SOLITON CELLULAR AUTOMATA

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1. Introduction

This is a lecture note on soliton cellular automata associated with crystal bases for quantized affine Lie algebras. It is partly based on my talks at the CAIS workshop, but the expositions are considerably different.

A cellular automaton (CA) is a discrete dynamical system in which site variables take on values in a finite set. The study of solitons in CAs has some long history (See e. g. chapter 3 of [AC]). Among other CAs our soliton cellular automata (SCAs) have the following particular properties:

(1) They are time reversible.
(2) All entities are solitons.
(3) Well separated solitons travel at their lengths per a unit time.
(4) There exist phase shifts caused by collisions of solitons.
(5) There are infinite number of conserved quantities.

The simplest example is Takahashi-Satsuma's automaton [TS]. In this CA the site variables take on two values $\{0,1\}$, only finite number of them taking 1 at a time [1]. It is a filter type CA such that the variable x_i^t at time t and site i is determined by the rule

$$x_i^t = \min\left(1 - x_i^{t-1}, \sum_{j=-\infty}^{i-1} (x_j^{t-1} - x_j^t)\right). \tag{1.1}$$

It gives a nonlinear dynamical system. This system aroused a lot of curiosity among people in the field of soliton theory because it is related to the other nonlinear dynamical systems such as those described by discrete KP or Toda equations [2].

There are generalizations of this CA to the systems in which the variables take on more than two values [T, TNS, TTM]. It was found that these systems can be described by crystal bases [K] for affine Lie algebra $A_n^{(1)}$ [FOY, HHIKTT] [3]. Furthermore in [HKT1] a class of SCAs associated with the crystal bases [KKM] of non-exceptional affine Lie algebras $A_{2n-1}^{(2)}, A_{2n}^{(2)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$ and $D_{n+1}^{(2)}$ [Kac] was introduced. Among those SCAs that associated with $D_n^{(1)}$ is of fundamental importance, because all the other ones (including $A_n^{(1)}$ SCA) can be embedded into this one [KTT]. Thus in a sense it is enough to explain only this case. However we also know that beginning the story by type A case is ideal for novices.

On regarding that the author planned the layout of this note: We discuss SCAs associated with crystal bases of both type A and type D affine Lie algebras. Now the reader may guess the role of each section by its name in the table of contents.

[1]In the main text we shall use $\{1,2\}$ instead of $\{0,1\}$.

[2]The reader who is interested in this aspect can consult [To].

[3]Crystal bases was born in the study of solvable lattice models [B, KMN1]. From the viewpoint of crystallization of lattice models a family of SCAs was derived in [HIK]

In the type A case the SCAs are also called *box-ball systems* since they are described by boxes and balls. Before going further we give a pair of examples. The following examples show the factorization of scattering of three solitons (Yang-Baxter relation) in a box-ball system.

Example 1.1.

```
t=0   554322......433..........6..............................
t=1   ......554322...433........6.............................
t=2   ............554322433......6............................
t=3   ..................322554433.6...........................
t=4   .....................322...5564433......................
t=5   ........................322..5....654433................
t=6   ...........................3225.........654433..........
t=7   ..............................3522............654433....
t=8   ...............................3..522...............6544
t=9   ................................3...522.................
```

Example 1.2.

```
t=0   554322..............433...6.............................
t=1   ......554322...........433.6............................
t=2   ............554322........4363..........................
t=3   ..................554322....4.633.......................
t=4   ........................5543242..633....................
t=5   .............................3.554226433................
t=6   ..............................3.....522.654433..........
t=7   ...............................3.......522....654433....
t=8   ................................3.........522.......6544
t=9   .................................3...........522........
```

The "$.(=1)$" denotes an empty box and each number ≥ 2 denotes a box containing a ball with index of that number. The updating rule of CA will (in the "original" algorithm) be given just below Example 4.11 in the main text. In the examples there are three solitons of lengths $6, 3$ and 1 before and after the collisions. We also see that the "particles" belonging to these solitons were reshuffled by the scattering processes.

In this SCA the crystal basis theory plays the following roles:

(1) The time evolution is given by the isomorphism of crystals.
(2) The reshuffling of particles caused by collisions of two solitons is also given by the isomorphism of crystals.
(3) The phase shift produced by collisions of two solitons is given by the energy function of crystals.

In the type D case there are "anti-particles" as well as particles, and we have their pair annihilation and pair creation processes in the SCA.

We recognize that the most important tool for the SCAs is the isomorphism of crystals (combinatorial R). Thus sections 2 and 3 are devoted to describe various formulas for the combinatorial R. Then the properties of the SCAs are presented in sections 4 and 5.

This note contains some topics developed after the workshop. They are related to the color separation scheme. In box-ball systems there are several kinds (colors) of balls distinguished each other by the numerical indices (See Examples 1.1 and 1.2). In other words this dynamical system has the color degree of freedom. However this degree of freedom can be separated from dynamics, i. e. it turns out to be freezing after a particular variable change. By the scheme it leaves the original Takahashi-Satsuma's automaton (1.1), or a more general monochrome system [TM].

This separation scheme uses (a special case of) the combinatorial R between crystals of symmetric and antisymmetric tensor representations. In type A case this combinatorial R is presented in section 2.4 and the separation scheme is explained in section 4.7. It is based on the author's recent work [Tg2]. About the scheme for the type A case a new result will be presented in section 4.10, concerning the box-ball system associated with antisymmetric tensor crystals [Yd]. In type D case that particular combinatorial R is presented in section 3.2, and the separation scheme is described in sections 5.6-5.7 which is also a new result.

The Takahashi-Satsuma's automaton (1.1) is a nonlinear dynamical system. Indeed, the existence of phase shifts in the collision processes in this CA signifies the nonlinearity of the system. However, the dynamics of this CA can be linearized [Tg1]. The algorithm for the linearization is presented in section 4.8. Combining with the color separation scheme, we may also say that the dynamics of SCAs associated with those type A and D crystals can be linearized. Throughout this note the author tries to treat the subjects from this viewpoint, rather than from those in the original works.

Acknowledgements I am grateful to Professors A. Kuniba and M. Okado, the organizers of the workshop "Combinatorial Aspects of Integrable Systems", RIMS, Kyoto University in July 2004, for giving me an opportunity to deliver lectures on this topic. I also thank the participants of my lectures at the workshop for their valuable questions and comments.

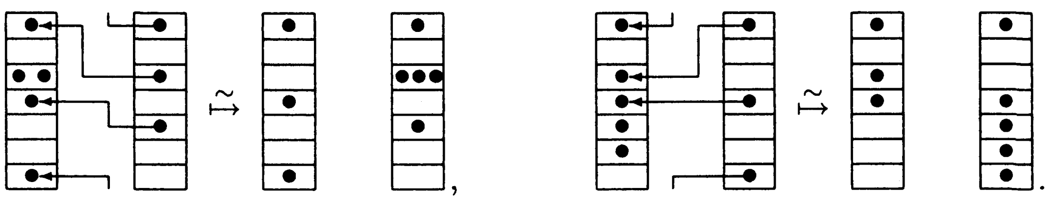
2. A brief review of combinatorial R: type A case

2.1. Diagrammatic algorithms for (anti)symmetric tensors. Let $B^{k,l}$ be the crystal of k by l rectangle shape semistandard Young tableaux [KMN2]. The *combinatorial R* (isomorphism of crystals) $R: B^{k,l} \otimes B^{k',l'} \stackrel{\sim}{\to} B^{k',l'} \otimes B^{k,l}$ is given by the rectangle switching bijection by Shimozono [Sh] or Schilling and Warnaar [SW].

First consider two special cases where we do not have to treat generic rectangles. Suppose $k = k' = 1$ or $l = l' = 1$. These are for the crystals of symmetric or antisymmetric representations. In these cases there are simple diagrammatic algorithms for the combinatorial R invented by Nakayashiki and Yamada [NY]. For example we have

$$[1\,3\,3\,4\,7] \otimes [1\,3\,5] \stackrel{\sim}{\mapsto} [1\,4\,7] \otimes [1\,3\,3\,3\,5], \qquad \begin{bmatrix}1\\3\\4\\5\\6\end{bmatrix} \otimes \begin{bmatrix}1\\4\\7\end{bmatrix} \stackrel{\sim}{\mapsto} \begin{bmatrix}1\\3\\4\end{bmatrix} \otimes \begin{bmatrix}1\\4\\5\\6\\7\end{bmatrix}.$$

Here we wrote Young tableaux by matrices. For their left hand sides, we obtain their right hand sides by using the following diagrams



The correspondence between the tableaux above and these diagrams should be clear. Starting at the dots in the right pile of boxes we find their partner dots in the left pile of boxes by the following rule and connect them. Then we transfer the unconnected dots from the left boxes to the right ones. For any dot in the right boxes, the rule to find its partner is as follows. In the symmetric (resp. antisymmetric) tensor case its partner is an unconnected dot in the lowest (resp. highest) position that is higher (resp. not lower) than its position. If there is no such dot we jump to the bottom (resp. top) and continue to search for, i. e. the top of the diagram should be identified with the bottom.

2.2. Rectangle switching bijections. Next we consider the case of generic rectangles. For any pair of semistandard tableaux T, U we denote their product tableau by $T * U$ [4]. Denote $\mathcal{T}(b)$ the rectangle tableau associated with the element of crystal b. Then we have $R(b \otimes b') = c \otimes c'$ if and only if $\mathcal{T}(b) * \mathcal{T}(b') = \mathcal{T}(c) * \mathcal{T}(c')$. Again we illustrate it only by examples. For a more complete treatment, see [Sh].

[4] The order of product in our notation is opposite to the conventional one. This is equal to $U \cdot T$ in [Fl].

Consider the following example on $R : B^{3,4} \otimes B^{2,2} \simeq B^{2,2} \otimes B^{3,4}$

$$R : \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 3 & 6 \\ 4 & 5 & 5 & 7 \end{bmatrix} \otimes \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \overset{\sim}{\mapsto} \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 2 & 5 & 5 \\ 3 & 4 & 6 & 7 \end{bmatrix}. \tag{2.1}$$

Recall that for $b \in B^{k,l}$ and $b' \in B^{k',l'}$ there are two equivalent ways to construct its associated product tableau $\mathcal{T}(b) * \mathcal{T}(b')$.

(1) Successive column insertion of letters of the column word of $\mathcal{T}(b')$ into $\mathcal{T}(b)$.
(2) Successive row insertion of letters of the row word of $\mathcal{T}(b)$ into $\mathcal{T}(b')$.

If we regard the LHS of (2.1) as $b \otimes b'$, the column word of $\mathcal{T}(b')$ is 2613 and the row word of $\mathcal{T}(b)$ is 4557 2336 1123. We illustrate method (1) to construct the product tableau (For method (2) see [SW]):

$$\left(6 \to \left(2 \to \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 3 & 6 \\ 4 & 5 & 5 & 7 \end{bmatrix}\right)\right) = \begin{bmatrix} 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 3 & 6 & \\ 4 & 5 & 5 & 7 & \\ 6 & & & & \end{bmatrix}, \tag{2.2}$$

$$\left(3 \to \left(1 \to \begin{bmatrix} 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 3 & 6 & \\ 4 & 5 & 5 & 7 & \\ 6 & & & & \end{bmatrix}\right)\right) = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 3 & 5 & 6 & \\ 3 & 4 & 5 & 7 & & \\ 6 & & & & & \end{bmatrix}. \tag{2.3}$$

In the RHS we have that

$$\left(7 \to \left(5 \to \left(3 \to \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}\right)\right)\right) = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 6 & \\ 5 & & \\ 7 & & \end{bmatrix}, \tag{2.4}$$

$$\left(6 \to \left(5 \to \left(1 \to \begin{bmatrix} 2 & 3 & 3 \\ 3 & 6 & \\ 5 & & \\ 7 & & \end{bmatrix}\right)\right)\right) = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 3 & 5 & 6 & \\ 5 & 7 & & \\ 6 & & & \end{bmatrix}, \tag{2.5}$$

$$\left(4 \to \left(2 \to \left(1 \to \begin{bmatrix} 1 & 2 & 3 & 3 \\ 3 & 5 & 6 & \\ 5 & 7 & & \\ 6 & & & \end{bmatrix}\right)\right)\right) = \begin{bmatrix} 1 & 1 & 2 & 3 & 3 \\ 2 & 3 & 5 & 6 & \\ 4 & 5 & 7 & & \\ 6 & & & & \end{bmatrix}, \tag{2.6}$$

$$\left(3 \to \left(2 \to \left(1 \to \begin{bmatrix} 1 & 1 & 2 & 3 & 3 \\ 2 & 3 & 5 & 6 & \\ 4 & 5 & 7 & & \\ 6 & & & & \end{bmatrix}\right)\right)\right) = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 3 & 5 & 6 & \\ 3 & 4 & 5 & 7 & & \\ 6 & & & & & \end{bmatrix}. \tag{2.7}$$

Thus we checked their product tableaux coincide each other.

Suppose we are given a product tableau. Then there is a problem to find a pair of rectangle tableaux of specified shapes whose product coincides with the tableau. It requires us to do inverse of column insertions (bumpings). In order to do an inverse bumping we need to specify a node of the tableau with which we begin. For example consider the following mapping:

$$R: \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 4 \\ 4 & 4 & 5 \\ 5 & 5 & 6 \end{bmatrix} \otimes \begin{bmatrix} 2 & 2 & 2 & 6 \\ 3 & 6 & 6 & 7 \\ 5 & 8 & 8 & 9 \end{bmatrix} \overset{\sim}{\mapsto} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 4 & 4 \\ 3 & 4 & 5 & 6 \end{bmatrix} \otimes \begin{bmatrix} 2 & 2 & 5 \\ 5 & 5 & 6 \\ 6 & 6 & 7 \\ 8 & 8 & 9 \end{bmatrix}. \tag{2.8}$$

Both sides give a common product tableau (the first one below). We added a pair of diagrams associated with the Littlewood-Richardson rule [Sa]:

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 4 & 4 & 6 & \\ 3 & 4 & 5 & 5 & & \\ 5 & 5 & 6 & & & \\ 6 & 6 & 9 & & & \\ 7 & 8 & & & & \\ 8 & & & & & \end{bmatrix}, \begin{bmatrix} \bullet & \bullet & \bullet & 1_3 & 1_2 & 1_1 \\ \bullet & \bullet & \bullet & 2_2 & 2_1 & \\ \bullet & \bullet & \bullet & 3_1 & & \\ \bullet & \bullet & \bullet & & & \\ 1_4 & 2_3 & 3_2 & & & \\ 2_4 & 3_3 & & & & \\ 3_4 & & & & & \end{bmatrix}, \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & 1_2 & 1_1 \\ \bullet & \bullet & \bullet & \bullet & 2_1 & \\ \bullet & \bullet & \bullet & \bullet & & \\ 1_3 & 2_2 & 3_1 & & & \\ 2_3 & 3_2 & 4_1 & & & \\ 3_3 & 4_2 & & & & \\ 4_3 & & & & & \end{bmatrix}.$$

For two partitions $\mu = (3^4)$ and $\nu = (4^3)$ (or any other rectangle shape Young diagrams) the Littlewood-Richardson coefficient $c^{\lambda}_{\mu\nu}$ satisfies $c^{\lambda}_{\mu\nu} \le 1$. For this example one has $\lambda = (6543321)$. First consider the middle diagram that has a skew tableau of shape λ/μ and content ν in it. The subscripts are counting letters of the same kinds from right to left. Neglecting the subscripts the row word of this skew tableau reads as 323123322111. It is the only way to arrange these letters (of content ν) in this shape such that its row word is in reverse lattice order (hence we have $c^{\lambda}_{\mu\nu} = 1$). The LHS of (2.8) is given by successive inverse bumpings from the product tableau by choosing the starting points as $3_1, 2_1, 1_1, \ldots, 3_4, 2_4, 1_4$. In the same way the RHS is obtained by using the right diagram where we have a skew tableau of shape λ/ν and content μ. And the starting points are so chosen as $4_1, 3_1, 2_1, 1_1, \ldots, 4_3, 3_3, 2_3, 1_3$.

The space of states for any soliton cellular automaton in this note is realized as a tensor product of many crystals. Consider a tensor product of $\mathcal{N}$ crystals $\mathcal{B} := B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_{\mathcal{N}}}$ where each λ_i $(1 \le i \le \mathcal{N})$ is a rectangle. For $\mathbf{p} = b_1 \otimes \cdots \otimes b_{\mathcal{N}} \in \mathcal{B}$ we define R_i $(1 \le i \le \mathcal{N} - 1)$ by

$$R_i(\mathbf{p}) = b_1 \otimes \cdots \otimes R(b_i \otimes b_{i+1}) \otimes \cdots \otimes b_{\mathcal{N}}.$$

Then we have

Proposition 2.1 ([Sh, SW]). The R_i's generate the symmetric group, i. e.

$$\begin{aligned} &R_i^2 = \mathrm{Id}, \\ &R_i R_j = R_j R_i \quad \text{for} \quad |i-j| \ge 2, \\ &R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}. \end{aligned}$$

The last identity is called the Yang-Baxter relation. It ensures the commutativity of "time evolution" operators in SCA.

Finally we briefly explain the notion of *energy function* in crystal basis [KKM]. It is related to 0-actions in crystals associated with affine Lie algebras. The importance of this function in SCA was first pointed out by Fukuda, Okado and Yamada [FOY]. For $b \otimes b' \in B^{k,l} \otimes B^{k',l'}$ the energy function $H(b \otimes b')$ is defined as [SW]

$$H(b \otimes b') = -\#[\text{nodes under } \max(k,k')\text{th row of } \mathcal{T}(b) * \mathcal{T}(b')], \tag{2.9}$$

with a suitable normalization (see below). For example it takes -6 for the element in (2.8). In this note we shall denote the energy function for $B^{k,l} \otimes B^{1,1}$ by $H^{k,l}$ (and set $H_l = H^{1,l}$) which takes on values $0, -1$.

For each $B^{k,l}$ there is a unique highest weight element that is characterized by all the letters in the first row being 1's, those in the second row 2's, and so on. We denote the highest weight element of $B^{k,l}$ by $u^{k,l}$. In the above normalization of the energy function we have $H(u^{k,l} \otimes u^{k',l'}) = 0$.

2.3. **Formulas for the symmetric tensors.** In what follows we occasionally denote $B^{1,l}$ by B_l, and $B^{2,1}$ by $B_{\natural}$. The notations $u_l = u^{1,l}$ and $u_{\natural} = u^{2,1}$ will be also used.

We consider the type A crystals in [KKM]. In this case there is a piecewise linear formula for the combinatorial R. Fix an integer $n \geq 2$ for sl_n. As a set the crystal B_l (l is any positive integer) is given by

$$B_l = \left\{ (x_1, \cdots, x_n) \in \mathbb{Z}^n_{\geq 0} \,\middle|\, \sum_{i=1}^{n} x_i = l \right\}. \tag{2.10}$$

For the other properties of this crystal, see Okado's lecture in this volume. The B_l is identified with the set of single row Young tableaux with length l by interpreting x_i as the number of letter i's in the tableaux. Below the subscripts of variables should be interpreted in modulo n.

Definition 2.2. For a pair of variables $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$, let R be the piecewise linear map

$$R : (x, y) \mapsto (x', y'), \tag{2.11}$$

which is defined as $x' = (x'_1, \ldots, x'_n), y' = (y'_1, \ldots, y'_n)$, $x'_i = y_i + P_i - P_{i-1}$, $y'_i = x_i + P_{i-1} - P_i$, and

$$P_i = \max_{1 \leq k \leq n} \left(\sum_{j=k}^{n} x_{i+j} + \sum_{j=1}^{k} y_{i+j} \right). \tag{2.12}$$

For $x = (x_1, \ldots, x_n)$ we denote $\sum_{k=1}^{n} x_k$ by $\ell(x)$.

Proposition 2.3 ([HHIKTT]). The map R in Definition 2.2 [5] gives the combinatorial R for the isomorphism $B_l \otimes B_{l'} \xrightarrow{\sim} B_{l'} \otimes B_l$ with $l = \ell(x) = \ell(y')$ and $l' = \ell(y) = \ell(x')$.

Thus we have used the same symbol R. The functions $P_i(x, y)$ take values on the integers between $\max(\ell(x), \ell(y))$ and $\ell(x) + \ell(y)$. As a valid expression for the combinatorial R, they can be simultaneously shifted by any common function of $\ell(x)$ and $\ell(y)$. Here we consider another three cases.

Case 1. If we replace (2.12) by

$$\tilde{P}_i = P_i - \ell(x) = \max_{1\le k\le n}\left(\sum_{j=1}^{k-1}(y_{i+j} - x_{i+j}) + y_{i+k}\right), \tag{2.13}$$

they take values on between $\max(0, \ell(y) - \ell(x))$ and $\ell(y)$. The expression for the energy function $\tilde{P}_0$ was first appeared in [KKM]. The energy function of the same normalization is obtained in the diagrammatic algorithm for the combinatorial R in subsection 2.1: It is identical to the number of winding lines. Conversely, because of the symmetry under $i \to i+1$ in the type A case one can retrieve the full formulas (2.11),(2.12) for the combinatorial R only from this expression for the energy function and that diagrammatic algorithm.

Case 2. If we replace (2.12) by

$$\hat{P}_i = P_i - |\ell(x) - \ell(y)|, \tag{2.14}$$

they take values on between $\min(\ell(x), \ell(y))$ and $2\min(\ell(x), \ell(y))$. The energy function $\hat{P}_0$ is suitable for the description of the *phase shifts* caused by the collisions of solitons in SCA.

Case 3. If we replace (2.12) by

$$\check{P}_i = P_i - \ell(x) - \ell(y), \tag{2.15}$$

they take values on between $-\min(\ell(x), \ell(y))$ and 0. The energy function $\check{P}_0$ is suitable for the description of *conserved quantities* of SCA (section 4.6). It coincides with the energy function H in (2.9).

In this case we would rather define the functions

$$Q_i = -\check{P}_i = \min_{1\le k\le n}\left(\sum_{j=1}^{k-1} x_{i+j} + \sum_{j=k+1}^{n} y_{i+j}\right), \tag{2.16}$$

to write the formula as $x'_i = y_i - Q_i + Q_{i-1}$, $y'_i = x_i - Q_{i-1} + Q_i$. It was pointed out by Y. Yamada that this expression can be derived from a solution of the discrete Toda equation $x_i y_i = x'_i y'_i$, $x_i + y_{i+1} = x'_i + y'_{i+1}$ by the substitutions $(\times \to +)$ and $(+ \to \min)$ [Yy].

Here we present the most important formula to describe the *basic* SCA, or the box-ball system in which all the boxes (cells) have capacity one. Recall

[5] The above expression is from [KOTY1] which is equivalent to the original formula in [HHIKTT].

the diagrammatic algorithm for the combinatorial R in subsection 2.1. Let $R : B_l \otimes B_1 \xrightarrow{\sim} B_1 \otimes B_l$ be the combinatorial R. In tableau notation it is given by

$$(2.17) \quad R : \boxed{\alpha_1}\boxed{\alpha_2}\boxed{\quad\cdots\quad}\boxed{\alpha_l} \otimes \boxed{\beta} \overset{\sim}{\mapsto} \begin{cases} \boxed{\alpha_l} \otimes \boxed{\beta}\boxed{\alpha_1}\boxed{\quad\cdots\quad}\boxed{\alpha_{l-1}} & \text{if } \beta \leq \alpha_1, \\ \boxed{\alpha_p} \otimes \boxed{\cdots}\boxed{\alpha_{p-1}}\boxed{\beta}\boxed{\alpha_{p+1}}\boxed{\cdots} & \text{if } \beta > \alpha_1, \end{cases}$$

where p is determined by the condition $\alpha_p < \beta \leq \alpha_{p+1}$. In the upper case we have $H_l = 0$ and in the lower case $H_l = -1$, where H_l is the energy function defined below (2.9).

2.4. Formulas for the color separation. Let $B_\natural (= B^{1,2})$ be the crystal of height two column shape Young tableaux. It is given by

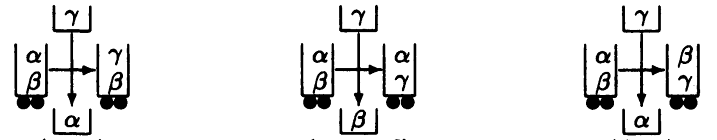
$$B_\natural = \left\{ \begin{array}{|c|}\hline \alpha \\ \hline \beta \\ \hline \end{array} \;\middle|\; 1 \leq \alpha < \beta \leq n \right\},$$

as a set. Let $R : B_\natural \otimes B_1 \xrightarrow{\sim} B_1 \otimes B_\natural$ be the combinatorial R. It is given by

$$R : \begin{array}{|c|}\hline \alpha \\ \hline \beta \\ \hline \end{array} \otimes \boxed{\gamma} \overset{\sim}{\mapsto} \begin{cases} \boxed{\alpha} \otimes \begin{array}{|c|}\hline \gamma \\ \hline \beta \\ \hline \end{array} & \text{if } \gamma \leq \alpha, \\ \boxed{\beta} \otimes \begin{array}{|c|}\hline \alpha \\ \hline \gamma \\ \hline \end{array} & \text{if } \alpha < \gamma \leq \beta, \\ \boxed{\alpha} \otimes \begin{array}{|c|}\hline \beta \\ \hline \gamma \\ \hline \end{array} & \text{if } \beta < \gamma. \end{cases}$$

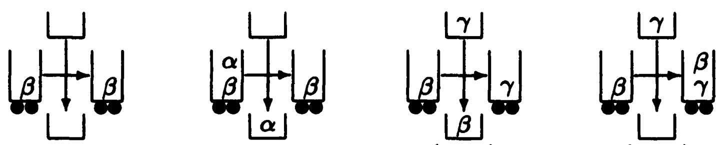
This R can be described by loading/unloading processes of balls into/from a special carrier of balls [Tg2]. The processes are depicted as follows.

(2.18)



$(\gamma \leq \alpha)$ $\quad$ $(\alpha < \gamma \leq \beta)$ $\quad$ $(\beta < \gamma)$

(2.19)



$(\gamma \leq \beta)$ $\quad$ $(\beta < \gamma)$

Here we assume $\alpha, \beta, \gamma > 1$ and let the vacancy denote letter 1.

Remark 2.4. This carrier is special because it describes the crystal for an *anti-symmetric* tensor representation. Usually carriers are used to describe *symmetric* tensor representations in SCA [F, TM, TTM], in which the combinatorial R in (2.17) is interpreted as loading/unloading processes by a carrier of capacity l.

3. A BRIEF REVIEW OF COMBINATORIAL R: TYPE D CASE

3.1. Formulas for the symmetric tensors. We consider $D_n^{(1)}$ crystals in [KKM]. As a set the crystal B_l (l is any positive integer) is given by

$$B_l = \left\{ (x_1, \cdots, x_n, \overline{x}_n, \cdots, \overline{x}_1) \in \mathbb{Z}_{\geq 0}^{2n} \,\middle|\, \sum_{i=1}^{n} (x_i + \overline{x}_i) = l, x_n \overline{x}_n = 0 \right\}. \tag{3.1}$$

For the other properties of this crystal, see Okado's lecture in this volume. The B_l can be regarded as a set of single row Young tableaux of length l by interpreting x_i (resp. $\overline{x}_i$) as the number of letter i's (resp. $\bar{i}$'s). An example of the crystal graph of B_1 is given in FIGURE 1 in which the arrow with index i denotes the action of crystal operator $\tilde{f}_i$.

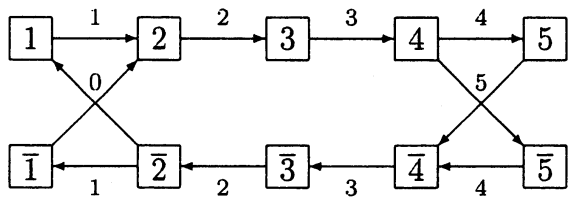


FIGURE 1. The crystal graph of B_1 for $D_5^{(1)}$.

The time evolution operator for the SCA associated with type D crystal was defined by the combinatorial R between these crystals [HKT1, HKOTY]. There is a piecewise linear formula for the combinatorial R and the energy function as in the type A case [KOTY1]. Since the former is rather involved we only quote an expression for the latter. For $x = (x_1, \ldots, \overline{x}_1)$ we denote $\sum_{i=1}^{n}(x_i + \overline{x}_i)$ by $\ell(x)$. Then the expression is given by

$$P_0^D(x, y) = \max_{1 \leq j \leq n-1} (\alpha_j, \alpha'_j),$$

$$\alpha_j = \max(\delta_{j,n-1}(x_n - \overline{y}_n), \overline{y}_j - x_j) + \ell(y) + \sum_{k=1}^{j} (\overline{x}_k - \overline{y}_k),$$

$$\alpha'_j = \max(\delta_{j,n-1}(\overline{y}_n - x_n), x_j - \overline{y}_j) + \ell(x) + \sum_{k=1}^{j} (y_k - x_k).$$

It takes values on between $|\ell(x) - \ell(y)|$ and $\ell(x) + \ell(y)$. If $\ell(x) = \ell(y)$ it coincides with the $D_n^{(1)}$ energy function in [KKM]. We can recognize that if all the barred variables vanish then this P_0^D reduces to the P_0 in section 2.3.

Another expression, which is suitable for the description of phase shifts, is given by

$$\hat{P}_0^D(x, y) := P_0^D(x, y) - |\ell(x) - \ell(y)|. \tag{3.2}$$

Since its minimal value is zero, there is a scattering of two solitons with no phase shift (section 5.4).

Example 3.1. Consider the energy function $\hat{P}_0$ on $B_1 \otimes B_1$. In the type A case it takes on two values, say $\hat{P}_0(1,1) = 2$ and $\hat{P}_0(1,2) = 1$ which are common with the type D case. In the type D case it takes on three values, i.e. we also have $\hat{P}_0^D(1,\bar{1}) = 0$.

In accordance with the symbols in the type A case we present one more expression which is suitable for the description of the conserved quantities

$$\check{P}_0^D(x,y) := P_0^D(x,y) - \ell(x) - \ell(y). \tag{3.3}$$

By the same reason we denote the energy function $\check{P}_0^D$ on $B_l \otimes B_1$ by H_l^D (section 5.5).

3.2. **Formulas for the color separation.** We define a partial order $\prec$ as

$$1 \prec 2 \prec \cdots \prec \begin{matrix} n \\ \bar{n} \end{matrix} \prec \cdots \prec \bar{2} \prec \bar{1}, \tag{3.4}$$

between the letters in the Young tableaux for type D crystals [KN] [6]. The $D_n^{(1)}$ crystal $B_\natural$ is given by

$$B_\natural = \left\{ \begin{array}{|c|} \hline \alpha \\ \hline \beta \\ \hline \end{array} \;\middle|\; \begin{array}{c} 1 \preceq \alpha \prec \beta \preceq \bar{1}, (\alpha,\beta) \neq (1,\bar{1}) \\ \text{or } (\alpha,\beta) = (n,\bar{n}), (\bar{n},n) \end{array} \right\} \cup \{\phi\}, \tag{3.5}$$

as a set. It is isomorphic to $B(\Lambda_2) \oplus B(0)$ as D_n crystals; The elements of $B(\Lambda_2)$ are represented by the column tableaux, and the sole element of $B(0)$ is denoted by ϕ. We give an example of crystal graph for $B_\natural$ (FIGURE. 2).

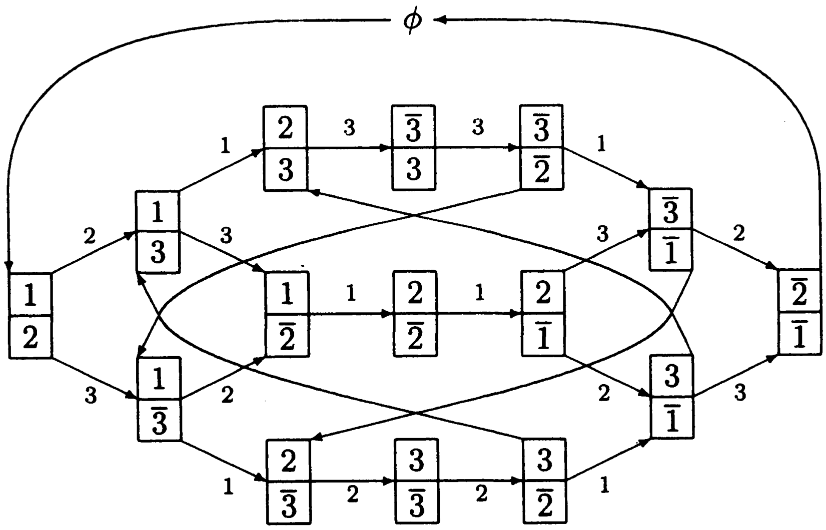


FIGURE 2. The crystal graph of $B_\natural$ for $D_3^{(1)}$. The arrows without index are for $\tilde{f}_0$ actions.

[6] We do not impose the order between n and $\bar{n}$.

The crystal actions are defined according to a general scheme for the crystals of antisymmetric tensor representations. As D_n crystals the actions of $\tilde{f}_i \, (i \neq 0)$ are defined in [KN]. As a $D_n^{(1)}$ crystal the $\tilde{f}_0$ actions are defined in [Ko, Sc]. [7]

Let us consider the $D_n^{(1)}$ crystal isomorphism $R : B_\natural \otimes B_l \xrightarrow{\sim} B_l \otimes B_\natural$. Given a partition λ, let B_λ denote the D_n crystal associated with λ. As D_n crystals the tensor product decomposition

$$B_\natural \otimes B_l \simeq B_{(l,1,1)} \oplus B_{(l+1,1)} \oplus (B_{(l)})^{\oplus 2},$$

is given by the Littlewood-Richardson rule [N]. If $x \otimes y \in B_\natural \otimes B_l$ falls into $B_{(l,1,1)}$ or $B_{(l+1,1)}$, then $R(x \otimes y)$ is uniquely determined by a column insertion scheme [Bk, HKOT]. We do not give a thorough treatment of this topic (See Lecouvey's lecture in this volume) but show some examples of the column insertion

$$(\bar{1} \to [1]) = [\ ], \left(\bar{1} \to \begin{bmatrix} 1 \\ \alpha \end{bmatrix}\right) = [\alpha], \left(\bar{2} \to \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = [1], \left(\bar{1} \to \begin{bmatrix} 2 \\ \bar{2} \end{bmatrix}\right) = [\bar{1}],$$

where α is any letter satisfying $1 \prec \alpha \prec \bar{1}$. Also we have

$$\left(1 \to \begin{bmatrix} 3 \\ \bar{1} \end{bmatrix}\right) = \begin{bmatrix} 2 & 3 \\ \bar{2} & \end{bmatrix}, \left(3 \to \begin{bmatrix} 2 \\ \bar{2} \end{bmatrix}\right) = \begin{bmatrix} 1 & \bar{1} \\ 3 & \end{bmatrix}.$$

If $x \otimes y \in B_\natural \otimes B_l$ falls into $B_{(l)}$ the insertion scheme seems not powerful enough to determine $R(x \otimes y)$ uniquely because there are two $B_{(l)}$s in the decomposition of $B_\natural \otimes B_l$. However the following Lemma allows us to determine it uniquely.

Lemma 3.2. Let $x' \otimes y' = R(x \otimes y) \in B_l \otimes B_\natural$. (1) If $x = \phi$, then $y' \neq \phi$, (2) If $y' = \phi$, then $x \neq \phi$.

Proof. We prove claim (1). We can find a sequence $i_1, \ldots, i_k$, none of them is 0, for some k such that $\tilde{f}_{i_k} \cdots \tilde{f}_{i_1}(y)$ is the tableau with all entries being 3. Since ϕ is not affected by these actions we have $\tilde{f}_{i_k} \cdots \tilde{f}_{i_1}(x \otimes y) = x \otimes \tilde{f}_{i_k} \cdots \tilde{f}_{i_1}(y)$. Then

$$\begin{array}{rcl} \phi \otimes \boxed{3\,|\,3\,|\cdots|\,3} & \overset{\tilde{f}_0}{\mapsto} & \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \otimes \boxed{3\,|\,3\,|\cdots|\,3} \\ & \overset{\sim}{\mapsto} & \boxed{1\,|\,3\,|\cdots|\,3} \otimes \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array} \\ & \overset{\tilde{e}_0}{\mapsto} & \boxed{1\,|\,3\,|\cdots|\,3} \otimes \begin{array}{|c|}\hline 3 \\ \hline \bar{1} \\ \hline \end{array}. \end{array}$$

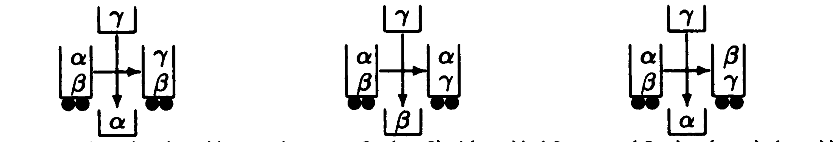
Then we apply $\tilde{e}_{i_1} \cdots \tilde{e}_{i_k}$ on it to obtain $x' \otimes y'$ which can not make y' be ϕ. In the same way claim (2) is also verified. □

[7]The $\tilde{f}_0$ actions in $B_\natural$ are also available in Appendix C of [HKOTY].

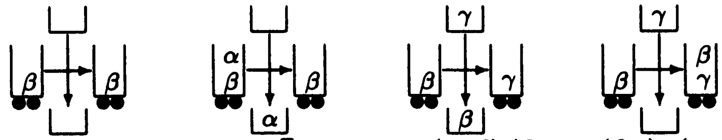
As in the type A case we give a description of the combinatorial R by loading/unloading processes in (3.6)–(3.12). Let the carrier represent a column tableau of height two or ϕ. For example we have

$$R:\phi\otimes\boxed{1}\overset{\sim}{\mapsto}\boxed{1}\otimes\begin{array}{|c|}\hline 2\\ \hline \bar{2}\\ \hline\end{array},$$

which is depicted by the first case of (3.11); The $\phi\in B(0)$ is represented by a carrier with the letter ϕ inside it. The letter 1 is represented by a vacancy as in the type A case. In (3.9)–(3.10) we denote $x\pm 1$ by $x_\pm$ and $\overline{x\pm 1}$ by $\bar{x}_\pm$.

(3.6)


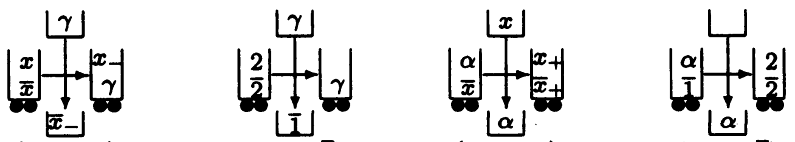
$(\gamma\preceq\alpha:(\gamma,\beta)\neq(x,\bar{x}))$ $(\alpha\prec\gamma\preceq\beta:(\alpha,\beta)\neq(x,\bar{x}))$ $(\beta\prec\gamma$ or $(\beta,\gamma)=(\bar{n},n),(n,\bar{n}))$

(3.7)


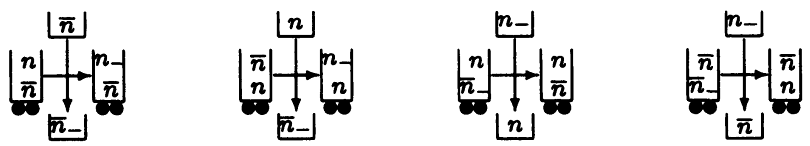
$(\beta\neq\bar{1})$ $(\gamma\preceq\beta)$ $(\beta\prec\gamma$ or $(\beta,\gamma)=(\bar{n},n),(n,\bar{n}))$

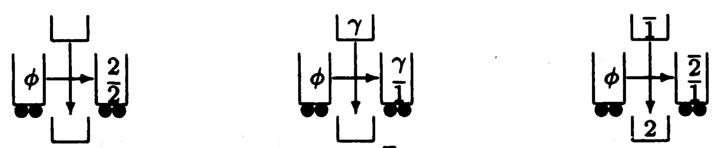
(3.8)


$(1\prec x\prec n)$

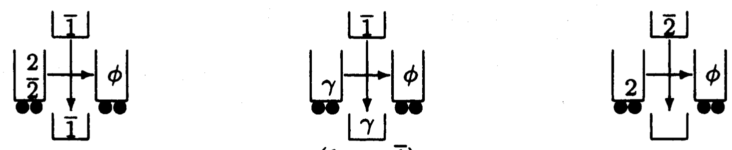
(3.9)


$(x\preceq\gamma\preceq\bar{x})$ $(1\prec\gamma\prec\bar{1})$ $(x\prec\alpha\prec\bar{x})$ $(1\prec\alpha\prec\bar{1})$

(3.10)


(3.11)


$(1\prec\gamma\prec\bar{1})$

(3.12) 

$(1 \prec \gamma \prec \bar{1})$

3.3. **Particle/anti-particle numbers.** For any letter x we define its *particle number* $P(x)$ and *anti-particle number* $A(x)$ as in TABLE 1.

x	particle number $P(x)$	anti-particle number $A(x)$
1	0	0
$\bar{1}$	1	1
$2, \ldots, n$	1	0
$\bar{2}, \ldots, \bar{n}$	0	1

TABLE 1. Particle and anti-particle number.

These numbers will be used in the proof of the color separation scheme (section 5.7). Next we define these numbers for the elements of $B_\natural$. For any element

$$b = \begin{array}{|c|}\hline \alpha \\ \hline \beta \\ \hline \end{array} \in B_\natural,$$

we set $P(b) = P(\alpha) + P(\beta)$ and $A(b) = A(\alpha) + A(\beta)$. We also set $P(\phi) = A(\phi) = 1$. The combinatorial R between $B_\natural$ and B_1 preserves these numbers separately.

Lemma 3.3. For any $(x', y') = R(x, y)$ the relations $P(x) + P(y) = P(x') + P(y')$ and $A(x) + A(y) = A(x') + A(y')$ hold.

4. Cellular automata: type A case

4.1. Time evolution operators. The highest weight element of B_l is given by $u_l = (l, 0, \dots, 0)$. Define the set of basic paths by

$$\mathcal{P} = \left\{ \mathbf{p} = p_1 \otimes p_2 \otimes \cdots \in B_1^{\otimes\infty} | p_i = \boxed{1} \text{ for } i \gg 1 \right\}. \tag{4.1}$$

In the terminology of the box-ball system [T, TNS], a basic path is regarded as an infinite array of boxes of capacity one with finite number of balls scattered among them, where $\boxed{1}$ represents an empty box and $\boxed{\alpha}$ $(\alpha \geq 2)$ a box containing a ball with index α. Practically one can replace the symbol ∞ in $B_1^{\otimes\infty}$ by any sufficiently large positive integer.

We introduce operators T_l $(l \geq 1)$ on $\mathcal{P}$ as follows. For any $\mathbf{p} \in \mathcal{P}$ we define $T_l(\mathbf{p}) \in \mathcal{P}$ by using the maps for the sl_n crystal isomorphism (2.17) as

$$u_l \otimes \mathbf{p} \overset{\sim}{\mapsto} T_l(\mathbf{p}) \otimes u_l. \tag{4.2}$$

Here we used (2.17) repeatedly as $B_l \otimes \mathcal{P} \overset{\sim}{\to} B_1 \otimes B_l \otimes \mathcal{P} \overset{\sim}{\to} B_1^{\otimes 2} \otimes B_l \otimes \mathcal{P} \overset{\sim}{\to} \cdots \overset{\sim}{\to} \mathcal{P} \otimes B_l$ to send u_l from left to right. Because of the boundary condition in (4.1) we always have the highest weight element u_l in the right hand side.

The operator T_l gives a time evolution of the automaton that can be described by a carrier of capacity l [TM, FOY]. Due to the Yang-Baxter relation they commute, $T_l T_{l'} = T_{l'} T_l$ for any l and l'. For $l = \infty$ we denote T_∞ by T. Clearly these operators are invertible. The inverse of T_l is given by

$$\mathbf{p} \otimes u_l \overset{\sim}{\mapsto} u_l \otimes T_l^{-1}(\mathbf{p}).$$

But if we would like to use these inverse operators we may have to replace the set of half-infinite paths (4.1) by that of paths extending to both ends.

4.2. Solitons. If well separated from the others a set of successive integers arranged in decreasing order[8] behaves like a soliton. Under the time evolution T it travels in the speed of its length.

Example 4.1. Here we have a one-soliton state under T.

```
t=0   ...........554322.................................
t=1   .................554322...........................
t=2   .......................554322.....................
t=3   .............................554322...............
t=4   ...................................554322.........
```

Here and hereafter we denote $\boxed{1}$ by "." and $\boxed{\alpha}$ $(\alpha \geq 2)$ by α in the Examples. Also we omit the symbol $\otimes$.

Exercise 4.2. Find a formula for the speed of a soliton under T_l.

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[8] Our convention is opposite to that in [T, TNS].

4.3. **Scattering rules.** If there are two solitons of different lengths, a collision can occur.

Example 4.3.

```
t=0  554322.........422......................................
t=1  ......554322......422...................................
t=2  ............554322...422................................
t=3  ..................5543..42222...........................
t=4  ......................553....442222.....................
t=5  .........................553.......442222...............
t=6  ............................553..........442222.........
t=7  ...............................553.............442222...
```

It seems natural to identify the one-soliton 554322 with an element of sl_5 crystal $(0,2,1,1,2)$. But in our convention there is no 1 in any soliton because 1 is used for an empty box. Thus we identify it with an element of sl_4 crystal $(2,1,1,2)$. Then in the above example we see the scattering of two solitons $(2,1,1,2)\times(2,0,1,0)\mapsto(0,1,0,2)\times(4,0,2,0)$.

Exercise 4.4. Show that the scattering in this example coincides with the result of the map in Definition 2.2.

In general we have the following:

Theorem 4.5 ([FOY, HKT1, TNS]). The scattering rules of any two solitons are given by the combinatorial R.

Of course there are scattering processes involving more than two solitons. See Examples 1.1-1.2.

Theorem 4.6 ([FOY]). The scattering of solitons is factorized into two body scatterings.

Remark 4.7. We do not present proofs of these theorems in line with original works, because one can regard them as results of the color separation scheme (subsection 4.7) and the linearization (subsection 4.8).

4.4. **Phase shifts.** In a collision of two solitons there is a phase shift. The existence of phase shifts signifies the nonlinearity of the system. The amount of a phase shift depends on constituent letters of the solitons.

Example 4.8.

```
t=0  554322.........553......................................
t=1  ......554322......553...................................
t=2  ............554322...553................................
t=3  ..................554322553.............................
t=4  ........................432555532.......................
t=5  ...........................432...555532.................
t=6  ..............................432......555532...........
t=7  .................................432.........555532.....
```

Exercise 4.9. Compare the phase shifts in Examples 4.3 and 4.8. Compute the values of the function $\hat{P}_0$ in (2.14) for $(x,y) = ((2,1,1,2),(2,0,1,0))$ and that for $(x,y) = ((2,1,1,2),(0,1,0,2))$. Check that these $\hat{P}_0$ values are equal to the phase shifts in the above Examples.

A clever idea of describing the scattering rules with the data of phase shifts is to use the *affinization* of crystals [FOY]

$$Aff(B_l) = \{z^d b | d \in \mathbb{Z}, b \in B_l\},$$

where z is a parameter. We redefine the combinatorial R as a map $\hat{R} : Aff(B_l) \otimes Aff(B_{l'}) \to Aff(B_{l'}) \otimes Aff(B_l)$ which reads as

$$\hat{R}(z^{\gamma_1} x \otimes z^{\gamma_2} y) = z^{\gamma_1 + \delta} x' \otimes z^{\gamma_2 - \delta} y',$$

where $\delta = \hat{P}_0(x,y)$ and the x', y' are given by the map $R : (x,y) \mapsto (x',y')$ in Definition 2.2. In this formalism we can say that the claim on the property of the phase shifts is included in the statement of Theorem 4.5

4.5. **Factorization.** We have defined the time evolution T by the crystal isomorphism. Now we show its factorized expression. Consider the following example.

Example 4.10.

```
t=0     554422....632....5................................
t=1     ......554422.632..5...............................
t=2     ............552.6445322...........................
t=3     ..............552.4...654322......................
```

Using a factorization $T = K_2K_3K_4K_5K_6$ given below, the updating process of the state from $t = 1$ to $t = 2$ is described as follows.

Example 4.11.

```
t=1       ......554422.632..5...............................
t=1+1/5   ......554422..326.5...............................
t=1+2/5   ........442255326..5..............................
t=1+3/5   ..........2255326445..............................
t=1+4/5   ..........2255.264453.............................
t=2       ............552.6445322...........................
```

In general we can write $T = K_2K_3 \cdots K_n$, where K_i is an operator for moving the letter i. Precisely the K_i's are so defined to work as:

(1) Move every letter i only once.
(2) Exchange the leftmost i with its nearest right 1.
(3) Exchange the leftmost i among the rest of the i's with its nearest right 1.
(4) Repeat this procedure until all of the i's are moved.

Another definition of K_i's will be given (as a specialized form of the type D case) in subsection 5.1. This algorithm is knows as the *original algorithm* (in the terminology of [F]), because it was originally used for the box-ball system. We note that the time evolution T_l with finite l does not have this property.

4.6. **Soliton content.** Fukuda, Okado and Yamada introduced a family of conserved quantities of the SCA, which was defined by the energy function [FOY]. Given $\mathbf{p} = p_1 \otimes p_2 \otimes p_3 \otimes \cdots \in \mathcal{P}$, define $b^{(i)}$ $(i \geq 1)$ by

$$
\begin{aligned}
u_l \otimes (p_1 \otimes p_2 \otimes p_3 \otimes \cdots) &\overset{\sim}{\mapsto} \tilde{p}_1 \otimes (b^{(1)} \otimes p_2 \otimes p_3 \otimes \cdots) \\
&\overset{\sim}{\mapsto} \tilde{p}_1 \otimes \tilde{p}_2 \otimes (b^{(2)} \otimes p_3 \otimes \cdots) \\
&\overset{\sim}{\mapsto} \cdots \overset{\sim}{\mapsto} (\tilde{p}_1 \otimes \tilde{p}_2 \otimes \tilde{p}_3 \otimes \cdots) \otimes u_l.
\end{aligned}
$$

Here $\tilde{p}_i$'s are those in $T_l(\mathbf{p}) = \tilde{p}_1 \otimes \tilde{p}_2 \otimes \tilde{p}_2 \otimes \cdots$. Consider the following quantities

$$E_l(\mathbf{p}) = -\sum_{j=1}^{\infty} H_l(b^{(j-1)}, p_j), \tag{4.3}$$

$$N_l(\mathbf{p}) = -E_{l-1}(\mathbf{p}) + 2E_l(\mathbf{p}) - E_{l+1}(\mathbf{p}). \tag{4.4}$$

Here $b^{(0)} = u_l$ and H_l is the energy function defined below (2.17). Because of the commutativity of T_l's they become conserved quantities of the system, i.e. $E_l(T(\mathbf{p})) = E_l(\mathbf{p})$ and $N_l(T(\mathbf{p})) = N_l(\mathbf{p})$ for any l. We call $(N_1, N_2, \ldots)$ the *soliton content*, because one can recognize that N_l is the number of solitons of length l in the state $\mathbf{p}$ [9]. In order to see this we apply the operator T sufficiently many times on $\mathbf{p}$. Then the solitons will finally separate each other even in the real space due to the difference of their speed, so we can read off the soliton content.

An extended family of conserved quantities is defined by simply generalizing this construction. Recall the energy function $H^{k,l}$ defined below (2.9). Denote $u^{k,l}$ (the highest weight element of $B^{k,l}$) by $c^{(0)}$ and define $c^{(i)}$ $(i \geq 1)$ by

$$
\begin{aligned}
u^{k,l} \otimes (p_1 \otimes p_2 \otimes p_3 \otimes \cdots) &\overset{\sim}{\mapsto} \tilde{p}_1 \otimes (c^{(1)} \otimes p_2 \otimes p_3 \otimes \cdots) \\
&\overset{\sim}{\mapsto} \tilde{p}_1 \otimes \tilde{p}_2 \otimes (c^{(2)} \otimes p_3 \otimes \cdots) \\
&\overset{\sim}{\mapsto} \cdots \overset{\sim}{\mapsto} (\tilde{p}_1 \otimes \tilde{p}_2 \otimes \tilde{p}_3 \otimes \cdots) \otimes b(\mathbf{p}).
\end{aligned} \tag{4.5}
$$

Here $b(\mathbf{p}) \in B^{k,l}$ is an element such that all entries in the first row of $\mathcal{T}(b(\mathbf{p}))$ are 1's (not identical to $u^{k,l}$ in general). We introduce the following quantities

$$E^{k,l}(\mathbf{p}) = -\sum_{j=1}^{\infty} H^{k,l}(c^{(j-1)}, p_j), \tag{4.6}$$

$$N^{k,l}(\mathbf{p}) = -E^{k,l-1}(\mathbf{p}) + 2E^{k,l}(\mathbf{p}) - E^{k,l+1}(\mathbf{p}). \tag{4.7}$$

Thus $E^{1,l}(\mathbf{p}) = E_l(\mathbf{p})$ and $N^{1,l}(\mathbf{p}) = N_l(\mathbf{p})$. Due to the Yang-Baxter relation (Proposition 2.1) these quantities are invariant under the time evolution of the SCA.

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[9] For the CA (1.1) an equivalent set of conserved quantities was first found by Torii, Takahashi and Satsuma [TTS].

Example 4.12. Denote $(E^{k,1}, E^{k,2}, \ldots)$ by $\mathbf{E}^k$ and $(N^{k,1}, N^{k,2}, \ldots)$ by $\mathbf{N}^k$. Then for the states at each time step of Examples 1.1-1.2 we have

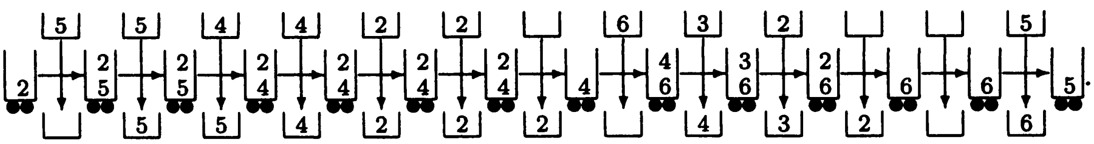
$$\begin{aligned}
\mathbf{E}^1 &= (3,5,7,8,9,10,10,\ldots), & \mathbf{N}^1 &= (1,0,1,0,0,1,0,\ldots),\\
\mathbf{E}^2 &= (3,4,5,6,7,8,8,\ldots), & \mathbf{N}^2 &= (2,0,0,0,0,1,0,\ldots),\\
\mathbf{E}^3 &= (2,3,4,5,5,\ldots), & \mathbf{N}^3 &= (1,0,0,1,0,\ldots),\\
\mathbf{E}^4 &= (2,3,3,\ldots), & \mathbf{N}^4 &= (1,1,0,\ldots),\\
\mathbf{E}^5 &= (1,1,\ldots), & \mathbf{N}^5 &= (1,0,\ldots).
\end{aligned}$$

4.7. **Color separation scheme.** For any $\mathbf{p} \in \mathcal{P}$ we define $T^{k,l}(\mathbf{p}) \in \mathcal{P}$ by using (4.5) as $T^{k,l}(\mathbf{p}) = \tilde{p}_1 \otimes \tilde{p}_2 \otimes \tilde{p}_2 \otimes \cdots$. In particular we denote $T^{2,1}$ by $T_\natural$. Therefore we have

$$u_\natural \otimes \mathbf{p} \overset{\sim}{\mapsto} T_\natural(\mathbf{p}) \otimes b(\mathbf{p}), \tag{4.8}$$

by the isomorphism of the crystals. Here $u_\natural = u^{2,1}$ is the highest weight element of $B_\natural$, and $b(\mathbf{p}) \in B_\natural$ is an element such that the *upper* entry of the tableau $\mathcal{T}(b(\mathbf{p}))$ is 1. Recall the description of combinatorial R by a carrier (2.18)-(2.19). Then one recognizes that the *lower* entry of $\mathcal{T}(b(\mathbf{p}))$ is identified with the letter taken off by the carrier. Let us apply $T_\natural$ repeatedly on a state and observe what happens there.

Example 4.13. Apply $T_\natural$ on the state at $t = 1$ of Example 4.10. In terms of the carrier description we have



By repeating the same procedure we obtain the following data.

```
s=0   ......554422.632..5.............................. 5
s=1   .......554222.432.6.............................. 6
s=2   ........552222443..2............................. 2
s=3   .........522225442.3............................. 3
s=4   ..........2222554224............................. 4
s=5   ..........2222.552242............................ 4
s=6   ..........2222..522522........................... 5
s=7   ..........2222...225222.......................... 5
s=8   ..........2222...22.2222.........................
```

Here s is the number of times $T_\natural$ is applied. The number at the end of each row is the letter which will be taken off from that row by the carrier.

The result is a state for a *monochrome system*, i.e. the box-ball system with only one kind of balls. Denote the first line of the above example by $\mathbf{p}$ and the last line by $\tilde{\mathbf{p}}$. Also denote $\mathbf{y}$ the word made of the removed letters arranged from bottom to top, i. e. $\mathbf{y}$ =55443265. Then we may write symbolically as $\mathbf{p} = \tilde{\mathbf{p}} \oplus \mathbf{y}$.

Example 4.14. By the same procedure on the state at $t = 2$ of Example 4.10 we have that:

```
s=0   ............552.6445322........................... 5
s=1   .............522.5464322.......................... 6
s=2   ..............22255.44322......................... 2
s=3   ..............222.5254432......................... 3
s=4   ..............222..2554422........................ 4
s=5   ..............222..2.554222....................... 4
s=6   ..............222..2..552222...................... 5
s=7   ..............222..2...522222..................... 5
s=8   ..............222..2....222222....................
```

For the state at $t = 3$ we have that:

```
s=0   ...............552.4...654322..................... 5
s=1   ................5225....644322.................... 6
s=2   .................2252....544322................... 2
s=3   .................22.22..554432.................... 3
s=4   .................22.22...554422................... 4
s=5   .................22.22....554222.................. 4
s=6   .................22.22.....552222................. 5
s=7   .................22.22......522222................ 5
s=8   .................22.22.......222222...............
```

We can recognize that the word $\mathbf{y}$ is invariant under T and the path $\tilde{\mathbf{p}}$ evolves according to T.

This property holds generally [Tg2].

Proposition 4.15.

(1) Any state $\mathbf{p}$ of the SCA admits a decomposition $\mathbf{p} = \tilde{\mathbf{p}} \oplus \mathbf{y}$ in which $\tilde{\mathbf{p}}$ is a state of the underlying monochrome system and $\mathbf{y}$ is a word.

(2) On applying T it evolves as

$$T(\mathbf{p}) = T(\tilde{\mathbf{p}}) \oplus \mathbf{y}. \tag{4.9}$$

We give a sketch of the proof [10]. The non-trivial problem associated with claim (1) is that whether we can always remove all the letters ≥ 3 from *any* state, by applying $T_\natural$ only finitely many times. It can be done by an induction. To establish claim (2), it suffices to prove the relation $T_l(\mathbf{p}) = T_l(\tilde{\mathbf{p}}) \oplus \mathbf{y}$ for any l. It is derived from the commutativity $T_l T_\natural = T_\natural T_l$, which comes from the Yang-Baxter relation on $B_l \otimes B_1 \otimes B_\natural$.

[10] In this note I shall show a generalization of this proposition in section 5.6 and give its proof there.

From that commutativity we also see that the conserved quantity (4.4) is invariant under $T_\natural$, i.e. $N_l(\mathbf{p}) = N_l(\tilde{\mathbf{p}})$. In other words the soliton contents of $\mathbf{p}$ and $\tilde{\mathbf{p}}$ are identical. Therefore the conserved quantities of the SCA consist of the word $\mathbf{y}$ and the soliton content. It also means that an equivalent data of the conserved quantities $N^{k,l}$ $(k > 1, l \geq 1)$ in subsection 4.6 should be contained in the word $\mathbf{y}$.

Remark 4.16. Since $\mathbf{y}$ is a conserved quantity, the P,Q-tableaux made from $\mathbf{y}$ by the Robinson-Schensted-Knuth (RSK) correspondence are also conserved quantities. The P-tableau here is not identical to that in [F] (which is also a conserved quantity) since the latter is made from the original path $\mathbf{p}$.

4.8. **Linearization.** It was conjectured in [KOTY2] and proved in [Tg1] that the time evolution $T(= T_\infty)$ of the monochrome system (Takahashi-Satsuma's CA) is linearized in the *rigged configuration.* A rigged configuration is a Young diagram with numbers (which we call *riggings*) attached to its rows [11]. The Young diagram is invariant under the time evolution of SCA, and the riggings increase linearly. The former is equivalent to the soliton content (subsection 4.6), i. e. the set of conserved quantities defined in [TTS].

We do not repeat the proof of [Tg1]. From now on we rewrite the method of the construction of rigged configurations into another one that seems suitable for an actual computation. Consider the state $\tilde{\mathbf{p}}$ we have got in Example 4.13.

```
..... ..... 2222. ..22. 2222. ..... ..... ..... ..... .....
```

Here we inserted the blanks every five letters. Call the pattern ".2" a wave tail and "2." a wave front. For this example we can read off the positions of wave tails and fronts as

```
{{10, 17, 20}, {14, 19, 24}}.
```

In general we can write such data as $\{M_1, M_2\}$ where $M_1 = (a_1, a_3, \dots, a_{2l-1})$ denotes the positions for wave tails, and $M_2 = (a_2, a_4, \dots, a_{2l})$ denotes those for wave fronts. Clearly they satisfy $a_1 < a_2 < \cdots < a_{2l}$. Taking the size l into account we set $A_1 = (1, 3, \dots, 2l-1)$ and $A_2 = (2, 4, \dots, 2l)$. Let $b = \{\}$. We repeat the following procedure until we have $M_1 = M_2 = \emptyset$.

- Let $i = 0$.
- While $M_1 \cap M_2 = \emptyset$, replace M_1 by $M_1 - A_1$, M_2 by $M_2 - A_2$, and i by $i+1$.
- When $M_1 \cap M_2 \neq \emptyset$ is attained, append $\{M_1 \cap M_2, i\}$ to b, and replace M_1 by $M_1 \setminus (M_1 \cap M_2)$ and M_2 by $M_2 \setminus (M_1 \cap M_2)$.

In the above procedure the multiplicity of the elements should be respected. For instance $\{1,2,2,3\} \cap \{2,2,4,5\}$ is equal to $\{2,2\}$, not to $\{2\}$. And $\{1,2,2,3\} \setminus \{2,3\}$ is equal to $\{1,2\}$, not to $\{1\}$.

---

[11] More generally, a rigged configuration is *a set of* Young diagrams with numbers attached to their rows [KKR, KR]. It is used in calculations of fermionic formulas. See Schilling's lecture in this volume.

Then we obtain such type of data $b = \{\{S_1, i_1\}, \{S_2, i_2\}, \ldots, \{S_k, i_k\}\}$ for some k, where S_a are sets of integers and i_a are positive integers. We replace it by $b' = \{\{S_1, j_1\}, \{S_2, j_2\}, \ldots, \{S_k, j_k\}\}$ with $j_1 = i_1, j_2 = i_1 + i_2, \ldots, j_k = i_1 + \cdots + i_k$. This S_a can be regarded as the set of linearized position coordinates of solitons of length j_a.

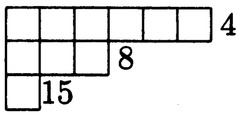
For the above example the procedure for linearization goes as follows.

```
{{10, 17, 20}, {14, 19, 24}},  b={}
{{9, 14}, {12, 18}},         b={{{15}, 1}}
{{7}, {10}},               b={{{15}, 1},{{8},  2}}
{{}, {}},              b={{{15}, 1},{{8},  2},{{4},  3}}
```

Therefore we have

```
b'={{{15}, 1}, {{8},  3}, {{4},  6}}
```

that is equivalent to the rigged configuration:

$b' =$ 

There are solitons of lengths $1, 3, 6$, and their "position" coordinates are $15, 8, 4$.

Exercise 4.17. Show that under the time evolution T these data evolves in the following way.

```
t=1      {{15}, 1}, {{8},  3}, {{4},  6}}
t=2      {{16}, 1}, {{11}, 3}, {{10}, 6}}
t=3      {{17}, 1}, {{14}, 3}, {{16}, 6}}
```

We give another example which is more involved.

Example 4.18.

```
t=0   2222....22222...222.222...222....2..................
t=1   ....2222.....222...2...222...2222.2222..............
t=2   ........2222....222.2.....222....2....2222222.......
t=3   ............2222...2.222.....222..2..........2222222
```

For the state at $t = 0$ of this example

```
2222. ...22 222.. .222. 222.. .222. ...2. ..... ..... .....
```

we read off the positions of wave tails/fronts.

```
{{0, 8, 16, 20, 26, 33}, {4, 13, 19, 23, 29, 34}}
```

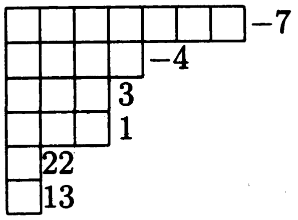
The linearization goes as follows:

```
{{0, 8, 16, 20, 26, 33}, {4, 13, 19, 23, 29, 34}},      b={}
{{-1, 5, 11, 17}, {2, 9, 15, 19}},         b={ {{13, 22}, 1}}
{{-3, -1}, {-2, 3}},                 b=Append[b,{{1, 3},   2}]
{{{-4}, {-1}},                       b=Append[b,{{-4},     1}]
{{{}, {}},                           b=Append[b,{{-7},     3}]
```

Here the command `Append[b,X]` means that it appends an element `X` to the list `b`. As a result we have

```
b'={{{13, 22}, 1},{{1, 3}, 3},{{-4}, 4},{{-7}, 7}}
```

that is equivalent to the rigged configuration:

$b' =$ 

The data (rigged configuration) evolves as

```
t=0   {{13, 22}, 1}, {{1, 3},   3}, {{-4}, 4}, {{-7}, 7}},
t=1   {{14, 23}, 1}, {{4, 6},   3}, {{0},  4}, {{0},  7}},
t=2   {{15, 24}, 1}, {{7, 9},   3}, {{4},  4}, {{7},  7}},
t=3   {{16, 25}, 1}, {{10, 12}, 3}, {{8},  4}, {{14}, 7}},
```

which should be compared with Example 4.18.

To close this subsection we give a comment on a slightly extended result on the relation between SCA and the rigged configuration [KOTY2]; The riggings associated with rows of width k increase by $\min(k,l)$ under T_l. This result can be derived from the time evolution of the rigged configuration under $T(= T_\infty)$. First note that if all the solitons are well separated in the state, this statement is rather obvious. Suppose we are given any state $\mathbf{p}$ for the system. By taking a sufficiently large integer N we can make all the solitons be well separated in $T^N(\mathbf{p})$. Since the T_ls commute each other we have

$$(T^{-1})^N \circ T_l \circ T^N(\mathbf{p}) = T_l(\mathbf{p}).$$

By this relation we can obtain the evolution of the riggings under T_l with finite l's.

4.9. **Extension to symmetric tensor crystals.** So far we have only treated *basic* automata in which all the cells are of capacity one. Now we give an overview of studies on generalized automata with cells of capacity ≥ 1 and even the *inhomogeneous* automata in which the capacities can vary site by site [F, HHIKTT, TM, TTM].

Given a set of positive integers $\{l_1, l_2, \ldots\}$, we define a set of inhomogeneous paths by

$$\mathcal{P} = \{\mathbf{p} = p_1 \otimes p_2 \otimes \cdots \in B_{l_1} \otimes B_{l_2} \otimes \cdots | p_i = u_{l_i} \text{ for } i \gg 1\}. \tag{4.10}$$

The time evolution of the automaton is defined in the same way as in subsection 4.1. It was first pointed out by Fukuda [F] that the time evolution $T(\doteq T_\infty)$ of this inhomogeneous automaton can be described by the *original algorithm* i. e. the factorized dynamics of basic automaton (subsection 4.5), if one puts partitioning walls into the state of the latter.

The conserved quantities and N-soliton solutions of the automata of this kind were studied from the viewpoint of ultradiscretization [TTM, HHIKTT] and that of the Robinson-Schensted-Knuth correspondence [F]. The color separation scheme in subsection 4.7 works also in this inhomogeneous case [Tg2] and it reduces these colored systems into the inhomogeneous monochrome systems [TM]. It seems interesting to reconsider results of those previous works on the conserved quantities and N-soliton solutions from this point of view.

4.10. **Extension to antisymmetric tensor crystals.** This system was studied by D. Yamada [Yd]. For any positive integer k we define a set of paths by

$$\mathcal{P} = \left\{ \mathbf{p} = p_1 \otimes p_2 \otimes \cdots \in (B^{k,1})^{\otimes\infty} | p_i = u^{k,1} \text{ for } i \gg 1 \right\}. \tag{4.11}$$

Given a positive integer l, we define the operator T_l on the path $\mathbf{p} \in \mathcal{P}$ as $u^{k,l} \otimes \mathbf{p} \overset{\sim}{\mapsto} T_l(\mathbf{p}) \otimes u^{k,l}$. Here we used the rectangle switching bijection in subsection 2.2 repeatedly as $B^{k,l} \otimes \mathcal{P} \overset{\sim}{\to} B^{k,1} \otimes B^{k,l} \otimes \mathcal{P} \overset{\sim}{\to} (B^{k,1})^{\otimes 2} \otimes B^{k,l} \otimes \mathcal{P} \overset{\sim}{\to} \cdots \overset{\sim}{\to} \mathcal{P} \otimes B^{k,l}$ to send $u^{k,l}$ from left to right. Because of the boundary condition in (4.11) we always have the $u^{k,l}$ in the right hand side. Let $T = T_\infty$ give the time evolution of this system. For $k = 1$ and $k = n-1$ (for sl_n) this system reduces to the basic automaton.

A characterization of one-soliton states and scattering rules of solitons in this CA were proposed in [Yd]. For example, a *generic* rectangle tableau of height k can *not* be embedded into the system to realize a one-soliton state. We can understand these results by a generalization of the color separation scheme in subsection 4.7. Assume $2 \le k \le n-2$ for sl_n.

Proposition 4.19. For any state $\mathbf{p} \in \mathcal{P}$ there is an integer N such that

$$(u^{k-1,1})^{\otimes N} \otimes (u^{k+1,1})^{\otimes N} \otimes \mathbf{p} \overset{\sim}{\mapsto} (u^{k-1,1})^{\otimes N} \otimes \mathbf{p}^* \otimes \mathbf{b}(\mathbf{p}) \overset{\sim}{\mapsto} \tilde{\mathbf{p}} \otimes \mathbf{c}(\mathbf{p}) \otimes \mathbf{b}(\mathbf{p}),$$

where $\mathbf{p}^*, \tilde{\mathbf{p}} \in \mathcal{P}$, $\mathbf{b}(\mathbf{p}) \in (B^{k+1,1})^{\otimes N}$, $\mathbf{c}(\mathbf{p}) \in (B^{k-1,1})^{\otimes N}$ are characterized by the following conditions.

(1) The $\mathbf{p}^*$ is for sl_{k+1}.
(2) The $\tilde{\mathbf{p}}$ is also for sl_{k+1}, but consists of only $u^{k,1}$ and ${}^t[1 \ \cdots \ k-1 \ \ k+1]$.
(3) If written as $\mathbf{b}(\mathbf{p}) = b_1 \otimes \cdots \otimes b_N, \mathbf{c}(\mathbf{p}) = c_1 \otimes \cdots \otimes c_N$ the b_i, c_i $(1 \le i \le N)$ have the forms

$$b_i = \begin{bmatrix} 1 \\ \vdots \\ k \\ y_i \end{bmatrix}, \qquad c_i = \begin{bmatrix} c_{i,1} \\ \vdots \\ c_{i,k-1} \end{bmatrix},$$

$$y_i \in \{k+1, \ldots, n\}, \quad 1 \le c_{i,1} < \cdots < c_{i,k-1} \le k.$$

To give a proof of this proposition by using the diagrammatic algorithm in section 2.1 is left for the interested readers.

For the above $\mathbf{c}(\mathbf{p})$ we define $z_i \in \{1, 2, \dots, k\}$ $(1 \le i \le N)$ as

$$\{z_i\} = \{1, 2, \dots, k\} \setminus \{c_{i,1}, \dots, c_{i,k-1}\}.$$

Then for the above $\mathbf{b}(\mathbf{p}), \mathbf{c}(\mathbf{p})$ we define a pair of words $\mathbf{y}, \mathbf{z}$ as $\mathbf{y} = y_1 \dots y_N$ and $\mathbf{z} = z_1 \dots z_N$. We write symbolically as

$$\mathbf{p} = \tilde{\mathbf{p}} \oplus \mathbf{z} \oplus \mathbf{y}. \tag{4.12}$$

Then by Proposition 2.1 we have that

Proposition 4.20. The relation $T(\mathbf{p}) = T(\tilde{\mathbf{p}}) \oplus \mathbf{z} \oplus \mathbf{y}$ holds.

Thus the words $\mathbf{y}$ and $\mathbf{z}$ are conserved quantities of the system. Note that we can regard the path $\tilde{\mathbf{p}}$ as a state of the basic monochrome system, i. e. the Takahashi-Satsuma's CA. Thus a characterization of generic N-soliton states of this CA is as follows; An N-soliton state of the SCA in [Yd] has an N-soliton state $\tilde{\mathbf{p}}$ of the basic monochrome system in the decomposition (4.12).

Example 4.21. Let $k = 2$ and consider the following state

$$\mathbf{p} = \begin{array}{|c|}\hline 2\\ \hline 4\\ \hline\end{array}\,\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}\,\begin{array}{|c|}\hline 2\\ \hline 4\\ \hline\end{array}\,\begin{array}{|c|}\hline 3\\ \hline 5\\ \hline\end{array}\,\begin{array}{|c|}\hline 2\\ \hline 5\\ \hline\end{array}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

which we quoted from [Yd]. Here $u^{2,1}$ is denoted by "$\cdot$". Then we have

$$\mathbf{p}^* = \cdot\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}\,\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}\cdot\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}\,\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}\cdot\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}\,\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\tilde{\mathbf{p}} = \cdot\cdot\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}\,\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}\cdot\cdot\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}\,\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}\cdot\cdot\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}\,\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

and $\mathbf{z} = 11121$, $\mathbf{y} = 44355$. Thus $\mathbf{p}$ is a three-soliton state.

4.11. **Comments on a typographical matter.** In the next section we consider SCAs associated with type D affine Lie algebra. Usually we use $\bar{1}, \bar{2}, \bar{3}, \dots$ as the letters that have weights of opposite signs to those of $1, 2, 3, \dots$. For typographical reason we shall denote $2, 3, 4, 5$ by `a,b,c,d` and $\bar{2}, \bar{3}, \bar{4}, \bar{5}$ by `A,B,C,D` etc. in Examples. We can adopt this convention even in the type A case.

Example 4.22.

```
t=0   solitons................................
t=1   ....sliotson............................
t=2   .......lsoi.tson........................
t=3   ........l..soi..tson....................
t=4   .........l....soi...tson................
t=5   ..........l......soi....tson............
t=6   ...........l........soi.....tson........
t=7   ............l..........soi......tson.....
```

There are three solitons at each time step of this example.

In this typographical convention the color separation scheme is shown in the following way. At time $t = 0$ we have:

Example 4.23.

```
s=0   solitons.................................. s
s=1   .solatoin................................. n
s=2   ..soaatloi................................ o
s=3   ...saaaotli............................... t
s=4   ....aaasaoli.............................. i
s=5   ....aaa.asola............................. l
s=6   ....aaa.a.soaa............................ o
s=7   ....aaa.a..saaa........................... s
s=8   ....aaa.a...aaaa..........................
```

In the next time step we have:

Example 4.24.

```
s=0   ....sliotson.............................. s
s=1   .....slaitoon............................. n
s=2   ......salatooi............................ o
s=3   .......asaatoli........................... t
s=4   .......a.aaasoli.......................... i
s=5   .......a.aaa.sola......................... l
s=6   .......a.aaa..soaa........................ o
s=7   .......a.aaa...saaa....................... s
s=8   .......a.aaa....aaaa......................
```

And in the next time step we have:

Example 4.25.

```
s=0   .......lsoi.tson.......................... s
s=1   ........asli.toon......................... n
s=2   ........a.sla.tooi........................ o
s=3   ........a..saa.toli....................... t
s=4   ........a...aaa.soli...................... i
s=5   ........a...aaa..sola..................... l
s=6   ........a...aaa...soaa.................... o
s=7   ........a...aaa....saaa................... s
s=8   ........a...aaa.....aaaa..................
```

We see that the underlying monochrome system evolves by the usual updating rule of SCA, and the word made of the removed letters (`solitons` from bottom to top) is not affected by the time evolution.

5. Cellular automata: type D case

5.1. Time evolution operators. Based on the crystals for type D affine Lie algebra [KKM] we can construct its associated SCA as in the type A case. Define the set of basic paths as in (4.1) but now B_1 is that for $D_n^{(1)}$ crystal. Let u_l be the highest weight element of $D_n^{(1)}$ crystal B_l. Then the operator T_l is defined as (4.2). In this note we mainly use the time evolution $T = T_\infty$. Also the operator $T_\natural$ is defined as (4.8) by using the crystal $B_\natural$ in (3.5).

We briefly review the algorithm for the factorized dynamics of the automaton under the time evolution T [HKT2, HKT3]. For $\gamma \in \{2, \dots, n, \overline{n}, \dots \overline{2}\}$ we introduce a map $L_\gamma : (\mathbb{Z}_{\geq 0}) \times B_1 \to B_1 \times (\mathbb{Z}_{\geq 0})$ as follows. Let the diagram

$$\begin{tikzcd}[row sep=small, column sep=small] & \beta \arrow[dd] & \\ m \arrow[rr] & & m' \\ & \beta' & \end{tikzcd} \tag{5.1}$$

depict the relation $L_\gamma : (m, \beta) \mapsto (\beta', m')$. Then under the assumption $m \in \mathbb{Z}_{\geq 0}$ and $\beta \in B_1 \backslash \{\gamma, \overline{\gamma}, 1, \overline{1}\}$, (we interpret $\overline{\overline{\gamma}} = \gamma$) the following list specifies the map L_γ completely:

$$\begin{tikzcd}[row sep=small, column sep=small] & 1 \arrow[dd] & \\ m+1 \arrow[rr] & & m \\ & \gamma & \end{tikzcd} \qquad \begin{tikzcd}[row sep=small, column sep=small] & \gamma \arrow[dd] & \\ m \arrow[rr] & & m+1 \\ & 1 & \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small, column sep=small] & \overline{\gamma} \arrow[dd] & \\ m+1 \arrow[rr] & & m \\ & \overline{1} & \end{tikzcd} \qquad \begin{tikzcd}[row sep=small, column sep=small] & \overline{1} \arrow[dd] & \\ m \arrow[rr] & & m+1 \\ & \overline{\gamma} & \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small, column sep=small] & 1 \arrow[dd] & \\ 0 \arrow[rr] & & 0 \\ & 1 & \end{tikzcd} \qquad \begin{tikzcd}[row sep=small, column sep=small] & \overline{\gamma} \arrow[dd] & \\ 0 \arrow[rr] & & 0 \\ & \overline{\gamma} & \end{tikzcd} \qquad \begin{tikzcd}[row sep=small, column sep=small] & \beta \arrow[dd] & \\ m \arrow[rr] & & m \\ & \beta & \end{tikzcd}$$

For $\gamma \in \{2, \dots, n, \overline{n}, \dots \overline{2}\}$ we construct an operator $K_\gamma : \mathcal{P} \to \mathcal{P}$ as a composition of L_γs. Let $L_\gamma^{(i)}$ be the map L_γ shown as the i-th arrow of the sequence

$$\begin{aligned} &(\mathbb{Z}_{\geq 0}) \times (B_1 \times B_1 \times B_1 \times \cdots) \\ \to &B_1 \times (\mathbb{Z}_{\geq 0}) \times (B_1 \times B_1 \times \cdots) \\ \to &(B_1 \times B_1) \times (\mathbb{Z}_{\geq 0}) \times (B_1 \times \cdots) \\ &\cdots \\ \to &(B_1 \times B_1 \times B_1 \times \cdots) \times (\mathbb{Z}_{\geq 0}). \end{aligned}$$

For $\mathbf{p} \in \mathcal{P}$ we define the map K_γ as

$$\prod_{i \geq 1} L_\gamma^{(i)} : (0, \mathbf{p}) \mapsto (K_\gamma(\mathbf{p}), 0).$$

The right hand side has certainly this form since the action of $L_\gamma^{(i)}$ stabilizes to $(0,1) \mapsto (1,0)$ for large i due to the boundary condition of $\mathcal{P}$ in (4.1). The time evolution operator T for the automaton associated with D_n is given by

$$T = K_2 K_3 \cdots K_n K_{\overline{n}} \cdots K_{\overline{3}} K_{\overline{2}}. \tag{5.2}$$

If there is no anti-particle in $\mathbf{p}$ the time evolution operator is effectively equivalent to $T = K_2 K_3 \cdots K_n$ i. e. it reduces to T in the type A case (section 4.5).

5.2. **Solitons.** Recall the order between the letters of the type D crystals (3.4). Like in the type A case a soliton is a set of successive letters arranged in decreasing order. Under the time evolution T it propagates in the speed of its length.

For typographical reason we denote $\overline{1}$ by `@`. We also denote $2, 3, 4, \ldots$ by `a,b,c,`... and $\overline{2}, \overline{3}, \overline{4}, \ldots$ by `A,B,C,`.... We occasionally use such a notation as $T = K_a K_b \cdots K_B K_A$.

Here is an example of a two-soliton state.

Example 5.1.

```
t=0     .......ABCDcba......BBBbbb.......................
t=1     ..............ABCDcba.....BBBbbb.................
t=2     .....................ABCDcba....BBBbbb...........
t=3     ............................ABCDcba...BBBbbb.....
```

In terms of the factorized algorithm the time evolution

$$T = K_a K_b K_c K_d K_D K_C K_B K_A$$

from $t = 0$ to $t = 1$ is described as follows.

Example 5.2.

```
t=0/8   .......ABCDcba......BBBbbb.......................
t=1/8   ........BCDcb@......BBBbbb.......................
t=2/8   .........CDc@bB........@@@.......................
t=3/8   ..........D@cbBC.......cccCCC....................
t=4/8   ...........dcbBCDD.....cccCCC....................
t=5/8   ............cbBC@D.....cccCCC....................
t=6/8   .............bB@CDc.......@@@....................
t=7/8   ..............@BCDcb......BBBbbb.................
t=8/8   ..............ABCDcba.....BBBbbb.................
```

For instance, from $t = 0/8$ to $t = 1/8$ we moved A to the place of a in which a pair annihilation $A + a \to @$ occurred. From $t = 1/8$ to $t = 2/8$ there are a pair creation $@ \to b + B$ and four pair annihilations $B + b \to @$.

Exercise 5.3. Show that the following state is not a one-soliton state for $n = 5$, but is a one-soliton state for $n > 5$.

```
............ABCDdcba.............................
```

5.3. **Scattering rules.** The scattering rules of solitons are given by the combinatorial R [HKOTY]. A new feature (compared with the type A case) is the pair annihilations/creations of the particles and anti-particles.

Example 5.4.

```
t=0   ABCdcba..............cb.................................
t=1   .......ABCdcba.........cb...............................
t=2   ..............ABCdcba....cb.............................
t=3   .....................ABCdbaccb..........................
t=4   ...........................cb.BCCdccb...................
t=5   .............................cb......BCCdccb............
t=6   ...............................cb...........BCCdccb.....
t=7   .................................cb................BCCdc
```

The soliton `ABCdcba` is identified with $D_4^{(1)}$ crystal element $(1,1,1,1,0,1,1,1)$. The scattering is written as

$$
\begin{aligned}
&(1,1,1,1,0,1,1,1) \times (0,1,1,0,0,0,0,0) \\
&\mapsto (0,1,1,0,0,0,0,0) \times (0,1,2,1,0,2,1,0).
\end{aligned}
$$

In the process of collision a pair of `A` and `a` disappeared, and that of `C` and `c` appeared. This coincides with the combinatorial R for the $D_4^{(1)}$ crystal.

We see the pair annihilation/creation process in Example 5.4 in more detail by the factorized algorithm.

Example 5.5.

```
t=3     ...................ABCdbaccb..........................
t=3+1/8 ....................BCdb@ccb..........................
t=3+2/8 .....................Cd@bcc@..........................
t=3+3/8 ......................dcb@@cC.........................
t=3+4/8 ......................dcbddcCDD.......................
t=3+5/8 .......................cb..cC@@d......................
t=3+6/8 ........................bc..@CCdcc....................
t=3+7/8 .........................cb.BCCdccb...................
t=4     .........................cb.BCCdccb...................
```

The time evolution operator is $T = K_a K_b K_c K_d K_D K_C K_B K_A$. When K_A is applied there is a pair annihilation $A + a \to$ @. At time $t = 3 + 7/8$ there is no @. So there is no pair creation @ $\to A + a$ when K_a is applied on it.

5.4. **Phase shifts.** As in the type A case the phase shifts are given by the energy function. The energy function $\hat{P}_0^D$ in (3.2) has the correct normalization to describe the phase shifts. In the type A case the amount of phase shift was at least the length of the smaller soliton. In the type D case less amount, or even no phase shift (See the following example) is possible.

Example 5.6.

```
t=0   ......aaaaaa.....AAA...................................
t=1   ............aaaaaa..AAA................................
t=2   ..................aaaaa@AA.............................
t=3   ........................aa@@@a.........................
t=4   ..............................a@@Aaaa..................
t=5   .................................aaa.AAAaaa............
t=6   ....................................aaa....AAAaaa......
```

5.5. **Soliton content.** The definition of the soliton content is similar to that in the type A case (section 4.6): We simply replace the H_l in (4.3) by the energy function H_l^D which has been defined just below (3.3). In some cases we have to be careful to read off the soliton content from the asymptotic state. See the following example.

Example 5.7.

```
t=0   @@@@..........@@.........@..............................
t=1   ....@@@@........@@........@.............................
t=2   ........@@@@......@@.......@............................
t=3   ............@@@@....@@......@...........................
t=4   ................@@@@..@@.....@..........................
t=5   ....................@@..@@@@..@.........................
t=6   ......................@@....@@.@@@......................
t=7   ........................@@....@...@@@@..................
t=8   ..........................@@...@......@@@@..............
t=9   ............................@@..@.........@@@@..........
```

Exercise 5.8. Compute the soliton content of the state in this example. Is it equal to $N_1 = N_2 = N_4 = 1$ and $N_k = 0\,(k \neq 1, 2, 4)$?

In fact, the @@@@ is not *a* soliton but a composite state of *two* solitons. In order to see this we "bombard" it by the other solitons.

Example 5.9.

```
t=0   @@@@......aa....aa......................................
t=1   ....@@@@....aa....aa....................................
t=2   ........@@@@..aa....aa..................................
t=3   ............@@aaAAaa..aa................................
t=4   ..............aaAAaaAA..aaaa............................
t=5   ................aa..AA@@....aaaa........................
t=6   ..................aa....@@AA....aaaa....................
t=7   ....................aa....aaAAAA....aaaa................
t=8   ......................aa....aa..AAAA....aaaa............
t=9   ........................aa....aa....AAAA....aaaa........
```

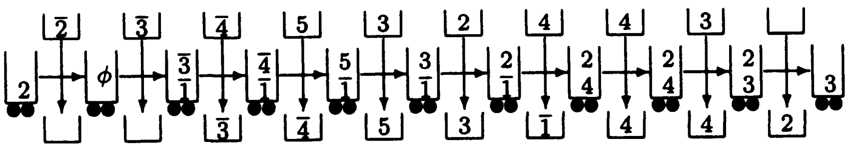
We recognize that the @@@@ was decomposed into AAAA and aaaa. The opposite is possible, i. e. there is also a process in which a composite state of two solitons of the same length is created.

Example 5.10.

```
t=0  AAAA....aaaa.....AA.....AA.........................
t=1  ....AAAA....aaaa...AA.....AA.......................
t=2  ........AAAA....aaaa.AA.....AA.....................
t=3  ............AAAA....aaa@A.....AA...................
t=4  ................AAAA....a@@a....AA.................
t=5  ....................AAAA...a@Aaa..AA...............
t=6  ........................AAA..AaaAAaaAA.............
t=7  ...........................AA...AAaaAA@@...........
t=8  .............................AA.....AA..@@@@.......
t=9  ...............................AA.....AA....@@@@...
```

5.6. **Color separation scheme.** This scheme works also in type D case. In this subsection we observe this fact by examples. In the next subsection we shall verify it.

The operator $T_\natural$ was defined as (4.8). Recall Example 5.4. We apply $T_\natural$ on the state at $t = 3$. Note that ABCdbaccb. is equal to $\bar{2}\bar{3}\bar{4}5324431$. In terms of the carrier description (section 3.2) we have

 .

The result is $11\bar{3}\bar{4}53\bar{1}442$ that is equal to ..BCdb@cca, and the carrier takes off the letter 3(=b). In the same way we apply $T_\natural$ on the state repeatedly.

Example 5.11.

```
s=0  ....................ABCdbaccb......................... b
s=1  ......................BCdb@cca........................ c
s=2  .......................BCda@cba....................... A
s=3  ........................BCaadcbaa..................... b
s=4  .........................BaaCdcaaa.................... c
s=5  ..........................aaBCdaaaa................... d
s=6  ..........................aa.BCaaaaa.................. C
s=7  ..........................aa..Baaaaaa................. B
s=8  ..........................aa...aaaaaaa................
```

Denote the first line of this example by $\mathbf{p}$ and the last line by $\tilde{\mathbf{p}}$. Also denote $\mathbf{y}$ the word made of the removed letters arranged from bottom to top, i. e. $\mathbf{y}$ =BCdcbAcb. Then we may write symbolically as $\mathbf{p} = \tilde{\mathbf{p}} \oplus \mathbf{y}$. As in the type A case the $\tilde{\mathbf{p}}$ is a state for the monochrome system (Takahashi-Satsuma's CA).

Now we apply $T_\natural$ repeatedly on the state at time $t = 4$.

Example 5.12.

```
s=0   ...........................cb.BCCdccb................... b
s=1   ............................ca.ACCdcca.................. c
s=2   .............................aa.ABCdcba................. A
s=3   .............................aa...BCdcbaa............... b
s=4   .............................aa....BCdcaaa.............. c
s=5   .............................aa.....BCdaaaa............. d
s=6   .............................aa......BCaaaaa............ C
s=7   .............................aa.......Baaaaaa........... B
s=8   .............................aa........aaaaaaa..........
```

As in the type A case we recognize the property $T(\mathbf{p}) = T(\tilde{\mathbf{p}}) \oplus \mathbf{y}$.

Let us apply this scheme to the state in Example 5.7 at $t = 0$.

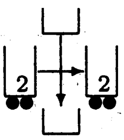
Example 5.13.

```
s=0   @@@@..........@@.........@.............................. A
s=1   aa@@Aa........AaAa.......A.a............................ a
s=2   aaaaAAaa......AA..aa.....A.A............................ A
s=3   aaaa..Aaaa....AA..Aa.......@............................ A
s=4   aaaa....aaaa..AA..AA.......a.a.......................... a
s=5   aaaa....aaaa....AaAA.......A.a.......................... a
s=6   aaaa....aaaa......@@.......A.A.......................... A
s=7   aaaa....aaaa......aaAa.....A.A.......................... A
s=8   aaaa....aaaa......aa..aa...A.A.......................... A
s=9   aaaa....aaaa......aa..aa.....@.......................... A
s=10  aaaa....aaaa......aa..aa.....a.a........................
```

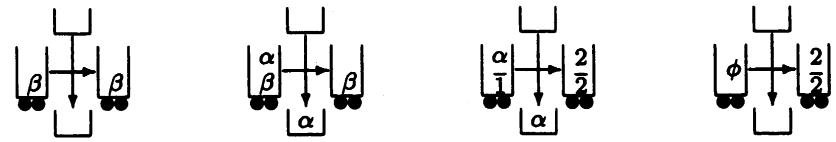
Now the soliton content is clear in the underlying monochrome system.

5.7. **Proof of the separation scheme.** In this subsection we establish the color separation scheme in the previous subsection by generalizing the arguments in [Tg2]. Let $\mathbf{p}$ be a state and N a positive integer. If $T_\natural^N$ is applied on $\mathbf{p}$ and all the carriers take off the letter 2(=a) (i. e. if $(u_\natural)^N \otimes \mathbf{p} \overset{\sim}{\mapsto} T_\natural^N(\mathbf{p}) \otimes (u_\natural)^N$ holds), then we call $\mathbf{p}$ *N-trivial.* Denote by $F = F(\mathbf{p})$ the position of the rightmost "non-empty box" (letter $\neq 1$) in the path $\mathbf{p}$.

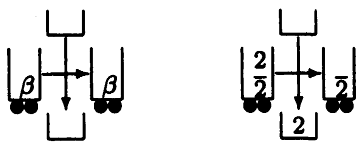
Lemma 5.14. Suppose $\mathbf{p}$ is 1-trivial. If $T_\natural$ is applied on $\mathbf{p}$ then the process that occurs at $F + 1$ is the following



Proof. Since the box at $F+1$ is empty there are four possible processes that can occur there.

(5.3) 

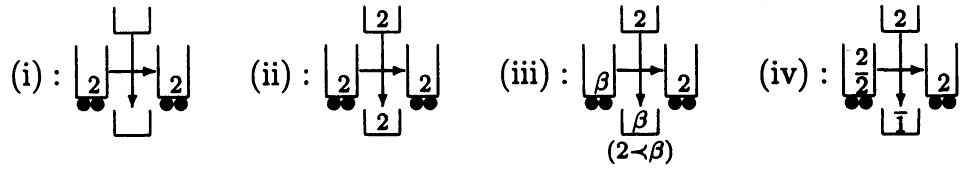
Therefore there are two possible processes at $F+2$.



The latter case contradicts our assumption of 1-triviality. Thus the third and forth cases of (5.3) are not acceptable at $F+1$. Note that the condition $2 \prec \beta$ is assumed in the second case of (5.3), so it is not acceptable either. $\square$

By case by case check we have that

Lemma 5.15. The following pictures exhaust all the cases in which the outgoing carrier contains "2" only.

(5.4) 

Theorem 5.16. Suppose $\mathbf{p}$ is k-trivial. Then there are only 1 and 2 at the positions $\geq F-k+1$ in $\mathbf{p}$.

Proof. It is sufficient to prove that if $\mathbf{p}$ is k-trivial then (when $T_\natural$ is applied on $\mathbf{p}$) the process that occurs at $\geq F-k+2$ is (i) or (ii), and the process at $F-k+1$ is one of (i), (ii), (iii), (iv). We prove it by induction on k. Suppose $\mathbf{p}$ is 1-trivial. Apply $T_\natural$ on $\mathbf{p}$. By Lemmas 5.14 and 5.15 the possible process at F is one of (ii), (iii), (iv). Thus $k=1$ case is proved.

Suppose $\mathbf{p}$ is k-trivial. Apply $T_\natural$ on $\mathbf{p}$. If the process at $F-k+1$ is (iii) or (iv), then $T_\natural(\mathbf{p})$ is not k-trivial by the assumption of induction. So if $\mathbf{p}$ is $k+1$-trivial then the process at $F-k+1$ is (i) or (ii), hence the process at $F-k$ is one of (i), (ii), (iii), (iv). The proof follows by induction. $\square$

Corollary 5.17. If $\mathbf{p}$ is F-trivial, then there are only 1 and 2 in $\mathbf{p}$.

Recall the definition of the particle/anti-particle numbers for letters in the type D crystals (TABLE 1). Given a path $\mathbf{p} = p_1 \otimes p_2 \otimes \cdots \in \mathcal{P}$, define its particle/anti-particle numbers as $P(\mathbf{p}) = \sum_{i=1}^{\infty} P(p_i)$ and $A(\mathbf{p}) = \sum_{i=1}^{\infty} A(p_i)$. By Lemma 3.3 we have that

Lemma 5.18. The following relations hold

$$P(T_\natural(\mathbf{p})) + P(b(\mathbf{p})) = P(\mathbf{p}) + 1,$$
$$A(T_\natural(\mathbf{p})) + A(b(\mathbf{p})) = A(\mathbf{p}).$$

Note that for any $b(\mathbf{p})$ its particle/anti-particle number is 0 or 1.

Theorem 5.19. For any $\mathbf{p} \in \mathcal{P}$ there is an integer N such that $(T_\natural)^N(\mathbf{p})$ has only 1 and 2.

Proof. By Lemma 5.18 the anti-particle number of the path does not increase under the application of $T_\natural$. If all the anti-particles can be removed, i. e. if $A((T_\natural)^{N'}(\mathbf{p})) = 0$ for some N' it reduces to the type A case where the claim follows immediately by Corollary 5.17. Suppose it is not. Then we can find a path which (by renewing the definition) we call $\mathbf{p}$ that satisfies $A(\mathbf{p}) \neq 0$ and $A((T_\natural)^N(\mathbf{p})) = A(\mathbf{p})$ for any N. From this assumption and by Corollary 5.17 we deduce that we can take off particles $\in \{3, \dots, n\}$ infinitely many times, by applying $T_\natural$ repeatedly. In order not to exhaust such particles in the state, the process $\overline{2} + 2 \to 3 + \overline{3}$ should occur infinitely many times. Then, except the opposite process $3 + \overline{3} \to \overline{2} + 2$, the $\overline{2}$'s can be provided only from those originally contained in the path, or by the process $\overline{1} \to 2 + \overline{2}$. But it is impossible to continue that, because the number of $\overline{2}$ and $\overline{1}$ in $\mathbf{p}$ is finite. □

Thus for any $\mathbf{p} \in \mathcal{P}$ we can choose an integer N such that the path $\tilde{\mathbf{p}} = (T_\natural)^N(\mathbf{p})$ has only 1 and 2. And we have that

Proposition 5.20. The claim of Proposition 4.15 also holds in the type D case.

This comes from the commutativity $T_l T_\natural = T_\natural T_l$ which is guaranteed by

Lemma 5.21. The Yang-Baxter relation holds on $B_l \otimes B_1 \otimes B_\natural$.

We give an elementary proof of this lemma which uses the crystal structure of D_n rather than $D_n^{(1)}$. As D_n crystals the $B_l \otimes B_1 \otimes B_\natural$ decomposes as [N]

$$\begin{aligned} B_l \otimes B_1 \otimes B_\natural \simeq\ & B_{(l+2,1)} \oplus B_{(l+1,2)} \oplus B_{(l,2,1)} \oplus B_{(l,1,1,1)} \oplus \left(B_{(l+1,1,1)}\right)^{\oplus 2} \\ & \oplus B_{(l-1,2)} \oplus \left(B_{(l+1)}\right)^{\oplus 2} \oplus \left(B_{(l-1,1,1)}\right)^{\oplus 2} \oplus \left(B_{(l,1)}\right)^{\oplus 4} \\ & \oplus B_{(l-2,1)} \oplus \left(B_{(l-1)}\right)^{\oplus 2}. \end{aligned}$$

It is sufficient to prove the relation on D_n highest weight elements of all the components in this tensor product decomposition. More precisely it is enough to verify the relation on every component whose multiplicity is greater than one . We give that case by case check in Appendix A.

Appendix A. Proof of Lemma 5.21

For each highest element u of the shape $\lambda = (l+1,1,1)$ the Yang-Baxter relation can be verified as (we have set $l = 3$ here)

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \boxed{1} \otimes \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}$$

$$\overset{\sim}{\mapsto} \boxed{1} \otimes \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} \overset{\sim}{\mapsto} \boxed{1} \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&3\\\hline\end{array} \overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \boxed{1} \otimes \begin{array}{|c|c|c|}\hline 1&1&3\\\hline\end{array}$$

$$\overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \boxed{3} \overset{\sim}{\mapsto} \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \boxed{3} \overset{\sim}{\mapsto} \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \boxed{1} \otimes \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array},$$

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \boxed{2} \otimes \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}$$

$$\overset{\sim}{\mapsto} \boxed{1} \otimes \begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \overset{\sim}{\mapsto} \boxed{1} \otimes \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \boxed{3} \otimes \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array}$$

$$\overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&3\\\hline\end{array} \otimes \boxed{1} \overset{\sim}{\mapsto} \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} \otimes \boxed{1} \overset{\sim}{\mapsto} \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \boxed{2} \otimes \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}.$$

For the shape $\lambda = (l+1)$ it can be verified as

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \boxed{2} \otimes \begin{array}{|c|}\hline 1\\\hline \bar{2}\\\hline\end{array}$$

$$\overset{\sim}{\mapsto} \boxed{1} \otimes \begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline \bar{2}\\\hline\end{array} \overset{\sim}{\mapsto} \boxed{1} \otimes \begin{array}{|c|}\hline 2\\\hline \bar{2}\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \overset{\sim}{\mapsto} \phi \otimes \boxed{1} \otimes \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array}$$

$$\overset{\sim}{\mapsto} \phi \otimes \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \boxed{1} \overset{\sim}{\mapsto} \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline \bar{2}\\\hline\end{array} \otimes \boxed{1} \overset{\sim}{\mapsto} \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \boxed{2} \otimes \begin{array}{|c|}\hline 1\\\hline \bar{2}\\\hline\end{array},$$

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \boxed{1} \otimes \phi$$

$$\overset{\sim}{\mapsto} \boxed{1} \otimes \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \phi \overset{\sim}{\mapsto} \boxed{1} \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&\bar{2}\\\hline\end{array} \overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \boxed{1} \otimes \begin{array}{|c|c|c|}\hline 1&1&\bar{2}\\\hline\end{array}$$

$$\overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \boxed{\bar{2}} \overset{\sim}{\mapsto} \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \boxed{\bar{2}} \overset{\sim}{\mapsto} \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \boxed{1} \otimes \phi.$$

For the shape $\lambda = (l-1,1,1)$ it can be verified as

$$
\begin{aligned}
&\boxed{1}\boxed{1}\boxed{1} \otimes \boxed{2} \otimes \begin{array}{|c|}\hline 3\\\hline \bar{1}\\\hline\end{array}\\
&\overset{\sim}{\mapsto} \boxed{1} \otimes \boxed{1}\boxed{1}\boxed{2} \otimes \begin{array}{|c|}\hline 3\\\hline \bar{1}\\\hline\end{array} \overset{\sim}{\mapsto} \boxed{1} \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \boxed{1}\boxed{3}\boxed{\bar{1}} \overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \boxed{1} \otimes \boxed{1}\boxed{3}\boxed{\bar{1}}\\
&\overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \boxed{1}\boxed{1}\boxed{3} \otimes \boxed{\bar{1}} \overset{\sim}{\mapsto} \boxed{1}\boxed{1}\boxed{1} \otimes \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} \otimes \boxed{\bar{1}} \overset{\sim}{\mapsto} \boxed{1}\boxed{1}\boxed{1} \otimes \boxed{2} \otimes \begin{array}{|c|}\hline 3\\\hline \bar{1}\\\hline\end{array},
\end{aligned}
$$

$$
\begin{aligned}
&\boxed{1}\boxed{1}\boxed{1} \otimes \boxed{\bar{1}} \otimes \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\\
&\overset{\sim}{\mapsto} \boxed{1} \otimes \boxed{1}\boxed{1}\boxed{\bar{1}} \otimes \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} \overset{\sim}{\mapsto} \boxed{1} \otimes \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} \otimes \boxed{1}\boxed{3}\boxed{\bar{3}} \overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \boxed{3} \otimes \boxed{1}\boxed{3}\boxed{\bar{3}}\\
&\overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \boxed{1}\boxed{2}\boxed{\bar{2}} \otimes \boxed{3} \overset{\sim}{\mapsto} \boxed{1}\boxed{1}\boxed{1} \otimes \begin{array}{|c|}\hline 2\\\hline \bar{1}\\\hline\end{array} \otimes \boxed{3} \overset{\sim}{\mapsto} \boxed{1}\boxed{1}\boxed{1} \otimes \boxed{\bar{1}} \otimes \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}.
\end{aligned}
$$

Here the second mapping is based on the following column insertion [12]

$$
\left(3 \longrightarrow \begin{bmatrix} 2\\ 3\\ \bar{3} \end{bmatrix}\right) = \begin{bmatrix} 1 & \bar{1}\\ 2 & \\ 3 & \end{bmatrix}.
$$

For the shape $\lambda = (l-1)$ it can be verified as

$$
\begin{aligned}
&\boxed{1}\boxed{1}\boxed{1} \otimes \boxed{2} \otimes \begin{array}{|c|}\hline \bar{2}\\\hline \bar{1}\\\hline\end{array}\\
&\overset{\sim}{\mapsto} \boxed{1} \otimes \boxed{1}\boxed{1}\boxed{2} \otimes \begin{array}{|c|}\hline \bar{2}\\\hline \bar{1}\\\hline\end{array} \overset{\sim}{\mapsto} \boxed{1} \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \boxed{1}\boxed{\bar{2}}\boxed{\bar{1}} \overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \boxed{1} \otimes \boxed{1}\boxed{\bar{2}}\boxed{\bar{1}}\\
&\overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \boxed{1}\boxed{1}\boxed{\bar{2}} \otimes \boxed{\bar{1}} \overset{\sim}{\mapsto} \boxed{1}\boxed{1}\boxed{1} \otimes \phi \otimes \boxed{\bar{1}} \overset{\sim}{\mapsto} \boxed{1}\boxed{1}\boxed{1} \otimes \boxed{2} \otimes \begin{array}{|c|}\hline \bar{2}\\\hline \bar{1}\\\hline\end{array},
\end{aligned}
$$

$$
\begin{aligned}
&\boxed{1}\boxed{1}\boxed{1} \otimes \boxed{\bar{1}} \otimes \phi\\
&\overset{\sim}{\mapsto} \boxed{1} \otimes \boxed{1}\boxed{1}\boxed{\bar{1}} \otimes \phi \overset{\sim}{\mapsto} \boxed{1} \otimes \begin{array}{|c|}\hline 2\\\hline \bar{2}\\\hline\end{array} \otimes \boxed{1}\boxed{1}\boxed{\bar{1}} \overset{\sim}{\mapsto} \phi \otimes \boxed{1} \otimes \boxed{1}\boxed{1}\boxed{\bar{1}}\\
&\overset{\sim}{\mapsto} \phi \otimes \boxed{1}\boxed{1}\boxed{1} \otimes \boxed{\bar{1}} \overset{\sim}{\mapsto} \boxed{1}\boxed{1}\boxed{1} \otimes \begin{array}{|c|}\hline 2\\\hline \bar{2}\\\hline\end{array} \otimes \boxed{\bar{1}} \overset{\sim}{\mapsto} \boxed{1}\boxed{1}\boxed{1} \otimes \boxed{\bar{1}} \otimes \phi.
\end{aligned}
$$

[12] In the terminology of [Bk] this case falls into "Type IIb special bump".

For the shape $\lambda = (l, 1)$ it can be verified as

$$
\begin{aligned}
&\begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline \bar{1}\\\hline\end{array}\\
&\overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline \bar{1}\\\hline\end{array} \overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&2&\bar{2}\\\hline\end{array} \overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&2&\bar{2}\\\hline\end{array}\\
&\overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&\bar{2}\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline\end{array} \overset{\sim}{\mapsto} \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \phi \otimes \begin{array}{|c|}\hline 2\\\hline\end{array} \overset{\sim}{\mapsto} \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline \bar{1}\\\hline\end{array},
\end{aligned}
$$

$$
\begin{aligned}
&\begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline \bar{2}\\\hline\end{array}\\
&\overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline \bar{2}\\\hline\end{array} \overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline\end{array} \otimes \phi \otimes \begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array} \overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \begin{array}{|c|}\hline \bar{2}\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array}\\
&\overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&2&\bar{2}\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline\end{array} \overset{\sim}{\mapsto} \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline \bar{1}\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline\end{array} \overset{\sim}{\mapsto} \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline \bar{2}\\\hline\end{array},
\end{aligned}
$$

$$
\begin{aligned}
&\begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline \bar{1}\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\\
&\overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&\bar{1}\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline \bar{2}\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array} \overset{\sim}{\mapsto} \phi \otimes \begin{array}{|c|}\hline 1\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array}\\
&\overset{\sim}{\mapsto} \phi \otimes \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline\end{array} \overset{\sim}{\mapsto} \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline \bar{2}\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline\end{array} \overset{\sim}{\mapsto} \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline \bar{1}\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array},
\end{aligned}
$$

$$
\begin{aligned}
&\begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline\end{array} \otimes \phi\\
&\overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array} \otimes \phi \overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&\bar{1}\\\hline\end{array} \overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&\bar{1}\\\hline\end{array}\\
&\overset{\sim}{\mapsto} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline \bar{1}\\\hline\end{array} \overset{\sim}{\mapsto} \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \begin{array}{|c|}\hline \bar{1}\\\hline\end{array} \overset{\sim}{\mapsto} \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline\end{array} \otimes \phi.
\end{aligned}
$$

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