Chapter 10

Gauss-Manin Connections

10.1 Fibration

Let $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$ be essential. We fix $\mathcal{A}$ in the rest of this section and write $\mathsf{B} = \mathsf{B}_{\mathcal{A}}$. We saw in the previous section that B may be considered a moduli space of the family of essential simple affine ℓ-arrangements which are combinatorially equivalent to $\mathcal{A}$. Recall that $\mathbf{t}$ are homogeneous coordinates for $((\mathbb{CP}^\ell)^*)^n$. Let $\mathbf{u} = (u_1, \dots, u_\ell)$ be standard coordinates for $\mathbb{C}^\ell$. Define

$$\mathsf{M} = \{(\mathbf{u}, \mathbf{t}) \in \mathbb{C}^\ell \times ((\mathbb{CP}^\ell)^*)^n \mid \mathbf{t} \in \mathsf{B},\ t_i^{(0)} + \sum_{j=1}^{\ell} t_i^{(j)} u_j \neq 0\ (i = 1, \dots, n)\}.$$

Let

$$\pi : \mathsf{M} \longrightarrow \mathsf{B}$$

be the projection defined by $\pi(\mathbf{u}, \mathbf{t}) = \mathbf{t}$. Then the fiber $\mathsf{M}_{\mathbf{t}} = \pi^{-1}(\mathbf{t})$ is the complement of the affine arrangement $\mathcal{A}_{\mathbf{t}}$ whose hyperplanes are defined by $\alpha_i = t_i^{(0)} + \sum_{j=1}^{\ell} t_i^{(j)} u_j$ $(i = 1, \dots, n)$. Thus $\pi : \mathsf{M} \longrightarrow \mathsf{B}$ is the complete family of essential simple affine arrangements in $\mathbb{C}^\ell$ which are combinatorially equivalent to $\mathcal{A}$. A result of Randell [Ra] implies that π is a fiber bundle over (the smooth part of) B.

Recall that d is the exterior differential operator with respect to the coordinates $\mathbf{u} = (u_1, \dots, u_\ell)$ of $\mathbb{C}^\ell$ in the fiber, $\omega_i = d \log \alpha_i = d\alpha_i / \alpha_i$ for $1 \le i \le n$ and

$$\omega_\lambda = \sum_{i=1}^{n} \lambda_i \omega_i, \qquad \nabla_\lambda : \Omega_M^p \to \Omega_M^{p+1}, \qquad \nabla_\lambda \eta = d\eta + \omega_\lambda \wedge \eta.$$

In this section we compute covariant derivatives of differential forms in the fiber along the direction of the base.

Definition 10.1.1. *Let d' be the exterior differential operator with respect to the homogeneous coordinates* $\mathbf{t}$ *of* $((\mathbb{CP}^\ell)^*)^n$. *For* $1 \le i \le n$ *define*

$$\omega_i' = d' \log(\frac{\alpha_i}{t_i^{(0)}}) = \frac{d'\alpha_i}{\alpha_i} - \frac{d't_i^{(0)}}{t_i^{(0)}}$$

and

$$\omega_\lambda' = \sum_{i=1}^n \lambda_i \omega_i', \qquad \nabla_\lambda' : \Omega_M^p \to \Omega_M^{p+1}, \qquad \nabla_\lambda' \eta = d'\eta + \omega_\lambda' \wedge \eta.$$

Our next aim is to compute the operator ∇_λ' explicitly. For $S = (j_1, \ldots, j_m)$, $j_1 < \cdots < j_m$, write $S_k = (j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_m)(1 \le k \le m)$ and $(S, j) = (j_1, \ldots, j_m, j)$ for $j \in [n+1] \setminus S$.

Definition 10.1.2. *Let* $T = (i_1, \ldots, i_\ell)$, *with* $i_k \in [n]$ $(1 \le k \le \ell)$. *Write*

$$\omega_T = \omega_{i_1} \wedge \cdots \wedge \omega_{i_\ell}, \qquad \zeta_T = \sum_{k=1}^\ell (-1)^{k+1} \omega_{i_k}' \wedge \omega_{T_k}.$$

The following computation was suggested by a method employed in [AK].

Proposition 10.1.3. *Recall that* $\Delta_S = \det(\mathsf{T}_S)$ *when* $|S| = \ell + 1$. *Then*

$$\nabla_\lambda' \omega_T = -\nabla_\lambda \zeta_T + \sum_{j \in [n] \setminus T} \lambda_j \sum_{k=1}^{\ell+1} (-1)^{\ell+k+1} d' \log\left(\frac{\Delta_{(T,j)}}{\Delta_{((T,j)_k, n+1)}}\right) \wedge \omega_{(T,j)_k}.$$

This result is an immediate consequence of the following two lemmas.

Lemma 10.1.4.

$$\nabla_\lambda' \omega_T + \nabla_\lambda \zeta_T = \sum_{j \in [n] \setminus T} \lambda_j \sum_{k=1}^{\ell+1} (-1)^{\ell+k+1} \omega_{i_k}' \wedge \omega_{(T,j)_k}.$$

Proof. Since d and d' operate in different variables, $dd' + d'd = 0$. This gives $d'\omega_T + d\zeta_T = 0$ used in the calculation below.

$$\begin{aligned}
\nabla_\lambda' \omega_T + \nabla_\lambda \zeta_T &= d'\omega_T + \omega_\lambda' \wedge \omega_T + d\zeta_T + \omega_\lambda \wedge \zeta_T \\
&= \sum_{k=1}^\ell (-1)^{k+1} (d'\omega_{i_k}) \wedge \omega_{T_k} + \omega_\lambda' \wedge \omega_T + \sum_{k=1}^\ell (-1)^{k+1} (d\omega_{i_k}') \wedge \omega_{T_k} \\
&\quad - \sum_{k=1}^\ell \lambda_{i_k} \omega_{i_k}' \wedge \omega_T + \sum_{j \in [n] \setminus T} \lambda_j \sum_{k=1}^\ell (-1)^{\ell+k+1} \omega_{i_k}' \wedge \omega_{(T_k, j)}
\end{aligned}$$

$$= \left(\sum_{j \in [n] \setminus T} \lambda_j \omega'_j \right) \wedge \omega_T + \sum_{j \in [n] \setminus T} \lambda_j \sum_{k=1}^{\ell} (-1)^{\ell+k+1} \omega'_{i_k} \wedge \omega_{(T_k, j)}$$

$$= \sum_{j \in [n] \setminus T} \lambda_j \sum_{k=1}^{\ell+1} (-1)^{\ell+k+1} \omega'_{i_k} \wedge \omega_{(T,j)_k}.$$

□

Lemma 10.1.5. *For* $S \in \left(\binom{[n+1]}{\ell+1} \right)$, *we have*

$$\sum_{k=1}^{\ell+1} (-1)^{k+1} \omega'_{j_k} \wedge \omega_{S_k} = \sum_{k=1}^{\ell+1} (-1)^{k+1} d' \log \left(\frac{\Delta_S}{\Delta_{(S_k, n+1)}} \right) \wedge \omega_{S_k}.$$

Proof. Note that

$$\Delta_S = \sum_{k=1}^{\ell+1} (-1)^{k+1} t_{j_k}^{(0)} \Delta_{(S_k, n+1)} = \sum_{k=1}^{\ell+1} (-1)^{k+1} \alpha_{j_k} \Delta_{(S_k, n+1)}$$

by the Laplace expansion. Let

$$\alpha_S = \alpha_{j_1} \cdots \alpha_{j_{\ell+1}}, \qquad d\mathbf{u} = du_1 \wedge \cdots \wedge du_\ell.$$

We compute

$$\sum_{k=1}^{\ell+1} (-1)^{k+1} (d' \log \Delta_S) \wedge \omega_{S_k}$$

$$= \frac{1}{\alpha_S} \sum_{k=1}^{\ell+1} (-1)^{k+1} \alpha_{j_k} \Delta_{(S_k, n+1)} (d' \log \Delta_S) \wedge (d\mathbf{u})$$

$$= \frac{1}{\alpha_S} \Delta_S (d' \log \Delta_S) \wedge (d\mathbf{u}) = \frac{1}{\alpha_S} (d' \Delta_S) \wedge (d\mathbf{u})$$

$$= \frac{1}{\alpha_S} d' \left(\sum_{k=1}^{\ell+1} (-1)^{k+1} \alpha_{j_k} \Delta_{(S_k, n+1)} \right) \wedge (d\mathbf{u})$$

$$= \frac{1}{\alpha_S} \left[\sum_{k=1}^{\ell+1} (-1)^{k+1} \left\{ (d' \alpha_{j_k}) \Delta_{(S_k, n+1)} + \alpha_{j_k} \left(d' \Delta_{(S_k, n+1)} \right) \right\} \right] \wedge (d\mathbf{u})$$

$$= \sum_{k=1}^{\ell+1} (-1)^{k+1} \left\{ \omega'_{j_k} \wedge \omega_{S_k} + \left(d' \log \Delta_{(S_k, n+1)} \right) \wedge \omega_{S_k} \right\}.$$

This proves the lemma. □

10.2 General Formulas

For $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mathbf{t} \in \mathsf{B}$, recall the rank-one local system $\mathcal{L}_\lambda$ on $\mathsf{M}_{\mathbf{t}} = \pi^{-1}(\mathbf{t})$ defined in Proposition 2.1.3. Assume that $\lambda \in \mathbf{W}$ of Definition 6.5.1. It follows from Theorem 6.2.3 that

(1) $H^p(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}}) = 0$ for $p \neq \ell$ and $\dim H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}}) = \beta(\mathcal{A})$.

It follows from Theorem 4.2.6 that

(2) there exisits a natural (twisted) de Rham isomorphism

$$B^\ell / \omega_\lambda \wedge B^{\ell-1} \xrightarrow{\sim} H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}}),$$

where $B^\cdot = \oplus_{q=0}^{\ell} B^q$ is the Brieskorn algebra of $\mathcal{A}_{\mathbf{t}}$.

Since $\nabla_\lambda \circ \nabla'_\lambda + \nabla'_\lambda \circ \nabla_\lambda = 0$ and

$$H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}}) \simeq B^\ell / \omega_\lambda \wedge B^{\ell-1} = B^\ell / \nabla_\lambda B^{\ell-1},$$

the operator ∇'_λ induces a $\mathbb{C}$-linear map

$$\nabla'_\lambda : H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}}) \to \Omega^1(\log \mathsf{D}) \otimes H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}})$$

by Proposition 10.1.3. Here $\Omega^1(\log \mathsf{D})$ is the space of meromorphic 1-forms on (the smooth part of) $\bar{\mathsf{B}}$ with logarithmic poles along $\mathsf{D} = \bar{\mathsf{B}} \setminus \mathsf{B}$. Let

$$\mathsf{D} = \bigcup_{s=1}^{t} \mathsf{D}_s$$

be the irreducible decomposition. For each irreducible component D_s and $S' \in \mathcal{J}(\mathcal{A}_\infty)^c$, define

$$\mathrm{mult}(S', \mathsf{D}_s) = \text{the order of zeros of } \ \Delta_{S'}|_{\bar{B}} \text{ along } \mathsf{D}_s$$

and

$$\begin{aligned} \Gamma(\mathsf{D}_s) &= \{S' \in \mathcal{J}(\mathcal{A}_\infty)^c \mid \mathrm{mult}(S', \mathsf{D}_s) \geq 1\} \\ &= \{S' \in \mathcal{J}(\mathcal{A}_\infty)^c \mid \Delta_{S'}|_{\bar{B}} \text{ vanishes on } \mathsf{D}_s\}. \end{aligned}$$

We denote the logarithmic 1-form on $\bar{B}$ with simple logarithmic pole along D_s by $d' \log \mathsf{D}_s$ by abuse of notation. It can be expressed locally as $d \log f$ where $f = 0$ is a local defining equation for D_s. For $\omega \in B^\ell$, let $[\omega] \in H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}})$ be its (twisted) de Rham cohomology class. Proposition 10.1.3 implies:

Theorem 10.2.1. *We have*

$$\nabla'_\lambda = \sum_{s=1}^{t} d' \log \mathsf{D}_s \otimes \nabla'_{\lambda,s},$$

where $\nabla'_{\lambda,s} \in \mathrm{End}(H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}}))$. *For* $T \in \left(\binom{[n+1]}{\ell+1}\right)$,

$$\begin{aligned}\nabla'_{\lambda,s}[\omega_T] = &\sum_{(T,j)\in\Gamma(\mathsf{D}_s)} \mathrm{mult}((T,j),\mathsf{D}_s)\lambda_j \sum_{k=1}^{\ell+1}(-1)^{\ell+k+1}[\omega_{(T,j)_k}] \\ &- \sum_{((T,j)_k,n+1)\in\Gamma(\mathsf{D}_s)} \mathrm{mult}(((T,j)_k,n+1),\mathsf{D}_s)(-1)^{\ell+k+1}\lambda_j[\omega_{(T,j)_k}].\end{aligned}$$

Although Theorem 10.2.1 determines ∇'_λ and $\nabla'_{\lambda,s}$ completely, it is desirable to express each $\nabla'_{\lambda,s}$ explicitly in terms of a basis for $H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}})$ and thereby exhibit the Gauss-Manin connection matrix Ω. We propose to use the β**nbc** basis of Section 6.3 for this purpose. In general, it is not difficult to see from [FT, 3.9] that $[\omega_T] \in H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}})$ is uniquely expressed as a linear combination of the β**nbc** basis $[\Xi_1], \ldots, [\Xi_\beta] \in H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}})$ with coefficients lying in $\widetilde{\mathbb{Q}[\lambda]} = \mathbb{Q}[\lambda_1, \ldots, \lambda_n, \{\lambda_X^{-1}\}]$, where $\lambda_X = \sum_{X\subseteq H_j} \lambda_j$ runs over the set $\{X \mid X \text{ is a dense edge }\}$. Recall that $\mathcal{H}_\ell$ is the rank β local system coming from the topological fibration $\pi : \mathsf{M} \to \mathsf{B}$. Then we have

Theorem 10.2.2. *The $\beta \times \beta$-matrix Ω, which satisfies the system of differential equations*

$$d' \begin{pmatrix} \int_\sigma \Phi_\lambda \Xi_1 \\ \cdot\cdot \\ \cdot\cdot \\ \int_\sigma \Phi_\lambda \Xi_\beta \end{pmatrix} = \Omega \wedge \begin{pmatrix} \int_\sigma \Phi_\lambda \Xi_1 \\ \cdot\cdot \\ \cdot\cdot \\ \int_\sigma \Phi_\lambda \Xi_\beta \end{pmatrix}$$

*for any (local) section σ of $\mathcal{H}_\ell$, the β**nbc** basis $[\Xi_1], \ldots, [\Xi_\beta] \in H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}})$ and $\Phi_\lambda = \alpha_1^{\lambda_1} \ldots \alpha_n^{\lambda_n}$, has logarithmic poles along D with coefficients lying in $\widetilde{\mathbb{Q}[\lambda]}$.*

Proof. The integral $\int_\sigma \Phi_\lambda \Xi$ depends only on the cohomology class $[\Xi] \in H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}})$. By Theorem 10.2.1, there exists a unique $\beta \times \beta$-matrix Ω such that

$$\begin{pmatrix} \nabla'_{\lambda,s}[\Xi_1] \\ \cdot\cdot \\ \cdot\cdot \\ \nabla'_{\lambda,s}[\Xi_\beta] \end{pmatrix} = \Omega \wedge \begin{pmatrix} [\Xi_1] \\ \cdot\cdot \\ \cdot\cdot \\ [\Xi_\beta] \end{pmatrix}.$$

Since $\Phi\nabla'_\lambda\eta = d'(\Phi\eta)$, we get $\int_\sigma \Phi\nabla'_\lambda\eta = \int_\sigma d'(\Phi\eta) = d'\int_\sigma \Phi\eta$ because σ is a section. Thus Ω satisfies the desired equation. □

It follows that the connection $d' - \Omega\wedge$ on $\mathcal{O}_{\mathsf{B}}^\beta$ is a logarithmic Gauss-Manin connection and its flat sections are given by

$$\left\{ \begin{pmatrix} \int_\sigma \Phi_\lambda \Xi_1 \\ \cdot\cdot \\ \cdot\cdot \\ \int_\sigma \Phi_\lambda \Xi_\beta \end{pmatrix} \mid \sigma \text{ is a local section of } \mathcal{H}_\ell \right\}.$$

In the rest of this chapter it is convenient to use the notation

$$\epsilon(T,T') = (-1)^{p+q}$$

if $T, T' \subseteq [n]$, $|T| = |T'| = \ell$, $|T \cap T'| = \ell - 1$, $U = T \cup T'$, $T = U_p$ and $T' = U_q$, where the subscript notation was introduced just before Definition 10.1.2. Define $\epsilon(T,T') = 1$ if $T = T'$. For example, $\epsilon(23,35) = 1$ because $U = 235, T = 23 = U_3, T' = 35 = U_1$.

10.3 Codimension Zero

Suppose that the codimension of $\mathsf{B}_{\mathcal{A}}$ in $((\mathbb{CP}^\ell)^*)^n$ is zero. In other words, $\mathcal{A}$ is in general position. The set

$$\{[\eta_T] \in H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}}) \mid T \in \beta\mathbf{nbc}\}$$

is a $\beta\mathbf{nbc}$ basis, where

$$\beta\mathbf{nbc} = \{(j_1, \ldots, j_\ell) \mid 2 \le j_1 < \cdots < j_\ell \le n\}, \quad \eta_T = \lambda_{i_1} \cdots \lambda_{i_\ell} \omega_T.$$

This basis was first obtained in [A1, p.292] in a slightly different form. The expression of each $\nabla'_{\lambda,s}$ in terms of the $\beta\mathbf{nbc}$ basis was essentially given in [AK, Ch. 3 §8]. Let $\mathsf{D}_s = \mathsf{C}_{\{S\}}$. Then $\Gamma(\mathsf{D}_s) = \{S\}$ and $\mathrm{mult}(S, \mathcal{D}_s) = 1$. There are four cases distinguished by $S \cap \{1, n+1\}$. We will express $\nabla'_{\lambda,s}[\eta_T]$, $T \in \beta\mathbf{nbc}$ as a linear combination of $\{[\eta_{T'}] \in H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}}) \mid T' \in \beta\mathbf{nbc}\}$ with coefficients in $\sum_{i=1}^n \mathbb{Z}\lambda_i$. The following formulas are obtained from Theorem 10.2.1 through routine computations.

Case 1: Suppose $S \cap \{1, n+1\} = \emptyset$.
If $S \supset T \in \beta\mathbf{nbc}$, then

$$\nabla'_{\lambda,s}[\eta_T] = \sum_{k=1}^{\ell+1} \epsilon(T, S_k) \lambda_{S \setminus S_k} [\eta_{S_k}],$$

where $S_k = (i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_{\ell+1})$ and $\lambda_{S \setminus S_k} = \lambda_{i_k}$ if $S = (i_1, \ldots, i_{\ell+1})$. Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case 2: Suppose $S \cap \{1, n+1\} = \{n+1\}$.
If $S \supset T \in \beta\mathbf{nbc}$, then $T = S_{\ell+1} = S \setminus \{n+1\}$ and

$$\nabla'_{\lambda,s}[\eta_T] = -\left(\sum_{j \in [n] \setminus T} \lambda_j \right) [\eta_T].$$

If $T \in \beta\mathbf{nbc}$ with $|T \cap S| = \ell - 1$, then

$$\nabla'_{\lambda,s}[\eta_T] = -\epsilon(T, S_{\ell+1}) \lambda_{T \setminus S} [\eta_{S_{\ell+1}}].$$

Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case 3: Suppose $S \cap \{1, n+1\} = \{1\}$.
If $S \supset T \in \beta\mathbf{nbc}$, then $T = S_1 = S \setminus \{1\}$ and

$$\nabla'_{\lambda,s}[\eta_T] = \left(\sum_{j\in S} \lambda_j\right) [\eta_T] - \sum_{\substack{T'\in\beta\mathbf{nbc} \\ |T\cap T'|=\ell-1}} \epsilon(T,T')\lambda_{T\setminus T'}[\eta_{T'}].$$

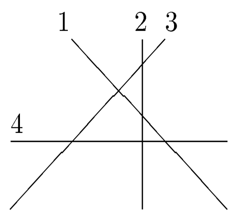
Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case 4: Suppose $S \cap \{1, n+1\} = \{1, n+1\}$.
If $T \in \beta\mathbf{nbc}$ with $|T \cap S| = \ell - 1$, then $S \setminus \{1, n+1\} \subset T$ and

$$\nabla'_{\lambda,s}[\eta_T] = -\lambda_{T\setminus S}[\eta_T] + \lambda_{T\setminus S} \sum_{\substack{T'\in\beta\mathbf{nbc} \\ T\cap T'=T\cap S}} \epsilon(T,T')[\eta_{T'}].$$

Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

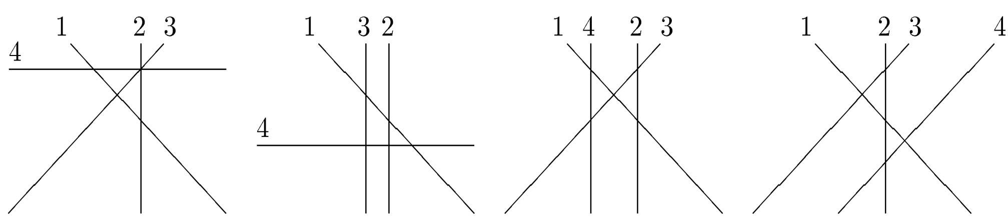
Example 10.3.1. *Suppose that $\mathcal{A}$ is in general position with $\ell = 2$ and $n = 4$ shown in Figure 10.1.*



$\mathcal{A}$

Figure 10.1: A Codimension Zero Arrangement

Write 123 *for* $(1,2,3)$ *etc. The boundary divisor* $\mathsf{D} = \overline{\mathsf{B}}_{\mathcal{A}} \setminus \mathsf{B}_{\mathcal{A}}$ *has ten irreducible components shown in Figure 10.2. The connection matrices* $\Omega_s (s = 1, \ldots, 10)$ *in terms of the* $\beta\mathbf{nbc}$ *basis* $\{[\eta_{23}], [\eta_{24}], [\eta_{34}]\}$*, are given below.*



$\mathsf{D}_1 = \mathsf{C}_{\{234\}}$ $\mathsf{D}_2 = \mathsf{C}_{\{235\}}$ $\mathsf{D}_3 = \mathsf{C}_{\{245\}}$ $\mathsf{D}_4 = \mathsf{C}_{\{345\}}$

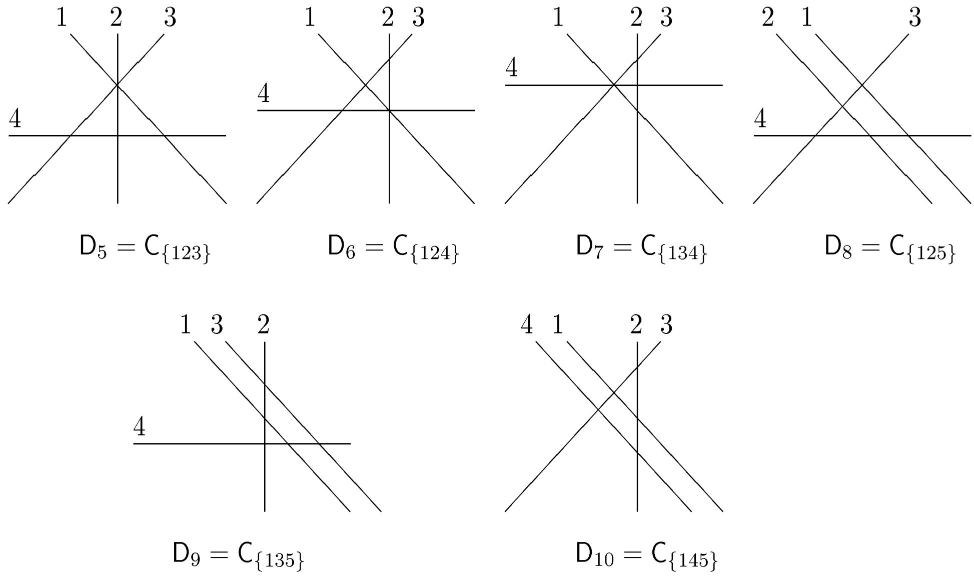


Figure 10.2: Ten Boundary Components

$$\Omega_1 = \begin{pmatrix} \lambda_4 & -\lambda_3 & \lambda_2 \\ -\lambda_4 & \lambda_3 & -\lambda_2 \\ \lambda_4 & -\lambda_3 & \lambda_2 \end{pmatrix} \text{(Case1)}, \quad \Omega_2 = \begin{pmatrix} -\lambda_1-\lambda_4 & 0 & 0 \\ \lambda_4 & 0 & 0 \\ -\lambda_4 & 0 & 0 \end{pmatrix} \text{(Case2)},$$

$$\Omega_3 = \begin{pmatrix} 0 & \lambda_3 & 0 \\ 0 & -\lambda_1-\lambda_3 & 0 \\ 0 & \lambda_3 & 0 \end{pmatrix} \text{(Case2)}, \quad \Omega_4 = \begin{pmatrix} 0 & 0 & -\lambda_2 \\ 0 & 0 & \lambda_2 \\ 0 & 0 & -\lambda_1-\lambda_2 \end{pmatrix} \text{(Case2)},$$

$$\Omega_5 = \begin{pmatrix} \lambda_1+\lambda_2+\lambda_3 & \lambda_3 & -\lambda_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{(Case3)}, \quad \Omega_6 = \begin{pmatrix} 0 & 0 & 0 \\ \lambda_4 & \lambda_1+\lambda_2+\lambda_4 & \lambda_2 \\ 0 & 0 & 0 \end{pmatrix} \text{(Case3)},$$

$$\Omega_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\lambda_4 & \lambda_3 & \lambda_1+\lambda_3+\lambda_4 \end{pmatrix} \text{(Case3)}, \quad \Omega_8 = \begin{pmatrix} -\lambda_3 & -\lambda_3 & 0 \\ -\lambda_4 & -\lambda_4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{(Case4)},$$

$$\Omega_9 = \begin{pmatrix} -\lambda_2 & 0 & \lambda_2 \\ 0 & 0 & 0 \\ \lambda_4 & 0 & -\lambda_4 \end{pmatrix} \text{(Case4)}, \quad \Omega_{10} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\lambda_2 & -\lambda_2 \\ 0 & -\lambda_3 & -\lambda_3 \end{pmatrix} \text{(Case4)}.$$

Using the notation in Theorem 10.2.2, we have

$$d' \begin{pmatrix} \int_\sigma \Phi_\lambda \eta_{23} \\ \int_\sigma \Phi_\lambda \eta_{24} \\ \int_\sigma \Phi_\lambda \eta_{34} \end{pmatrix} = \Omega \wedge \begin{pmatrix} \int_\sigma \Phi_\lambda \eta_{23} \\ \int_\sigma \Phi_\lambda \eta_{24} \\ \int_\sigma \Phi_\lambda \eta_{34} \end{pmatrix},$$

where Ω is the Gauss-Manin connection matrix given by

$$\Omega = \sum_{s=1}^{10} d' \log \mathsf{D}_s \otimes \Omega_s.$$

10.4 Codimension One

Suppose that the codimension of $\mathsf{B}_{\mathcal{A}}$ in $((\mathbb{CP}^\ell)^*)^n$ is one. Then $\mathcal{J}(\mathcal{A}_\infty) = \{S\}$ for some $S \in \left(\binom{[n+1]}{\ell+1}\right)$. There are two cases : $n+1 \notin S$ (Case A) or $n+1 \in S$ (Case B). By permuting the hyperplanes if necessary, we can assume that $S = (1, 2, \ldots, \ell+1)$ (Case A) or $S = (n-\ell+1, n-\ell+2, \ldots, n+1)$ (Case B). It is easy to see that the β**nbc** basis for $H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}})$ is given by $\{[\eta_T] \in H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}}) \mid T \in \beta\mathbf{nbc}\}$, where

$$\beta\mathbf{nbc} = \{(j_1, \ldots, j_\ell) \mid 2 \leq j_1 < \cdots < j_\ell \neq \ell+1\} \quad \text{(Case A)}$$

or

$$\beta\mathbf{nbc} = \{(j_1, \ldots, j_\ell) \mid 2 \leq j_1 < \cdots < j_\ell, j_1 \neq n-\ell+1\} \quad \text{(Case B)}.$$

We will express $\nabla'_{\lambda,s}[\eta_T]$, $T \in \beta\mathbf{nbc}$ as a linear combination of $\{[\eta_{T'}] \in H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}}) \mid T' \in \beta\mathbf{nbc}\}$ with coefficients in $\widetilde{\mathbb{Q}[\lambda]}$. (It will turn out that all the coefficients lie in $\sum_{i=1}^n \mathbb{Z}\lambda_i$.) The following formulas are obtained from Theorem 10.2.1 through routine computations.

Case A: Let $S = (1, 2, \ldots, \ell+1)$.

Type A.I: Let $\mathsf{D}_s = \mathsf{C}_{\{S,S'\}}$ for $S' \in \left(\binom{[n+1]}{\ell+1}\right)$ with $|S \cap S'| \leq \ell - 1$ (Proposition 9.3.3 (iii)). In this case, $\Gamma(\mathsf{D}_s) = \{S, S'\}$ and $\mathrm{mult}(S', \mathsf{D}_s) = 1$ because the ideal $(\Delta_S, \Delta_{S'})$ is prime by Lemma 9.3.2 (ii).

Case A.I.1: Suppose $S' \cap \{1, n+1\} = \emptyset$.
If $S' \supset T \in \beta\mathbf{nbc}$, then

$$\nabla'_{\lambda,s}[\eta_T] = \sum_{k=1}^{\ell+1} \epsilon(T, S'_k)\lambda_{S' \setminus S'_k}[\eta_{S'_k}],$$

where $S'_k = (i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_{\ell+1})$ and $\lambda_{S' \setminus S'_k} = \lambda_{i_k}$ if $S' = (i_1, \ldots, i_{\ell+1})$. Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case A.I.2: Suppose $S' \cap \{1, n+1\} = \{n+1\}$.

If $S' \supset T \in \beta\mathbf{nbc}$, then $T = S'_{\ell+1} = S' \setminus \{n+1\}$ and

$$\nabla'_{\lambda,s}[\eta_T] = -\left(\sum_{j\in[n]\setminus T} \lambda_j\right)[\eta_T].$$

If $T \in \beta\mathbf{nbc}$ with $|T \cap S'| = \ell - 1$, then

$$\nabla'_{\lambda,s}[\eta_T] = -\epsilon(T, S'_{\ell+1})\lambda_{T\setminus S'}[\eta_{S'_{\ell+1}}].$$

Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case A.I.3: Suppose $S' \cap \{1, n+1\} = \{1\}$.
If $S' \supset T \in \beta\mathbf{nbc}$, then $T = S'_1 = S' \setminus \{1\}$ and

$$\nabla'_{\lambda,s}[\eta_T] = \left(\sum_{j\in S'} \lambda_j\right)[\eta_T] - \sum_{\substack{T'\in\beta\mathbf{nbc} \\ |T\cap T'|=\ell-1}} \epsilon(T,T')\lambda_{T\setminus T'}[\eta_{T'}].$$

Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case A.I.4: Suppose $S' \cap \{1, n+1\} = \{1, n+1\}$.
If $T \in \beta\mathbf{nbc}$ with $|T \cap S'| = \ell - 1$, then $S' \setminus \{1, n+1\} \subset T$ and

$$\nabla'_{\lambda,s}[\eta_T] = -\lambda_{T\setminus S'}[\eta_T] + \lambda_{T\setminus S'} \sum_{\substack{T'\in\beta\mathbf{nbc} \\ T'\cap S'=T\cap S'}} \epsilon(T,T')[\eta_{T'}].$$

Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Type A.II: Suppose $\ell \geq 2$. Let $\mathsf{D}_s = \mathsf{C}_{\langle S-p\rangle}$ where $p \in S = (1, 2, \ldots, \ell+1)$, $S - p = S \setminus \{p\}$, and $\langle S-p\rangle = \{S' \in \left(\binom{[n+1]}{\ell+1}\right) \mid S' \supset S - p\}$. In this case, $\Gamma(\mathsf{D}_s) = \langle S-p\rangle$ and $\mathrm{mult}(S', \mathsf{D}_s) = 1$ for $S' \in \langle S-p\rangle$, $S' \neq S$, because the ideal $(\Delta_S, \Delta_{S'})$ is a radical ideal by Lemma 9.3.2 (i).

Case A.II.1: Suppose $p \neq 1$.
If $T \in \beta\mathbf{nbc}$ with $|T \cap (S-p)| = \ell - 1$, then

$$\nabla'_{\lambda,s}[\eta_T] = \left(\sum_{j\in S-p} \lambda_j\right)[\eta_T] - \sum_{\substack{T'\in\beta\mathbf{nbc} \\ |T\cap T'|=\ell-1 \\ |T'\cap(S-p)|=\ell-2}} \epsilon(T,T')\lambda_{T\setminus T'}[\eta_{T'}].$$

Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case A.II.2: Suppose $p = 1$.

If $T \in \beta\mathbf{nbc}$ with $|T \cap (S-1)| = \ell - 1$, then

$$\nabla'_{\lambda,s}[\eta_T] = \lambda_{(S-1)\setminus T}[\eta_T] + \sum_{\substack{T' \in \beta\mathbf{nbc} \\ |T \cap T'| = \ell-1 \\ T' \subset S \cup T}} \epsilon(T,T')\lambda_{T\setminus T'}[\eta_{T'}].$$

Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Type A.III: Let $\mathsf{D}_s = \mathsf{C}_{\langle S+q\rangle}$ where $q \in [n+1] \setminus S = (\ell+2, \ell+3, \ldots, n+1)$, and $S+q = S \cup \{q\}$, and $\langle S+q\rangle = \{S' \in \left(\binom{[n+1]}{\ell+1}\right) \mid S' \subset S+q\}$. In this case, $\Gamma(\mathsf{D}_s) = \langle S+q\rangle$ and $\mathrm{mult}(S', \mathsf{D}_s) = 1$ for $S' \in \langle S+q\rangle$, $S' \neq S$, because the ideal $(\Delta_S, \Delta_{S'})$ is a radical ideal by Lemma 9.3.2 (i).

Case A.III.1: Suppose $q \neq n+1$.
If $T \in \beta\mathbf{nbc}$ with $T \subset S+q$, then

$$\nabla'_{\lambda,s}[\eta_T] = \left(\sum_{j \in S+q} \lambda_j\right)[\eta_T] - \sum_{\substack{T' \in \beta\mathbf{nbc} \\ |T \cap T'| = \ell-1 \\ |T' \cap (S+q)| = \ell-1}} \epsilon(T,T')\lambda_{T\setminus T'}[\eta_{T'}].$$

Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case A.III.2: Suppose $q = n+1$.
If $T \in \beta\mathbf{nbc}$ with $|T \cap S| = \ell - 1$, then

$$\nabla'_{\lambda,s}[\eta_T] = -\lambda_{T\setminus S} \sum_{\substack{T' \in \beta\mathbf{nbc} \\ T' \cap S = T \cap S}} [\eta_{T'}].$$

Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case B: Let $S = (n-\ell+1, \ldots, n+1)$.

Type B.I: Let $\mathsf{D}_s = \mathsf{C}_{\{S,S'\}}$ for $S' \in \left(\binom{[n+1]}{\ell+1}\right)$ with $|S \cap S'| \leq \ell - 1$. For this type, we have the exact same formulas as Case A.I.

Type B.II: Suppose $\ell \geq 2$. Let $\mathsf{D}_s = \mathsf{C}_{\langle S-p\rangle}$.

Case B.II.1: Suppose $p \neq n+1$.
If $T \in \beta\mathbf{nbc}$ with $|T \cap (S-p)| = \ell - 1$, then

$$\nabla'_{\lambda,s}[\eta_T] = -\left(\sum_{j \notin S-p} \lambda_j\right)[\eta_T].$$

If $T \in \beta\mathbf{nbc}$ with $|T \cap (S-p)| = \ell - 2$, then

$$\nabla'_{\lambda,s}[\eta_T] = - \sum_{\substack{T' \in \beta\mathbf{nbc} \\ |T \cap T'| = \ell - 1 \\ |T' \cap (S-p)| = \ell - 1}} \epsilon(T,T') \lambda_{T \setminus T'} [\eta_{T'}].$$

Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case B.II.2: Suppose $p = n+1$.
If $T \in \beta\mathbf{nbc}$ with $|T \cap (S-(n+1))| = \ell - 1$, then

$$\nabla'_{\lambda,s}[\eta_T] = \lambda_{(S-(n+1)) \setminus T}[\eta_T] + \sum_{\substack{T' \in \beta\mathbf{nbc} \\ |T \cap T'| = \ell - 1 \\ T' \subset S \cup T}} \epsilon(T,T') \lambda_{T \setminus T'} [\eta_{T'}].$$

Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Type B.III: Let $\mathsf{D}_s = \mathsf{C}_{\langle S+q \rangle}$ where $q \in [n+1] \setminus S = (1, 2, \ldots, n-\ell)$.

Case B.III.1: Suppose $q \neq 1$.
If $T \in \beta\mathbf{nbc}$ with $T \subset S + q$, then

$$\nabla'_{\lambda,s}[\eta_T] = - \left(\sum_{j \notin S+q} \lambda_j \right) [\eta_T].$$

If $T \in \beta\mathbf{nbc}$ with $|T \cap (S+q)| = \ell - 1$, then

$$\nabla'_{\lambda,s}[\eta_T] = - \sum_{\substack{T' \in \beta\mathbf{nbc} \\ |T \cap T'| = \ell - 1 \\ T' \subset (S+q)}} \epsilon(T,T') \lambda_{T \setminus T'} [\eta_{T'}].$$

Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case B.III.2: Suppose $q = 1$.
If $T \in \beta\mathbf{nbc}$ with $|T \cap S| = \ell - 1$, then

$$\nabla'_{\lambda,s}[\eta_T] = -\lambda_{T \setminus S} \sum_{\substack{T' \in \beta\mathbf{nbc} \\ T' \cap S = T \cap S}} [\eta_{T'}].$$

Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

We summarize Cases A and B.

Theorem 10.4.1. *Suppose that* $\mathsf{B} = \mathsf{B}_{\mathcal{A}}$ *has codimension one in* $((\mathbb{CP}^{\ell})^*)^n$. *Let* $\mathcal{J}(\mathcal{A}_\infty) = \{S\}$, $\mathsf{D} = \overline{\mathsf{B}} \setminus \mathsf{B}$ *and* $\mathsf{D} = \cup_{s=1}^{t} \mathsf{D}_s$ *be the irreducible decomposition. Then*

(1) the logarithmic Gauss-Manin connection matrix Ω *in Theorem 10.2.2 can be expressed as* $\Omega = \sum_{s=1}^{t} d' \log \mathsf{D}_s \otimes \Omega_s$ *such that each* Ω_s *has its entries in* $\sum_{i=1}^{n} \mathbb{Z}\lambda_i$.

(2) The eigenvalues of Ω_s *are:*

(i) $\sum_{j \in S'} \lambda_j$ *with multiplicity one and the rest are zero (if* $\mathsf{D}_s = \mathsf{C}_{\{S,S'\}}$ *is of type I in Proposition 9.3.3),*

(ii) $\sum_{j \in S-p} \lambda_j$ *with multiplicity* $n - \ell - 1$ *and the rest are zero (if* $\mathsf{D}_s = \mathsf{C}_{\langle S-p \rangle}$ *is of type II), or*

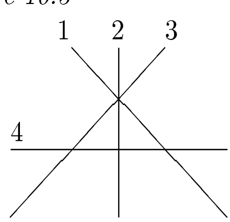
(iii) $\sum_{j \in S+q} \lambda_j$ *with multiplicity* ℓ *and the rest are zero (if* $\mathsf{D}_s = \mathsf{C}_{\langle S+q \rangle}$ *is of type III),*

where we define $\lambda_{n+1} = -\lambda_1 - -\lambda_2 - \cdots - \lambda_n$.

(The explicit formulas for Ω_s *are given above when* $S = (1, 2, \ldots, \ell + 1)$ *(Case A) or* $S = (n - \ell + 1, n - \ell + 2, \ldots, n + 1)$ *(Case B).)*

Proof. Although the β**nbc** basis depends on the linear order on $\mathcal{A}$, it is known [FT, 3.11] that two β**nbc** bases are connected by an integral unimodular matrix (without λ). Thus one can assume that $S = (1, 2, \ldots, \ell + 1)$ (when $n + 1 \notin S$) or $S = (n - \ell + 1, n - \ell + 2, \ldots, n + 1)$ (when $n + 1 \in S$). Use the above-mentioned explicit formulas for Cases A and B. $\square$

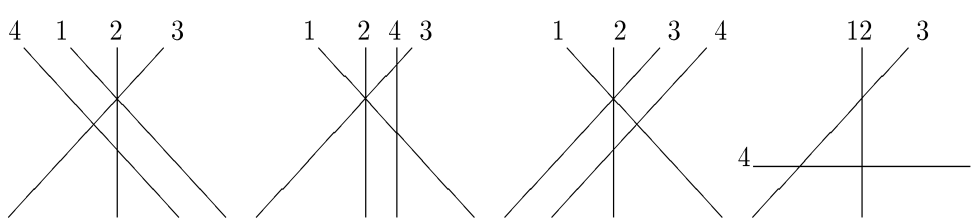
Example 10.4.2. *Let* $\ell = 2$, $n = 4$, $S = (1, 2, 3)$ *and* $\mathcal{J}(\mathcal{A}_\infty) = \{S\}$. *The arrangement is shown in Figure 10.3*



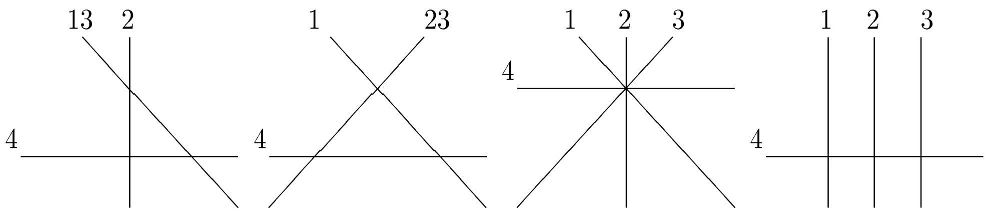
$\mathcal{A}$

Figure 10.3: A Codimension One Arrangement

Write 123 *for* $(1, 2, 3)$ *etc. The boundary divisor* $\mathsf{D} = \overline{\mathsf{B}}_{\mathcal{A}} \setminus \mathsf{B}_{\mathcal{A}}$ *has eight irreducible components shown in Figure 10.4.*



$\mathsf{D}_1 = \mathsf{C}_{\{123,145\}}$ $\mathsf{D}_2 = \mathsf{C}_{\{123,245\}}$ $\mathsf{D}_3 = \mathsf{C}_{\{123,345\}}$ $\mathsf{D}_4 = \mathsf{C}_{\langle 123-3 \rangle}$



$\mathsf{D}_5 = \mathsf{C}_{\langle 123-2\rangle}$ $\mathsf{D}_6 = \mathsf{C}_{\langle 123-1\rangle}$ $\mathsf{D}_7 = \mathsf{C}_{\langle 123+4\rangle}$ $\mathsf{D}_8 = \mathsf{C}_{\langle 123+5\rangle}$

Figure 10.4: Eight Boundary Components

The matrices $\Omega_s (s = 1, \ldots, 8)$ in terms of the $\beta\mathbf{nbc}$ basis $\{[\eta_{24}], [\eta_{34}]\}$, are

$$\Omega_1 = \begin{pmatrix} -\lambda_2 & -\lambda_2 \\ -\lambda_3 & -\lambda_3 \end{pmatrix} \text{(CaseA.I.4)}, \quad \Omega_2 = \begin{pmatrix} -\lambda_1 - \lambda_3 & 0 \\ \lambda_3 & 0 \end{pmatrix} \text{(CaseA.I.2)},$$
$$\Omega_3 = \begin{pmatrix} 0 & \lambda_2 \\ 0 & -\lambda_1 - \lambda_2 \end{pmatrix} \text{(CaseA.I.2)}, \quad \Omega_4 = \begin{pmatrix} \lambda_1 + \lambda_2 & \lambda_2 \\ 0 & 0 \end{pmatrix} \text{(CaseA.II.1)},$$
$$\Omega_5 = \begin{pmatrix} 0 & 0 \\ \lambda_3 & \lambda_1 + \lambda_3 \end{pmatrix} \text{(CaseA.II.1)}, \quad \Omega_6 = \begin{pmatrix} \lambda_3 & -\lambda_2 \\ -\lambda_3 & \lambda_2 \end{pmatrix} \text{(CaseA.II.2)},$$
$$\Omega_7 = \begin{pmatrix} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 & 0 \\ 0 & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \end{pmatrix} \text{(CaseA.III.1)},$$
$$\Omega_8 = \begin{pmatrix} -\lambda_4 & 0 \\ 0 & -\lambda_4 \end{pmatrix} \text{(CaseA.III.2)}.$$

Using the notation in Theorem 10.2.2, we have

$$d' \begin{pmatrix} \int_\sigma \Phi_\lambda \eta_{24} \\ \int_\sigma \Phi_\lambda \eta_{34} \end{pmatrix} = \Omega \wedge \begin{pmatrix} \int_\sigma \Phi_\lambda \eta_{24} \\ \int_\sigma \Phi_\lambda \eta_{34} \end{pmatrix},$$

where Ω is the Gauss-Manin connection matrix given by

$$\Omega = \sum_{s=1}^{8} d' \log \mathsf{D}_s \otimes \Omega_s.$$

For an arbitrary arrangement $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$ and $\mathsf{B} = \mathsf{B}_\mathcal{A}$, it seems to be difficult to find explicit matrix presentations for ∇'_λ. Based upon our result for the codimension one case, it might be natural to ask the following questions:

Question 1. Does each entry of the matrix Ω_s lie in $\sum_{i=1}^{n} \mathbb{Z}\lambda_i$?

Question 2. Is B smooth?