CHAPTER 2

Introduction to Part 1 and Part 2.

Chapter 2 consists of two parts, Part 1 (§1-§17) and Part 2 (§18-§36). The subject here is what we call a "G_p-field", where $G_p = PSL_2(k_p)$. The definition is as follows. A field L is called a G_p-field over a subfield k if $\dim_k L = 1$ and if G_p acts effectively on L as a group of field automorphisms over k, fulfilling the following conditions [1]:

(i) For each open compact subgroup $V \subset G_p$, its fixed field L_V is finitely generated over k, and L/L_V is normally and separably algebraic. Moreover, V is topologically isomorphic to the Krull's Galois group of L/L_V.
(ii) Almost all prime divisors of L_V over k are unramified in L.
(iii) The fixed field of G_p is k. (k is called the *constant field* of L.)

The motivation for the study of such a field is this:
— If Γ is a discrete subgroup of $G = G_{\mathbf{R}} \times G_p$ with finite-volume-quotient such that the projections $\Gamma_{\mathbf{R}}, \Gamma_p$ are dense in $G_{\mathbf{R}}, G_p$ respectively, then Γ *defines a G_p-field L over the complex number field $\mathbf{C}$, and conversely* (Theorem 1, §9). Thus Γ *and L (over $\mathbf{C}$) are equivalent notions.* Moreover, it seems that the study of G_p-fields over algebraic number fields [2] is crucial for the solution of our problems. Thus we meet our first problem: *"Is every G_p-field L over $\mathbf{C}$ a constant field extension of a G_p-field L_0 over an algebraic number field ?"* This problem is solved affirmatively in Part 2 (Theorem 4, §18). The readers note, however, that this would not be remarkable enough without *"essential uniqueness" of L_0, which is guaranteed by Theorems 5, 6, 7 (§18, §32, §33) under a certain condition on L.* Namely, by Theorem 5, under a condition on L which is always satisfied if Γ is maximal (see §10), there is a unique [3] G_p-field L_{k_0} over an algebraic number field k_0 such that

(i) L is a constant field extension of L_{k_0}, and
(ii) if L is a constant field extension of another G_p-field L_k over a field $k \subset \mathbf{C}$, then k contains k_0 and $L_k = L_{k_0} \cdot k$.

Thus if Γ is maximal, then Γ defines a unique G_p-field L_{k_0} over an algebraic number field k_0. Theorems 6, 7 are some variations of Theorem 5.

[1] See also §1. We do not assume that G_p is the full automorphism group of L over k.

[2] By an algebraic number field, we always mean a *finite* extension of the field of rationals $\mathbf{Q}$.

[3] L_{k_0} is unique not only up to isomorphisms, but also as a G_p-invariant subfield of L.

In the last two sections (§35, §36), we shall prove that under a certain condition on Γ (to which no counterexample is known), the field k_0 contains the field F defined by $F = \mathbf{Q}((\mathrm{tr}\,\gamma_{\mathbf{R}})^2 | \gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}})$ (Theorem 8, §36). The idea of the proof is

(i) to consider the $\Gamma_{\mathbf{R}}$-fixed points (on $\mathfrak{H}$) and the rotation arguments of the stabilizers *in algebraic terms,* and
(ii) to prove that F is generated over $\mathbf{Q}$ only by $(\mathrm{tr}\,\gamma_{\mathbf{R}})^2$ of *elliptic elements* $\gamma_{\mathbf{R}}$ of $\Gamma_{\mathbf{R}}$.

The proof of (ii) is given in Chapter 3 (§11). A further study of the relations between $k_0, F,$ and $\mathbf{Q}((\mathrm{tr}\,\gamma_{\mathfrak{p}})^2 | \gamma_{\mathfrak{p}} \in \Gamma_{\mathfrak{p}})$ will be left to the next stage of this chapter.

Part 1 is rather a preliminary to Part 2. In Theorem 1 (§9), the one-to-one correspondence $\Gamma \leftrightarrow L$ (over $\mathbf{C}$) is established. In Theorem 2 (§10), some "Galois theory" between Γ and L is proved.[4] In particular, it is shown that L is irreducible (see §10) if and only if Γ is maximal. In Theorem 3 (§11), it is proved that $G_{\mathfrak{p}}$ is of finite index in $\mathrm{Aut}_{\mathbf{C}}\,L$, a fact needed in Part 2.

A large part of Part 2 is devoted to the proof of Theorem 4 (i.e., §21 ~ §31). Two basic lemmas for this proof are :

(i) *$G_{\mathfrak{p}}$ is a certain free product with amalgamation* (Lemma 7, §28), and
(ii) *homomorphisms of $\Gamma_{\mathbf{R}}$ into $G_{\mathbf{R}}$ satisfying some conditions are induced by inner automorphisms of $G'_{\mathbf{R}} = PL_2(\mathbf{R})$* (Lemma 8, §29).

As an example of $G_{\mathfrak{p}}$-fields, we shall treat the $G_{\mathfrak{p}}$-field L over $\mathbf{C}$ that corresponds to the group $\Gamma = PSL_2(\mathbf{Z}^{(p)})$ (see §2). This field is treated in connection with Theorems 1, 3 and 5, in §2, §17 and §34, respectively.

Although the "$G_{\mathfrak{p}}$-field" can be defined for any locally compact, non-compact, and totally disconnected group $G_{\mathfrak{p}}$, our main results after §11 are essentially based on the particular structure of the group $G_{\mathfrak{p}} = PSL_2(k_{\mathfrak{p}})$ (see Lemmas 1, 4 and 6). Moreover, the only examples of $G_{\mathfrak{p}}$-fields that we know at present are those for $G_{\mathfrak{p}} \supset PSL_2(k_{\mathfrak{p}})$ with $(G_{\mathfrak{p}} : PSL_2(k_{\mathfrak{p}})) < \infty$; and for such $G_{\mathfrak{p}}$, we can obtain results similar to ours immediately from our results (e.g., by using Proposition 4 (§12) and Theorem 6 (§32)). Therefore, we shall assume throughout the chapter that $G_{\mathfrak{p}}$ is the group $PSL_2(k_{\mathfrak{p}})$.

[4] In Piatetski-Shapiro and Shafarevich **[24]** (in Russian), it seems that a certain transcendental Galois theory is developed, which seems quite interesting. However, the results (resp. ideas) are different (resp. independent).