2n-1}(ba)^{-2n+1}$ is a "staircase" of height $2n-1$. In this language, the induction step can be expressed as saying that a staircase of height $2n-1$ can be written as the product of a commutator with a staircase of height $2n-3$. Since a staircase of height 1 is just the commutator $[a, b]$, this completes the proof. This can be expressed graphically in the following way:



2.3. Examples

In this section we discuss some fundamental examples of quasimorphisms. These examples can all be generalized considerably, as we shall see in later Chapters.

2.3.1. de Rham quasimorphisms. The following construction is due to Barge–Ghys [**6**].

Let M be a closed hyperbolic manifold, and let α be a 1-form. Define a quasimorphism $q_\alpha : \pi_1(M) \to \mathbb{R}$ as follows. Choose a basepoint $p \in M$. For each $\gamma \in \pi_1(M)$, let L_γ be the unique oriented geodesic arc with both endpoints at p which as a based loop represents γ in $\pi_1(M)$. Then define

$$q_\alpha(\gamma) = \int_{L_\gamma} \alpha$$

If γ_1, γ_2 are two elements of $\pi_1(M)$, there is a geodesic triangle T whose oriented boundary is the union of $L_{\gamma_1}, L_{\gamma_2}, L_{\gamma_2^{-1}\gamma_1^{-1}}$. By Stokes' theorem we can calculate

$$q_\alpha(\gamma_1) + q_\alpha(\gamma_2) - q_\alpha(\gamma_1\gamma_2) = \int_T d\alpha$$

A geodesic triangle in a hyperbolic manifold has area at most π. It follows that the defect of q_α is at most $\pi \cdot \|d\alpha\|$.

Note that the homogenization $\overline{q}_\alpha$ satisfies

$$\overline{q}_\alpha(\gamma) = \int_{l_\gamma} \alpha$$

where l_γ is the *free* geodesic loop corresponding to the *conjugacy* class of γ in $\pi_1(M)$. For, changing the basepoint p changes q_α by a bounded amount, and therefore does not change the homogenization. Then this formula is obviously true when p is chosen (for each γ) so that $L_\gamma = l_\gamma$.

A similar construction makes sense for closed manifolds M of variable negative curvature.

2.3.2. Counting quasimorphisms.

Definition 2.25. Let F be a free group on a symmetric generating set S. Let w be a reduced word in S. The *big counting function* $C_w(g)$ is defined by

$$C_w(g) = \text{number of copies of } w \text{ in the reduced representative of } g$$

and the *little counting function* $c_w(\cdot)$ is defined by

$$c_w(g) = \text{max. number of } \textit{disjoint} \text{ copies of } w \text{ in the reduced representative of } g$$

A *big counting quasimorphism* is a function of the form

$$H_w(g) := C_w(g) - C_{w^{-1}}(g)$$

and a *little counting function* is a function of the form

$$h_w(g) = c_w(g) - c_{w^{-1}}(g)$$

Big counting functions were introduced by Brooks in [**27**]. We sometimes refer to C_w or H_w (and even c_w or h_w) as *Brooks functions* or *Brooks quasimorphisms.* The little counting functions, and variations on them, were introduced by Epstein–Fujiwara [**78**], who generalized them to arbitrary hyperbolic groups (although the big counting functions also generalize easily to hyperbolic groups). These two functions are related, but different, and have different advantages in different situations. We shall see that the big counting quasimorphisms are computationally simpler, and easier to deal with, whereas the little counting quasimorphisms (and their generalizations) have *uniformly* small defects, and are therefore more "powerful".

Remark 2.26. Suppose no proper suffix of w is equal to a proper prefix. Then copies of w in any reduced word are necessarily disjoint, and $h_w = H_w$. Grigorchuk [**95**] uses the terminology "no overlapping property" to describe such words.

Every H_w and h_w is a quasimorphism. In fact, we will explicitly calculate their defects in what follows. First we must prove some preliminary statements.

Lemma 2.27. *Let $u \in F$ be reduced. Copies of w in u are disjoint from copies of w^{-1}.*

Proof. Suppose not, so that without loss of generality some suffix of w is equal to some prefix of w^{-1}. But in this case $w = w_1 w_2$ where $w_2 = w_2^{-1}$ which is impossible. □

Let $u \in F$ be reduced, and let $u = u_1 u_2$ as a reduced expression (i.e. there is no cancellation of the suffix of u_1 with the prefix of u_2). Say that a copy of w intersects the *juncture* of u if it overlaps both the suffix of u_1 and the prefix of u_2. By Lemma 2.27, at most one of w, w^{-1} can intersect the juncture of u.

Definition 2.28. Given a reduced expression $u = u_1 u_2$ and a reduced word w, the *sign* of the expression, denoted s, is

$$s = \begin{cases} 1 & \text{if } w \text{ intersects the juncture} \\ -1 & \text{if } w^{-1} \text{ intersects the juncture} \\ 0 & \text{otherwise} \end{cases}$$

Lemma 2.29. *Let $u = u_1 u_2$ be a reduced expression with sign s. Then*

$$h_w(u) - h_w(u_1) - h_w(u_2) = 0 \text{ or } s$$

and

$$0 \le s(H_w(u) - H_w(u_1) - H_w(u_2)) \le |w| - 1$$

Proof. At most $|w| - 1$ copies of w or w^{-1} can intersect the juncture, proving the second inequality.

To prove the first equality, for $i = 1, 2$ let U_i be a maximal disjoint configuration of copies of w in u_i. Then $U_1 \cup U_2$ is contained in $u_1 u_2$, so $c_w(u) - c_w(u_1) - c_w(u_2) \ge 0$. Conversely, let U be a maximal disjoint configuration of copies of w in $u_1 u_2$. Then either U contains one copy of w which intersects the juncture, or else it is disjoint from the juncture and decomposes as $U = U_1 \cup U_2$. Hence $c_w(u) - c_w(u_1) - c_w(u_2) \le 1$ if $s = 1$ and $c_w(u) - c_w(u_1) - c_w(u_2) \le 0$ otherwise. □

It follows that $D(H_w) \le 3(|w| - 1)$. One cannot do better than $O(|w|)$ in general, as an example like $w = ababababababa$ shows. However, for little counting quasimorphisms, one obtains $D(h_w) \le 3$, and with more work one can find an even sharper estimate.

Proposition 2.30. *Let w be a reduced word. Then*

1. $D(h_w) = 0$ *if and only if* $|w| = 1$
2. $D(h_w) = 2$ *if and only if w is of the form* $w = w_1 w_2 w_1^{-1}$, $w = w_1 w_2 w_1^{-1} w_3$ *or* $w = w_1 w_2 w_3 w_2^{-1}$ *as reduced expressions*
3. $D(h_w) = 1$ *otherwise*

Proof. If $|w| = 1$, the subgroup $\langle w \rangle$ generated by w is a $\mathbb{Z}$ summand of F, and h_w is just projection from F onto this summand; i.e. it is a homomorphism. Otherwise, if $w = w_1 w_2$ is a reduced expression, $h_w(w) = 1$ whereas $h_w(w_1) = h_w(w_2) = 0$. This proves the first statement.

Let $u, v \in F$ be reduced. Then we can uniquely write $u = u'x, v = x^{-1}v'$ where $u'v'$ is the reduced representative of uv. Let s_1, s_2, s_3 be the signs of the reduced expressions $u'x, x^{-1}v', u'v'$ respectively. We calculate

$$\begin{aligned} h_w(uv) - h_w(u) - h_w(v) &= h_w(uv) - h_w(u) - h_w(v) \\ &\quad - h_w(u') + h_w(u') - h_w(v') + h_w(v') + h_w(x) - h_w(x^{-1}) \\ &= (0 \text{ or } s_3) - (0 \text{ or } s_1) - (0 \text{ or } s_2) \end{aligned}$$

After possibly replacing w with w^{-1} and reversing the order of the strings, there are only nine possibilities for (s_1, s_2, s_3):

$$|h_w(uv) - h_w(u) - h_w(v)| \le \begin{cases} 0 & \text{for } (0,0,0) \\ 1 & \text{for } (1,0,0),\ (0,0,1),\ (1,-1,0),\ (1,0,1) \\ 2 & \text{for } (1,1,0),\ (1,1,1),\ (1,0,-1) \\ 3 & \text{for } (1,1,-1) \end{cases}$$

Case $((1,0,-1))$. If w overlaps $u'x$ and w^{-1} overlaps $u'v'$ then either some prefix of w is equal to a substring of w^{-1} or some prefix of w^{-1} is equal to a substring of w. In either case w has the form asserted by bullet (2).

Case $((1,1,s))$. Since w overlaps both $u'x$ and $x^{-1}v'$ we can write $w = w_1 w_2 w_3$ where either $w_2 w_3$ is the prefix of x and $w_1 w_2$ is the suffix of x^{-1} or w_3 is the prefix of x and w_1 is the suffix of x^{-1}. In the first case, $w_2^{-1} w_1^{-1}$ is the prefix of x so $w_2 = w_2^{-1}$ which is absurd. Hence we must be in the second case, and one of w_1^{-1}, w_3 is a prefix of the other.

In either case w has the form asserted by bullet (2), so we are done unless $s = -1$.

Subcase $((1,1,-1))$. Without loss of generality, we can assume w is of the form $w = w_1 w_2 w_3 w_2^{-1}$ where $w_1 w_2 w_3$ is the terminal string of u' and $w_3 w_2^{-1}$ is the initial string of v'. By hypothesis, a copy of $w^{-1} = w_2 w_3^{-1} w_2^{-1} w_1^{-1}$ overlaps $y := w_1 w_2 w_3 w_3 w_2^{-1}$.

By Lemma 2.27, the subword $w_3^{-1} w_2^{-1} w_1^{-1}$ cannot overlap $w_1 w_2 w_3$ in y. Also, the subword $w_2 w_3^{-1}$ of w^{-1} cannot overlap $w_3 w_2^{-1}$ in y. Hence the w_3^{-1} in w^{-1} cannot overlap $w_1 w_2 w_3 w_3 w_2^{-1}$ at all. So if there is any overlap, either the suffix $w_2^{-1} w_1^{-1}$ of w^{-1} intersects the prefix $w_1 w_2$ of y or the prefix w_2 of w^{-1} intersects the suffix w_2^{-1} of y. But neither case can occur, again by Lemma 2.27. Hence this subcase cannot occur.

One can check that if w has the form asserted by bullet (2) then $D(h_w) \ge 2$ by example. This completes the proof. ☐

Example 2.31 (monotone words).

Definition 2.32. A word w is *monotone* if for each $a \in S$, at most one of a and a^{-1} appears in w.

By Proposition 2.30, for any reduced monotone word w, there is an inequality $D(h_w) \le 1$ where $D(h_w) = 1$ whenever $|w| > 1$. Notice that any reduced word of length 2 is monotone.

It is also interesting to study linear combinations of counting quasimorphisms. If w_i is a sequence of words, and t_i is a sequence of real numbers with $\sum_i |t_i| < \infty$ then $\sum_i t_i h_{w_i}$ is a quasimorphism with defect at most $2\sum_i |t_i|$. However, even if $\sum_i |t_i|$ is infinite, the function $\sum_i t_i h_{w_i}$ might still be a quasimorphism.

Definition 2.33. A family of reduced words W is *compatible* if there are words $\overline{u}, \overline{v}$ (possibly left- and right-infinite respectively) so that for each $w \in W$ there is a factorization $w = uv$ (not necessarily unique) for which each u is a suffix of $\overline{u}$ and each v is a prefix of $\overline{v}$.

Proposition 2.34. *Let $\phi = \sum_{w \in W} t(w) h_w$ for some real numbers $t(w)$. Suppose there is a finite T such that for every compatible family $V \subset W$ there is an inequality*

$$\sum_{w \in V} |t(w)| \le T$$

Then ϕ is a quasimorphism with $D(\phi) \le 3T$.

Proof. Given $u = u'x$ and $v = x^{-1}v'$, the size of $\phi(u) + \phi(v) - \phi(uv)$ can be estimated by counting copies of words $w \in W$ which overlap $u'x, x^{-1}v'$ or $u'v'$. The family of words which contribute at each overlap is a compatible family, so the claim follows. ☐

Example 2.35. The function

$$H := H_{aba} + H_{abba} + H_{abbba} + \cdots$$

satisfies $D(H) = 1$ (by monotonicity, and the fact that the big and small counting quasimorphisms are equal for these particular words).

Example 2.36. Let W be the family of all words in a, b (but not their inverses). There are 2^n words of length n. Define $\phi = \sum_{w \in W} 2^{-|w|} |w|^{-1} h_w$. In a compatible family, there are at most n words of length n for each n, so $D(\phi) \le 3$. On the other hand, $\sum_w 2^{-|w|} |w|^{-1} = \sum_n n^{-1} = \infty$.

Remark 2.37. Similar examples and a discussion of limits of sums of quasimorphisms are found in [**95**].

2.3.3. Rotation number.

Poincaré [**167**] introduced rotation numbers in his study of 1-dimensional dynamical systems. Let $\text{Homeo}(S^1)$ denote the group of homeomorphisms of the circle, and $\text{Homeo}^+(S^1)$ its orientation-preserving subgroup. Let G be a subgroup of $\text{Homeo}^+(S^1)$. Let $\widehat{G}$ be the preimage of G in $\text{Homeo}^+(\mathbb{R})$ under the covering projection $\mathbb{R} \to S^1$.

Note that $\widehat{G}$ is a (possibly trivial) central extension of G, and is centralized (in $\text{Homeo}^+(\mathbb{R})$) by the subgroup generated by a translation $Z : x \to x + 1$.

Definition 2.38 (Poincaré's rotation number). Given $g \in \widehat{G}$, define the *rotation number* to be

$$\text{rot}(g) = \lim_{n \to \infty} \frac{g^n(0)}{n}$$

Remark 2.39. Many authors also use the terminology "translation number" or "translation quasimorphism" for rot on $\widehat{G}$.

Rotation number is a quasimorphism:

Lemma 2.40. rot *is a quasimorphism on* $\widehat{G}$.

Proof. Since Z is central, $\text{rot}(Z^n a) = n + \text{rot}(a)$ for all a. Given arbitrary a, b, write $a = Z^n a', b = Z^m b'$ where $0 \le a'(0) < 1$ and $0 \le b'(0) < 1$. Of course this implies $ab = Z^{m+n} a'b'$. Then

$$0 \le \text{rot}(a') + \text{rot}(b') \le 2,\ 0 \le \text{rot}(a'b') \le 2$$

and one obtains the estimate $D(\text{rot}) \le 2$. □

In fact, one can obtain more precise information.

Lemma 2.41. *For all* $p \in \mathbb{R}$ *and* $a, b \in \widehat{G}$ *there is an inequality*

$$p - 2 < [a, b](p) < p + 2$$

Proof. For any p, after multiplying a, b by elements of the center if necessary (which does not change $[a, b]$) we can assume $p \le a(p), b(p) < p+1$. Then we obtain two inequalities

$$p \le a(p) \le ab(p) < a(p+1) < p+2$$
$$p \le b(p) \le ba(p) < b(p+1) < p+2$$

Let $q = ba(p)$. Then from the second inequality we obtain

$$p \le q < p + 2$$

and therefore from the first inequality,

$$q - 2 < p \le ab(p) = aba^{-1}b^{-1}(q) < p + 2 \le q + 2$$

Since p was arbitrary, so was q (up to multiplication by an element of the center). But the center commutes with $aba^{-1}b^{-1}$, so we obtain an inequality

$$q - 2 < aba^{-1}b^{-1}(q) < q + 2$$

valid for any $q \in \mathbb{R}$. This proves the Lemma. □

Remark 2.42. Lemma 2.41 is well-known; the proof given above is essentially the same as that of Proposition 3.1 from [**197**].

It follows that there is an estimate $\text{scl}(a) \ge |\text{rot}(a)|/2$ for any $a \in \widehat{G}$. It turns out that this estimate is sharp.

Theorem 2.43. *Let* $\text{Homeo}^+(\mathbb{R})^{\mathbb{Z}}$ *denote the full preimage of* $\text{Homeo}^+(S^1)$ *in* $\text{Homeo}^+(\mathbb{R})$. *Then* $\text{scl}(a) = |\text{rot}(a)|/2$ *in* $\text{Homeo}^+(\mathbb{R})^{\mathbb{Z}}$.

Proof. Let b be an element which translates some elements in the positive direction and some elements in the negative direction. Then for any $p \in \mathbb{R}$ and any small $\epsilon > 0$, some conjugate of b takes p to $p + 1 - \epsilon$. Similarly, some conjugate of b^{-1} takes $b(p)$ to $b(p) + 1 - \epsilon$. It follows that for any $p \in \mathbb{R}$ and any small $\epsilon > 0$ there is a commutator which takes p to $p + 2 - 2\epsilon$.

Given a with $|\text{rot}(a)| = r$, the power a^n moves every point a distance less than $nr + 1$. It turns out that the estimate in Lemma 2.41 is sharp, in the sense that for any $p \in \mathbb{R}$ and any $|s| < 2$ one can find a commutator g such that $g(p) - p = s$. Therefore a^n can be written as a product of at most $\lfloor (nr + 1)/2 \rfloor + 1$ commutators with an element a' which fixes some point. The dynamics of a' on every complementary interval to $\text{fix}(a')$ is topologically conjugate to a translation of $\mathbb{R}$, which is the commutator of two dilations. Therefore any element a' of $\text{Homeo}^+(\mathbb{R})^{\mathbb{Z}}$ with a fixed point is a commutator. So $\mathrm{cl}(a^n) \le \lfloor (nr+1)/2 \rfloor + 2$. Dividing both sides by n, and taking the limit as $n \to \infty$ we get an inequality $\mathrm{scl}(a) \le |\mathrm{rot}(a)|/2$.

On the other hand, since a^n moves every point a distance at least $nr+1$, and by Lemma 2.41 every commutator moves every point a distance at most 2, we get an inequality $n|\mathrm{rot}(a)| \le 2 \cdot \mathrm{cl}(a^n) + 1$ and therefore $|\mathrm{rot}(a)|/2 \le \mathrm{scl}(a)$. This proves the Theorem. □

See e.g. [**70**] for more details and an extensive discussion.

Remark 2.44. Note that the group $\mathrm{Homeo}^+(S^1)$ is uniformly perfect — every element can be written as a product of at most two commutators. For, every element can be written as a product of two elements both of which have a fixed point, and (as observed in the proof of Theorem 2.43) every element of $\mathrm{Homeo}^+(S^1)$ with a fixed point is a commutator. In fact, a more detailed argument shows that every element of $\mathrm{Homeo}^+(S^1)$ is a commutator.

2.4. Bounded cohomology

2.4.1. Bar complex.

Definition 2.45. Let G be a group. The *bar complex* $C_*(G)$ is the complex generated in dimension n by n-tuples $(g_1,\dots,g_n)$ with $g_i \in G$ and with boundary map ∂ defined by the formula

$$\partial(g_1,\dots,g_n) = (g_2,\dots,g_n) + \sum_{i=1}^{n-1} (-1)^i (g_1,\dots,g_i g_{i+1},\dots,g_n) + (-1)^n (g_1,\dots,g_{n-1})$$

For a coefficient group R, we let $C^*(G;R)$ denote the terms in the dual cochain complex $\mathrm{Hom}(C_*(G),R)$, and let δ denote the adjoint of ∂. The homology groups of $C^*(G;R)$ are called the *group cohomology of G with coefficients in R*, and are denoted $H^*(G;R)$.

If R is a subgroup of $\mathbb{R}$, a cochain $\alpha \in C^n(G)$ is *bounded* if

$$\sup |\alpha(g_1,\dots,g_n)| < \infty$$

where the supremum is taken over all generators. This supremum is called the *norm* of α, and is denoted $\|\alpha\|_\infty$. The set of all bounded cochains forms a subcomplex $C^*_b(G)$ of $C^*(G)$, and its homology is the so-called *bounded cohomology* $H^*_b(G)$.

The norm $\|\cdot\|_\infty$ makes $C^n_b(G)$ into a Banach space for each n. There is a natural function on $H^*_b(G)$ defined as follows: if $[\alpha] \in H^*_b(G)$ is a cohomology class, set

$$\|[\alpha]\|_\infty = \inf \|\sigma\|_\infty$$

where the infimum is taken over all cocycles σ in the class of $[\alpha]$. If the bounded coboundaries $B^n_b(G)$ are a closed subspace of $C^n_b(G)$, this function defines a Banach norm on $H^n_b(G)$. However, it should be pointed out that $B^n_b(G)$ is *not* typically closed in $C^n_b(G)$.

There is an obvious L^1 norm on $C_*(G;\mathbb{R})$ defined in the same way as the Gromov norm for singular chains from Definition 1.11, so these chain groups may be thought of as (typically incomplete) normed vector spaces.

2.4.2. Amenable groups. Let G be a group. Recall that a *mean* on G is a linear functional on $L^\infty(G)$ which maps the constant function $f(g) = 1$ to 1, and maps non-negative functions to non-negative numbers.

DEFINITION 2.46. A group G is *amenable* if there is a G-invariant mean $\pi : L^\infty(G) \to \mathbb{R}$ where G acts on $L^\infty(G)$ by

$$g \cdot f(h) = f(g^{-1}h)$$

for all $g, h \in G$ and $f \in L^\infty(G)$.

Examples of amenable groups are finite groups, solvable groups, and Grigorchuk's groups of intermediate growth.

Bounded cohomology behaves well under amenable covers:

THEOREM 2.47 (Johnson, Trauber, Gromov). *Let*

$$1 \to H \to G \to A \to 1$$

be exact, where A is amenable. Then the natural homomorphisms $H_b^*(G;\mathbb{R}) \to H_b^*(H;\mathbb{R})^A$ are isometric isomorphisms in each dimension.

Here $H_b^*(H;\mathbb{R})^A$ denotes the A-invariant part of $H_b^*(H;\mathbb{R})$ under the action of A on H by outer automorphisms. In particular, if $H_b^*(H;\mathbb{R})$ vanishes, so does $H_b^*(G;\mathbb{R})$. We give the sketch of a proof (also see Proposition 2.65):

PROOF. Replace groups by spaces, so that X is a $K(G,1)$, and $\tilde{X}$ is a $K(H,1)$ thought of as a covering space of X with deck group A. The complex of singular bounded cochains $C_b^*(X)$ on X can be naturally identified with the complex of A-invariant singular bounded cochains $C_b^*(\tilde{X})^A$ on $\tilde{X}$. Since A is amenable, averaging over orbits defines an A-invariant projection $\pi : C_b^*(\tilde{X}) \to C_b^*(X)$. The projection π commutes with the coboundary, and is a left inverse to the pullback homomorphism defined by $\tilde{X} \to X$, and therefore the pullback homomorphism induces an isometric embedding $H_b^*(X) \to H_b^*(\tilde{X})$. The image is clearly contained in $H_b^*(\tilde{X})^A$, and in fact by averaging can be shown to coincide with it.

The proof is completed by showing that bounded group cohomology $H_b^*(G;\mathbb{R})$ is isometrically isomorphic to bounded singular cohomology $H_b^*(K(G,1);\mathbb{R})$ for any G. □

See [**117**] or [**97**] pp. 38–44 for more details.

REMARK 2.48. Theorem 2.47 is only valid for $\mathbb{R}$ coefficients, since the maps depend on averaging, which does not make sense over other coefficient groups. In particular, bounded cohomology over other coefficient groups (e.g. $\mathbb{Z}$) can be nontrivial, and even quite interesting, for some amenable groups.

An important corollary is the case that $G = A$ amenable. Since H_b^* of the trivial group is trivial, this implies that $H_b^*(A;\mathbb{R})$ vanishes identically when A is amenable.

Fibrations with amenable fiber are not so well-behaved, since spectral sequences for bounded cohomology are complicated. However, in dimension two, one has the following useful theorem of Bouarich [**19**]:

THEOREM 2.49 (Bouarich [**19**]). *Let*

$$K \to G \to H \to 1$$

be exact. Then the induced sequence on second bounded cohomology is (left) exact:

$$0 \to H_b^2(H;\mathbb{R}) \to H_b^2(G;\mathbb{R}) \to H_b^2(K;\mathbb{R})$$

In particular, if K is amenable, $H_b^2(H) \to H_b^2(G)$ is an isomorphism. We will give a proof of this theorem in § 2.7.2.

For a more detailed introduction to bounded cohomology, see Gromov's paper [**97**] or either of the references [**115**], [**157**].

2.4.3. Exact sequences and filling norms.

I am grateful to Shigenori Matsumoto who provided elegant proofs of many results in this section. In the sequel, we use some of the elements of abstract functional analysis; Rudin [**180**] is a general reference.

Recall our notation $\widehat{Q}(G)$ for the vector space of all quasimorphisms on G, and $Q(G)$ for the vector subspace of homogeneous quasimorphisms. Recall that $D(\cdot)$ defines pseudo-norms on both $\widehat{Q}(G)$ and $Q(G)$ which vanish exactly on the subspace spanned by *homomorphisms* $G \to \mathbb{R}$. This subspace may be naturally identified with $H^1(G;\mathbb{R})$.

A real-valued function φ on G may be thought of as a 1-cochain, i.e. as an element of $C^1(G;\mathbb{R})$. The coboundary δ of such a function is defined by the formula

$$\delta\varphi(a,b) = \varphi(a) + \varphi(b) - \varphi(ab)$$

At the level of norms, there is an equality, $\|\delta\varphi\|_\infty = D(\varphi)$. It follows that the coboundary of a quasimorphism is a *bounded* 2-cocycle.

Theorem 2.50 (Exact sequence). *There is an exact sequence*

$$0 \to H^1(G;\mathbb{R}) \to Q(G) \to H_b^2(G;\mathbb{R}) \to H^2(G;\mathbb{R})$$

Proof. There is an exact sequence of chain complexes

$$0 \to C_b^* \to C^* \to C^*/C_b^* \to 0$$

and an associated long exact sequence of cohomology groups. A bounded homomorphism to $\mathbb{R}$ is trivial, hence $H_b^1(G;\mathbb{R}) = 0$ for any group G. A function φ on G is in $\widehat{Q}(G)$ if and only if $\delta\varphi$ is in C_b^2. Moreover, any two quasimorphisms which differ by a bounded amount have the same homogenization. Hence

$$H^1(C^*/C_b^*) = \widehat{Q}/C_b^1 \cong Q$$

□

Example 2.51. Recall from § 2.3.3 that rot is a homogeneous quasimorphism on the group $\text{Homeo}^+(\mathbb{R})^{\mathbb{Z}}$, which is our notation for the group of homeomorphisms of $\mathbb{R}$ which are periodic with period 1. Further recall that this group is the universal central extension of $\text{Homeo}^+(S^1)$. The function rot does not descend to a well-defined real-valued function on $\text{Homeo}^+(S^1)$, but it is well-defined mod $\mathbb{Z}$. However, the coboundary $[\delta\text{rot}]$, as a class in $H_b^2(\text{Homeo}^+(\mathbb{R})^{\mathbb{Z}})$, can be pulled back from a class in $H_b^2(\text{Homeo}^+(S^1))$. By Theorem 2.50, the image of this class in $H^2(\text{Homeo}^+(S^1))$ is a nontrivial class, called the *Euler class*. The L^∞ norm of this class is $1/2$ (compare with Theorem 2.43). This fact is otherwise known as the *Milnor–Wood inequality* ([**154**],[**204**]), and is usually stated in the following way:

THEOREM 2.52 (Milnor–Wood inequality). *Let S be a closed, oriented surface of genus g, and let $\rho : \pi_1(S) \to \mathrm{Homeo}^+(S^1)$ be an action of $\pi_1(S)$ on a circle by homeomorphisms. Let $[e] \in H^2(S)$ be the pullback of the generator of $H^2(\mathrm{Homeo}^+(S^1);\mathbb{Z})$. Then there is an inequality*

$$|[e](S)| \le -\chi^-(S)$$

For ease of notation, we abbreviate $C_*(G;\mathbb{R})$ in what follows by C_*. Similarly, denote cycles and boundaries with real coefficients by Z_* and B_* respectively. Then

$$0 \to Z_2 \to C_2 \to B_1 \to 0$$

is exact. Since C_2 is normed, and Z_2 is a normed subspace, B_1 inherits a quotient norm.

Observe that if $a \in [G,G]$ then $a \in B_1$ when thought of as a generator of C_1. For example, if $a = [x,y]$ then

$$\partial((xyx^{-1},x) + ([x,y],y) - (x,y)) = [x,y]$$

In general, a one-vertex triangulation of a surface of genus g with one boundary component exhibits a product of g commutators as an element of B_1.

DEFINITION 2.53. Let $a \in B_1(G;\mathbb{R})$. The *Gersten boundary norm* (or just the *Gersten norm* or the *boundary norm*) of a, denoted $\|a\|_B$, is defined by

$$\|a\|_B = \inf_{\partial A = a} \|A\|_1$$

where the infimum is taken over all 2-chains $A \in C_2(G;\mathbb{R})$ with boundary a, and $\|A\|_1$ denotes the usual L^1 norm.

REMARK 2.54. Gersten calls his norm a *filling norm* in [**90**]. However, we reserve this name for a suitable homogenization of $\|\cdot\|_B$.

It is important to note that this quotient is really a norm and not just a pseudonorm, since ∂ is a bounded operator on C_2 of norm 3, and therefore $\|a\|_1 \le 3\|a\|_B$. In particular, Z_2 is closed in C_2 in the L^1 norm.

REMARK 2.55. We can define $C_*^{l_1}$ to be the completion of C_* with respect to the L^1 norm. The boundary map ∂ extends continuously to $C_*^{l_1}$, and we let $Z_*^{l_1}$ and $B_*^{l_1}$ denote the kernel and image of ∂ respectively. The exact sequence

$$0 \to Z_2^{l_1} \to C_2^{l_1} \to B_1^{l_1} \to 0$$

defines a quotient norm on $B_1^{l_1}$ and thereby on B_1 under inclusion $B_1 \to B_1^{l_1}$. However, in general there is a *strict* inclusion $\overline{Z}_2 \subset Z_2^{l_1}$, where $\overline{Z}_*$ denotes the completion of Z_* in the L^1 norm, and therefore the norm B_1 inherits as a subspace of $C_2^{l_1}/Z_2^{l_1}$ will be typically *smaller* than $\|\cdot\|_B$.

In fact there is an important special case in which the two norms on B_1 are the same. Matsumoto–Morita [**150**] say that the chain complex C_* satisfies condition 1-UBC if there is a positive constant $K > 0$ such that $K\|a\|_B \le \|a\|_1$ for all $a \in B_1$. Note that this is equivalent to the condition that the norms $\|\cdot\|_1$ and $\|\cdot\|_B$ induce the same topology on B_1. Under this circumstance, there is an equality $\overline{Z}_2 = Z_2^{l_1}$. In fact, Theorem 2.8 from [**150**] implies that condition 1-UBC is equivalent to injectivity of the map $H_b^2 \to H^2$. By Theorem 2.50, this is equivalent to $Q(G)/H^1(G) = 0$, a situation which is largely orthogonal to the focus of this book.

We now identify the dual space of B_1 with respect to the norm $\|\cdot\|_B$.

Lemma 2.56. *The dual of* B_1 *with respect to the* $\|\cdot\|_B$ *norm is* $\widehat{Q}(G)/H^1(G;\mathbb{R})$, *and the operator norm on the dual is equal to* $D(\cdot) = \|\delta\cdot\|_\infty$.

Proof. In the sequel, if V is a normed vector space, we denote the space of bounded linear functionals on V with the operator norm by V'.

By definition of the quotient norm, an element f of B_1' determines $F \in C_2'$ with the same operator norm, vanishing on Z_2, by the formula $F(A) = f(\partial A)$. Since F vanishes on Z_2, it is a coboundary; hence $F = \delta\phi$ where $\phi \in C^1$ is unique up to an element of H^1. Since F is bounded, ϕ is a quasimorphism, and we have defined $B_1' \to \widehat{Q}/H^1$ (note that the restriction of ϕ to B_1 is equal to f). This map is evidently injective and surjective, and is therefore an isomorphism of vector spaces.

It remains to identify the norm. Let $b \in B_1$ be an element with $\|b\|_B = 1$, so there is $A \in C_2$ with $\partial A = b$ and $\|A\|_1 - 1 < \epsilon$. Express A as $A = \sum_j r_j(g_j, h_j)$ with $r_j \in \mathbb{R}$, and $\sum_j |r_j| - 1 < \epsilon$. By the triangle inequality,

$$\begin{aligned}|F(A)|/(1+\epsilon) &\le \sup_j |F(g_j,h_j)| = \sup_j |\delta\phi(g_j,h_j)| = \sup_j |\phi(\partial(g_j,h_j))| \\ &= \sup_j |\phi(g_jh_j) - \phi(g_j) - \phi(h_j)| \le D(\phi)\end{aligned}$$

so we deduce that the operator norm of F (and therefore that of f) is $\le D(\phi)$.

Conversely, let $g_1, g_2 \in G$ be arbitrary. Then (except in degenerate cases) $\partial(g_1,g_2) = g_1 + g_2 - g_1g_2$ has L^1 norm equal to 3, and therefore

$$1 \ge \|\partial(g_1,g_2)\|_B \ge \frac{1}{3}\|\partial(g_1,g_2)\|_1 = 1$$

But $F(g_1,g_2) = \phi(g_1) + \phi(g_2) - \phi(g_1g_2)$, so by the definition of the defect there are $g_1, g_2 \in G$ with $\|\partial(g_1,g_2)\|_B = 1$ for which $|F(g_1,g_2)|$ is arbitrarily close to $D(\phi)$. This implies that the operator norm of F is at least equal to $D(\phi)$, and together with the previous inequality, this shows that the operator norm of F is exactly equal to the defect of ϕ, as claimed. ☐

We deduce the following corollary:

Corollary 2.57. *The space* $\widehat{Q}/H^1$ *with its defect norm is a Banach space, and is isometric to the dual of* $C_2^{l_1}/\overline{Z}_2$ *with its* L^1 *norm.*

Proof. By Lemma 2.56, we know that $\widehat{Q}/H^1$ with its defect norm is the dual of B_1 with its $\|\cdot\|_B$ norm, which by definition is equal to the dual of C_2/Z_2 with its L_1 norm. If X is a normed vector space, and Y is a closed normed vector subspace, the dual $(X/Y)'$ is isometrically isomorphic to the dual $(\overline{X}/\overline{Y})'$ where the overline denotes completion with respect to the norm. In our case, $C_2^{l_1}$ and $\overline{Z}_2$ are the completions of C_2 and Z_2 in the L^1 norm, so the second claim of the corollary follows.

The dual space of a normed vector space is always a Banach space. Hence the first claim follows already from Lemma 2.56. ☐

Since homogeneity is a closed condition, the quotient Q/H^1 is a Banach subspace of $\widehat{Q}/H^1$. We refer to the Banach topology on this space as the *defect topology. A priori*, there is a natural pseudo-norm on H_b^2. We will see shortly that this pseudo-norm is actually a norm (this fact is due to Matsumoto–Morita [**150**]). Theorem 2.50 shows that δ is an injection of Q/H^1 into H_b^2. The next lemma describes how the norm behaves under δ :

LEMMA 2.58. *Let $\phi \in Q(G)$. Then*

$$D(\phi) \ge \|[\delta\phi]\|_\infty \ge \frac{1}{2}D(\phi)$$

PROOF. By definition, $\|[\delta\phi]\|$ is the infimum of the L^∞ norm of all bounded 2-cocycles A which are cohomologous to $\delta\phi$. Now any such A is of the form δf for some unique (not necessarily homogeneous) quasimorphism f for which $f - \phi \in C_b^1$. In particular, ϕ is the homogenization of f, and we have an inequality

$$\|[\delta\phi]\|_\infty = \inf_{f-\phi \in C_b^1} D(f) \le D(\phi)$$

Since any quasimorphism can be antisymmetrized without increasing its defect, it suffices to take the infimum over antisymmetric f.

Let $a, b \in G$ be such that $|\delta\phi(a,b)|$ is very close to $D(\phi)$. Recall from the proof of Lemma 2.24 that $a^{2n}b^{2n}(ab)^{-2n}$ can be written as a product of at most n commutators. Since f is antisymmetric, it follows that $|f(a^{2n}b^{2n}(ab)^{-2n})| \le (4n-1)D(f)$. Since $f - \phi \in C_b^1$, there is a constant C, independent of a, b and n, so that $|f(a^{2n}b^{2n}(ab)^{-2n}) - \phi(a^{2n}b^{2n}(ab)^{-2n})| \le C$. Moreover, by homogeneity, $|\phi(a^{2n}b^{2n}(ab)^{-2n}) - 2n\delta\phi(a,b)| \le 2D(\phi)$ and therefore

$$\lim_{n\to\infty} \frac{|\phi(a^{2n}b^{2n}(ab)^{-2n})|}{2n} = |\delta\phi(a,b)|$$

which is arbitrarily close to $D(\phi)$. Putting this together, we get an estimate

$$D(\phi) \le 2D(f)$$

and the lemma is proved. □

It is convenient to explicitly record the following corollary:

COROLLARY 2.59. *Let $f \in \widehat{Q}(G)$ with homogenization $\phi \in Q(G)$. Then*

$$D(f) = \|\delta f\|_\infty \ge \|[\delta\phi]\|_\infty \ge \frac{1}{2}D(\phi)$$

REMARK 2.60. Lemma 2.58 and its Corollary can be restated in homological language. The following argument is due to Shigenori Matsumoto. Since $C_b^1 \cap H^1 = 0$, we can think of C_b^1 as a subspace of $\widehat{Q}/H^1$. We have already shown in Corollary 2.57 that $\widehat{Q}/H^1$ can be identified with the dual $(C_2^{l_1}/\overline{Z}_2)'$. What is the image $\delta(C_b^1)$ in this dual space? First we make an observation.

LEMMA 2.61. *The boundary map $\partial : C_2^{l_1} \to C_1^{l_1}$ has a (bounded) cross-section σ defined by the formula*

$$\sigma(g) = \frac{1}{2}(g,g) + \frac{1}{4}(g^2,g^2) + \cdots$$

PROOF. The proof is immediate. □

From this it follows that $B_1^{l_1} = C_1^{l_1}$ as abstract vector spaces. Moreover, Lemma 2.61 shows that $\|b\|_B \le \|b\|_1$ for $b \in C_1^{l_1}$. Since we also have $\|b\|_B \ge \frac{1}{3}\|b\|_1$, this shows that the quotient norm and the L^1 norm on $C_1^{l_1}$ are equivalent (though not necessarily isometric).

The dual of $C_1^{l_1}$ with its L^1 norm is C_b^1 with its L^∞ norm. Dualizing $Z_2^{l_1} \to C_2^{l_1} \to C_1^{l_1}$ shows that the image $\delta(C_b^1)$ is equal to $(C_2^{l_1}/Z_2^{l_1})'$. Since $\widehat{Q}/H^1 = (C_2^{l_1}/\overline{Z}_2)'$, if we give $\widehat{Q}/(C_b^1 \oplus H^1) = (\widehat{Q}/H^1)/C_b^1$ its quotient norm, we obtain an isometric isomorphism

$$\widehat{Q}/(C_b^1 \oplus H^1) \xrightarrow{\delta} (Z_2^{l_1}/\overline{Z}_2)'$$

As vector spaces, Q/H^1 and $\widehat{Q}/(C_b^1 \oplus H^1)$ are naturally isomorphic; in this language, Lemma 2.58 says that their norms differ at most by a factor of 2.

Unfortunately, the Banach space $Q(G)/H^1(G;\mathbb{R})$ is typically very big, even if G is finitely presented. We give some examples to illustrate this phenomenon for the case that G is free.

Example 2.62 (Free group). Let F denote the free group on two generators a, b. Let $w_n = ab^n a$ for each positive integer n. For each $f : \mathbb{N} \to \{0, 1\}$ define

$$\overline{H}_f = \sum_n f(n)\overline{H}_{w_n}$$

where each H_{w_n} is the big counting function (see Definition 2.25), and the overline denotes homogenization. Since the words are not nested, $D(H_f) = 1$ for each f (compare with Example 2.35), and therefore $D(\overline{H}_f) \le 2$ by Corollary 2.59. If $f \ne g$ then if n is in the support of f but not g (say), we have

$$(\overline{H}_f - \overline{H}_g)(ab^n a) = 1$$

so the difference is nontrivial. On the other hand, since $\overline{H}_f$ and $\overline{H}_g$ vanish on both a and b, they are not in H^1. It follows that $D(\overline{H}_f - \overline{H}_g)$ is positive, and since they are both integer valued, the defect is at least 1. In other words, we have constructed a subset of $Q(F)/H^1(F)$ of cardinality $2^{\aleph_0}$ which is discrete in the defect topology. In particular, Q/H^1 is not separable.

Example 2.63 (Density). Jason Manning constructed an explicit example of a vector in $Q(F)/H^1(F)$ which is not in the closure (in the defect topology) of the span of Brooks quasimorphisms. For each n let $w_n = [a^n b^n a^{-n}, b^{-n}]$. Then $\overline{H}_v(w_n) = \overline{h}_v(w_n) = 0$ where $\overline{H}_v$ and $\overline{h}_v$ denote the homogenizations of the big and small counting functions, whenever v is a word of length $\le n$. Now, define

$$\overline{H} = \sum_i \overline{H}_{w_i}$$

Since the w_i and their inverses do not overlap, one can estimate $D(\overline{H}) \le 6$. Now suppose $\overline{H}'$ is a finite linear combination of homogenized counting quasimorphisms (of either sort). Then there is an n such that $\overline{H}'(w_n) = 0$ but $\overline{H}(w_n) = 1$. Since each w_n is a commutator, by Lemma 2.24 it follows that $D(\overline{H}' - \overline{H}) \ge 1$.

Example 2.64 (Pullbacks). Let $F_3 = \langle a, b, c\rangle$ and $F_2 = \langle a, b\rangle$. Let $p : F_3 \to F_2$ be the obvious retraction, obtained by killing c. Let $h \in Q(F_2)$ be the homogenization of the Brooks function h_{ab}, and let $p^*h \in Q(F_3)$ denote the pullback. Then p^*h is not in the closure of the span of Brooks quasimorphisms. To see why, consider the elements $w_n := a^n c a^{-n} b^{-1} a^n c^{-1} a^{-n} b$ and $w_n' := a^{n-1} c a^{-n} b^{-1} a^n c^{-1} a^{1-n} b$. The element w_n is in the kernel of p, but $p(w_n') = a^{-1}b^{-1}ab$ so $p^*h_{ab}(w_n) = 0$ whereas $p^*h_{ab}(w_n') = 1$. Note further that each w_n is a commutator, and each w_n' is a product of two commutators, and therefore satisfies $\mathrm{scl}(w_n') \le 3/2$. Notice that for any word v we must have $h_v(w_n) = h_v(w_n')$ for sufficiently large n (and similarly for H_v). It follows that p^*h cannot be approximated in defect by the homogenization of a finite linear combination of Brooks quasimorphisms (of either kind). This example is obviously not sporadic; a similar argument shows that if $p : F \to G$ is surjective with nontrivial kernel, and $h \in Q(G)$ is not in $H^1(G)$, then p^*h is never in the closure of the span of Brooks quasimorphisms.

If G is amenable, Theorem 2.47 shows that $H^2_b(G;\mathbb{R}) = 0$ and therefore $Q(G) = H^1(G;\mathbb{R})$; in other words, every homogeneous quasimorphism on an amenable group is a homomorphism to $\mathbb{R}$. For completeness, we give a self-contained proof of this fact.

Proposition 2.65. *Let G be amenable. Then every homogeneous quasimorphism on G is a homomorphism to $\mathbb{R}$.*

Proof. Let $\phi : G \to \mathbb{R}$ be a quasimorphism. We will construct a homomorphism which differs from ϕ by a bounded amount; this is enough to prove the proposition. Let $\mathbb{R}^{G\times G}$ be the space of real valued functions on $G \times G$, with the topology of pointwise convergence. A function $\phi : G \to \mathbb{R}$ determines an element $\Phi : G \times G \to \mathbb{R}$ by the formula

$$\Phi(a, b) = \phi(a) - \phi(b)$$

The group G acts on $G \times G$ diagonally: $g(a, b) = (ga, gb)$ and thus on $\mathbb{R}^{G\times G}$. For any $g \in G$, we have $g\Phi(a, b) = \phi(ga) - \phi(gb)$ and therefore

$$|g\Phi(a, b) - \Phi(a, b)| \le 2D(\phi)$$

Hence the convex hull of the orbit $G\Phi$ is a compact, convex, G-invariant subset of $\mathbb{R}^{G\times G}$. Note that Φ has the property that $\Phi(a, b) + \Phi(b, c) = \Phi(a, c)$ for any $a, b, c \in G$. In particular, Φ vanishes on any (a, a) and is antisymmetric in its arguments. This property is invariant under the action of G, and preserved under linear combinations and limits, and therefore holds for any element of the closed convex hull of $G\Phi$. This part of the argument does not use the fact that G is amenable.

If G is amenable, any linear action by G on a topological vector space which leaves invariant a compact, convex subset must have a global fixed point in that set; basically, the barycenter of any bounded orbit, weighted by the invariant mean, is G-invariant. If Ψ is such a G-invariant function we can define $\psi : G \to \mathbb{R}$ by $\psi(a) = \Psi(a, \mathrm{id})$. Since Ψ is G-invariant, $\psi(ab) = \Psi(ab, \mathrm{id}) = \Psi(b, a^{-1})$. But $\Psi(b, a^{-1}) + \Psi(a^{-1}, \mathrm{id}) = \Psi(b, \mathrm{id})$ so $\psi(ab) = \psi(b) - \psi(a^{-1})$. Since

$$\psi(a^{-1}) = \Psi(a^{-1}, \mathrm{id}) = \Psi(\mathrm{id}, a) = -\Psi(a, \mathrm{id}) = -\psi(a)$$

we are done. □

2.4.4. Antisymmetrization and orientations. In singular homology, simplices are *marked* by a total ordering of the vertices. Similarly, in group homology, generators of the bar complex are *ordered* tuples of group elements. Given a simplex Δ^n, the symmetric group S_{n+1} acts on Δ^n by permuting the vertices. There is a chain map $s : C_* \to C_* \otimes \mathbb{Q}$ defined on a generator σ of C_n by

$$s(\sigma) = \frac{1}{(n+1)!} \sum_{g \in S_{n+1}} \mathrm{sign}(g)\sigma \circ g$$

where $\mathrm{sign}(g)$ is ± 1 depending on whether $g : \Delta^n \to \Delta^n$ is orientation preserving or reversing. We can define a similar chain map from the bar complex $C_*(G) \otimes \mathbb{Q}$ to itself.

The chain map s is chain homotopic to id, and therefore induces an isomorphism in homology over $\mathbb{Q}$ or $\mathbb{R}$. Moreover, this chain map has operator norm 1 in each dimension with respect to the L^1 norm.

In dimension 1, the map s replaces an element $a \in G$ with the sum

$$s : a \to \frac{1}{2}(a - a^{-1})$$

It follows that if f' is the antisymmetrization of a 1-cochain f, there is an equality

$$f'(a) = f(s(a))$$

that is, $f' = s^* f$ where s^* is the adjoint of s in dimension 1. The observation in § 2.2.2 that antisymmetrization of quasimorphisms does not increase defect is dual to the the observation that s has operator norm 1.

This discussion is most relevant when one considers bounded cohomology over other coefficient groups, for instance over $\mathbb{Z}$. One can neither (anti)symmetrize chains nor cochains over $\mathbb{Z}$, and therefore some of the estimates we obtain in this section are no longer valid in greater generality.

2.5. Bavard's Duality Theorem

2.5.1. Banach duality and filling norms. In the last section, we defined the Gersten boundary norm, and identified its dual space. By an application of the Hahn–Banach Theorem, Lemma 2.56 lets us reinterpret the Gersten boundary norm in terms of quasimorphisms.

Corollary 2.66. *Let $a \in [G,G]$ so that $a \in B_1$ as a cycle. Then*

$$\|a\|_B = \sup_{\phi \in \widehat{Q}(G)/H^1(G;\mathbb{R})} \frac{|\phi(a)|}{D(\phi)}$$

To relate the Gersten norm to stable commutator length, we must homogenize.

Definition 2.67. Define the *filling norm*, denoted $\text{fill}(a)$ to be the homogenization of $\|a\|_B$. That is,

$$\text{fill}(a) = \lim_{n\to\infty} \frac{\|a^n\|_B}{n}$$

Remark 2.68. Some authors refer to $\text{fill}(\cdot)$ as the *stable* filling norm, to distinguish it from the Gersten filling norm.

It is not quite true that the function $\|a^n\|_B$ is subadditive in n. However, for any r, s there is an identity $\partial(a^r, a^s) = a^r + a^s - a^{r+s}$ and therefore $\|a^{r+s}\|_B \le \|a^r\|_B + \|a^s\|_B + 1$. This is enough to show that the limit exists in Definition 2.67.

Using the estimates proved in Chapter 1, we can relate scl and $\text{fill}(\cdot)$ in a straightforward manner:

Lemma 2.69 (Bavard, Prop. 3.2. [**8**]). *There is an equality*

$$\text{scl}(a) = \frac{1}{4}\text{fill}(a)$$

Proof. An expression of a^n as a product of commutators lets us construct an orientable surface S with one boundary component, and a homomorphism $\varphi : \pi_1(S) \to G$ with $\varphi_* \partial S = a^n$ in π_1. We can find a triangulation of S with $4 \cdot \text{genus}(S) - 1$ triangles, where one edge maps to the boundary, and therefore

$$\|a^n\|_B \le 4 \cdot \text{cl}(a^n) - 1$$

Dividing both sides by n, and taking the limit as $n \to \infty$ gives the inequality

$$\text{fill}(a) \le 4 \cdot \text{scl}(a)$$

Conversely, let A be a chain with $\partial A = a$ with $\|A\|_1$ close to $\|a\|_B$. Let V be the finite dimensional subspace of $C_2(G;\mathbb{R})$ consisting of 2-chains with support contained in the support of A. Since V is a rational subspace, and a is a rational chain, the subspace $V \cap \partial^{-1}(a)$ contains rational points arbitrarily close to A (compare with Remark 1.5). So we may assume A is rational, after changing its norm an arbitrarily small amount. After scaling by some integer, we may assume A is an integral chain with $\partial A = na$ for which the ratio $\|A\|_1/n\|a\|_B$ is very close to 1.

As in Example 1.4, there is an orientable surface S and a chain A_S representing the fundamental class of S, and a map $\varphi : \pi_1(S) \to G$ sending boundary components to powers of conjugates of a, and such that $\varphi_*(A_S) = A$. Moreover, by construction, $\|A_S\|_1 = \|A\|_1$.

By Theorem 1.14 and Lemma 2.10 we have an inequality

$$\frac{\|A_S\|_1}{n} \geq \frac{-2\chi(S)}{n} \geq 4 \cdot \mathrm{scl}(a)$$

But $\|A_S\|_1/n$ may be taken to be arbitrarily close to $\|a\|_B$. Homogenizing the left hand side (and using the fact that the right hand side is homogeneous by definition) we obtain

$$\mathrm{fill}(a) \geq 4 \cdot \mathrm{scl}(a)$$

Putting this together with the earlier inequality, we are done. □

2.5.2. Bavard's Duality Theorem. We are now in a position to relate quasimorphisms and stable commutator length by means of Bavard's Duality Theorem:

THEOREM 2.70 (Bavard's Duality Theorem, [**8**]). *Let G be a group. Then for any $a \in [G,G]$, we have an equality*

$$\mathrm{scl}(a) = \frac{1}{2} \sup_{\phi \in Q(G)/H^1(G;\mathbb{R})} \frac{|\phi(a)|}{D(\phi)}$$

PROOF. For the sake of legibility, we suppress G in our notation in what follows. By Corollary 2.66 there is a duality

$$\|a\|_B = \sup_{\phi \in \widehat{Q}/H^1} \frac{|\phi(a)|}{D(\phi)}$$

Homogenizing and applying Lemma 2.69, we obtain an equality

$$\mathrm{scl}(a) = \frac{1}{4} \lim_{n\to\infty} \left(\sup_{\phi \in \widehat{Q}/H^1} \frac{|\phi(a^n)|}{nD(\phi)} \right)$$

Recall that in Lemma 2.21 we obtained the estimate $|\phi(a^n) - \overline{\phi}(a^n)| \leq D(\phi)$ where $\overline{\phi}$ denotes the homogenization of ϕ. It follows that for each n and any $\phi \in \widehat{Q}$ there is an inequality

$$\frac{|\phi(a^n) - \overline{\phi}(a^n)|}{nD(\phi)} \leq n^{-1}$$

Parsing this, for each n let ϕ_{n_i} be a sequence of elements in $\widehat{Q}(G)$ such that $\phi_{n_m}(a^n)/nD(\phi_{n_m})$ is within m^{-1} of the supremum. Then $\overline{\phi}_{n_m}(a^n)/nD(\phi_{n_m})$ is within $m^{-1} + n^{-1}$ of the supremum. Using $\overline{\phi}(a^n)/n = \overline{\phi}(a)$ and passing to a diagonal subsequence, we obtain

$$\mathrm{scl}(a) = \frac{1}{4} \sup_{\phi \in \widehat{Q}/H^1} \frac{|\overline{\phi}(a)|}{D(\phi)}$$

By Corollary 2.59, we get an inequality

$$\mathrm{scl}(a) \le \frac{1}{2} \sup_{\phi \in Q/H^1} \frac{|\phi(a)|}{D(\phi)}$$

On the other hand, for any homogeneous quasimorphism ϕ, if a^n is a product of m commutators then

$$|\phi(a^n)| \le 2mD(\phi)$$

so we get an inequality in the other direction, and the theorem is proved. □

2.6. Stable commutator length as a norm

In this section we show that scl can be extended in a natural way to a pseudonorm on (a suitable quotient of) B_1. Moreover Bavard duality holds more generally in this broader context, thus revealing it as a genuine duality theorem (in the usual sense of functional analysis).

2.6.1. Definition.

Definition 2.71. Let G be a group, and $a_i \in G$ for $1 \le i \le m$ a finite collection of elements. If the product of the a_i is in $[G, G]$, then define $\mathrm{cl}(a_1 + a_2 + \cdots + a_m)$ to be the smallest number of commutators whose product is equal to an expression of the form

$$a_1 t_1 a_2 t_1^{-1} \cdots t_{m-1} a_m t_{m-1}^{-1}$$

for some elements $t_i \in G$. Then define

$$\mathrm{scl}(\sum_i a_i) = \lim_{n \to \infty} \frac{\mathrm{cl}(\sum_i a_i^n)}{n}$$

Geometrically, if $\pi_1(X) = G$, and γ_i is a loop in X representing the conjugacy class of a_i, then $\mathrm{cl}(\sum_i a_i)$ is the least genus of a surface with m boundary components which maps to X in such a way that the ith boundary component wraps once around γ_i.

Remark 2.72. If the product of the a_i has order n in $H_1(G; \mathbb{Z})$, define $\mathrm{scl}(\sum a_i) = \frac{1}{n}\mathrm{scl}(\sum a_i^n)$, and otherwise define $\mathrm{scl}(\sum a_i) = \infty$.

In fact, it it not immediately obvious that the limit in Definition 2.71 exists, since the function $\mathrm{cl}_n(\sum a_i) := \mathrm{cl}(\sum a_i^n)$ is not subadditive as a function of n. We address this issue in the next lemma.

Lemma 2.73. *The limit in Definition 2.71 exists when it is defined (i.e. when the product of the a_i are in $[G, G]$).*

Proof. If $\sum a_i$ has m terms, define $\mathrm{cl}_{n,m} = \mathrm{cl}(\sum a_i^n) + (m-1)$. Then (for fixed m) the function $\mathrm{cl}_{n,m}$ *is* subadditive as a function of n. For, if S_{n_1}, S_{n_2} are surfaces with m boundary components, each of which wraps n_1 and n_2 times respectively around each of m loops, then they can be tubed together by adding m rectangles to produce a surface S' with m boundary components, each of which wraps $n_1 + n_2$ times around each of the m loops, and satisfies $\mathrm{genus}(S') = \mathrm{genus}(S_{n_1}) + \mathrm{genus}(S_{n_2}) + (m-1)$. On the other hand, for fixed m, there is an equality

$$\lim_{n \to \infty} \frac{\mathrm{cl}(\sum a_i^n)}{n} = \lim_{n \to \infty} \frac{\mathrm{cl}(\sum a_i^n) + (m-1)}{n}$$

the right hand limit exists by the subadditivity of $\mathrm{cl}_{n,m}$, and therefore the left hand side does too. □

Given a space X and loops $\gamma_i : S^1 \to X$ we say that a map $f : S \to X$ is *admissible* if there is a commutative diagram:

$$\begin{tikzcd}
\partial S \arrow[r, "i"] \arrow[d, "\partial f"'] & S \arrow[d, "f"] \\
\coprod_i S^1 \arrow[r, "\coprod \gamma_i"'] & X
\end{tikzcd}$$

for which there is an integer $n(S)$ such that

$$\partial f_*[\partial S] = n(S)[\coprod_i S^1]$$

(note that the existence of an integer $n(S)$ is not automatic from the commutativity of the diagram, when there is more than one γ_i).

One has the following analogue of Proposition 2.10.

Proposition 2.74. *Let $\pi_1(X) = G$, and for $1 \le i \le m$, let $\gamma_i : S^1 \to X$ be a loop representing the conjugacy class of $a_i \in G$. Then*

$$\mathrm{scl}(\sum_i a_i) = \inf_S \frac{-\chi^-(S)}{2n(S)}$$

where the infimum is taken over all admissible maps as above.

Proof. The proof is almost identical to that of Proposition 2.10. An inequality in one direction follows from the definition, at least if one uses the "corrected" function $\mathrm{cl}_{n,m}$ in place of cl_n (see Lemma 2.73). To obtain the inequality in the other direction, let $f : S \to X$ be an admissible map of a surface. Without loss of generality, one may restrict attention to the case that each component of S_i has at least one boundary component mapping with nontrivial degree to some γ_i. Fix some big (even) integer N, and construct connected covers T_i of each S_i of degree $2N$, each with at most twice as many boundary components as S_i. The T_i may be surgered to have exactly m boundary components, each mapping to some γ_i with degree $2Nn(S)$ by gluing on only a constant number of rectangles, and thereby raising $-\chi$ by an amount which is independent of N. The reverse inequality follows. ☐

A surface realizing the infimum in Proposition 2.74 is called *extremal* (compare with Definition 2.11).

From the geometric perspective it is clear that $\mathrm{scl}(\sum a_i)$ depends only on the conjugacy class of each term a_i, and is commutative in its arguments.

Lemma 2.75. scl *satisfies the identity*

$$\mathrm{scl}(a^n + \sum a_i) = \mathrm{scl}(\underbrace{a + \cdots + a}_{n} + \sum a_i)$$

for any non-negative integer n and any $a, a_i \in G$.

Proof. We use Proposition 2.74. Let X be a space with $\pi_1(X) = G$ and let γ be a loop representing the conjugacy class of a. Let S be a surface mapping to X, with n boundary components each wrapping around γ a total of m times, for some large m, and the rest wrapping around loops γ_i corresponding to the conjugacy classes of the a_i. The distinct boundary components wrapping around γ can be tubed together at the cost of raising $-\chi^-(S)$ by $n-1$, which can be taken to be arbitrarily small compared to m. This establishes an inequality in one direction.

Conversely, if S is a surface mapping to X with one boundary component wrapping some number of times around γ^n and the rest around the γ_i, take n copies of S to obtain the inequality in the other direction. □

Similarly we have the following.

LEMMA 2.76. scl *satisfies the identity*

$$\mathrm{scl}(a + a^{-1} + \sum a_i) = \mathrm{scl}(\sum a_i)$$

for any $a, a_i \in G$.

PROOF. Let X, γ, γ_i be as before. Let S be a surface whose boundary wraps around the various γ_i. Let A be an annulus from γ to γ^{-1} and let S' be the disjoint union of S with some number of parallel copies of A. Then $-\chi^-(S) = -\chi^-(S')$.

Conversely, suppose S is a surface with one boundary component ∂_1 bounding γ^m and one component ∂_2 bounding γ^{-m}. Glue ∂_1 to ∂_2 to obtain a surface S' with $-\chi^-(S) = -\chi^-(S')$. □

By abuse of notation we define $\mathrm{scl}(\sum_i a_i - a) := \mathrm{scl}(\sum_i a_i + a^{-1})$. It follows from Lemma 2.75 and Lemma 2.76 that for any a, a_i and for any equality $n = \sum_i n_i$ over $\mathbb{Z}$ there is a corresponding equality

$$\mathrm{scl}(a^n + \sum_j a_j) = \mathrm{scl}(\sum_i a^{n_i} + \sum_j a_j)$$

Moreover, for any integer n, there is an equality

$$|n|\,\mathrm{scl}(\sum a_i) = \mathrm{scl}(\sum n a_i) = \mathrm{scl}(\sum a_i^n)$$

Consequently scl can be extended by linearity on rays to rational chains $\sum_i r_i a_i$ representing 0 in $H_1(G;\mathbb{Q})$. Since scl is subadditive on rational chains, it extends continuously in a unique way to a pseudo-norm on the real vector space $B_1(G)$.

Recall from § 2.5.1 that we defined the Gersten norm $\|\cdot\|_B$ on B_1 by the equality

$$\|a\|_B = \inf_{\partial A = a} \|A\|_1$$

where $a \in B_1$ and $A \in C_2$. Then for an element $g \in [G,G]$ we defined the (stable) filling norm by the formula

$$\mathrm{fill}(g) = \lim_{n\to\infty} \frac{\|g^n\|_B}{n}$$

One can extend fill to all of B_1. First extend fill to integral chains:

$$\mathrm{fill}(\sum_i g_i) = \lim_{n\to\infty} \frac{\|\sum_i g_i^n\|_B}{n}$$

and then by linearity to rational chains, and by continuity to arbitrary chains in B_1. To see that a continuous extension exists, observe that for each n, there is an inequality $\|\sum_i g_i^n + \sum_j f_j^n\|_B \le \|\sum_i g_i^n\|_B + \|\sum_j f_j^n\|_B$ and therefore fill is subadditive. Since fill is homogeneous, it is evidently a class function in each argument.

With this definition, one obtains the following analogue of Lemma 2.69:

Lemma 2.77. *For any finite linear chain* $\sum_i t_i a_i \in B_1$ *there is an equality*

$$\text{scl}(\sum_i t_i a_i) = \frac{1}{4}\text{fill}(\sum_i t_i a_i)$$

Proof. It suffices to prove the result for integral chains; i.e. chains of the form $\sum_i a_i$ for $1 \le i \le m$.

The proof is very similar to that of Lemma 2.69; the only complication is the issue of basepoints. A surface S realizing $\text{cl}(\sum_i a_i^n)$ can be efficiently triangulated, as in Theorem 1.14, with $4\text{cl}(\sum_i a_i^n) + 3m - 4$ triangles, with exactly one vertex on each boundary component. Let T be an embedded spanning tree in the 1-skeleton, connecting up the boundary vertices (T has $m-1$ edges). We obtain a simplicial 2-complex with one vertex by collapsing T to a point, and then further collapsing degenerate triangles. Denote this 2-complex by S/T. The triangulation of S determines a triangulation of the complex S/T, with fewer triangles. Since this complex has only one vertex, it determines a (group) 2-chain A with $\|A\|_1 \le 4\text{cl}(\sum_i a_i^n) + 3m - 4$, and satisfying $\partial A = \sum_i b_i^n$ where each b_i is conjugate to a_i. Since m is fixed, and fill is a class function in each argument, as $n \to \infty$ we obtain an inequality in one direction.

Conversely, a 2-chain A with $\partial A = \sum_i a_i^n$ and $\|A\|_1$ close to $\|\sum_i a_i^n\|_B$ can be approximated by a rational chain. After multiplying through by a big integer to clear denominators one obtains an (approximating) integral chain. Gluing up triangles, one obtains a "collapsed surface" of the form S/T as above, with one vertex on each boundary component. This collapsed surface can be thickened to a genuine surface by adding a cylindrical collar to each boundary component, at the cost of adding a further $2m$ triangles. Since m is fixed but n is arbitrarily large, the desired inequality follows by applying Proposition 2.74, and Theorem 1.14. □

2.6.2. Generalized Bavard duality.

Definition 2.78. Let G be a group. Let $H(G)$ (for "homogeneous") be the subspace of $B_1(G)$ spanned by elements of the form $g - hgh^{-1}$ and $g^n - ng$ for $g, h \in G$ and $n \in \mathbb{Z}$. Denote the quotient space as $B_1^H(G) := B_1(G)/H(G)$ or B_1^H for short, if G is understood.

By construction, scl vanishes on the subspace $H(G)$, and therefore descends to a pseudo-norm on B_1^H. With this notation, we obtain the following statement of generalized Bavard duality:

Theorem 2.79 (Generalized Bavard Duality). *Let G be a group. Then for any* $\sum_i t_i a_i \in B_1^H(G)$ *there is an equality*

$$\text{scl}(\sum_i t_i a_i) = \frac{1}{2} \sup_{\phi \in Q/H^1} \frac{\sum_i t_i \phi(a_i)}{D(\phi)}$$

Proof. The proof is the same as that of Theorem 2.70 with Lemma 2.77 in place of Lemma 2.69. □

This mixture of group theoretic and homological language is convenient for deriving some interesting corollaries.

Proposition 2.80 (Finite index formula). *Let G be a group, and H a subgroup of finite index. Let $g_1, \cdots, g_m \in G$. Suppose $\pi_1(X) = G$, and let $\gamma_1, \cdots, \gamma_m$ be* *loops in X representing the conjugacy classes of the g_i. Let $p:\widehat{X}\to X$ be a finite cover corresponding to the subgroup H. Let $\beta_1,\cdots,\beta_l$ be the covers of the γ_i which lift to $\widehat{X}$, and $h_1,\cdots,h_l$ the corresponding conjugacy classes in H. Then*

$$\mathrm{scl}_G(\sum_i g_i)=\frac{1}{[G:H]}\cdot \mathrm{scl}_H(\sum_i h_i)$$

Proof. We use Proposition 2.74. Given a map of a surface $f:(S,\partial S)\to(X,\cup_i\gamma_i)$ there is a finite covering map $\pi:(\widehat{S},\partial\widehat{S})\to(S,\partial S)$ such that $f\pi$ lifts to $\widehat{f}:(\widehat{S},\partial\widehat{S})\to(\widehat{X},\cup_i\beta_i)$ in such a way that $p\widehat{f}=f\pi$. One way to construct such a π is to let $K<H$ be normal in G of finite index, and then take $\widehat{S}$ to be the regular cover of S corresponding to the kernel of the map $\pi_1(S)\to G/K$. Conversely, given $g:(S,\partial S)\to(\widehat{X},\cup_i\beta_i)$ the composition pg maps S to X, wrapping the boundary around the various γ_i. The result follows. □

In the case that H is normal and g is a single element in H, the finite index formula takes the following form:

Corollary 2.81. *Let G be a group, and let H be a normal subgroup of finite index, with (finite) quotient group $A=G/H$. Let $h\in H$. Then*

$$\mathrm{scl}_G(h)=\frac{1}{|A|}\cdot\mathrm{scl}_H(\sum_{a\in A}aha^{-1})$$

where for each $a\in A$, the expression aha^{-1} represents the corresponding (well-defined) conjugacy class in H.

Remark 2.82. One can give a more algebraic proof of Corollary 2.81 as follows. By Theorem 2.47, and the fact that finite groups are amenable, the map $H_b^2(G)\to H_b^2(H)$ is an isometric embedding with image equal to the A-invariant part of $H_b^2(H)$. If $\psi\in Q(H)$ then the projection ψ^A of ψ to $Q(H)^A$ is the sum $1/|A|\sum_a a^*\psi$. Here the group A acts on H by outer automorphisms: if $a=aH$ is a left coset of H, then aha^{-1} is a well-defined element of H up to an inner automorphism. In other words, $a^*\psi(h)=\phi(aha^{-1})$.

It follows that

$$\mathrm{scl}_G(h)=\sup_{\phi\in Q(G)}\frac{\phi(h)}{2D(\phi)}=\sup_{\psi\in Q(H)}\frac{\psi^A(h)}{2D(\psi^A)}$$

Now for any $\psi\in Q(H)$, one has

$$\psi^A(h)=\frac{1}{|A|}\sum_a\psi(aha^{-1})=\frac{1}{|A|}\sum_a\psi^A(aha^{-1})$$

Furthermore, $D(\psi^A)\le D(\psi)$ by convexity. It follows that

$$\frac{1}{|A|}\mathrm{scl}_H(\sum_a aha^{-1})=\sup_{\psi\in Q(H)}\frac{1}{|A|}\frac{\sum_a\psi(aha^{-1})}{2D(\psi)}=\sup_{\psi\in Q(H)}\frac{1}{|A|}\frac{\sum_a\psi^A(aha^{-1})}{2D(\psi^A)}$$

proving the formula.

Remark 2.83. Corollary 2.81 is useful even (especially?) when an element $h\in H$ is in $[G,G]$ but not in $[H,H]$.

One advantage of working with the space B_1^H over B_1 is that while scl is, except in trivial cases, never a genuine norm on B_1, it is sometimes a genuine norm on B_1^H.

Proposition 2.84. *Let F be a free group. Then* scl *is a genuine norm on the vector space $B_1^H(F)$.*

Proof. A chain c in $B_1^H(F)$ has a representative of the form $\sum_i t_i w_i$ where each w_i is a cyclically reduced primitive word in F, where all coefficients t_i are nonzero, and where no two $w_i^{\pm 1}, w_j^{\pm 1}$ are conjugate for distinct i, j. After reordering, assume that the length of $w := w_1$ is at least as big as that of any w_i. Let N be a sufficiently big integer (to be determined), and let φ be the homogenization of the big Brooks counting quasimorphism H_{w^N} associated to w^N. We claim that for sufficiently big N, there is equality $\varphi(w_i) = 0$ for any $i \neq 1$. Since $\varphi(w) = 1/N$, this shows that $\mathrm{scl}(c) \geq |t_1|/2ND(\varphi) > 0$.

To prove the claim, suppose to the contrary that for some $i \neq 1$ the infinite product w_i^∞ contains an arbitrarily big power w^N as a subword, where without loss of generality, we may assume N is positive. If $N = \mathrm{lcm}(|w|, |w_i|)/|w|$ then w^N is conjugate to w_i^M for some M. But elements in free groups have unique primitive roots, up to conjugacy, so this implies $M = N$ and w_i is conjugate to w, contrary to hypothesis. This establishes the claim, and the proposition. □

Remark 2.85. A similar argument using de Rham quasimorphisms in place of Brooks quasimorphisms works whenever G is equal to π_1 of a closed hyperbolic manifold. In fact, using generalized counting quasimorphisms § 3.5 one can show that scl is a norm on $B_1^H(G)$ whenever G is a hyperbolic group.

Remark 2.86. It is not true that fill is equal to the quotient norm on B_1^H under the exact sequence

$$H \to B_1 \to B_1^H$$

where B_1 and H have the $\|\cdot\|_B$ norm. For instance, in a free group, a (nontrivial) commutator $ghg^{-1}h^{-1}$ has scl norm 1/2, and therefore fill norm 2. On the other hand, the chains $ghg^{-1}h^{-1}$ and $ghg^{-1}h^{-1} + hgh^{-1} - g$ differ by an element of H, and

$$\partial(ghg^{-1}h^{-1}, hgh^{-1}) = ghg^{-1}h^{-1} + hgh^{-1} - g$$

so $\|ghg^{-1}h^{-1} + hgh^{-1} - g\|_B \leq 1$.

2.7. Further properties

In this section we enumerate some further properties of scl which will be used in the sequel.

2.7.1. Extremal quasimorphisms. Theorem 2.70 provides a method of calculating scl in some cases, especially when the dimension of $Q(G)$ is small. Given an element $a \in [G, G]$, it is natural to ask whether the supremum of $\phi(a)/D(\phi)$ is realized by some $\phi \in Q(G)$.

Definition 2.87. Let $a \in [G, G]$. An element $\phi \in Q(G)$ is *extremal* for a if

$$\mathrm{scl}(a) = \frac{\phi(a)}{2D(\phi)}$$

The union of 0 with the set of homogeneous quasimorphisms on G which are extremal for a is denoted $Q_a(G)$.

The next Proposition shows that extremal quasimorphisms always exist.

Proposition 2.88. *Let $a \in [G, G]$. Then $Q_a(G)$ is a nontrivial convex cone in $Q(G)$ which is closed both in the defect and the weak* topology.*

PROOF. Recall from Remark 2.60 that there is an isomorphism of vector spaces $Q/H^1 \cong (Z_2^{l_1}/\overline{Z}_2)'$. As a dual space, we can endow Q/H^1 with the weak* topology. A subset closed in the weak* topology is also closed in the defect topology.

The space

$$K := \{\varphi \in Q/H^1 \text{ such that } D(\phi) \le 1/2\}$$

is convex, closed and bounded with respect to the defect norm and therefore also with respect to the operator norm (since these two norms differ by a factor of at most 2). Hence K is weak* compact.

Fix an element $a \in [G,G]$ and for each n, define

$$K^n := \{\varphi \in K \text{ such that } \varphi(a) \ge \mathrm{scl}(a) - 1/n\}$$

Let us show that K^n is weak* closed. Since $[G,G] \subset B_1$, there is $A \in C^2$ such that $dA = a$. The element $A - \sigma(a)$, where σ is the section defined in Lemma 2.61, satisfies $A - \sigma(a) \in Z_2^{l_1}$ and further satisfies $\delta\varphi(A - \sigma(a)) = \varphi(a)$ for any homogeneous φ. This, together with the defining property of K_n, shows that K_n is weak* closed.

The K^n are closed and contained in K and are therefore weak* compact. By Bavard duality (Theorem 2.70), each K^n is nonempty, and therefore their intersection is nonempty. Any element $\varphi \in \cap_n K^n$ has $\varphi(a) = \mathrm{scl}(a)$ and $D(\varphi) = 1/2$. Conversely any $\varphi \in Q_a(G)$ can be scaled to have $D(\varphi) = 1/2$, and therefore $Q_a(G)$ is exactly equal to the cone on the weak* compact set $\cap_n K^n$. This completes the proof. □

REMARK 2.89. In a similar way we may define $Q_a(c)$ for any chain $c \in B_1$. The proof of Proposition 2.88 extends easily to this case.

2.7.2. Left exactness and Bouarich's Theorem.

For the convenience of the reader, we provide a proof of Bouarich's Theorem 2.49. Recall that Bouarich's Theorem says if

$$K \xrightarrow{\iota} G \xrightarrow{\rho} H \to 0$$

is an exact sequence of groups then the induced sequence

$$0 \to H_b^2(H;\mathbb{R}) \xrightarrow{\rho^*} H_b^2(G;\mathbb{R}) \xrightarrow{\iota^*} H_b^2(K;\mathbb{R})$$

is left exact. In fact, it is no more difficult to give a proof of Bouarich's theorem which is valid for any Abelian coefficient group; in particular, the proof we give below applies to bounded cohomology with $\mathbb{Z}$ coefficients.

PROOF. Without loss of generality, we can replace K by its image $\iota(K)$. So we can assume K is a subgroup of G, and ι is the inclusion homomorphism. Since $\rho\iota$ is the zero map, the composition $H_b^2(H) \to H_b^2(G) \to H_b^2(K)$ is zero. So we just need to check that ρ^* is injective, and that everything in $\ker(\iota^*)$ is in the image of the map ρ^*.

CLAIM. *The map $\rho^* : H_b^2(H) \to H_b^2(G)$ is an injection.*

PROOF. Suppose ψ be a bounded 2-cocycle on H whose image in $H_b^2(H)$ is nonzero, but for which $\rho^*\psi = \delta\phi$ on G, where ϕ is bounded. Observe that for all $a_1, a_2 \in G$ and $k_1, k_2 \in K$ that

$$\phi(a_1) + \phi(a_2) - \phi(a_1a_2) = \phi(a_1k_1) + \phi(a_2k_2) - \phi(a_1k_1a_2k_2)$$

In particular, $\phi(k^{n+1}) - \phi(k^n) = \phi(ak^{n+1}) - \phi(ak^n)$ for any $a \in G$, $k \in K$. Taking $a = k$ this implies $\phi(k^n) = n(\phi(k) - \phi(\mathrm{id})) + \phi(\mathrm{id})$. But ϕ is bounded, so $\phi(k) -$

$\phi(\mathrm{id}) = 0$ for all $k \in K$, and more generally, ϕ is constant on left cosets. This implies that ϕ descends to a bounded function ϕ_H on $H = G/K$ which by construction satisfies $\delta\phi_H = \psi$. □

CLAIM. *Let $[\psi] \in H_b^2(G)$ be in the kernel of $\iota^* : H_b^2(G) \to H_b^2(K)$. Then $[\psi]$ is in the image of $H_b^2(H)$.*

PROOF. By hypothesis, for any representative ψ of $[\psi]$ there is a bounded function ϕ on K such that $\delta\phi = \psi$ on K. If $\psi(\mathrm{id}, \mathrm{id}) = c \neq 0$ then we replace ψ by $\psi - \delta h_c$ where h_c is the constant bounded 1-cochain $h_c(g) = c$. So without loss of generality, we can assume that $\psi(\mathrm{id}, \mathrm{id}) = 0$. This leads to the convenient normalization $\phi(\mathrm{id}) = 0$.

We want to extend ϕ in a suitable way to a function ϕ_G on all of G. For each $h_i \in H$, choose a left coset representative g_i of h_i in G. For each h_i we define $\phi_G(g_i) = 0$. Then for each $k \in K$ we set $\phi_G(g_i k) = \psi(g_i, k) - \phi(k)$. Since ϕ and ψ are bounded, ϕ_G is bounded. Now define $\psi' = \psi + \delta\phi_G$. Since ϕ_G is bounded, ψ' and ψ represent the same cohomology class. Moreover, for any g in G and $k \in K$ we write $g = g_i k_i$ and calculate

$$\begin{aligned}\psi'(g, k) &= \psi(g_i k_i, k) + \phi_G(g_i k_i) + \phi_G(k) - \phi_G(g_i k_i k)\\ &= \psi(g_i k_i, k) + \psi(g_i, k_i) - \phi(k_i) + \psi(\mathrm{id}, k) - \phi(k) - \psi(g_i, k_i k) + \phi(k_i k)\end{aligned}$$

Since $\phi(\mathrm{id}) = 0$, we have $\psi(\mathrm{id}, k) = \delta\phi(\mathrm{id}, k) = \phi(\mathrm{id}) + \phi(k) - \phi(k) = 0$. Moreover, $-\phi(k_i) - \phi(k) + \phi(k_i k) = -\delta\phi(k_i, k) = -\psi(k_i, k)$. Therefore we can write

$$\psi'(g, k) = \psi(g_i k_i, k) + \psi(g_i, k_i) - \psi(k_i, k) - \psi(g_i, k_i k) = -\delta\psi(g_i, k_i, k) = 0$$

We claim that ψ' can be obtained by pulling back a bounded 2-cocycle from H. Let $g_1, g_2 \in G$ and $k \in K$. Since $\delta\psi'(g_1, g_2, k) = 0$, we calculate

$$\psi'(g_1, g_2 k) - \psi'(g_1, g_2) = \psi'(g_1 g_2, k) - \psi'(g_2, k) = 0$$

and therefore $\psi'(g_1, g_2 k) = \psi'(g_1, g_2)$ for any $g_1, g_2 \in G$ and any $k \in K$.

Similarly, since $\delta\psi'(g_1, k, g_2) = 0$ we have

$$\psi'(g_1, k g_2) - \psi'(g_1, k) = \psi'(g_1 k, g_2) - \psi'(k, g_2)$$

We have shown that $\psi'(g_1, k) = 0$. Moreover, $\psi'(g_1, k g_2) = \psi'(g_1, g_2(g_2^{-1} k g_2))$ which is equal to $\psi'(g_1, g_2)$ by our earlier calculation. Rearranging, we obtain

$$\psi'(g_1 k, g_2) - \psi'(g_1, g_2) = \psi'(k, g_2)$$

and therefore

$$\psi'(g_1 k^n, g_2) = \psi'(g_1, g_2) + n\psi'(k, g_2)$$

for any integer n. Since n is arbitrary but ψ' is bounded, we see that $\psi'(k, g_2) = 0$ for any $g_2 \in G$ and $k \in K$ and therefore also $\psi'(g_1 k, g_2) = \psi'(g_1, g_2)$. In particular, ψ' is constant on left cosets of K, and descends to a cocycle on H. □

This completes the proof of Bouarich's Theorem. □

REMARK 2.90. A similar but more straightforward argument proves the left exactness of Q.

REMARK 2.91. There is a more direct proof of Bouarich's Theorem using spectral sequences. In fact, the astute reader will recognize that the proof given above is really a spectral sequences argument in disguise, together with the observation that H_b^1 is always zero. However one must be careful in general, since bounded cohomology is typically not separated in degree 3 and higher (see the end of § 2.4.1 and § 2.5.1). This is a point which is sometimes overlooked in the literature on bounded cohomology. Nevertheless, in sufficiently low dimensions, such an argument can be made to work. See e.g. Chapter 12 of [**157**], especially Example 12.4.3.

2.7.3. Rotation numbers.

As an application of Theorem 2.70 we obtain a precise estimate of the defect of rotation number.

PROPOSITION 2.92. *Let G be a subgroup of* $\mathrm{Homeo}^+(S^1)$ *and let $\widehat{G}$ be the preimage in* $\mathrm{Homeo}^+(\mathbb{R})$*. Then $D(\mathrm{rot}) \le 1$ as a homogeneous quasimorphism on $\widehat{G}$.*

PROOF. For the sake of brevity, let $T = \mathrm{Homeo}^+(S^1)$ and let $\widehat{T} = \mathrm{Homeo}^+(\mathbb{R})^{\mathbb{Z}}$. By Remark 2.44 we see that $Q(T) = 0$. The exact sequence $\mathbb{Z} \to \widehat{T} \to T$ together with Bouarich's Theorem 2.49 and the vanishing of H_b^* for amenable groups implies that $H_b^2(T) \to H_b^2(\widehat{T})$ is an isomorphism. On the other hand, the map $H^2(T) \to H^2(\widehat{T})$ is not injective, and the kernel is generated by the class of the (universal) central extension $\widehat{T} \to T$. It follows that $Q(\widehat{T})$ is 1-dimensional, and generated exactly by rot. By Theorem 2.43 there is an equality $\mathrm{scl}(a) = |\mathrm{rot}(a)|/2$ for every $a \in \widehat{T}$ and therefore $D(\mathrm{rot}) = 1$, by Bavard's Theorem 2.70. It follows that $D(\mathrm{rot}) \le 1$ on any subgroup of $\widehat{T}$. □

2.7.4. Free products.

Bavard Prop. 3.7.2 [**8**] asserts that if G_1 and G_2 are two groups, and $G = G_1 * G_2$ is their free product, then for all nontrivial elements $g_i \in G_i$, there is an equality $\mathrm{scl}(g_1g_2) = \mathrm{scl}(g_1) + \mathrm{scl}(g_2) + 1/2$. This assertion is not quite true as stated. Nevertheless, it turns out that Bavard's assertion is true *when g_1 and g_2 have infinite order*, and can be suitably modified when one or both of them are torsion. We give the correct statement and proof, and defer a discussion of Bavard's argument and what can be salvaged from it to the sequel.

THEOREM 2.93 (Product formula). *Let G_1, G_2 be groups, and for $i = 1, 2$ let g_i be a nontrivial element in G_i of order n_i. Let $G = G_1 * G_2$. Then there is an equality*

$$\mathrm{scl}_G(g_1g_2) = \mathrm{scl}_{G_1}(g_1) + \mathrm{scl}_{G_2}(g_2) + \frac{1}{2}\left(1 - \frac{1}{n_1} - \frac{1}{n_2}\right)$$

where $1/n_i$ may be replaced by 0 when $n_i = \infty$.

PROOF. Build a space X as follows. Let X_1, X_2 be spaces with $\pi_1(X_i) = G_i$, and let γ_i be a loop in X_i representing the conjugacy class of g_i. Let P be a pair of pants. Let $X = X_1 \cup X_2 \cup P$ be obtained by gluing two boundary components of P to γ_1 and γ_2 respectively, and let γ_P denote the unglued boundary component of P.

Let S be a surface with one boundary component, and $f : S \to X$ a map sending ∂S to γ_P with degree n. We have $\mathrm{scl}(g_1g_2) \le -\chi(S)/2n$. Make f transverse to the γ_i. The surface is decomposed into *pieces*, which are the closures, in the path topology, of $S - f^{-1}(\gamma_1 \cup \gamma_2)$. We say that f is *efficient* if no piece has a boundary component which maps with degree zero to a γ_i, and if no piece is an annulus with both boundary components mapping to the same γ_i with opposite degree.

If S is not efficient, the Euler characteristic of S can be increased by surgering S along a circle which maps to some γ_i with degree 0 (and is therefore null-homotopic), or simplified by homotoping a trivial annulus. So without loss of generality, it suffices to consider the case that f is efficient. Let S_i denote the union of the pieces mapping to X_i, and S_P the union of pieces mapping to P. Let f_1, f_2, f_P be the restrictions of f to these unions. These maps are all proper. Since f is efficient, no piece mapping to P is a disk or annulus. In other words, S_P admits a hyperbolic metric. Moreover, the only disk pieces are components of S_i mapping with degree a multiple of n_i to γ_i, in the case g_i is torsion.

Since f_P is proper, it has a well-defined degree. Since $f_P^{-1}(\gamma_P)$ is equal to ∂S, the degree is n. By the definition of degree, the union of components of ∂S_P mapping to each γ_i maps with degree n, and therefore $n(S_1) = n(S_2) = n$ in the notation of Proposition 2.10. By replacing f_P by a pleated map (with respect to a choice of hyperbolic structures on S_P and on P) and Gauss–Bonnet, we obtain an inequality $-\chi(S_P)/2n \ge -\chi(P)/2 = 1/2$.

If each g_1, g_2 has infinite order, no component of S_i is a disk. In this case, $-\chi^-(S) = -\chi^-(S_1) - \chi^-(S_2) - \chi^-(S_P)$, and therefore

$$\frac{-\chi^-(S)}{2n} = \frac{-\chi^-(S_1)}{2n} + \frac{-\chi^-(S_2)}{2n} + \frac{-\chi^-(S_P)}{2n} \ge \mathrm{scl}(g_1) + \mathrm{scl}(g_2) + \frac{1}{2}$$

Since S was arbitrary, we obtain an inequality

$$\mathrm{scl}(g_1 g_2) \ge \mathrm{scl}(g_1) + \mathrm{scl}(g_2) + \frac{1}{2}$$

Conversely, by the proof of Lemma 2.24 the elements $(g_1 g_2)^{2n}$ and $g_1^{2n} g_2^{2n}$ differ by at most n commutators, and therefore we obtain the first inequality

$$\mathrm{scl}(g_1 g_2) \le \mathrm{scl}(g_1) + \mathrm{scl}(g_2) + \frac{1}{2}$$

This proves the theorem when the g_i have infinite order.

If g_i is torsion of order n_i, then S_i may have disk components whose boundaries map to g_i with degree a multiple of n_i. In this case, S_i might have as many as n/n_i disk components, and therefore $\chi(S_i)$ might be as big as n/n_i, so we obtain an inequality

$$-\chi^-(S) \ge -\chi^-(S_1) - \chi^-(S_2) - \chi^-(S_P) - \frac{n}{n_1} - \frac{n}{n_2}$$

which, after dividing by $2n$, and taking the infimum over all S, gives

$$\mathrm{scl}(g_1 g_2) \ge \mathrm{scl}(g_1) + \mathrm{scl}(g_2) + \frac{1}{2}\left(1 - \frac{1}{n_1} - \frac{1}{n_2}\right)$$

To obtain the reverse inequality, replace P by an orbifold with a cone point of order n_i in place of the γ_i boundary component(s) and take a finite cover which is a smooth surface. This completes the proof. □

Remark 2.94. The use of geometric language is really for convenience of exposition rather than mathematical necessity. A similar argument could be made by replacing maps to X with equivariant maps to a suitable Bass–Serre tree.

One drawback of the method of proof is that it does not exhibit an extremal homogeneous quasimorphism for the element $g_1 g_2$. In the next section we show how to construct such an extremal quasimorphism in the case that G_1 and G_2 are *left orderable*.

Remark 2.95. Bavard, in [**8**], exhibits a nontrivial quasimorphism for g_1g_2 arising from the structure of $G_1 * G_2$ as a free product and its action on a Bass–Serre tree, which is a special case of a construction that will be discussed in more detail in § 3.5. One can estimate the defect of the quasimorphism constructed in this way, but the estimate is not good enough to establish Theorem 2.93.

2.7.5. Left-orderability.

Definition 2.96. Let G be a group. G is *left orderable* (LO for short) if there is a total ordering $<$ on G which is invariant under left multiplication. That is, for all $a, b, c \in G$, the inequality $a < b$ holds if and only if $ca < cb$.

Right orderability is defined similarly. A group is left orderable if and only if it is right orderable. The difference is essentially psychological.

Example 2.97 (Locally indicable). A group is *locally indicable* if every nontrivial finitely generated subgroup admits a surjective homomorphism to $\mathbb{Z}$. For example, free groups are locally indicable. A more nontrivial example, due to Boyer–Rolfsen–Wiest [**22**] says that if M is an irreducible 3-manifold, and $H^1(M) \neq 0$ then $\pi_1(M)$ is locally indicable.

A theorem of Burns–Hale [**36**] says that every locally indicable group is left orderable.

Left orderability is intimately bound up with 1-dimensional dynamics. The following "folklore" theorem is very well-known.

Theorem 2.98 (Action on $\mathbb{R}$). *A countable group G is left orderable if and only if there is an injective homomorphism $G \to \mathrm{Homeo}^+(\mathbb{R})$.*

We give a sketch of a proof. For more details, see [**40**].

Proof. Suppose G acts faithfully on $\mathbb{R}$ by homeomorphisms. Suppose $p \in \mathbb{R}$ has trivial stabilizer. Then define $a > \mathrm{id}$ if and only if $a(p) > p$. Conversely, suppose G is left orderable. The order topology on G makes G order-isomorphic to a countable subset of $\mathbb{R}$. Include $G \hookrightarrow \mathbb{R}$ in an order-preserving way, compatibly with the order topology. Then the action of G on itself extends to an action on the closure of its image. The complement is a countable union of intervals; the action of G extends uniquely to a permutation action on these intervals. ☐

The first part of the next proposition is a special case of Theorem 2.93; however, the proof is different, and shows how to construct an explicit extremal quasimorphism for g_1g_2.

Proposition 2.99. *Let G_1, G_2 be left orderable, and suppose $g_i \in G_i$ are nontrivial. Then there is an equality*

$$\mathrm{scl}(g_1g_2) = \mathrm{scl}(g_1) + \mathrm{scl}(g_2) + \frac{1}{2}$$

Moreover there is an explicit construction of an extremal quasimorphism for g_1g_2 in terms of extremal quasimorphisms for g_1 and g_2.

Proof. Assume first that G_1, G_2 are countable. Using Theorem 2.98, construct an orientation-preserving action of $G_1 * G_2$ on S^1 where G_1 fixes the point -1 and G_2 fixes the point 1 (here we think of S^1 as the unit circle in $\mathbb{C}$). Since g_1, g_2 are nontrivial, without loss of generality we can assume $g_1(i) = -i$ and $g_2(-i) = i$.

But then g_1g_2 has a fixed point, and therefore its rotation number is trivial (in $\mathbb{R}/\mathbb{Z}$). We lift the action to an action on $\mathbb{R}$, which can be done by lifting each G_i individually to have a global fixed point. Then rot is a homogeneous quasimorphism on $G_1 * G_2$, which vanishes on G_1 and on G_2, and satisfies $\text{rot}(g_1g_2) = 1$. Proposition 2.92 shows that $D(\text{rot}) \le 1$. Adding to rot pullbacks of extremal quasimorphisms with defect 1 for g_1 and g_2 under the surjections $G_1 * G_2 \to G_1$ and $G_1 * G_2 \to G_2$, one obtains an explicit extremal quasimorphism for g_1g_2 which, by Bavard duality, proves the proposition.

If G_1, G_2 are not countable, one substitutes actions on circularly ordered sets for actions on circles. The distinction between these two contexts is more psychological than substantial. See e.g. [**40**], especially Chapter 2, for a discussion. ☐

Example 2.100 (Bavard, p. 146 [**8**]). In $F_2 = \langle u, v\rangle$ the element $[u, v]$ satisfies $\text{scl}([u, v]) = 1/2$, by Theorem 1.14 and Theorem 2.70. Let $G = \langle u_1, v_1, \cdots, u_k, v_k\rangle$. Then by Proposition 2.99 and induction,

$$\text{scl}(\prod_i [u_i, v_i]^{p_i}) = \frac{1}{2}\sum |p_i| + \frac{k-1}{2}$$

since free groups are locally indicable and therefore left orderable (see Example 2.97).

The interaction of left orderability and scl (especially in order to obtain sharp estimates in free groups) will be discussed again in § 4.3.4.

2.7.6. Self-products. There is an analogue of Theorem 2.93 with (free) HNN extensions in place of free products. For convenience, we state and prove the theorem only in the case that the elements in question are torsion free.

Theorem 2.101 (Self-product formula). *Let G be a group, and $g_1, g_2 \in G$ two elements of infinite order. Let $G' = G * \langle t\rangle$. Then there is an equality*

$$\text{scl}_{G'}(g_1tg_2t^{-1}) = \text{scl}_G(g_1 + g_2) + \frac{1}{2}$$

Proof. Let X be a space with $\pi_1(X) = G$. Let γ_1, γ_2 be loops representing the conjugacy classes of g_1, g_2 respectively. Let P be a pair of pants, and let $Y = X \cup P$ be obtained by gluing two boundary components of P to γ_1 and γ_2 respectively, and let γ_P denote the unglued boundary component of P.

Notice that $\pi_1(Y) = G'$ and γ_P represents the conjugacy class of $g_1tg_2t^{-1}$. If $f : S \to Y$ sends ∂S to γ_P with degree n, then after making f efficient, S decomposes into $f_X : S_X \to X$ and $f_P : S_P \to P$. The degree of f_P is n, so $-\chi^-(S_P)/2n \ge 1/2$, and $-\chi^-(S_X)/2n$ is an upper bound for $\text{scl}(g_1 + g_2)$. Since the g_i have infinite order, no component of S_X is a disk, and therefore $-\chi^-(S) = -\chi^-(S_P) - \chi^-(S_X)$. The proof now follows, as in the proof of Theorem 2.93, from Proposition 2.10 and Proposition 2.74. ☐

Remark 2.102. Note that the same proof shows

$$\text{scl}_{G'}(g_1tg_2t^{-1} + \sum t_ig_i) = \text{scl}_G(g_1 + g_2 + \sum t_ig_i) + \frac{1}{2}$$

for any $\sum t_ig_i \in B_1^H$ where we sum over $i \ge 3$.

Remark 2.103. By Remark 2.102 and by the linearity and continuity of scl on B_1^H, the calculation of scl on B_1^H can be reduced to calculations of scl on "ordinary" elements of $G * F$ for sufficiently large free groups F.

2.7.7. LERF and injectivity. Recall Proposition 2.10, which says that if X is a space with $\pi_1(X) = G$, and γ is a loop in X representing the conjugacy class of a, then

$$\mathrm{scl}(a) = \inf_S \frac{-\chi^-(S)}{2n(S)}$$

where the infimum is taken over all maps of oriented surfaces $f : S \to X$ whose boundary components all map to γ with sum of degrees equal to $n(S)$. Recall (Definition 2.11) that f, S is said to be *extremal* if it realizes the infimum. The following proposition says that extremal surfaces must be π_1-injective.

PROPOSITION 2.104 (injectivity). *Let X, γ be as above. Suppose f, S as above is extremal. Then the map $f : S \to X$ induces a monomorphism $\pi_1(S) \to \pi_1(X)$.*

Before we prove the proposition, we must discuss the property LERF for surface groups.

DEFINITION 2.105. Let G be a group. Then G is *locally extended residually finite* (or LERF for short) if all of its finitely generated subgroups are separable. That is, for all finitely generated subgroups H and all elements $a \in G - H$ there is a subgroup H' of G of finite index which contains H but not a.

EXAMPLE 2.106 (Malcev; polycyclic groups). A solvable group is *polycyclic* if all its subgroups are finitely generated. Malcev [**142**] showed that polycyclic groups are LERF.

EXAMPLE 2.107 (Hall; free groups). Marshall Hall [**102**] showed that free groups are LERF. In fact, he showed that free groups satisfy the stronger property that finitely generated subgroups are *virtual retracts.* We sketch an illuminating topological proof of this fact due to Stallings [**191**].

Let F be free, and let G be a finitely generated proper subgroup. Represent $F = \pi_1(X)$ where X is a wedge of circles, and let $\tilde{X}$ be a cover of X corresponding to the subgroup G. Since G is a finitely generated subgroup of a free group, it is free of finite rank, so $\tilde{X}$ deformation retracts to a compact subgraph X_G with $\pi_1(X_G) = G$. Each directed edge of X_G is labeled by a generator of F. Let X'_G be another copy of X_G with each directed edge labeled by the inverse of the corresponding label in X_G. For each vertex v of X_G, let v' be the corresponding vertex of X'_G. Join v to v' by a collection of edges, one for each generator of $\pi_1(X)$ not represented by an edge in X_G with a vertex at v. Let the result be X''_G. Then by construction, X''_G is a finite covering of X, and therefore corresponds to a finite index subgroup H of F. Moreover, by construction, G is a free summand of H.

EXAMPLE 2.108 (Scott; surface groups). Peter Scott [**185**] showed that surface groups are LERF. For surfaces with boundary, this is a special case of Example 2.107, but even in this case, Scott's proof is different and illuminating.

Let S be a surface with $\chi(S) < 0$. Observe that S can be tiled by right-angled hyperbolic pentagons, for some choice of hyperbolic structure on S. Let G be a finitely generated subgroup of $\pi_1(S)$, and let $\tilde{S}$ be the covering corresponding to G. The surface $\tilde{S}$ deformation retracts to a compact subsurface X with $\pi_1(X) = G$. This subsurface can be engulfed by a convex union Y of right-angled hyperbolic pentagons. Since all the pentagons are right-angled, Y is a surface with right-angled corners. There is a hyperbolic orbifold obtained from Y by adding mirrors to the non-boundary edges. This orbifold has a finite index subgroup, containing G, which is also finite index in $\pi_1(S)$.

A geometric corollary of property LERF for free and surface groups is the fact that for any hyperbolic surface S and any geodesic loop γ in S there is a finite cover $\tilde{S}$ of S to which γ lifts as an embedded loop. Using this fact, we now prove Proposition 2.104:

PROOF. Suppose S minimizes $-\chi(S)/2n(S)$ but $f : S \to X$ is not injective in π_1. Let $a \in \pi_1(S)$ be in the kernel. Choose a hyperbolic structure on S, and represent the conjugacy class of a by a geodesic loop γ in S. If γ is embedded, compress S along γ to produce a surface S' satisfying $-\chi(S') < -\chi(S)$. The compression factors through f, and there is a map $f' : S' \to X$ satisfying $n(S') = n(S)$, contrary to the minimality of S.

If γ is not embedded, let $\tilde{S}$ be a finite cover of S to which γ lifts as an embedded loop. Let $\pi : \tilde{S} \to S$ be the covering map. Since both χ and $n(\cdot)$ are multiplicative under covers, there is an equality $-\chi(S)/2n(S) = -\chi(\tilde{S})/2n(\tilde{S})$. But $\tilde{S}$ can be compressed along γ to produce a new surface $\tilde{S}'$. The compression factors through $f\pi$, contradicting the minimality of S, as before. This contradiction shows that f is injective on $\pi_1(S)$, as claimed. □

This lets us give a short proof of the following corollary. Note that this corollary is easy to prove in many other ways. For instance, it follows from the fact that every subgroup of a free group is free, and from the theorem of Malcev [**141**] that free groups are Hopfian (i.e. surjective self-maps are injective).

COROLLARY 2.109. *Let $\rho : F_2 \to F$ be a homomorphism from F_2, the free group on two elements, to F, a free group. If the image is not Abelian, ϕ is injective.*

PROOF. Let X be a wedge of circles with $\pi_1(X) = F$. Let $F_2 = \langle a, b\rangle$. The map ρ defines a map from a punctured torus S into X, taking the boundary to $\rho([a, b])$. By hypothesis, this element is nontrivial in F. If ρ is not injective, Proposition 2.104 implies $\text{scl}(\rho([a, b])) < 1/2$. But we will show in § 4.3.4 that every nontrivial element in a free group satisfies $\text{scl} \geq 1/2$. □