Chapter 9

Moduli Spaces

9.1 Combinatorially Equivalent Arrangements

Fix a pair (ℓ, n) with $\ell \geq 1$ and $n \geq 0$. Recall from Section 3.1 that we compactify $\mathbb{C}^\ell$ by adding the infinite hyperplane $\bar{H}_\infty$ to get complex projective space $\mathbb{CP}^\ell$. In order to understand the moduli space of arrangements, we must consider their degeneration. There are two possibilities when we move a single hyperplane. Either it moves into an already existing intersection, thereby creating more dependencies (or parallelism), or it coincides with an existing hyperplane as was the case in 1-arrangements. In the former case the result is still an arrangement. In the latter case we want to register the coincidence. We do this by using the following notion.

Definition 9.1.1. *A multiset is a set which allows repetitions. A multiset $\mathcal{M}$ is a projective* **multiarrangement** *if $\mathcal{M}$ is a finite multiset of projective hyperplanes of $\mathbb{CP}^\ell$. Let*

$$\mathcal{M}_n(\mathbb{CP}^\ell) = \{\textit{projective multiarrangements of } n+1 \textit{ linearly ordered hyperplanes of } \mathbb{CP}^\ell \textit{ where } \bar{H}_\infty \textit{ is the last hyperplane}\}.$$

Let $(\mathbb{CP}^\ell)^*$ be the dual projective space of $\mathbb{CP}^\ell$. Each point of $(\mathbb{CP}^\ell)^*$ corresponds to a hyperplane of $\mathbb{CP}^\ell$. Thus we indentify $\mathcal{M}_n(\mathbb{CP}^\ell)$ with $((\mathbb{CP}^\ell)^*)^n$:

$$\mathcal{M}_n(\mathbb{CP}^\ell) = ((\mathbb{CP}^\ell)^*)^n$$

so $\mathcal{M}_n(\mathbb{CP}^\ell)$ is a compact complex manifold isomorphic to $(\mathbb{CP}^\ell)^n$.

Let

$$\mathbf{t} = \Big((t_1^{(0)} : \cdots : t_1^{(\ell)}), (t_2^{(0)} : \cdots : t_2^{(\ell)}), \ldots, (t_n^{(0)} : \cdots : t_n^{(\ell)})\Big).$$

be homogeneous coordinates for $((\mathbb{CP}^\ell)^*)^n$. Let $\mathbf{u} = (u_0 : u_1 : \cdots : u_\ell)$ be standard coordinates for $\mathbb{CP}^\ell$. The linear forms $\alpha_i = t_i^{(0)} u_0 + \sum_{j=1}^{\ell} t_i^{(j)} u_j$ $(i = 1, \ldots, n)$

together with the hyperplane at infinity define a multiarrangement with coefficient matrix

$$\mathsf{T} = \begin{pmatrix} t_1^{(0)} & \cdots & t_n^{(0)} & 1 \\ t_1^{(1)} & \cdots & t_n^{(1)} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ t_1^{(\ell)} & \cdots & t_n^{(\ell)} & 0 \end{pmatrix}.$$

Let $\mathcal{M} \in \mathcal{M}_n(\mathbb{CP}^\ell)$. Write $\mathcal{M} = \{\bar{H}_1, \bar{H}_2, \ldots, \bar{H}_{n+1}\}$. We say that $\mathcal{M}$ is **essential** if $\bigcap_{H\in\mathcal{M}} H = \emptyset$. Denote the set $\{1, 2, \ldots, n+1\}$ by $[n+1]$. Define

$$\left(\binom{[n+1]}{\ell+1}\right) = \{ \text{ subsets of } [n+1] \text{ of cardinality } \ell+1 \ \}.$$

Let $\wp$ denote the power set. Let

$$\mathcal{J} : \mathcal{M}_n(\mathbb{CP}^\ell) \longrightarrow \wp\left(\left(\binom{[n+1]}{\ell+1}\right)\right)$$

be the map defined by

$$\mathcal{J}(\mathcal{M}) = \{\{i_1, \ldots, i_{\ell+1}\} \in \left(\binom{[n+1]}{\ell+1}\right) \mid \bar{H}_{i_1} \cap \cdots \cap \bar{H}_{i_{\ell+1}} \neq \emptyset\}.$$

Let $S = (i_1, \ldots, i_{\ell+1})$ and let T_S denote the submatrix of T consisting of the columns labeled by S. Let $\Delta_S = \det(\mathsf{T}_S)$ be the corresponding minor. Then $S \in \mathcal{J}(\mathcal{M})$ if and only if $\Delta_S = 0$.

Example 9.1.2. *Let $\mathcal{M} \in \mathcal{M}_n(\mathbb{CP}^\ell)$. Suppose that $n > \ell$. Then $\mathcal{M}$ is in general position if and only if $\mathcal{J}(\mathcal{M}) = \emptyset$. Similarly, $\mathcal{M}$ is essential if and only if*

$$\mathcal{J}(\mathcal{M}) \neq \left(\binom{[n+1]}{\ell+1}\right)$$

because $\bigcap_{H\in\mathcal{M}} H = \emptyset$ implies that there exist $\ell+1$ hyperplanes $\bar{H}_{i_1}, \ldots, \bar{H}_{i_{\ell+1}} \in \mathcal{M}$ such that $\bar{H}_{i_1} \cap \cdots \cap \bar{H}_{i_{\ell+1}} = \emptyset$.

For $\mathcal{M} \in \mathcal{M}_n(\mathbb{CP}^\ell)$, we define the intersection poset $L(\mathcal{M})$ by

$$L(\mathcal{M}) = \{ \bigcap_{H\in\mathcal{N}} H \mid \mathcal{N} \subseteq \mathcal{M}\}.$$

Here we agree that $\cap_{H\in\emptyset} H = \mathbb{CP}^\ell \in L(\mathcal{M})$. A (multi) subset $\mathcal{N}$ is said to be **linearly independent** if $\operatorname{codim}_{\mathbb{CP}^\ell}(\cap_{H\in\mathcal{N}} H) = |\mathcal{N}|$, where we agree that $\operatorname{codim}_{\mathbb{CP}^\ell} \emptyset = \ell + 1$.

Proposition 9.1.3. *Let $\mathcal{M}_i \in \mathcal{M}_n(\mathbb{CP}^\ell)$ for $i = 1, 2$ and let $\iota : \mathcal{M}_1 \to \mathcal{M}_2$ be an order-preserving bijection. Suppose that both $\mathcal{M}_1$ and $\mathcal{M}_2$ are essential. Then the following three conditions are equivalent:*

(i) ι induces an isomorphism $L(\mathcal{M}_1) \to L(\mathcal{M}_2)$,

(ii) $\mathcal{N}_1 \subseteq \mathcal{M}_1$ is linearly independent if and only if $\iota(\mathcal{N}_1)$ is linearly independent,

(iii) $\mathcal{J}(\mathcal{M}_1) = \mathcal{J}(\mathcal{M}_2)$

Proof. It is obvious that conditions (i) and (ii) are equivalent. Note that $\mathcal{M}_1$ and $\mathcal{M}_2$ contain $\ell + 1$ linearly independent hyperplanes. In general, let $v_1, v_2, \ldots, v_n$ span an $(\ell + 1)$-dimensional vector space W. Then it is an elementary fact that any linearly independent subset of the set $\{v_1, v_2, \ldots, v_n\}$ is contained in a basis for W. This implies that (ii) and (iii) are equivalent. □

Definition 9.1.4. *Call $\mathcal{J}(\mathcal{M})$ the* **combinatorial type** *of $\mathcal{M} \in \mathcal{M}_n(\mathbb{CP}^\ell)$. Two essential arrangements $\mathcal{M}_1$ and $\mathcal{M}_2$ in $\mathcal{M}_n(\mathbb{CP}^\ell)$ are called* **combinatorially equivalent** *if they have the same combinatorial type.*

It follows from Proposition 9.1.3 that two essential arrangements $\mathcal{M}_1$ and $\mathcal{M}_2$ in $\mathcal{M}_n(\mathbb{CP}^\ell)$ are combinatorially equivalent if and only if there is a natural isomorphism $L(\mathcal{M}_1) \to L(\mathcal{M}_2)$. We see in the next example that the map $\mathcal{J}$ is not surjective.

Definition 9.1.5. *Define*

$$\mathsf{B}_{\mathcal{S}} = \mathcal{J}^{-1}(\mathcal{S})$$

for $\mathcal{S} \subseteq \left(\binom{[n+1]}{\ell+1}\right)$. We say that $\mathcal{S} \subseteq \left(\binom{[n+1]}{\ell+1}\right)$ is **realizable** *when $\mathsf{B}_{\mathcal{S}} \neq \emptyset$.*

Example 9.1.6. *Example 9.1.2 shows that the sets $\emptyset$ and $\left(\binom{[n+1]}{\ell+1}\right)$ are realizable. However, when $\ell = 1$, $\mathcal{S} = \{\{1, 2\}, \{2, 3\}\}$ is not realizable. The smallest realizable set containing $\mathcal{S}$ is $\mathcal{S}' = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$. Then $\mathcal{M}$ has combinatorial type $\mathcal{S}'$ if and only if $\mathcal{M}$ is an arrangement of $n + 1$ points in $\mathbb{CP}^1$ such that the first three points coincide and all the other points are distinct.*

By definition we have

$$\mathcal{M}_n(\mathbb{CP}^\ell) = \bigcup_{\mathcal{S}} \mathsf{B}_{\mathcal{S}},$$

where $\mathcal{S}$ runs over the set of all realizable subsets of $\left(\binom{[n+1]}{\ell+1}\right)$ and the union is disjoint. Given a realizable set $\mathcal{S} \subseteq \left(\binom{[n+1]}{\ell+1}\right)$, define

$$\mathsf{C}_{\mathcal{S}} = \bigcup_{\mathcal{S}' \supseteq \mathcal{S}} \mathsf{B}_{\mathcal{S}'}.$$

Then $\mathsf{C}_{\mathcal{S}}$ is defined by the vanishing of the minors specified by $\mathcal{S}$. These are homogeneous polynomial equations of degree $\ell+1$ and thus $\mathsf{C}_{\mathcal{S}}$ is a closed subvariety of $\mathcal{M}_n(\mathbb{CP}^\ell)$. Since we have

$$\mathsf{B}_{\mathcal{S}} = \mathsf{C}_{\mathcal{S}} \setminus \bigcup_{\mathcal{S}' \supset \mathcal{S}} \mathsf{C}_{\mathcal{S}'},$$

$\mathsf{B}_{\mathcal{S}}$ is a locally closed set of $\mathcal{M}_n(\mathbb{CP}^\ell)$. Let $\bar{\mathsf{B}}_{\mathcal{S}}$ be the closure of $\mathsf{B}_{\mathcal{S}}$ in $\mathcal{M}_n(\mathbb{CP}^\ell)$. It is known that $\bar{\mathsf{B}}_{\mathcal{S}}$ can have singularities. It is conjectured that $\mathsf{B}_{\mathcal{S}}$ is a smooth manifold.

Proposition 9.1.7. *Suppose that* $\mathcal{S} \subseteq \left(\binom{[n+1]}{\ell+1}\right)$ *is realizable. Define*

$$\mathsf{D}_{\mathcal{S},T} = \bar{\mathsf{B}}_{\mathcal{S}} \cap \mathsf{C}_{\mathcal{S}\cup\{T\}}$$

for $T \in \mathcal{S}^c = \left(\binom{[n+1]}{\ell+1}\right) \setminus \mathcal{S}$, *and* $\mathsf{D}_{\mathcal{S}} = \bigcup_{T\in\mathcal{S}^c} \mathsf{D}_{\mathcal{S},T}$. *Then*

(i) $\bar{\mathsf{B}}_{\mathcal{S}} \setminus \mathsf{B}_{\mathcal{S}} = \mathsf{D}_{\mathcal{S}}$,

(ii) for any $T \in \mathcal{S}^c$, $\mathsf{D}_{\mathcal{S},T}$ *is a hypersurface in* $\bar{\mathsf{B}}_{\mathcal{S}}$.

Proof. (i) We have

$$\bar{\mathsf{B}}_{\mathcal{S}} \setminus \mathsf{D}_{\mathcal{S}} = \bar{\mathsf{B}}_{\mathcal{S}} \setminus \bigcup_{T\in\mathcal{S}^c} \mathsf{C}_{\mathcal{S}\cup\{T\}} = \bar{\mathsf{B}}_{\mathcal{S}} \setminus \bigcup_{\mathcal{S}'\supset\mathcal{S}} \mathsf{C}_{\mathcal{S}'} \subseteq \mathsf{C}_{\mathcal{S}} \setminus \bigcup_{\mathcal{S}'\supset\mathcal{S}} \mathsf{C}_{\mathcal{S}'} = \mathsf{B}_{\mathcal{S}}.$$

On the other hand, it is clear that $\mathsf{B}_{\mathcal{S}} \subseteq \bar{\mathsf{B}}_{\mathcal{S}} \setminus \mathsf{D}_{\mathcal{S}}$.

(ii) Note that $\mathsf{C}_{\mathcal{S}\cup\{T\}}$ $(T \in \mathcal{S}^c)$ is defined by a single equation in $\mathsf{C}_{\mathcal{S}}$. If $\mathsf{D}_{\mathcal{S},T}$ is not of codimension one in $\bar{\mathsf{B}}_{\mathcal{S}}$, then there exists an irreducible component C_0 of $\bar{\mathsf{B}}_{\mathcal{S}}$ which lies in $\mathsf{D}_{\mathcal{S},T}$. Thus $C_0 \cap \mathsf{B}_{\mathcal{S}} = \emptyset$ by (i). On the other hand, since $\mathsf{B}_{\mathcal{S}}$ is dense in $\bar{\mathsf{B}}_{\mathcal{S}}$, $\mathsf{B}_{\mathcal{S}}$ meets any irreducible component of $\bar{\mathsf{B}}_{\mathcal{S}}$. This is a contradiction, which proves (ii). □

The pure braid space of Chapter 8 is the space $\mathsf{B}_\emptyset$ for $\ell = 1$. In Chapter 8 we considered $\mathsf{B}_\emptyset$ inside $\mathbb{C}^n$ while here we view it inside $(\mathbb{CP}^1)^n$, so we see a compactification of $\mathsf{B}_\emptyset$. The additional components in this compactification correspond to points moving to infinity.

9.2 Realizable Arrangements

So far in this chapter we have considered only multiarrangements in $\mathbb{CP}^\ell$. Next let $\mathcal{A}_n(\mathbb{C}^\ell)$ be the set of affine arrangements of n linearly ordered hyperplanes in $\mathbb{C}^\ell$. When we want to emphasize that repetitions are *not* allowed, we call an arrangement *simple*. Let $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$. Recall the projective closure $\mathcal{A}_\infty$ from Section 3.1:

$$\mathcal{A}_\infty = \{\bar{H} \mid H \in \mathcal{A}\} \cup \{\bar{H}_\infty\},$$

where $\bar{H}$ is the closure of H in $\mathbb{CP}^\ell$. Then $\mathcal{A}_\infty$ is a projective arrangement of $n+1$ hyperplanes of $\mathbb{CP}^\ell$. The hyperplanes of $\mathcal{A}_\infty$ are naturally linearly ordered by regarding the infinite hyperplane $\bar{H}_\infty$ as the $(n+1)$st hyperplane. Thus $\mathcal{A}_\infty \in \mathcal{M}_n(\mathbb{CP}^\ell)$ and there is an injective map

$$\mathcal{A}_n(\mathbb{C}^\ell) \longrightarrow \mathcal{M}_n(\mathbb{CP}^\ell)$$

which sends $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$ to its projective closure $\mathcal{A}_\infty \in \mathcal{M}_n(\mathbb{CP}^\ell)$. Through this injection, we identify $\mathcal{A}_n(\mathbb{C}^\ell)$ with its image in $\mathcal{M}_n(\mathbb{CP}^\ell)$. Then the subset $\mathcal{A}_n(\mathbb{C}^\ell)$ is open and dense in $\mathcal{M}_n(\mathbb{CP}^\ell)$ with respect to the Zariski topology because it is characterized by the open condition that no two hyperplanes are equal.

Proposition 9.2.1. *Let $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$ and recall the cone construction from Section 3.1. The following five conditions are equivalent:*

(i) $\mathcal{A}$ is essential so $r(\mathcal{A}) = \ell$,

(ii) $\mathbf{c}\mathcal{A}$ is essential so $r(\mathbf{c}\mathcal{A}) = \ell + 1$,

(iii) the intersection of all hyperplanes of $\mathbf{c}\mathcal{A}$ contains no point other than the origin,

(iv) $\mathcal{A}_\infty$ is essential so the intersection of all hyperplanes of the projective closure $\mathcal{A}_\infty$ is empty,

(v) $\mathcal{J}(\mathcal{A}_\infty) \neq \left(\binom{[n+1]}{\ell+1}\right)$.

Proof. It is clear that conditions (ii) and (iii) are equivalent. Example 9.1.2 shows that (iv) and (v) are equivalent. The implication (iv) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i): Let H_∞ be the "infinite" hyperplane in $\mathbf{c}\mathcal{A}$. Then there exist hyperplanes $H_{i_1}, \ldots, H_{i_\ell}$ in $\mathcal{A}$ such that the intersection $\mathbf{c}H_{i_1} \cap \cdots \cap \mathbf{c}H_{i_\ell} \cap H_\infty$ consists of the origin only. Thus $\mathbf{c}H_{i_1} \cap \cdots \cap \mathbf{c}H_{i_\ell}$ is a line not entirely inside H_∞. Therefore $H_{i_1} \cap \cdots \cap H_{i_\ell}$ is a point.

(i) $\Rightarrow$ (iv): Let $\bar{H}_\infty$ be the infinite hyperplane (=last hyperplane) in $\mathcal{A}_\infty$. By the assumption there exist hyperplanes $H_{i_1}, \ldots, H_{i_\ell}$ such that the intersection $H_{i_1} \cap \cdots \cap H_{i_\ell}$ is a point. Thus $\bar{H}_{i_1} \cap \cdots \cap \bar{H}_{i_\ell} \cap \bar{H}_\infty$ is empty. □

Proposition 9.2.2. *Let $\mathcal{A}_1$, $\mathcal{A}_2$ be essential simple ℓ-arrangements with an order-preserving bijection $\iota : \mathcal{A}_1 \to \mathcal{A}_2$. Then the following three conditions are equivalent:*

(i) ι induces an isomorphism $L(\mathcal{A}_1) \to L(\mathcal{A}_2)$,

(ii) ι induces an isomorphism $L((\mathcal{A}_1)_\infty) \to L((\mathcal{A}_2)_\infty)$,

(iii) $\mathcal{J}((\mathcal{A}_1)_\infty) = \mathcal{J}((\mathcal{A}_2)_\infty)$

Proof. It follows from Proposition 9.1.3 that conditions (ii) and (iii) are equivalent.

Since $L(\mathcal{A})$ is obtained from $L(\mathcal{A}_\infty)$ by deleting everything above or equal to the infinite hyperplane $\bar{H}_\infty$, (ii) implies (i).

Now it is sufficient to prove that (i) implies (iii). We will show that the poset structure of $L(\mathcal{A})$ completely determines $\mathcal{J}(\mathcal{A}_\infty)$. Let $S \subseteq \mathcal{A}_\infty$ with $|S| = \ell + 1$.

Case 1) Suppose $S = \{\bar{H}_{i_1}, \dots, \bar{H}_{i_\ell}, \bar{H}_\infty\}$ with $1 \le i_1 < \cdots < i_\ell \le n$. Let $\mathcal{B} = \{H_{i_1}, \dots, H_{i_\ell}\}$. Then $S = \mathcal{B}_\infty \in \mathcal{J}(\mathcal{A}_\infty)$ if and only if $\mathcal{B}$ is not essential by Proposition 9.2.1.

Case 2) Suppose $S = \{\bar{H}_{i_1}, \dots, \bar{H}_{i_{\ell+1}}\}$ with $1 \le i_1 < \cdots < i_{\ell+1} \le n$. Let $\mathcal{B} = \{H_{i_1}, \dots, H_{i_{\ell+1}}\}$. Then

$$\begin{aligned} S \in \mathcal{J}(\mathcal{A}_\infty) &\iff \bar{H}_{i_1} \cap \cdots \cap \bar{H}_{i_{\ell+1}} \neq \emptyset \\ &\iff \text{either } H_{i_1} \cap \cdots \cap H_{i_{\ell+1}} \neq \emptyset \text{ or } \bar{H}_{i_1} \cap \cdots \cap \bar{H}_{i_{\ell+1}} \cap \bar{H}_\infty \neq \emptyset \\ &\iff \text{either } \mathcal{B} \text{ is central or } \mathcal{B}_\infty \text{ is nonessential} \\ &\iff \mathcal{B} \text{ is either central or nonessential .} \end{aligned}$$

The last equivalence follows from Proposition 9.2.1. These arguments imply that the condition for S to be in $\mathcal{J}(\mathcal{A}_\infty)$ can be stated in terms of $L(\mathcal{A})$. □

Definition 9.2.3. *For $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$, we say that $\mathcal{A}$ has* **combinatorial type** *$\mathcal{J}(\mathcal{A}_\infty)$. Two essential affine arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ in $\mathcal{A}_n(\mathbb{C}^\ell)$ are* **combinatorially equivalent** *if they have the same combinatorial type.*

It follows from Proposition 9.2.2 that two essential simple affine arrangements $\mathcal{A}_1$, $\mathcal{A}_2 \in \mathcal{A}_n(\mathbb{C}^\ell)$ are combinatorially equivalent if and only if there is an isomorphism $L(\mathcal{A}_1) \to L(\mathcal{A}_2)$.

Definition 9.2.4. *We say that $\mathcal{S} \subseteq \left(\binom{[n+1]}{\ell+1}\right)$ is* **affine realizable** *if there exists a simple affine arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$ with $\mathcal{J}(\mathcal{A}_\infty) = \mathcal{S}$ hence*

$$\mathcal{A}_n(\mathbb{C}^\ell) \cap \mathsf{B}_{\mathcal{S}} \neq \emptyset.$$

It is clear that $\mathsf{B}_{\mathcal{S}} \subseteq \mathcal{A}_n(\mathbb{C}^\ell)$ if $\mathcal{S} \neq \left(\binom{[n+1]}{\ell+1}\right)$ is affine realizable. In other words, if $\mathcal{M} \in \mathcal{M}_n(\mathbb{CP}^\ell)$ is combinatorially equivalent to (the projective closure of) an essential simple affine arrangement, then $\mathcal{M}$ is (the projective closure of) an essential simple affine arrangement.

9.3 Codimension ≤ 1

Let $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$ be essential in the rest of this section. In particular, $\ell \le n$. We write $\mathsf{B}_{\mathcal{A}} = \mathsf{B}_{\mathcal{J}(\mathcal{A}_\infty)}$. By Propostion 9.2.2, we can regard $\mathsf{B}_{\mathcal{A}}$ as a moduli space of the affine arrangements which are combinatorially equivalent to $\mathcal{A}$. When the codimension of $\mathsf{B}_{\mathcal{A}}$ in $((\mathbb{CP}^\ell)^*)^n$ is less than two, we can describe explicitly the geometry of $\mathsf{B}_{\mathcal{A}}$ and $\mathsf{D}_{\mathcal{A}}$.

Codimension Zero

The moduli space $\mathsf{B}_{\mathcal{A}}$ has codimension zero in $((\mathbb{CP}^\ell)^*)^n$ if and only if $|\mathcal{J}(\mathcal{A}_\infty)| = 0$. Recall that an affine arrangement $\mathcal{A}$ is in general position if $\mathcal{J}(\mathcal{A}_\infty) = \emptyset$. Thus $\mathsf{B}_{\mathcal{A}}$ is a moduli space of general position arrangements. In this case $\mathsf{B}_{\mathcal{A}}$ is a dense open subset of $((\mathbb{CP}^\ell)^*)^n$ and

$$\mathsf{D}_{\mathcal{A}} = \bigcup_T \mathsf{C}_{\{T\}},$$

where T runs over $\left(\binom{[n+1]}{\ell+1}\right)$. Since $\mathsf{C}_{\{T\}}$ is defined by the single equation $\Delta_T = 0$ and the determinant function is an irreducible polynomial (a special case of Theorem 9.3.1), each $\mathsf{C}_{\{T\}}$ is an irreducible hypersurface. Therefore $\mathsf{D}_{\mathcal{A}}$ is composed of $\binom{n+1}{\ell+1}$ irreducible components.

When $\ell = 1$, $\mathsf{B}_{\mathcal{A}} = \{(t_1, \ldots, t_n) \in \mathbb{C}^n \mid t_i \neq t_j \ (i \neq j)\}$ is the pure braid space.

Codimension One

We need the following fundamental result on determinantal ideals.

Theorem 9.3.1 (Hochster-Eagon[HE]). *Let $X = (X_{ij})$ be a matrix of indeterminates over an integral domain R of size $m \times n$. Let $I_t(X)$ be the ideal in the polynomial ring $R[X_{ij}]$ generated by the t-minors of X. Then $I_t(X)$ is a prime ideal of height $(m-t+1)(n-t+1)$.* □

Recall the $(\ell+1)\times(n+1)$-matrix T. Let $\mathbb{C}[\mathsf{T}]$ be the polynomial ring over $\mathbb{C}$ with indeterminates $\{t_j^{(i)}\}_{0\leq i\leq \ell, 1\leq j\leq n}$. For $S \subseteq [n+1]$, define T_S to be the submatrix of T consisting of the columns corresponding to S. Recall that for $|S| = \ell+1$, we defined $\Delta_S = \det(\mathsf{T}_S)$.

Lemma 9.3.2. *Let $S, S' \in \left(\binom{[n+1]}{\ell+1}\right)$ and $I = (\Delta_S, \Delta_{S'})\mathbb{C}[\mathsf{T}]$. Define I_t as in Theorem 9.3.1.*

(i) (cf. Andrade-Simis [AS, Corollary 1.2]) If $|S \cap S'| = \ell$, then

$$I = I_\ell(\mathsf{T}_{S\cap S'}) \cap I_{\ell+1}(\mathsf{T}_{S\cup S'}).$$

(ii) If $|S \cap S'| \leq \ell - 1$, then I is a prime ideal of height two.

Proof. (i) Let $A = \mathsf{T}_{S\cap S'}, B = \mathsf{T}_{S\cup S'}$. Write $B = (b_{ij})_{0\leq i\leq \ell, 0\leq j\leq \ell+1}$. Define

$$\Delta_j = (-1)^j \det(B_j) \quad (j = 0, \ldots, \ell+1),$$

where B_j is obtained from B by deleting the jth column of B. We may assume that $\Delta_S = \Delta_0$ and $\Delta_{S'} = \Delta_{\ell+1}$. Let $P_1 = I_\ell(A)$ and $P_2 = I_{\ell+1}(B) = (\Delta_0, \ldots, \Delta_{\ell+1})$. We will show that $I = P_1 \cap P_2$. If $\ell = 1$ and $S \cap S' = \{n+1\}$, then $P_1 = \mathbb{C}[\mathsf{T}]$. In this case $I = P_2$ and (i) holds true. In the other cases, both P_1 and P_2 are prime ideals of height two by Theorem 9.3.1. By elementary linear algebra, we have $\sum_{j=0}^{\ell+1} b_{ij}\Delta_j = 0 \quad (i = 0,\dots,\ell)$. Thus $\sum_{j=1}^{\ell} b_{ij}\Delta_j \in I \quad (i = 0,\dots,\ell)$. By applying Cramer's rule, we get $P_1P_2 \subseteq I$. Since I is generated by two irreducible polynomials, every associated prime of I is of height two or less. If P is an associated prime of I, then $P_1P_2 \subseteq I \subseteq P$. Thus either $P_1 \subseteq P$ or $P_2 \subseteq P$. So we have $P \in \{P_1, P_2\}$. Write a primary decomposition of I as $I = Q_1 \cap Q_2$ with $\sqrt{Q_i} = P_i \quad (i = 1, 2)$. Note that there is no inclusion relation between P_1 and P_2. Since $P_1P_2 \subseteq Q_i$, we have $P_i = Q_i \quad (i = 1, 2)$.

(ii) We thank K. Kurano for the following argument. Case 1): Suppose $n+1 \notin S\cap S'$. Choose $S'' \in \left(\binom{[n+1]}{\ell+1}\right)$ such that $S\cap S' \subset S'' \subset S\cup S'$ and $|S\cap S''| = \ell$. Let $\Delta = \Delta_S$, $\Delta' = \Delta_{S'}$, and $\Delta'' = \Delta_{S''}$. By abuse of notation, let a matrix also denote the set of its entries. So the ring $R = \mathbb{C}\left[\mathsf{T}_{S''}, (\Delta'')^{-1}\right]$ stands for the subring of $\mathbb{C}(\mathsf{T}_{S''})$ generated by $(\Delta'')^{-1}$ and the entries of $\mathsf{T}_{S''}$ over $\mathbb{C}$. Let $\mathsf{Z} = (\mathsf{T}_{S''})^{-1}$. Then each entry of Z lies in R. Let $S''' = (S \cup S') \setminus S''$. Since the entries of $\mathsf{T}_{S'''}$ are algebraically independent over $\mathbb{C}(\mathsf{T}_{S''})$, so are the entries of $\mathsf{Z}\mathsf{T}_{S'''}$. Note that there exists an entry of $\mathsf{Z}\mathsf{T}_{S'''}$ which is equal either to $\det(\mathsf{Z}\mathsf{T}_S)$ or to $-\det(\mathsf{Z}\mathsf{T}_S)$ and that there exists a minor of $\mathsf{Z}\mathsf{T}_{S'''}$ which is equal either to $\det(\mathsf{Z}\mathsf{T}_{S'})$ or to $-\det(\mathsf{Z}\mathsf{T}_{S'})$. Thus the ideal

$$(\Delta, \Delta')R\left[\mathsf{T}_{S'''}\right] = (\det(\mathsf{T}_S), \det(\mathsf{T}_{S'}))R\left[\mathsf{T}_{S'''}\right] = (\det(\mathsf{Z}\mathsf{T}_S), \det(\mathsf{Z}\mathsf{T}_{S'}))R\left[\mathsf{Z}\mathsf{T}_{S'''}\right]$$

is a prime ideal of

$$R\left[\mathsf{Z}\mathsf{T}_{S'''}\right] = R\left[\mathsf{T}_{S'''}\right] = \mathbb{C}\left[\mathsf{T}_{S\cup S'}, (\Delta'')^{-1}\right]$$

by Theorem 9.3.1. On the other hand, the associated primes of $(\Delta, \Delta')R\left[\mathsf{T}_{S\cup S'}\right]$ are $I_\ell(\mathsf{T}_{S\cap S''})$ and $I_{\ell+1}(\mathsf{T}_{S\cup S''})$. Since $(S\cap S'')\setminus S' \neq \emptyset$ and $|(S\cup S'')\setminus S'| \geq 2$, we have $\Delta' \notin I_\ell(\mathsf{T}_{S\cap S''})$ and $\Delta' \notin I_{\ell+1}(\mathsf{T}_{S\cup S''})$. Therefore $(\Delta, \Delta'') : (\Delta') = (\Delta, \Delta'')$. This implies $(\Delta, \Delta') : (\Delta'') = (\Delta, \Delta')$. Thus Δ'' is a non-zero divisor of $\mathbb{C}\left[T_{S\cup S'}\right]/(\Delta, \Delta')$. Since the factor ring $\mathbb{C}\left[T_{S\cup S'}, (\Delta'')^{-1}\right]/(\Delta, \Delta')$ is a domain, so is the factor ring $\mathbb{C}\left[T_{S\cup S'}\right]/(\Delta, \Delta')$. This shows (ii).

Case 2): Suppose $n+1 \in S\cap S'$. Then this case reduces to Case 1).

Case 3): Suppose $n+1 \in S\setminus S'$. Choose $S'' \in \left(\binom{[n+1]}{\ell+1}\right)$ such that $S\cap S' \subset S'' \subset S\cup S'$, $|S\cap S''| = \ell$, and $n+1 \in S''$. The rest of the proof is exactly the same as Case 1). □

Proposition 9.3.3. *The moduli space $\mathsf{B}_{\mathcal{A}}$ has codimension one in $((\mathbb{CP}^\ell)^*)^n$ if and only if $|\mathcal{J}(\mathcal{A}_\infty)| = 1$. Suppose $\mathcal{J}(\mathcal{A}_\infty) = \{S\}$. Write $\mathsf{B} = \mathsf{B}_{\mathcal{A}}$, $\mathsf{C} = \mathsf{C}_{\{S\}}$ and $\mathsf{D} = \bar{\mathsf{B}} \setminus \mathsf{B}$. Then*

(i) $\bar{\mathsf{B}} = \mathsf{C}$ is irreducible,

(ii) B is smooth,

(iii) the irreducible components of D are:

type I: $\mathsf{C}_{\{S,S'\}}$ *for* $S' \in \left(\binom{[n+1]}{\ell+1}\right)$ *with* $|S \cap S'| \leq \ell - 1$,

type II: $\mathsf{C}_{\langle S-p\rangle}$ *for* $p \in S$, *where* $\langle S - p\rangle = \{S' \in \left(\binom{[n+1]}{\ell+1}\right) \mid S' \supseteq S \backslash \{p\}\}$,

type III: $\mathsf{C}_{\langle S+q\rangle}$ *for* $q \in [n+1] \setminus S$, *where* $\langle S + q\rangle = \{S' \in \left(\binom{[n+1]}{\ell+1}\right) \mid S' \subseteq S \cup \{q\}\}$.

In all, there exist $\binom{n+1}{\ell+1} - \ell(n-\ell-1)$ *irreducible components of* D. *When* $\ell = 1$, *type II does not appear and the number of irreducible components of* D *is equal to* $n(n-1)/2$.

Proof. Since $\mathcal{A}$ is essential and not in general position, $\ell + 1 \leq n$.

(i) By Theorem 9.3.1, Δ_S is an irreducible polynomial. Thus C is irreducible and $\bar{\mathsf{B}} = \mathsf{C}$.

(ii) Let $n + 1 \notin S$. Let J be the ideal generated by the partial derivatives of Δ_S. Because of the Laplace expansion formula for $\det(\mathsf{T}_S)$, J is generated by the ℓ-minors of T_S. Thus any singular point $\mathbf{t}$ of B lies in $\mathsf{C}_{\{S'\}}$ for any $S' \in \left(\binom{[n+1]}{\ell+1}\right)$ with $|S \cap S'| = \ell$. Thus $\mathbf{t} \notin \mathsf{B}$. We can similarly prove the assertion when $n+1 \in S$.

(iii) Let $S' \in \left(\binom{[n+1]}{\ell+1}\right) \setminus \{S\}$. Note $\mathsf{D}_{S'} = \mathsf{C}_{\{S,S'\}}$. If $|S \cap S'| \leq \ell - 1$, then $(\Delta_S, \Delta_{S'})$ is a prime ideal by Lemma 9.3.2 (i). Thus $\mathsf{D}_{S'} = \mathsf{C}_{\{S,S'\}}$ is irreducible. If $|S \cap S'| = \ell$, then $(\Delta_S, \Delta_{S'}) = I_\ell(\mathsf{T}_{S\cap S'}) \cap I_{\ell+1}(\mathsf{T}_{S\cup S'})$ by Lemma 9.3.2 (ii). If $\ell \geq 2$, this is a primary decomposition of $(\Delta_S, \Delta_{S'})$. Let $\{p\} = S \setminus S'$ and $\{q\} = S' \setminus S$. Then

$$\mathsf{D}_{S'} = \mathsf{C}_{\{S,S'\}} = \mathsf{C}_{\langle S-p\rangle} \cup \mathsf{C}_{\langle S+q\rangle}$$

is the decomposition of $\mathsf{D}_{S'}$ into irreducible components. The cardinality of the set $\{S' \in \left(\binom{[n+1]}{\ell+1}\right) \mid |S \cap S'| \leq \ell - 1\}$ is equal to $\binom{n+1}{\ell+1} - 1 - (n-\ell)(\ell+1)$. Thus the total number of irreducible components of $\mathsf{D} = \bigcup_{S' \in \mathcal{J}(\mathcal{A}_\infty)^c} \mathsf{D}_{S'}$ is equal to

$$\binom{n+1}{\ell+1} - 1 - (n-\ell)(\ell+1) + (\ell+1) + (n-\ell) = \binom{n+1}{\ell+1} - \ell(n-\ell-1).$$

If $\ell = 1 = |S \cap S'|$, then the ideal $I_\ell(\mathsf{T}_{S\cap S'})$ does not define a subvariety of $((\mathbb{CP}^\ell)^*)^n$. Thus $\mathsf{D}_{S'} = \mathsf{C}_{\langle S+q\rangle}$ where $\{q\} = S' \setminus S$. Therefore the total number of irreducible components of $\mathsf{D} = \bigcup_{S' \in \mathcal{J}(\mathcal{A}_\infty)^c} \mathsf{D}_{S'}$ is equal to

$$\binom{n+1}{2} - 1 - 2(n-1) + (n-1) = n(n-1)/2.$$

□