2. Cayley graphs.

A Cayley graph gives us a means by which a finitely generated group can be viewed as a geometric object. The starting point is a finite generating set. The dependence of the construction on the choice of generating set will be discussed in Section 3.

Many combinatorial constructions can be interpreted geometrically or topologically, and this often results in the most efficient means of proof. This is the main theme of this course. Nevertheless, the earlier combinatorial tools developed by Higman, Neumann etc, still remain a powerful resource, and a rich source of interesting examples.

2.1. Basic terminology and notation.

Let K be a graph. Formally this is thought of as a set, $V(K)$, of vertices together with a set $E(K)$ of edges. It will be convenient to allow multiple edges (edges connecting the same pair of vertices) and loops (edges starting and finishing at the same vertex). We recall some standard terminology from graph theory:

Definition : A (combinatorial) *path* consists of a sequence of edges with consecutive edges adjacent.

An *arc* is an embedded path.

A *cycle* is a closed path.

A *circuit* is an embedded cycle.

A graph is *connected* if every pair of vertices are connected by a path (and hence also by an arc).

The *valence* of a vertex is the number of incident edges (counting muliplicities of multiple edges, and counting each loop twice).

A graph is *locally finite* if each vertex has finite valence.

It is *n-regular* if every vertex has valence n.

We will often write a combinatorial path as a sequence of vertices rather than edges, though if there are loops or multiple edges one also needs to specify the edges connecting them.

2.2. Graphs viewed as metric and topological spaces.

Let K be a graph. We can realise K as a 1-complex. Each edge of K corresponds to a copy of the unit interval with vertices at its endpoints. (A loop gets bent around into a circle.) This gives a description of the 1-complex as a set, denoted $|K|$. We can define a distance on $|K|$ as follows. We first fix a parameterisation for each edge. This defines a length for each interval in an edge, so that the whole interval has length 1. We can generalise the notion of a path allowing it to start and finish in the interiors of edges as well as at a vertex. A path, π, thus has a well defined *length*, $l(\pi) \in [0, \infty)$. Given $x, y \in K$ we defined $d(x, y) \in [0, \infty]$ to be the minimum length of a path connecting x to y. We set this equal to ∞ if there is no such path. Otherwise, the minimum is attained, since the set of lengths of such arcs is discrete. If K is connected, then $(|K|, d)$ is a metric space. It thus induces a topology on $|K|$. This topology makes sense even if K is not connected. The space $|K|$ might be called the *realisation* of K. We note (exercise):

$|K|$ is compact $\Leftrightarrow$ K is finite.

$|K|$ is locally compact $\Leftrightarrow$ K is locally finite.

$|K|$ is (topologically) connected $\Leftrightarrow$ K is connected.

We frequently omit the $|.|$, and simply write $|K|$ as K. Thus, a graph K can be viewed in three ways: as a combinatorial object, as a geometric object (metric space) or as a topological object.

Remark : For most purposes here, we will only be interested in locally finite graphs. In this case, the topology described here is the only "sensible" one. In the non locally finite case, however, there are other natural topologies such as the "CW-topology" which is different. We will not need to worry about these issues in this course.

Suppose that a group Γ acts on the graph K. It then acts by isometries on its realisation. We say that the action on K is *free* if it is free on both $V(K)$ and $E(K)$. This means that no element of Γ fixes a vertex or inverts an edge (i.e. swaps the vertices of an edge).

In this case, we can form the quotient graph K/Γ.

Simple example : We can view the real line, $\mathbf{R}$, as the (realisation of) a graph with vertices at the integers. The group $\mathbf{Z}$ acts by translation. $\mathbf{R}/\mathbf{Z}$ consists of a single loop. $\mathbf{R}/n\mathbf{Z}$ is a circuit with n vertices. Both are topological circles.

Definition : A *tree* is a connected graph with no circuits.

Thus (exercise) a graph is a tree if and only if every pair of vertices are connected by a unique arc. Also (exercise) any two n-regular trees are isomorphic. The 2-regular tree is the real line described above.

2.3. Graphs associated to groups.

Let Γ be a group, and $S \subseteq \Gamma$ a subset not containing the identity. Let $\bar{S}$ be a set of "formal inverses" of elements of S (as in the construction of a free group) and let $A = S \sqcup \bar{S}$. If $S \cap S^{-1} = \emptyset$, then we can identify A as a subset of Γ, by identifying the formal inverse $\bar{a}$ of an element $a \in S$, with its actual inverse, a^{-1}, in Γ. Indeed, we can usually arrange that $S \cap S^{-1} = \emptyset$: if both a and a^{-1} lie in S, then just throw one of them away. This is only a problem if $a = a^{-1}$. In this case, we need to view the formal inverse of a as distinct from a. This is a somewhat technical point, and we will generally write a^{-1} for the inverse whether used formally or as a group element. Only if $a^2 = 1$ do we need to make this distinction.

We construct a graph $\Delta = \Delta(\Gamma; S)$ as follows. Let $V(\Delta) = \Gamma$. We connect vertices $g, h \in \Gamma$ by and edge in Δ if $g^{-1}h \in A$. In other words, for each $g \in \Gamma$ and $a \in A$ we have an edge connecting g to ga. We imagine the directed edge from g to ga as being labelled by the element a. The same edge with the opposite orientation is thus labelled by a^{-1}. (If it happens that $a^2 = 1$, then we connect g to ga by a pair of edges, labelled by a and its formal inverse.) Note that Δ is $|A|$-regular. It is locally finite if and only if $|S|$ is finite. (Note $|A| = 2|S|$.)

Now Γ acts on $\Gamma = V(\Delta)$ by left multiplicaton, and this extends to an action on Δ: if $g, h, k \in \Gamma$, with $g^{-1}h \in A$ then $(kg)^{-1}(kh) =$

$g^{-1}h \in A$. This action is free and preserves the labelling. The quotient graph consists of a single vertex and $|S|$ loops. Topologically this is a wedge of $|S|$ circles — a space we will meet again in relation to free groups in Section 4.

Let $W(A)$ be the set of words in A. There is a natural bijection between $W(A)$ and the set of paths in Δ starting from the identity. More precisely, if $w = a_1 a_2 \dots a_n \in W(A)$, then there is a unique path $\pi(w)$ starting at $1 \in V(\Delta)$ so that the ith directed edge of $\pi(w)$ is labelled by a_i. We see, inductively, that the final vertex of $\pi(w)$ is the group element obtained by multiplying together the a_i in Γ. Thus, by interpreting a word in the generators a group element, we retain only the final destination point in Δ, and forget about how we arrived there. We write $p : W(A) \longrightarrow \Gamma$ for the map obtained in this way. Thus $\pi(w)$ is a path from 1 to $p(w)$. We can similarly start from any group element $g \in \Gamma$, and get a path from g to $gp(w)$. It is precisely the image, $g\pi(w)$, of $\pi(w)$ by g in the above group action.

Lemma 2.1 : $\Gamma = \langle S \rangle \Leftrightarrow \Delta(\Gamma; S)$ *is connected.*

This follows from the above construction and the observation that $\langle S \rangle$ is precisely the set of elements expressible as a word in the alphabet A. (Note that, in general, $\langle S \rangle$ is the set of vertices of the connected component of Δ containing 1.)

Definition : If $S \subseteq \Gamma$ is a generating set of Γ, then $\Delta(\Gamma; S)$ is the *Cayley graph* of Γ with respect to S.

Note that, in summary, we have seen that any finitely generated group acts freely on a connected locally finite graph. Converses to this statement will be discussed in Section 3.

Examples.

(0) The Cayley graph of the trivial group with respect to the empty generating set is just a point.

(1a) $\mathbf{Z} = \langle a \rangle$. In this case Δ is the real line (Figure 2a). $\mathbf{Z}$ acts on it by translation with quotient graph a circle: a single vertex and a single loop.

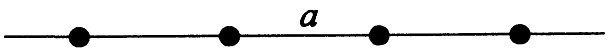

Figure 2a.

(b) If we add a generator $\mathbf{Z} = \langle a, a^2 \rangle$ we get an infinite "ladder" (Figure 2b).

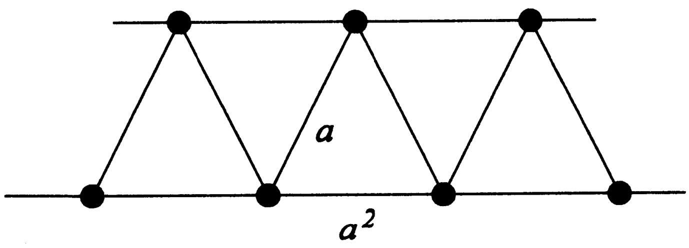

Figure 2b.

(c) $\mathbf{Z} = \langle a^2, a^3 \rangle$, see Figure 2c.

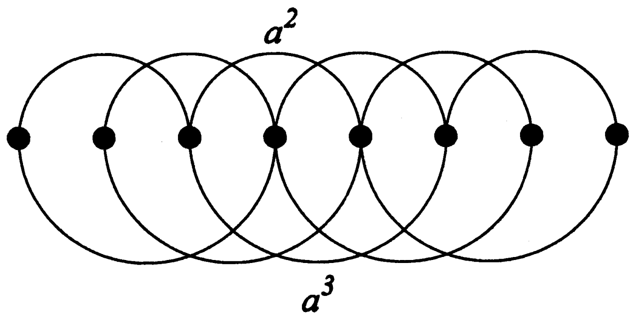

Figure 2c.

In these examples, the graphs are all combinatorially different, but they all "look like" the real line from "far away": in a sense that will be made precise later (see Section 3).

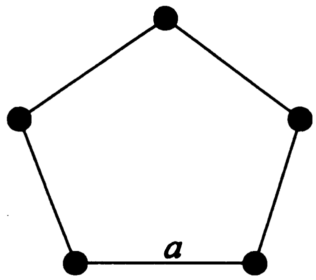

Figure 2d.

(2) $\mathbf{Z}_n = \langle a \mid a^n = 1 \rangle$. This is a circuit of length n (see Figure 2d, where $n = 5$).

(3) $\mathbf{Z} \oplus \mathbf{Z} = \langle a, b \mid ab = ba \rangle$. Here Δ is the 1-skeleton of the square tessellation of the plane, $\mathbf{R}^2$ (Figure 2e). It "looks like" $\mathbf{R}^2$ from "far away".

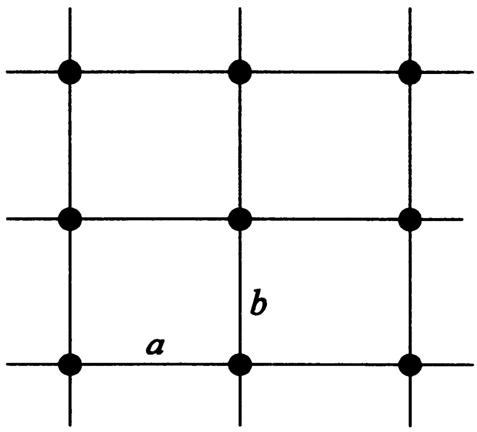

Figure 2e.

(4) The dihedral group $\langle a, b \mid b^2 = baba = a^n = 1 \rangle$
and the infinite dihedral group $\langle a, b \mid b^2 = baba = 1 \rangle$ (exercise).

(5) $\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (ca)^5 = (bc)^2 = 1 \rangle$ — the "icosahedral group" (exercise).

Note that in a Cayley graph, a word representing the identity gives a path starting and finishing at 1, in other words a cycle through 1. In particular, the relators give rise to cycles. For example, $aba^{-1}b^{-1}$ gives us the boundary of a square tile in example (3) above. Since a free group has no relations, the following should be no surprise:

Theorem 2.2 : *Suppose F is freely generated by $S \subseteq F$, then $\Delta(F; S)$ is a tree.*

Indeed the converse holds provided we assume, in that case, that $S \cap S^{-1} = \emptyset$.

Before we begin, we make the observation that if a word w corresponds to a path $\pi = \pi(w)$, then any subword of the form aa^{-1} would mean that we cross an edge labelled a, and then immediately cross back again, in other words, we *backtrack* along this edge. Cancelling this word corresponds to eliminating this backtracking. A word is therefore reduced if and only if the corresponding path has no backtracking.

Proof of Theorem 2.2 : We can assume that F has the form $F(S)$ as in the construction of free groups. We want to show that Δ has no circuits. We could always translate such a circuit under the action of F so that it passed through 1, and so can be thought of a path starting and ending at 1. Now such a circuit, σ, corresponds to a word in $S \cup S^{-1}$ representing the identity element in F. Thus, there is a finite sequence of reductions and inverse reductions that eventually transforms this word to the empty word. Reinterpreting this in terms of paths in the Cayley graph, we get a sequence of cycles, $\sigma = \sigma_0, \ldots, \sigma_n$, where σ_n is just the constant path based at 1, and each σ_i is obtained from σ_{i-1} by either eliminating or introducing a backtracking along an edge.

Now, given any cycle, τ, in Δ, let $O(\tau) \subseteq E(\Delta)$ be the set of edges through which τ passes an odd number of times. It is clear that $O(\tau)$ remains unchanged after eliminating a backtracking from τ. In particular, $O(\sigma_i)$ remains constant throughout. But $O(\sigma) = E(\sigma)$ since σ is a circuit, and $O(\sigma_n) = \emptyset$. Thus, $E(\sigma) = \emptyset$, and so σ could only have been the constant path at 1. $\diamondsuit$

Remark : We are really observing that the $\mathbf{Z}_2$-homology class of a cycle in $H_1(\Delta; \mathbf{Z}_2)$ remains unchanged under cancellation of back-tracking, and that the $\mathbf{Z}_2$-homology class of a non-trivial circuit is non-trivial.

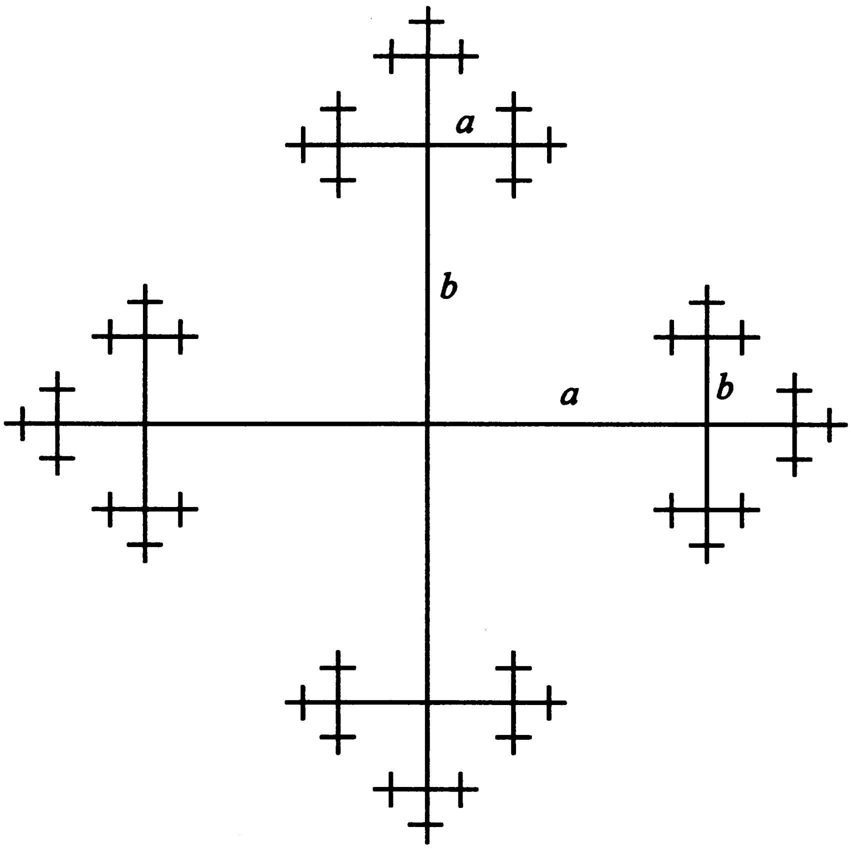

Figure 2f.

This is the main direction of interest to us. The converse is based on the observation that in a tree, any path with no backtracking is an arc. Moreover there is only one arc connecting 1 to any given element $g \in F$. This tells us that each element, g, of F has a unique representative as a reduced word. If we have a map, ϕ, from S into any group Γ, we can use such a reduced word to define an element, $\hat{\phi}(g)$, in Γ by multiplying together the ϕ-images of the letters in our reduced word. We need to check that $\hat{\phi}$ is a homomorphism from F into Γ, and that there was no choice in its definition. This then shows that S is, by definition, a free generating set. The details of this are left as an exercise.

Note that putting together Theorem 2.2 with the above observation on arcs in trees, we obtain the uniqueness part of Proposition 1.3: every element in a free group has a unique representative as a reduced word in the generators.

A picture of the Cayley graph of the free group with two generators, a and b, is given in Figure 2f. To represent it on the page, we have distorted distances. In reality, all edges have length 1.