Part 3B. Unique existence of an invariant S-operator on "arithmetic" algebraic function fields (including G_p-fields) over any field of characteristic zero.

Unique existence of invariant S-operator on ample (arithmetic) L/k.

§45.

[1]. In §41 (Part 3A), we considered the algebraic function fields $L/\mathbf{C}$ satisfying (L1), (L2), and proved Theorem 9 for such fields. In particular, we proved that if L is ample, then there exists a unique $\mathrm{Aut}_{\mathbf{C}} L$-invariant S-operator on L. Our purpose here is to generalize this result to the cases where the constant field k of L is an arbitrary field of characteristic zero (instead of $\mathbf{C}$). First, we must define the fields L/k. This is completely parallel to the definition of $L/\mathbf{C}$ (§41); namely, our object will be the following field L/k:

Definition. k is any field of characteristic 0, and L is any one-dimensional extension of k not assumed to be finitely generated over k, but assumed to satisfy:

$(L0)_k$ k is algebraically closed in L;

$(L1)_k$ Let $\mathcal{L}_0$ be the set of all finitely generated extensions L_0/k contained in L such that L/L_0 is normally algebraic. Then $\mathcal{L}_0$ is non-empty;

$(L2)_k$ For each $L_0 \in \mathcal{L}_0$ and a prime divisor P_0 of L_0/k, denote by $e_0(P_0)$ the ramification index of P_0 in L/L_0. Then $e_0(P_0) = 1$ for almost all P_0, and the quantity

$$V(L_0) = 2g_0 - 2 + \sum_{P_0}\left(1 - \frac{1}{e_0(P_0)}\right)\deg P_0 \tag{128}$$

is positive, where g_0 is the genus of L_0/k.

Remark 1. Remark 1 of §41 is also valid here.

Remark 2. If $k = \mathbf{C}$, this coincides with the definition of $L/\mathbf{C}$ of §41.

[2]. The arguments of [2] [3] of §41 are also applicable to this general case; so, all definitions and results of [2] [3] §41 are directly carried over to this case if we only replace $\mathbf{C}$ by k. In particular, $\mathcal{L}_0$ always contains a minimal element (with respect to $\subset$), and L is called *simple* if it is unique, and *ample* (or *arithmetic*) if it is not unique. Moreover, L is ample if and only if $\mathrm{Aut}_k L$ is non-compact. The definitions of $D(L)$ and $d : L \to D(L)$ are also exactly parallel to the case of $k = \mathbf{C}$ ([4] §41).

Remark 3. There is one point where we need a slight modification of our argument: In [3] §41, we used the finiteness of $\mathrm{Aut}\{L_0, e_0\}$ (to prove Proposition 14), and reduced this finiteness proof to the well-known finiteness of $N(\Delta)/\Delta$, where Δ is the fuchsian group corresponding to $\{L_0, e_0\}$, and $N(\Delta)$ is its normalizer in $G_{\mathbf{R}}$. For the general case, the finiteness of $\mathrm{Aut}\{L_0, e_0\}$ is proved in the following way: First, if the genus g_0 of L_0 is greater than one, then $\mathrm{Aut}_k L_0$ is finite; hence there is no problem. On the other hand, if $g_0 = 1$ resp. 0, then, by $V(L_0) > 0$, we have $\sum_{e(P)>1} \deg P \geq 1$ resp. ≥ 3. But if $g_0 = 1$ resp. 0, the group of automorphisms of L_0 that leave one (resp. three) prime divisors fixed is finite; hence the finiteness of $\mathrm{Aut}\{L_0, e_0\}$ follows.

[3]. Now the group $\mathrm{Aut}_k L$ acts on the set of all S-operators on L by $S \to S^\sigma$; $S^\sigma\langle\xi\rangle = S\langle\xi^{\sigma^{-1}}\rangle^\sigma$ ($\sigma \in \mathrm{Aut}_k L$). Our main purpose is to prove the following theorem:

THEOREM 10. *Let L/k be as above, k being any field of characteristic* 0. *Suppose that L is ample. Then there exists a unique* $\mathrm{Aut}_k L$*-invariant S-operator on L. More strongly, if Φ is any closed non-compact subgroup of* $\mathrm{Aut}_k L$*, then there exists a unique Φ- invariant S-operator on L.*

COROLLARY 1. *Let L be a $G_\mathfrak{p}$-field over any field k of characteristic* 0. *Then there is a unique $G_\mathfrak{p}$-invariant S-operator on L, and it is moreover* $\mathrm{Aut}_k L$*-invariant.*

DEFINITION . In the situation of Theorem 10, we shall call the unique $\mathrm{Aut}_k L$-invariant S-operator *the invariant S-operator on L/k.*

REMARK 4. If L is simple, there are also $\mathrm{Aut}_k L$-invariant S-operators (in fact, $S\langle\xi\rangle = \langle\xi, \zeta\rangle + C$ give such operators, where $\zeta \in D(L_{00})^\times$, $C \in D^2(L_{00})$; L_{00} being the minimal element of $\mathcal{L}_0$), but they are not at all unique.

REMARK 5. Theorem 10 is equivalent to the following assertion (♯):

(♯) *Let L/k and Φ be as in Theorem 10, and let $\zeta \in D(L)^\times$. Then there is a unique element $C \in D^2(L)$ such that*

$$\langle\zeta, \zeta^\sigma\rangle = C - C^\sigma \tag{129}$$

for all $\sigma \in \Phi$.

In fact, if we put $S\langle\xi\rangle = \langle\xi, \zeta\rangle + C$, then S is σ- invariant if and only if C satisfies (129). Since $\sigma \to \langle\zeta, \zeta^\sigma\rangle$ is a cocycle, the existence of such C is a consequence of $\mathrm{H}^1(\Phi, D^2(L)) = 0$. However, it turns out that the last cohomology group does not vanish generally (even if we consider continuous cocycles only). So, this method cannot be applied.

[4]. As in §41, the uniqueness proof for Theorem 10 is immediately reduced to the following lemma:

LEMMA 14_k. *Let Φ be any closed non-compact subgroup of* $\mathrm{Aut}_k L$*, and let $h \geq 1$. Then the only Φ-invariant element of $D^h(L)$ is* 0.

In the following, we shall prove Lemma 14_k and Theorem 10, by reducing them to the case of $k = \mathbf{C}$.

Proofs of Lemma 14_k and Theorem 10.

§46. Let k be any field of characteristic 0, and *let L/k be ample*. Our purpose is to prove Lemma 14_k and Theorem 10 for such general L/k. These proofs are reduced to the case of $k = \mathbf{C}$ (i.e., to Lemma 14 and Theorem 9 (§41)) by using the following Lemmas 16, 17:

[1]. First, let K be any overfield of k, and let L_K be the quotient field of $L \underset{k}{\otimes} K$. Then L_K/K also satisfies $(L0)_k$, $(L1)_k$, $(L2)_k$ of §45 (where k is replaced by K), and $\mathrm{Aut}_k L$ is regarded as an *open* subgroup of $\mathrm{Aut}_K L_K$ in a natural manner. Therefore, *if L/k is ample, so is L_K/K.*

Lemma 16. *Let K be algebraically closed. Then if Lemma 14_k and Theorem 10 are both valid for L_K/K, they are also valid for L/k.*

Proof. (i) Let Φ be any closed non-compact subgroup of $\mathrm{Aut}_k L$ and let $\omega \in D^h(L)$ $(h \geq 1)$ be Φ-invariant. Consider ω as an element of $D^h(L_K)$ and Φ as a subgroup of $\mathrm{Aut}_K L_K$. Then since $\mathrm{Aut}_k L$ is open in $\mathrm{Aut}_K L_K$, Φ is also closed (and non-compact) as a subgroup of $\mathrm{Aut}_K L_K$, and ω is Φ-invariant. Hence $\omega = 0$ by Lemma 14_k for L_K/K.

(ii) Let Φ be as in (i). Then by our assumption (Theorem 10 for L_K/K), there is a unique Φ-invariant S-operator on L_K; hence by Remark 5 (§45), there exists a unique element $C \in D^2(L_K)$ such that

$$\langle \zeta, \zeta^\sigma \rangle = C - C^\sigma \qquad (\forall \sigma \in \Phi), \tag{130}$$

where ζ is any fixed element of $D(L_K)^\times$. Now take ζ from $D(L)^\times$. Then we claim that $C \in D^2(L)$, which, by virtue of Remark 5 (§45), would settle our lemma. To prove $C \in D^2(L)$, let ρ be any element of $\mathrm{Aut}_k K$, and let $\tilde{\rho}$ be the unique element of $\mathrm{Aut}_L L_K$ that coincides with ρ on K. Then $\tilde{\rho}$ commutes with all elements of Φ. Moreover, the fixed field of the group $\{\tilde{\rho} \mid \rho \in \mathrm{Aut}_k K\}$ is L. This is checked exactly in the same manner as Lemma 2 (Part 2), by noting that the fixed field of $\mathrm{Aut}_k K$ is k (since K is algebraically closed), and that the fixed field of $\mathrm{Aut}_k L$ is also k (since L/k is ample). Now apply $\tilde{\rho}$ on both sides of (130). Then since $\langle \zeta, \zeta^\sigma \rangle \in D^2(L)$ are $\tilde{\rho}$-invariant, we obtain $\langle \zeta, \zeta^\sigma \rangle = C^{\tilde{\rho}} - C^{\sigma\tilde{\rho}} = C^{\tilde{\rho}} - C^{\tilde{\rho}\sigma}$ $(\sigma \in \Phi)$; hence by the uniqueness of C, we obtain $C^{\tilde{\rho}} = C$ for all ρ. Let $\xi \in D(L)^\times$ and put $C = a\xi^2 (a \in L_K)$. Then $a^{\tilde{\rho}} = a$ for all ρ; hence $a \in L$ by our above remark. Hence $C \in D^2(L)$, which settles our lemma. □

Corollary. *If $\dim_{\mathbf{Q}} k \leq \aleph$, then Lemma 14_k and Theorem 10 are valid for L/k.*

Proof. Since $\dim_{\mathbf{Q}} k \leq \aleph$, we can embed k into $\mathbf{C}$; hence by Lemma 16, we can reduce Lemma 14_k and Theorem 10 to the case of $k = \mathbf{C}$. □

[2].

Lemma 17. *Let k be algebraically closed. Then L contains an* Aut_k *L-invariant subfield L' such that $L'.k = L$ and that* $\dim_{\mathbf{Q}} k' \leq \aleph_0$, *where $k' = L' \cap k$.*

The proof of this lemma will be given in the next section (§47).

Remark . In the situation of Lemma 17, we see easily that L' and k are linearly disjoint over k', and that L'/k' also satisfies the conditions $(L0)_k$, $(L1)_k$, $(L2)_k$ of §45. (Consult the proof of Proposition 2 (Part 1).)

[3]. Completing the proofs of Lemma 14_k and Theorem 10, assuming Lemma 17. To prove Lemma 14_k and Theorem 10 for L/k, we may assume that k is algebraically closed (by Lemma 16). So, L contains an $\mathrm{Aut}_k L$-invariant subfield L' such that $L'.k = L$ and that $\dim k' \leq \aleph_0$, where $k' = L' \cap k$ (by Lemma 17). Let Φ be any closed non-compact subgroup of $\mathrm{Aut}_k L$, and let $\omega \in D^h(L)$ $(h \geq 1)$ be Φ-invariant. Take a finitely generated extension k'' of k' such that $\omega \in D^h(L'')$, where $L'' = L'.k''$. Since L' is $\mathrm{Aut}_k L$-invariant, L'' is also $\mathrm{Aut}_k L$-invariant, and since $L''.k = L$, $\mathrm{Aut}_k L$ acts effectively on L''. On the other hand, $\mathrm{Aut}_{k''} L''$ can be regarded as a subgroup of $\mathrm{Aut}_k L$ in a natural manner. Therefore, $\mathrm{Aut}_{k''} L'' = \mathrm{Aut}_k L$; hence $\Phi|_{L''}$ is a closed non-compact subgroup of $\mathrm{Aut}_{k''} L''$. But since $\dim_{\mathbf{Q}} k'' \leq \aleph_0 < \aleph$, Lemma 14_k is valid for L''/k'' (by the Corollary of Lemma 16); hence $\omega = 0$. This proves Lemma 14_k for L/k.

Now we shall prove Theorem 10 for L/k. In the same manner as above, we shall identify: $\mathrm{Aut}_{k'} L' = \mathrm{Aut}_k L$. Since $\dim_{\mathbf{Q}} k' \leq \aleph_0$, Theorem 10 is valid for L'/k'; hence there exists a unique element C in $D^2(L')$ such that $\langle \zeta, \zeta^\sigma \rangle = C - C^\sigma$ holds for all $\sigma \in \Phi$, where ζ is any fixed element of $D(L')^\times$ (by Remark 5, §45). Moreover, C is unique *in* $D^2(L)$ by Lemma 14_k for L/k. But then, by Remark 5 (§45) again, Theorem 10 is valid for L/k. □

§47. In this section, we shall give a proof of Lemma 17. For this proof, we need several preliminary considerations.

[1]. Let k be any field of characteristic 0, and let L/k be ample. Put $G = \mathrm{Aut}_k L$, and let $\mathfrak{V}$ be the set of all open compact subgroups of G. Then $\mathcal{L}_0$ and $\mathfrak{V}$ are in a natural one-to-one correspondence:

$$\begin{array}{ccc} \mathcal{L}_0 \ni L_V & \overset{}{\underset{1:1}{\longleftrightarrow}} & V \in \mathfrak{V} \\ \vdots & & \vdots \\ \text{the fixed field of } V & & \mathrm{Aut}_{L_V} L. \end{array} \tag{131}$$

Since $\mathcal{L}_0$ is inductive with respect to $\supset$, $\mathfrak{V}$ is also inductive with respect to $\subset$; hence each element of $\mathfrak{V}$ is contained in a maximal element of $\mathfrak{V}$.

Definition . We denote by G_0 the subgroup of G generated by all open compact subgroups V of G.

It is clear that G_0 is an open *non*-compact *normal* subgroup of G.

Proposition 22. *Let $V \in \mathfrak{V}$ and let $N(V)$ be its normalizer in G. Then $N(V) \in \mathfrak{V}$.*

Proof. Let L_V be the fixed field of V, and let $\sigma \in N(V)$. Then L_V^σ is the fixed field of $\sigma^{-1} V \sigma = V$; hence $L_V^\sigma = L_V$. Therefore, σ induces an automorphism $\bar{\sigma}$ of $\{L_V, e_V\}$ ($e_V(P)$ is the ramification index of P in L/L_V, where P is any prime divisor of L_V). But the group of automorphisms of $\{L_V, e_V\}$ is finite (Remark 3, §45). Hence the kernel of the homomorphism $N(V) \to \operatorname{Aut}\{L_V, e_V\}$ induced by $\sigma \to \bar{\sigma}$ is of finite index in $N(V)$. But this kernel is clearly V. Hence $(N(V) : V) < \infty$; hence $N(V) \in \mathfrak{B}$. □

COROLLARY 1. *Every compact subgroup of G is contained in some open compact subgroup of G.*

PROOF. Let K be any compact subgroup of G, and let V be any element of $\mathfrak{B}$. Put $V_0 = \bigcap_{k \in K}(k^{-1}Vk)$. Then since $(K : V \cap K) < \infty$, V_0 is open; hence $V_0 \in \mathfrak{B}$. Moreover, K normalizes V_0; hence $K \subset N(V_0)$. But $N(V_0) \in \mathfrak{B}$ by Proposition 22. □

COROLLARY 2. *Let $\mathfrak{Z}$ be the centralizer of G_0 in G. Then $\mathfrak{Z}$ is compact, and is contained in G_0.*

PROOF. Let $V \in \mathfrak{B}$. Then $\mathfrak{Z}$ centralizes V; hence $\mathfrak{Z} \subset N(V)$. But $\mathfrak{Z}$ is closed, and $N(V)$ is compact by Proposition 22. Therefore, $\mathfrak{Z}$ is compact. Since $N(V) \in \mathfrak{B}$, $N(V) \subset G_0$; hence $\mathfrak{Z} \subset G_0$. □

Now we shall prove the following proposition by applying the above Corollary 2.

PROPOSITION 23. *Assume that k is algebraically closed, and let k' be a given algebraically closed subfield of k. Suppose that L contains a G_0-invariant subfield L' with $L'.k = L$ and $L' \cap k = k'$. Then such L' is unique, and is moreover G-invariant.*

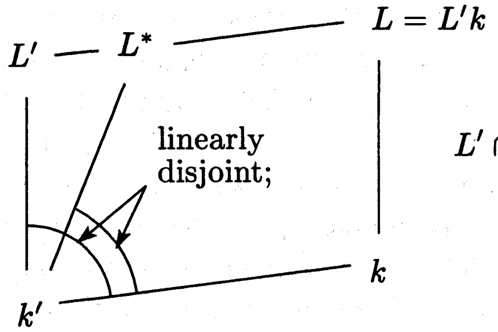
PROOF. Let $\mathfrak{Z}$ be the centralizer of G_0 in G. Then by Corollary 2 of Proposition 22, $\mathfrak{Z}$ is contained in G_0; hence L' is $\mathfrak{Z}$-invariant. Moreover, by the same corollary, $\mathfrak{Z}$ is compact. Hence if $L'_{\mathfrak{Z}}$ denotes the fixed field of $\mathfrak{Z}|_{L'}$ in L', then $L'/L'_{\mathfrak{Z}}$ is algebraic (in fact, normally algebraic with the Galois group $\mathfrak{Z}|_{L'}$).

$$\begin{array}{ccc} L' & \text{———} & L \\ \text{algebraic}\ \Big| & & \Big| \\ L'_{\mathfrak{Z}} & & \Big| \\ \Big| & & \Big| \\ k' & \text{———} & k \end{array} \qquad (132)$$

We shall show that $L'_{\mathfrak{Z}}$ *is independent of the choice of* L'. First, note that since $\operatorname{Aut}_{L'} L = \operatorname{id}_{L'} \otimes \operatorname{Aut}_{k'} k$, and since the fixed field of $\operatorname{Aut}_{k'} k$ is k' (by the algebraic closedness of k), we see that the fixed field of $\operatorname{Aut}_{L'} L$ is L'. Therefore, if we denote by $\tilde{\mathfrak{Z}}'$ the subgroup of $\operatorname{Aut}_{k'} L$ generated by $\mathfrak{Z}$ and $\operatorname{Aut}_{L'} L$, then $L'_{\mathfrak{Z}}$ is nothing but the fixed field of $\tilde{\mathfrak{Z}}'$ in L. We shall show that $\tilde{\mathfrak{Z}}'$ coincides with the centralizer of G_0 in $\operatorname{Aut}_{k'} L$, which would prove the independence of $L'_{\mathfrak{Z}}$ on L'. Let $\tilde{\mathfrak{Z}}$ denote the centralizer of G_0 in $\operatorname{Aut}_{k'} L$. Then it is clear that $\mathfrak{Z}, \operatorname{Aut}_{L'} L \subset \tilde{\mathfrak{Z}}$; hence $\tilde{\mathfrak{Z}}' \subset \tilde{\mathfrak{Z}}$. On the other hand, let $\sigma \in \tilde{\mathfrak{Z}}$. Then since k^{σ} is the fixed field of $\sigma^{-1}G_0\sigma = G_0$, we have $k^{\sigma} = k$ (from this follows that $\mathfrak{Z}$ is normal in $\tilde{\mathfrak{Z}}$). Let ρ be the unique element of $\operatorname{Aut}_{L'} L$ that coincides with σ on k. Then $\sigma \cdot \rho^{-1} \in \mathfrak{Z}$; hence $\tilde{\mathfrak{Z}} \subset \mathfrak{Z} \cdot \operatorname{Aut}_{L'} L = \tilde{\mathfrak{Z}}'$. Hence $\tilde{\mathfrak{Z}}' = \tilde{\mathfrak{Z}}$. Therefore, the field $L'_{\mathfrak{Z}}$ is independent of the choice of L'.

Now let $\mathcal{L}'$ be the set of all L' satisfying the conditions of Proposition 23 (for the given k'). Then since G_0 is normal in G, $L' \in \mathcal{L}'$ implies $L'^g \in \mathcal{L}'$ for any $g \in G$. Therefore, the composite L^* of all $L' \in \mathcal{L}'$ is G-invariant. Moreover, since L'_3 is common for all $L' \in \mathcal{L}'$, and since L'/L'_3 are algebraic, we conclude that L^*/L'_3 is algebraic. Now put $L^* \cap k = k^*$. Then the elements of k^* are algebraic over L'_3 and hence over L'. But L' and k are linearly disjoint over k' (this can be proved exactly in the same manner as Proposition 2 of Part 1, since the fixed field of G_0 is k). Therefore, the elements of k^* are algebraic over k'. But since k' is algebraically closed by assumption, we conclude $k^* = k'$; hence $L^* \cap k = k'$. But then L^* and k are linearly disjoint over k':

(133)



$$L' \cap k = L^* \cap k = k'.$$

Therefore, $L^* = L'$; hence L' is unique, and is G-invariant. □

[2]. We shall also need the following proposition:

Proposition 24. *The cardinality of the set $\mathcal{L}_0$ is countable.*

To prove this, we need the following Lemma 18:—. Let L_0 be any finitely generated algebraic function field of dimension one over k, and let $e_0 = e_0(P_0)$ be a $\{1, 2, \ldots; \infty\}$-valued function defined on the set of all prime divisors P_0 of L_0/k, such that $e_0(P_0) = 1$ for almost all P_0 and that $V\{L_0, e_0\} \underset{\text{def.}}{=} 2g_0 - 2 + \sum_{P_0}\left(1 - \frac{1}{e_0(P_0)}\right) \deg P_0 > 0$, g_0 being the genus of L_0. Then a finite extension L'_0 of L_0 is called an *admissible extension* with respect to $\{L_0, e_0\}$ if for each P_0 and its factor P'_0 in L'_0, the ramification index of P'_0/P_0 divides $e_0(P_0)$. (In this case, if we define $e'_0(P'_0)$ as the quotient of $e_0(P_0)$ by the ramification index of P'_0/P_0, then $\{L'_0, e'_0\}$ may be called an admissible extension of $\{L_0, e_0\}$.) On the other hand, a subfield L^*_0 of L_0 with $[L_0 : L^*_0] < \infty$ is called an admissible subfield with respect to $\{L_0, e_0\}$ if for each prime divisor P^*_0 of L^*_0 and its prime factor P_0 in L_0, the product of $e_0(P_0)$ and the ramification index of P_0/P^*_0 depends only on P^*_0. (In this case, if we define $e^*_0(P^*_0)$ to be this product, we may call $\{L^*_0, e^*_0\}$ an admissible subfield of $\{L_0, e_0\}$. Thus the former is an admissible subfield of the latter if and only if the latter is an admissible extension of the former.)

Remark . The notations being as above, we have

$$\begin{cases} V\{L'_0, e'_0\} = V\{L_0, e_0\} \times [L'_0 : L_0.k'] \\ V\{L_0, e_0\} = V\{L^*_0, e^*_0\} \times [L_0 : L^*_0] \end{cases} \tag{134}$$

by Hurwitz formula, where k' denotes the algebraic closure of k in L'_0.

Lemma 18. *Let $\{L_0, e_0\}$ be given, and suppose that k is algebraically closed. Then* (i) *there exist only finitely many admissible subfields of L_0 with respect to $\{L_0, e_0\}$;* (ii) *for given n, there exist only finitely many admissible extensions of L_0 of degree n with respect to $\{L_0, e_0\}$.*

Proof of Lemma 18. First we note that this is a well-known fact when $k = \mathbf{C}$. In fact, if Δ is the fuchsian group corresponding to $\{L_0, e_0\}$ (see §40), then the admissible extensions of degree n with respect to $\{L_0, e_0\}$ correspond to the subgroups of Δ with index n, and the admissible subfields with respect to $\{L_0, e_0\}$ correspond to the fuchsian groups containing Δ. But as is well-known, they are finite in number. Hence the case $k = \mathbf{C}$ is settled. The general case is easily reduced to the $k = \mathbf{C}$ case. In fact, suppose that there are infinitely many admissible subfields with respect to $\{L_0, e_0\}$. Take any countable subset from the set of all such subfields, and call them L_i ($i = 1, 2, \ldots$). Let X_i ($i \geq 0$) be any complete non-singular model of L_i, and let f_i be the rational map $f_i : X_0 \to X_i$ defined by the inclusion $L_i \subset L_0$. For each i, let $k_i \subset k$ be a finitely generated extension of $\mathbf{Q}$ over which X_0, X_i and f_i are defined, and over which all ramifying prime divisors of X_0 (w.r.t. f_i) and all prime divisors P_0 of X_0 with $e_0(P_0) > 1$ are rational. Let k' be the composite of k_i for all $i \geq 1$, so that $\dim_{\mathbf{Q}} k' \leq \aleph_0$. Then by embedding k' into $\mathbf{C}$, we can immediately reduce our assertion (i) to the case of $k = \mathbf{C}$. A similar reduction is also valid for the assertion (ii). □

Proof of Proposition 24. It is enough to prove this proposition when k is algebraically closed. Let L_0 be any element of $\mathcal{L}_0$, and let e_0 be as in the condition $(L2)_k$ (§45). Then all finite extensions of L_0 contained in L are admissible extensions with respect to $\{L_0, e_0\}$; hence by Lemma 18, they are countable. Call them $\{L_i, e_i\}$ ($i \geq 0$). Let L_0' be any other element of $\mathcal{L}_0$. Then $L_0'.L_0 = L_i$ for some i, and L_0' is an admissible subfield of L_i with respect to $\{L_i, e_i\}$; but by Lemma 18, each $\{L_i, e_i\}$ contains only finitely many admissible subfields. Therefore, $\mathcal{L}_0$ is countable. □

[3]. Proof of Lemma 17. Now having Propositions 23, 24 on hand, we can prove Lemma 17 easily. Since $\mathcal{L}_0$ is countable, we may denote the elements of $\mathcal{L}_0$ as L_i ($i = 1, 2, 3, \ldots$). For each i, let X_i be a complete non-singular model of L_i/k, and for each i, j with $L_j \subset L_i$, let $f_{ij} : X_i \to X_j$ be the induced rational map. Since $\mathcal{L}_0$ is countable, there exists a subfield k' of k such that $\dim_{\mathbf{Q}} k' \leq \aleph_0$ and that all X_i, all f_{ij}, and all covering groups of f_{ij} (whenever f_{ij} is a Galois covering) are defined over k'. We may assume further that k' is algebraically closed. Let L_i' be the field of k'-rational functions on X_i, so that $L_j' \subset L_i'$ naturally whenever $L_j \subset L_i$, and let L_i'/L_j' is a Galois extension whenever L_i/L_j is so. Let L' be the union of all L_i' (with respect to these inclusions). Then it is clear that L' is a G_0-invariant subfield of L such that $L'.k = L$ and that $L' \cap k = k'$. Therefore, by Proposition 23, L' is moreover G-invariant. This proves Lemma 17, and hence completes the proofs of Lemma 14_k and Theorem 10 for the general L/k.

Some corollaries and applications of Theorem 10.

§48.

[1]. The following corollary is an immediate consequence of Theorem 10.

COROLLARY 2.[32] *Let L/k be ample, and let Φ be an open non-compact subgroup of* $\mathrm{Aut}_k L$. *Let L' be a Φ-invariant subfield of L satisfying $L'k = L$ and put $k' = L' \cap k$, so that L'/k' is also ample. Let S resp. S' be the invariant S-operators on L/k resp. L'/k'. Then*

$$S\langle\xi\rangle = S'\langle\xi'\rangle + \langle\xi, \xi'\rangle \tag{135}$$

holds for all $\xi \in D(L)^\times$ and $\xi' \in D(L')^\times$.

PROOF. Fix any $\xi' \in D(L')^\times$, and put $S_1\langle\xi\rangle = S'\langle\xi'\rangle + \langle\xi, \xi'\rangle$, so that S_1 is an S-operator on L. Let $\sigma \in \Phi$. Then $S_1^\sigma\langle\xi\rangle = \{S'\langle\xi'\rangle\}^\sigma + \langle\xi, \xi'^\sigma\rangle = S'\langle\xi'^\sigma\rangle + \langle\xi, \xi'^\sigma\rangle = S'\langle\xi'\rangle + \langle\xi'^\sigma, \xi'\rangle + \langle\xi, \xi'^\sigma\rangle = S'\langle\xi'\rangle + \langle\xi, \xi'\rangle = S_1\langle\xi\rangle$. Therefore, S_1 is Φ-invariant. Therefore, by Theorem 10, S_1 must be the unique $\mathrm{Aut}_k L$-invariant S-operator on L/k. □

EXAMPLE . Let L be a G_p-field over $\mathbf{C}$, and let S be the canonical (hence the invariant) S-operator on $L/\mathbf{C}$. By Theorem 4 (Part 2), L contains a full G_p-subfield L_k over an algebraic number field k of finite degree. Therefore, $S\langle\xi\rangle = S'\langle\xi'\rangle + \langle\xi, \xi'\rangle$ ($\xi \in D(L)^\times$, $\xi' \in D(L_k)^\times$), where S' is the invariant S-operator on L_k/k. Therefore, in a sense, S is "defined over an algebraic number field."

[2]. Now consider any field k (of characteristic 0) and a pair $\{L_0, e_0\}/k$, where L_0 is a finitely generated algebraic function field of dimension one over k, and $e_0 = e_0(P_0)$ is a $\{1, 2, \dots; \infty\}$-valued function defined on the set of all prime divisors of L_0/k such that $e_0(P_0) = 1$ for almost all P_0 and that $V\{L_0, e_0\} > 0$ (see [2] of §47). For each overfield K of k, we denote by $\{L_0.K, e_0\}/K$ the constant field extension of $\{L_0, e_0\}/k$. We shall say that $\{L_0, e_0\}/k$ is "ample" if there exists a normally algebraic extension L of L_0 such that

(a) k is algebraically closed in L;
(b) for each P_0, $e_0(P_0)$ coincides with the ramification index of P_0 in L/L_0;
(c) L/k is ample (in the sense of §45 [2]).

Now let k be a subfield of $\mathbf{C}$ and consider any $\{L_0, e_0\}/k$, so that $\{L_0\mathbf{C}, e_0\}/\mathbf{C}$ satisfies the conditions of §40. Let S be the canonical S-operator attached to $\{L_0\mathbf{C}, e_0\}$ (see §40). We shall say that S *is k-rational* if $S\langle D(L_0)^\times\rangle \subset D^2(L_0)$. Then the following is a criterion for the k-rationality of S:

CRITERION . *S is k-rational if there exists a family $\{K_\lambda\}_\lambda$ of intermediate fields of $\mathbf{C}/k$ such that $\bigcap_\lambda K_\lambda = k$ and that $\{L_0K_\lambda, e_0\}/K_\lambda$ are ample for all λ.*

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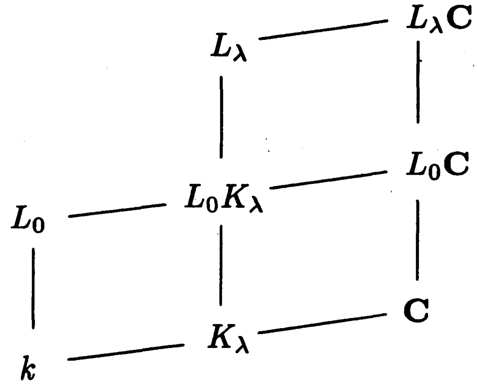
[32]As is seen in the proof, the condition $L'.k = L$ may be replaced by a weaker condition $L' \not\subset k$.

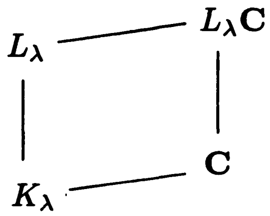
Proof. Let $\xi \in D(L_0)^\times$ and put $S\langle\xi\rangle = a\xi^2$ ($a \in L_0\mathbf{C}$). It is enough to prove $a \in L_0$. For each λ, let L_λ be an extension of L_0k_λ showing the amplitude of $\{L_0K_\lambda, e_0\}/K_\lambda$.

(136)



By the definition of the canonical S-operator on the ample field (§41), the restriction of the canonical S-operator of $L_\lambda\mathbf{C}/\mathbf{C}$ to $L_0\mathbf{C}$ is nothing but S. Moreover, by applying Corollary 2 (of Theorem 10) to the "parallellogram"

(137)



we conclude that $a \in L_\lambda$; hence $a \in L_0.K_\lambda$. But since $\bigcap_\lambda K_\lambda = k$ by assumption, we have $\bigcap_\lambda L_0K_\lambda = L_0$; hence $a \in L_0$. □

[3]. Now we shall conclude Part 3B by an application to the canonical S-operators on Shimura curves.

Let F be a totally real algebraic number field, considered as a subfield of $\mathbf{R}$. Put $[F : \mathbf{Q}] = n$, and let $\mathfrak{p}_{\infty 1}, \cdots, \mathfrak{p}_{\infty n}$ be the infinite prime divisors of F, and let $\mathfrak{p}_{\infty 1}$ correspond to the given inclusion $F \subset \mathbf{R}$. Let $\mathfrak{c}$ be any integral ideal of F, and let $C(F, \mathfrak{c})$ denote the strahl-classfield of F modulo $\mathfrak{c}\prod_{i=1}^n \mathfrak{p}_{\infty i}$. Let B be a quaternion algebra over F in which $\mathfrak{p}_{\infty 1}$ is unramified and all other $\mathfrak{p}_{\infty i}$ ($2 \le i \le n$) are ramified. Let $\mathfrak{o}$ be a maximal order of B, and let $\Delta = \Delta(\mathfrak{c})$ be the group of units in $\mathfrak{o}$ which is congruent 1 modulo $\mathfrak{c}\mathfrak{o}$ and whose reduced norm over F is totally positive. Then by the isomorphism $B \otimes_F \mathbf{R} \simeq M_2(\mathbf{R})$, Δ is considered as a fuchsian group. Let $\{L_\mathbf{C}, e\} = \{L^\mathfrak{c}_\mathbf{C}, e^\mathfrak{c}\}$ be the pair corresponding to Δ (see §40), so that $L_\mathbf{C}$ may be regarded as the field of automorphic functions with respect to Δ. Now by Shimura **[32]**, $L_\mathbf{C}$ has a nice model V over $k = C(F, \mathfrak{c})$ (which is characterized arithmetically up to biregular isomorphisms over k). Let $L = L^\mathfrak{c}$ be the field of k-rational functions on V (so that $L_\mathbf{C} = L.\mathbf{C}$). Then it is easy to check that $e = e^\mathfrak{c}$ is actually a function of the prime divisors of L/K. We shall check, by using the results of **[32]**, that $\{L, e\}/k$ satisfies the above criterion for the k-rationality of the canonical S-operator attached to $\{L_\mathbf{C}, e\}$.

For this purpose, let $\mathfrak{p}$ be any finite prime divisor of F such that $\mathfrak{p} \nmid \mathfrak{c}D(B/F)$, where $D(B/F)$ is the discriminant of B/F. Put $\tilde{k}^\mathfrak{p} = \bigcup_{n=0}^\infty C(F, \mathfrak{c}\mathfrak{p}^n)$ and $\tilde{L}^\mathfrak{p} = \bigcup_{n=0}^\infty L^{\mathfrak{c}\mathfrak{p}^n}$. Here, for each $n \ge 0$, we identify $L^{\mathfrak{c}\mathfrak{p}^n}$ with a subfield of $L^{\mathfrak{c}\mathfrak{p}^{n+1}}$ in a natural manner. Let $F_\mathfrak{p}$ be the $\mathfrak{p}$-adic completion of F and put $G_\mathfrak{p} = PSL_2(F_\mathfrak{p})$. Then by the results of [**32**], it can be easily checked that $\tilde{L}^\mathfrak{p}/\tilde{k}^\mathfrak{p}$ is a $G_\mathfrak{p}$-field (hence ample), and that $\tilde{L}^\mathfrak{p}/L^\mathfrak{c}.\tilde{k}^\mathfrak{p}$ is normally algebraic. Moreover, we can check easily (by using Supplement §6) that the extension $\tilde{L}^\mathfrak{p}/L^\mathfrak{c}.\tilde{k}^\mathfrak{p}$ satisfies the above conditions (b) of [**2**].[33] Hence $\{L^\mathfrak{c}.\tilde{k}^\mathfrak{p}, e^\mathfrak{c}\}/\tilde{k}^\mathfrak{p}$ is ample. But in $C(F, \mathfrak{c}\mathfrak{p}^n)/C(F, \mathfrak{c})$, all prime factors of $\mathfrak{p}$ in $C(F, \mathfrak{c})$ are totally ramified and all other finite prime divisors of $C(F, \mathfrak{c})$ are unramified; hence $\bigcap_\mathfrak{p} \tilde{k}^\mathfrak{p} = k$. But this implies that $\{L, e\}/k$ satisfies the assumptions of our criterion. Hence we may summarize this result as:

COROLLARY 3. *The canonical S-operator attached to the Shimura's model $V/C(F, \mathfrak{c})$ of automorphic function field of $\Delta(\mathfrak{c})$ is rational over $C(F, \mathfrak{c})$.*

Here, we treated only the principal congruence subgroups $\Delta(\mathfrak{c})$. Results for other congruence subgroups can be obtained easily from this by using Proposition 10 (and [**32**]).

[33] That it satisfies the condition (a) of [**2**] is clear.