CHAPTER 4

Part 1. Examples of Γ.

In Part 1 of this chapter, we shall give some examples of Γ. They are obtained from quaternion algebras A over totally real algebraic number fields F; and up to commensurability, they are the only examples of Γ that we know at present. We shall also prove that if L is a quasi-irreducible $G_{\mathfrak{p}}$-field over $\mathbf{C}$ such that the corresponding discrete subgroup is commensurable with one obtained from a quaternion algebra A over F, then the field k_0 (defined by Theorem 5 of Chapter 2) contains F (see Theorem 1, §5).

Examples of Γ.

§1. Quaternion algebra. By a quaternion algebra over a field F, we mean a simple algebra A with center F and with $[A : F] = 4$. The simplest example is $A = M_2(F)$, and all other quaternion algebras are division algebras. In the following, we shall make no distinction between two quaternion algebras over F which are isomorphic over F. If F is algebraically closed (e.g., if $F = \mathbf{C}$), then $A = M_2(F)$ is the only quaternion algebra over F. If $F = \mathbf{R}$ or $F = k_{\mathfrak{p}}$ ($\mathfrak{p}$-adic number field), then there is a *unique* division quaternion algebra over F, which will be denoted by $D_{\mathbf{R}}$ or $D_{\mathfrak{p}}$ respectively.

Now let F be an algebraic number field, and let $\mathfrak{p}$ be a prime divisor (finite or infinite) of F. Denote by $F_{\mathfrak{p}}$ the $\mathfrak{p}$-adic completion of F, so that either $F_{\mathfrak{p}} \cong \mathbf{C}$, or $F_{\mathfrak{p}} \cong \mathbf{R}$, or $F_{\mathfrak{p}}$ is a $\mathfrak{p}$-adic number field. For each quaternion algebra A over F, put $A_{\mathfrak{p}} = A \otimes_F F_{\mathfrak{p}}$; hence $A_{\mathfrak{p}}$ is a quaternion algebra over $F_{\mathfrak{p}}$. Therefore, if $F_{\mathfrak{p}} \cong \mathbf{C}$, $A_{\mathfrak{p}}$ must be $M_2(\mathbf{C})$, and if $F_{\mathfrak{p}} \ncong \mathbf{C}$, then there are two possibilities for $A_{\mathfrak{p}}$; namely, $M_2(F_{\mathfrak{p}})$ or $D_{\mathfrak{p}}$ (or $D_{\mathbf{R}}$ if $F_{\mathfrak{p}} \cong \mathbf{R}$). A prime divisor $\mathfrak{p}$ of F is called *unramified* in A if $A_{\mathfrak{p}} \cong M_2(F_{\mathfrak{p}})$ holds, and *ramified* if $A_{\mathfrak{p}} \ncong M_2(F_{\mathfrak{p}})$. Denote by $\delta(A)$ the set of all prime divisors of F which are ramified in A. Then it is well-known that $\delta(A)$ is finite and that its cardinal number is even. Conversely, if δ is any finite set of prime divisors of F not containing complex prime divisors and having even cardinal number, then there exists a quaternion algebra A over F, unique up to an isomorphism over F, such that $\delta = \delta(A)$;

(1) $$A \underset{1:1}{\leftrightarrow} \delta.$$

In particular, $A = M_2(F)$ corresponds to $\delta = \phi$.

Let A be a quaternion algebra over an algebraic number field F. For each $\mathfrak{p}$, put

$$(2)\qquad A_{\mathfrak{p}}^1 = \{x_{\mathfrak{p}} \in A_{\mathfrak{p}} \mid N_{A_{\mathfrak{p}}/F_{\mathfrak{p}}} x_{\mathfrak{p}} = 1\}.$$

Then $A_{\mathfrak{p}}^1$ is a locally compact group under multiplication, and

$$(3)\qquad \begin{cases} A_{\mathfrak{p}}^1 \cong SL_2(F_{\mathfrak{p}}) & \cdots\ \mathfrak{p} \notin \delta, \\ \quad = \text{compact} & \cdots\ \mathfrak{p} \in \delta. \end{cases}$$

Let S_∞ be the set of all infinite prime divisors of F, and let S be any *finite* set of prime divisors of F *containing* S_∞. Put

$$(4)\qquad A_S^1 = \prod_{\mathfrak{p}\in S} A_{\mathfrak{p}}^1 \quad \text{(direct product)}.$$

Let $\mathfrak{o}$ be the ring of integers of F, and let $\mathfrak{o}^{(S)}$ be the ring of all elements of F of the form α/β with $\alpha, \beta \in \mathfrak{o}$ such that all prime factors of $\beta\mathfrak{o}$ are contained in S; or in short;[1]

$$(5)\qquad \mathfrak{o}^{(S)} = \bigcup_{n=0}^{\infty} (\mathfrak{p}_1 \cdots \mathfrak{p}_s)^{-n}, \quad S - S_\infty = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_s\}.$$

In particular, $\mathfrak{o}^{(S_\infty)} = \mathfrak{o}$. By an $\mathfrak{o}^{(S)}$-order $\mathcal{O}^{(S)}$ of A, we mean a subring of A containing 1 which is a finite $\mathfrak{o}^{(S)}$-module and which satisfies $\mathcal{O}^{(S)} \otimes_{\mathfrak{o}^{(S)}} F = A$. Then, it is easy to see that all $\mathfrak{o}^{(S)}$-orders are given by $\mathcal{O} \otimes_{\mathfrak{o}} \mathfrak{o}^{(S)}$ with some $\mathfrak{o}$-order $\mathcal{O}$. Now let $\mathcal{O}^{(S)}$ be an $\mathfrak{o}^{(S)}$-order of A, and put

$$(6)\qquad \Gamma^{(S)} = \{x \in \mathcal{O}^{(S)} \mid N_{A/F} x = 1\}.$$

By the diagonal embedding, we shall consider $\Gamma^{(S)}$ as a subgroup of A_S^1;

$$(7)\qquad \Gamma^{(S)} \subset A_S^1.$$

Then $\Gamma^{(S)}$ *is a discrete subgroup of* A_S^1; *the quotient* $A_S^1/\Gamma^{(S)}$ *has finite invariant volume and is compact if and only if* $A \neq M_2(F)$; *if* $\mathfrak{p}_0 \in S$, $\mathfrak{p}_0 \notin \delta$, *then the projection of* $\Gamma^{(S)}$ *to* $\prod_{\mathfrak{p}\in S-\{\mathfrak{p}_0\}} A_{\mathfrak{p}}^1$ *is dense in the latter.* These are special cases of more general theorems on arithmetic of algebraic groups (cf. [2], [9], [20]). Since $A_{\mathfrak{p}}^1$ for $\mathfrak{p} \in \delta$ are compact, it is clear that if we replace A_S^1 by $A_{S-\delta}^1 = \prod_{\mathfrak{p}\in S, \notin\delta} A_{\mathfrak{p}}^1$ and consider $\Gamma^{(S)}$ as a subgroup of $A_{S-\delta}^1$, then we still get the same results as those italicized above.

§2. Now let $k_{\mathfrak{p}}$ be a given $\mathfrak{p}$-adic number field, and let us construct discrete subgroups of $SL_2(\mathbf{R}) \times SL_2(k_{\mathfrak{p}})$ by the above method. Thus, the problem is to find F, S, and δ ($\leftrightarrow A$) such that $A_{S-\delta}^1 \cong SL_2(\mathbf{R}) \times SL_2(k_{\mathfrak{p}})$. First, $S - \delta$ cannot contain complex prime divisors. But S must contain all infinite prime divisors of F, and δ cannot contain complex prime divisors. Therefore, F cannot have complex prime divisors at all, so that F must be totally real. Since $S - \delta$ contains one and only one real prime divisor, δ contains all real prime divisors of F except one. Also, F must have a finite prime divisor $\mathfrak{p} \notin \delta$ such that $F_{\mathfrak{p}} \cong k_{\mathfrak{p}}$.

[1] We shall call this ring $\mathfrak{o}^{(S)}$ *the ring of all elements of* F *which are integral except at* S.

Therefore, the necessary and sufficient conditions (on F, δ, S) for $A^1_{S-\delta}$ to be isomorphic to $SL_2(\mathbf{R}) \times SL_2(k_\mathfrak{p})$ are the following:

$$\text{(8)} \qquad \begin{cases} F: & \text{totally real, } \exists \text{ a finite prime divisor } \mathfrak{p} \text{ of } F \\ & \text{such that } F_\mathfrak{p} \cong k_\mathfrak{p}; \\ \delta: & \text{contains all real prime divisors of } F \text{ but one,} \\ & \text{and } \delta \not\ni \mathfrak{p}; \\ S = & S_\infty \cup \{\mathfrak{p}\}. \end{cases}$$

It is clear that there exist such F, δ and S. Take any such F, δ, put $S = S_\infty \cup \{\mathfrak{p}\}$, and denote as $\mathfrak{o}^{(S)} = \mathfrak{o}^{(\mathfrak{p})}$. Then, by taking an $\mathfrak{o}^{(\mathfrak{p})}$-order $O^{(\mathfrak{p})}$ and defining $\Gamma^{(\mathfrak{p})}$ to be the subgroup of $O^{(\mathfrak{p})}$ formed of all elements of norm 1, we get a discrete subgroup $\Gamma^{(\mathfrak{p})}$ of $SL_2(\mathbf{R}) \times SL_2(k_\mathfrak{p})$ whose quotient has finite invariant volume and whose projection to each factor is dense in that factor. Therefore, we have proved the following proposition.

Proposition 1. *Let F be a totally real algebraic number field, and let A be a quaternion algebra over F in which all real prime divisors of F but one (denoted by $\mathfrak{p}_{\infty,1}$) are ramified. Let $\mathfrak{p}$ be a finite prime divisor of F which is unramified in A, and let $\mathfrak{o}^{(\mathfrak{p})}$ be the ring of all elements of F which are integral except at $\mathfrak{p}$. Let $O^{(\mathfrak{p})}$ be any $\mathfrak{o}^{(\mathfrak{p})}$-order of A, and put*

$$\text{(9)} \qquad \Gamma = \{x \in O^{(\mathfrak{p})} | N_{A/F}x = 1\}/\pm 1.$$

Then, by the diagonal embedding of Γ into

$$A^1_{\mathfrak{p}_{\infty,1}}/(\pm 1) \times A^1_\mathfrak{p}/(\pm 1) \cong PSL_2(\mathbf{R}) \times PSL_2(F_\mathfrak{p}),$$

Γ is regarded as a discrete subgroup of $G = PSL_2(\mathbf{R}) \times PSL_2(F_\mathfrak{p})$ whose quotient has finite invariant volume and whose projection to each factor is dense in that factor. The quotient is compact if and only if $A \not\cong M_2(\mathbf{Q})$.

Corollary. *Let $k_\mathfrak{p}$ be a $\mathfrak{p}$-adic number field. Then there exists a discrete subgroup Γ of $PSL_2(\mathbf{R}) \times PSL_2(k_\mathfrak{p})$ with compact quotient and with dense image of projection in each component of G.*

In particular, by taking $A = M_2(\mathbf{Q})$ and $O^{(\mathfrak{p})} = M_2(\mathbf{Z}^{(p)})$, where $\mathbf{Z}^{(p)} = \cup_{n=0}^{\infty} p^{-n}\mathbf{Z}$ (p: a prime number), we get $\Gamma = PSL_2(\mathbf{Z}^{(p)})$. This was the only example of Γ discussed in the preceding chapters (of Volume 1).

§3. Up to commensurability, the examples of Γ given in §2 are the only examples of Γ that we know at present. On the other hand, if Γ is such as given in §2, then we can define its congruence subgroups in the usual manner (the modulus must be coprime to $\mathfrak{p}$); and a problem arises whether all subgroups of Γ with finite indices contain some congruence subgroup. Recently, this problem was solved affirmatively for the group $\Gamma = PSL_2(\mathbf{Z}^{(p)})$ by J. Mennicke **[23]** and J. P. Serre **[26]**. But it remains open in the case $A \not\cong M_2(\mathbf{Q})$.

That k_0 contains F.

§4. To prove Theorem 1 (§5), we need the following proposition.

PROPOSITION 2.[2] *Let F, A and Γ be as in Proposition 1. Then we have $F = \mathbf{Q}((\mathrm{tr}\,\gamma_{\mathbf{R}})^2 \mid \gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}})$; and if $A \not\cong M_2(\mathbf{Q})$, then the quaternion algebra attached to Γ (defined in Chapter 3 (§12)) is nothing but A.*

PROOF. It is clear that if $A = M_2(\mathbf{Q})$, then $F = \mathbf{Q}((\mathrm{tr}\,\gamma_{\mathbf{R}})^2 | \gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}})$ holds. Now assume that $A \not\cong M_2(\mathbf{Q})$, so that A is a division algebra and G/Γ is compact. Let Γ^* be the intersection of all normal subgroups of Γ whose quotients are finite $(2, \cdots, 2)$ type, and let $A^* = \mathbf{Q}[\Gamma^*]$ be the subalgebra of A generated over $\mathbf{Q}$ by Γ^*. We shall prove $A^* = A$, which, by virtue of Proposition 6 (Chapter 3, §13), would prove our Proposition. For that purpose, put $F^* = F \cap A^*$ and let $x \mapsto \bar{x}$ be the canonical conjugation of A over F. Then since $\bar{\gamma}^* = \gamma^{*-1}$ for each $\gamma^* \in \Gamma^*$, we have $\bar{A}^* = A^*$; hence A^* is a division algebra. By the same reason, $A^* \cdot F$ is also a divison algebra and $F \subset A^* \cdot F \subset A$ holds. But since Γ^* is non-commutative, we get $A^* \cdot F = A$. Therefore, A^* contains four elements x_i $(1 \le i \le 4)$ that are linearly independent over F. Then we have $\det((\mathrm{tr}_{A/F}(x_i x_j))) \neq 0$, and $\mathrm{tr}_{A/F}(x_i x_j) = x_i x_j + \overline{x_i x_j} \in A^* \cap F = F^*$. This shows that $A^* = F^* x_1 + \cdots + F^* x_4$; hence we have $A \cong A^* \otimes_{F^*} F$ over F. But if $F^* \neq F$, then it cannot happen that all but one infinite prime divisor of F are ramified in $A^* \otimes_{F^*} F$ (the number of unramified infinite prime divisors must be divisible by $[F : F^*]$). Therefore, $F^* = F$; hence $A^* = A^* F = A$, which proves our Proposition. □

COROLLARY . *Let F, A, Γ, G be as in Proposition 1. Then all subgroups of G which are commensurable with Γ are contained in the image of the diagonal embedding of $A^\times/F^\times$ into $PL_2(\mathbf{R}) \times PL_2(F_{\mathfrak{p}})$.*

PROOF. Let $\varphi_{\mathbf{R}}, \varphi_{\mathfrak{p}}$, and φ be our embeddings $A \to M_2(\mathbf{R}), A \to M_2(F_{\mathfrak{p}})$, and $A \to M_2(\mathbf{R}) \times M_2(F_{\mathfrak{p}})$ (diagonal) respectively, and let $\varphi_{\mathbf{R}}^\times, \varphi_{\mathfrak{p}}^\times$, and $\varphi^\times$ be the embeddings $A^\times/F^\times \to PL_2(\mathbf{R}), \to PL_2(F_{\mathfrak{p}})$, and $\to PL_2(\mathbf{R}) \times PL_2(F_{\mathfrak{p}})$ that are induced by $\varphi_{\mathbf{R}}, \varphi_{\mathfrak{p}}$, and φ respectively. Let Γ' be a subgroup of $G = PSL_2(\mathbf{R}) \times PSL_2(F_{\mathfrak{p}})$ which is commensurable with Γ , and put $\Gamma'' = \Gamma \cap \Gamma'$. Then $\mathbf{Q}[\Gamma'^*_{\mathbf{R}}] = \mathbf{Q}[\Gamma''^*_{\mathbf{R}}] = \mathbf{Q}[\Gamma^*_{\mathbf{R}}]$ (Corollary of Proposition 6 of Chapter 3), and it is isomorphic over the center F to A. But then, it is clear that $\mathbf{Q}[\Gamma^*_{\mathbf{R}}] = \varphi_{\mathbf{R}}(A)$. Now, by $\Gamma'_{\mathbf{R}} \subset \mathbf{Q}[\Gamma'^*_{\mathbf{R}}]^\times/F^\times \subset PL_2(\mathbf{R})$ (Proposition 6 of Chapter 3), we get $\Gamma'_{\mathbf{R}} \subset \varphi_{\mathbf{R}}(A)^\times/F^\times = \varphi_{\mathbf{R}}^\times(A^\times/F^\times)$. Put $\Delta = \varphi_{\mathbf{R}}^{\times-1}(\Gamma'_{\mathbf{R}})$. Then $\varphi^\times(\Delta)$ is a discrete subgroup of $PSL_2(\mathbf{R}) \times PL_2(F_{\mathfrak{p}})$, and is commensurable with Γ. Therefore, $\varphi^\times(\Gamma)$ is commensurable with Γ' and $\varphi^\times(\Delta)_{\mathbf{R}} = \Gamma'_{\mathbf{R}}$. Therefore, by Supplement §3 (Remark 2), we get $\Gamma' = \varphi^\times(\Delta)$. □

The notations being as above, put $\varphi^{\times-1}(G) = A_0/F^\times$, so that

$$A_0 = \{x \in A \mid N_{A/F}(x) \in (\mathbf{R}^\times)^2, \in (F_{\mathfrak{p}}^\times)^2\}. \tag{10}$$

[2] To be more precise, we should write $\mathfrak{p}_{\infty,1}(F)$ instead of F. The only reason for excluding the case of $A \cong M_2(\mathbf{Q})$ is that "the quaternion algebra attached to Γ" was defined only when G/Γ is compact (see Chapter 3).

§5. Now we shall prove the following theorem.

Theorem 1. *Let F, A, Γ, G be as in Proposition 1, and let Γ' be a subgroup of G which is commensurable with Γ. Let L be the $G_{\mathfrak{p}}$-field over $\mathbf{C}$ which corresponds to Γ', and suppose that L contains a full $G_{\mathfrak{p}}$-subfield over a field $k \subset \mathbf{C}$. Then k contains F. In particular, if L is quasi-irreducible, then the field k_0 (defined by Theorem 5 of Chapter 2) contains F.*

(Here, to be more precise, we should write $\mathfrak{p}_{\infty,1}(F)$ instead of F (see Proposition 1 for the definition of $\mathfrak{p}_{\infty,1}$), but since it is always of this meaning whenever we consider F as a subfield of $\mathbf{R}$ or $\mathbf{C}$, we denote it simply as F.)

Remark. By the Corollary of Proposition 2, Γ' is of the form $\varphi^{\times}(\Delta)$ with $\Delta \subset A_0/F^{\times}$. By Corollary 4 of Theorem 3 in Chapter 2, we have $(N(\Gamma') : \Gamma') < \infty$; hence by the former corollary, $N(\Gamma')$ is also contained in $\varphi^{\times}(A_0/F^{\times})$. Therefore, *$L$ is quasi-irreducible if and only if the normalizer of Δ in $A_0/F^{\times}$ is Δ itself.*

Proof. Since Γ' is of the form $\varphi^{\times}(\Delta)$ and φ is the diagonal embedding, it is clear that Γ' satisfies the condition given in Lemma 12 of Chapter 2. Hence our Theorem is a direct consequence of Theorem 8 (Chapter 2, §36) and Proposition 2 of this chapter. □

Further study of these Γ will be left to the succeeding parts of this chapter.