Chapter 7

The Determinant of a Period Matrix

In this chapter we assume that $\mathcal{A}$ is an essential complexified real arrangement and follow [DT]. Using the β**nbc** bases and the hypergeometric pairing of Definition 2.3.3, we obtain a period matrix whose rows and columns are labeled by β**nbc**. The entries are hypergeometric integrals. In general these individual entries cannot be calculated in closed form. The main result is a formula for the determinant of this period matrix. The formula was conjectured by Varchenko in [V1] who proved it for arrangements of general position as well as arrangements in $\mathbb{R}^2$. We assume throughout this chapter that the weights are in

$$\mathbf{U}_{\mathbb{Z}} = \{\lambda \in \mathbb{C}^n \mid \lambda_X \notin \mathbb{Z},\ X \in \mathsf{D}(\mathcal{A}_\infty)\}$$

so the weights are nonresonant.

7.1 The Period Matrix

The next result is due to Kohno [Ko1].

Theorem 7.1.1. *If $\lambda \in \mathbf{U}_{\mathbb{Z}}$, then*

$$i_c : H^p_c(M, \mathcal{L}) \to H^p(M, \mathcal{L}) \quad i_h : H_p(M, \mathcal{L}^\vee) \to H^{lf}_p(M, \mathcal{L}^\vee)$$

are isomorphisms for all p.

Proof. Let $j : M \to \bar{X}$ denote the inclusion where $\bar{X}$ is the resolution constructed in Theorem 4.2.3. Since 1 is not an eigenvalue of the monodromy along any irreducible component of Y, $j_*\mathcal{L} = j_!\mathcal{L}$, where $j_!$ is extension by zero. This provides isomorphisms:

$$H^p(M, \mathcal{L}) \simeq H^p(\bar{X}, j_*\mathcal{L}) \simeq H^p(\bar{X}, j_!\mathcal{L}) \simeq H^p_c(M, \mathcal{L}).$$

□

Consider the β**nbc** bases Ψ, bch. Use Theorem 7.1.1 and write $\gamma_j = i_h^{-1}(\Delta_j)$ to get the associated linearly ordered basis for $H_\ell(M, \mathcal{L}^\vee)$, $G(\mathcal{A}) = \{\gamma_j\}_{j=1}^{\beta}$. An explicit description of $G(\mathcal{A})$, which is called Hadamard's finite part, requires considerable effort, see [AK], [Kt2], and Example 2.2.2. Choose a branch of $\alpha_p^{\lambda_p}$ on each chamber Δ_j. Note that this specifies a branch on γ_j and that the orientation of Δ_j orients γ_j. Given $\psi_i \in \Psi$, the pairing in Definition 2.3.3 provides a matrix of hypergeometric integrals

$$\mathsf{PM}^*(\mathcal{A}, \lambda)_{i,j} = \int_{\gamma_j} \Phi_\lambda \psi_i. \tag{1}$$

Since the integrand is a form with noncompact support, integrating over the noncompact domain Δ_j results in an improper integral whose convergence must be argued.

It follows from [LS, 4.2] that the integral (1) may be regarded as a meromorphic function on $\mathbb{C}^n$ whose poles lie on some hypersurfaces defined by equations $e^{2i\pi L(\lambda)} - \mu = 0$, where L is a linear form of λ_j and μ is a nonzero complex number. Since the hypergeometric pairing is nondegenerate, we may write $L(\lambda) = \lambda_X$ for some dense edge X and $\mu = 1$. If $\Re\lambda_p > 0$, for all p, then

$$\int_{\Delta_j} \Phi_\lambda \psi_i = \int_{\gamma_j} \Phi_\lambda \psi_i$$

meaning that the improper integral on the left exists and has the value of the integral on the right. Note that the integral representation of the classical hypergeomertric function of Gauss, formula (4) in the Introduction, requires a similar consideration.

Definition 7.1.2. *Define the* **hypergeometric period matrix** $\mathsf{PM}(\mathcal{A}, \lambda)$ *by*

$$\mathsf{PM}(\mathcal{A}, \lambda)_{i,j} = \int_{\Delta_j} \Phi_\lambda \psi_i.$$

It follows that $\det \mathsf{PM}(\mathcal{A}, \lambda)$ takes a finite nonzero value at each nonresonant λ. We formally define $\det \mathsf{PM}(\mathcal{A}, \lambda) = 1$ if $\beta(\mathcal{A}) = 0$.

7.2 The Main Theorem

Definition 7.2.1. *For $X \in L(\mathcal{A}_\infty)$, define*

$$\rho(X) = |e(M(\mathbb{P}(\mathcal{A}_\infty)_X))\, e(M((\mathcal{A}_\infty)^X))|.$$

Here $e(M)$ is euler characteristic and $\mathbb{P}(\mathcal{A}_\infty)_X$ is the projective quotient of the central arrangement $(\mathcal{A}_\infty)_X$.

Example 7.2.2. *In the projective closure of the Selberg arrangement 3.1.1, the six lines and the four triple points are dense. If X is a line, then $M(\mathbb{P}(\mathcal{A}_\infty)_X)$ is the projective line minus a point, while $M((\mathcal{A}_\infty)^X)$ is the projective line minus three points, so $\rho(X) = 1$. If X is a triple point, then $M(\mathbb{P}(\mathcal{A}_\infty)_X)$ is the projective line minus three points, while $(\mathcal{A}_\infty)^X = \emptyset$ and $M((\mathcal{A}_\infty)^X)$ is a point, so $\rho(X) = 1$.*

Lemma 7.2.3. *Let $\mathcal{A}$ be an essential arrangement with projective closure $\mathcal{A}_\infty$. Then*

$$\begin{aligned} |e(M(\mathbb{P}(\mathcal{A}_\infty)_X))| &= \beta(\mathbf{d}(\mathcal{A}_\infty)_X), \\ |e(M((\mathcal{A}_\infty)^X))| &= \begin{cases} 1 & \text{if } (\mathcal{A}_\infty)^X = \emptyset, \\ \beta((\mathcal{A}_\infty)_0^X) & \text{if } (\mathcal{A}_\infty)^X \neq \emptyset, \end{cases} \end{aligned}$$

where $(\mathcal{A}_\infty)_0^X$ denotes $(\mathcal{A}_\infty)^X$ with an arbitrary hyperplane removed.

It follows from Corollary 3.3.5 that $\rho(X) = 0$ if $X \notin \mathsf{D}(\mathcal{A}_\infty)$. There is a disjoint union $L(\mathcal{A}_\infty) = L_+(\mathcal{A}_\infty) \cup L_-(\mathcal{A}_\infty)$ where $L_+(\mathcal{A}_\infty) = L(\mathcal{A})$ consists of edges not in H_∞ and $L_-(\mathcal{A}_\infty) = L(\mathcal{A}_\infty)^{H_\infty}$ consists of edges in H_∞. Recall that the weight of H_∞ is $\lambda_\infty = -\sum_{H\in\mathcal{A}} \lambda_H$ and for $X \in L(\mathcal{A}_\infty)$, we define $\lambda_X = \sum_{H\in(\mathcal{A}_\infty)_X} \lambda_H$.

Definition 7.2.4 (Varchenko [V1]). *The* **beta function** *of $\mathcal{A}$ is the following product of gamma functions*

$$B(\mathcal{A}, \lambda) = \prod_{X\in L_+(\mathcal{A}_\infty)} \Gamma(\lambda_X + 1)^{\rho(X)} \prod_{X\in L_-(\mathcal{A}_\infty)} \Gamma(-\lambda_X + 1)^{-\rho(X)}.$$

Example 7.2.5. *If $\mathcal{A}$ consists of two points in the line with weights λ_1, λ_2, this function is Euler's original Beta function*

$$B(\mathcal{A}, \lambda) = \frac{\Gamma(\lambda_1 + 1)\Gamma(\lambda_2 + 1)}{\Gamma(\lambda_1 + \lambda_2 + 1)}.$$

For the Selberg arrangement we get

$$B(\mathcal{A}, \lambda) = \frac{(\prod_{i=1}^5 \Gamma(\lambda_i + 1))\Gamma(\lambda_1 + \lambda_3 + \lambda_5 + 1)\Gamma(\lambda_2 + \lambda_4 + \lambda_5 + 1)}{\Gamma(\sum_{i=1}^5 \lambda_i + 1)\Gamma(\lambda_1 + \lambda_2 + \lambda_5 + 1)\Gamma(\lambda_3 + \lambda_4 + \lambda_5 + 1)}.$$

The next theorem is one of the main results of [LS].

Theorem 7.2.6. *We have*

$$\det \mathsf{PM}^*(\mathcal{A}, \lambda) = c_1^{\lambda_1} \dots c_n^{\lambda_n} B(\mathcal{A}, \lambda) h(\lambda),$$

where $c_1, \cdots, c_n$ are nonzero constants and $h \in \mathbb{C}(\lambda_1, \cdots, \lambda_n)^*$.

Proof. By [LS, 4.2.10], we have

$$\det \mathsf{PM}^*(\mathcal{A},\lambda) = \varphi(e^{2i\pi\lambda_1},\cdots,e^{2i\pi\lambda_n})c_1^{\lambda_1}\dots c_n^{\lambda_n}B(\mathcal{A},\lambda)\hat{h}(\lambda),$$

where φ is a periodic function of $\lambda = (\lambda_1,\cdots,\lambda_n)$, $c_1,\cdots,c_n$ are nonzero constants and $\hat{h} \in \mathbb{C}(\lambda_1,\cdots,\lambda_n)^*$. Since the polynomials α_p take real values on each Δ_j and $\det \mathsf{PM}^*(\mathcal{A},\lambda)$ is holomorphic if $\Re\lambda_p > 0$ for all p, $\varphi(e^{2i\pi\lambda_1},\cdots,e^{2i\pi\lambda_n})$ is constant by the final remark of [LS]. Write $h(\lambda) = \varphi\hat{h}(\lambda)$. □

Remark 7.2.7. *Theorem7.2.6 can be also obtained as a consequence of a theorem by M. Sato [SSM, Theorem in Appendix].*

Definition 7.2.8 (Varchenko [V1]). *Fix a branch of $\alpha_p^{\lambda_p}$ on each Δ_j. Choose $x_{p,j} \in \bar{\Delta}_j$ so that $|\alpha_p^{\lambda_p}(x_{p,j})| \geq |\alpha_p^{\lambda_p}(y)|$ for all $y \in \bar{\Delta}_j$. Define the complex number*

$$R(\mathcal{A},\lambda) = \prod_{p=1}^{n}\prod_{j=1}^{\beta}\alpha_p^{\lambda_p}(x_{p,j}).$$

Write $I = \{1,\dots,n\}$. The rest of this chapter is devoted to a proof of the following theorem conjectured by Varchenko [V1].

Theorem 7.2.9. *Suppose $\Re\lambda_p > 0$ for all $p \in I$. Then we have*

$$\det \mathsf{PM}(\mathcal{A},\lambda) = R(\mathcal{A},\lambda)B(\mathcal{A},\lambda).$$

In view of Theorem 7.2.6, we must prove that $c_p^{\lambda_p} = \prod_{j=1}^{\beta}\alpha_p^{\lambda_p}(x_{p,j})$ and $h(\lambda) = 1$.

7.3 Deletion-Restriction Formulas

Let $P = \{j \mid H_j \in \mathcal{A}',\, H_j \cap H_n = \emptyset\}$ be the set of indices of hyperplanes parallel to H_n. If $j \in P$, then $\alpha_j^{\lambda_j}|_{H_n}$ is a nonzero constant. Given the weight λ_i of $H_i \in \mathcal{A}$, the weight of $H_i \in \mathcal{A}'$ is defined $\lambda_i' = \lambda_i$. If $B \in A''$, let $I_B = \{i \mid H_i \in \mathcal{A}',\, B \subset H_i\}$ and define $\lambda_B'' = \sum_{i \in I_B}\lambda_i$. The proof of the next result is analogous to the proof of [Lo, 6.3].

Theorem 7.3.1.

$$\begin{aligned} B(\mathcal{A},\lambda) &= B(\mathcal{A}',\lambda')B(\mathcal{A}'',\lambda'') \\ &\times \prod_{X \in L_+(\mathcal{A}_\infty),\, X \subseteq H_n} [\Gamma(\lambda_X+1)/\Gamma(\lambda_X-\lambda_n+1)]^{\rho(X)} \\ &\times \prod_{X \in L_-(\mathcal{A}_\infty),\, X \nsubseteq H_n} [\Gamma(-\lambda_X-\lambda_n+1)/\Gamma(-\lambda_X+1)]^{\rho(X)} \end{aligned}$$

$$R(\mathcal{A},\lambda) = R(\mathcal{A}',\lambda')R(\mathcal{A}'',\lambda'')\prod_{j=1}^{\beta(\mathcal{A})}\alpha_n^{\lambda_n}(x_{n,j})\,[\prod_{j \in P}\alpha_j^{\lambda_j}|_{H_n}]^{\beta(\mathcal{A}'')}.$$

Corollary 7.3.2. *We have*

$$B(\mathcal{A},\lambda)|_{\lambda_n=0} = B(\mathcal{A}',\lambda')B(\mathcal{A}'',\lambda'')$$

and

$$R(\mathcal{A},\lambda)|_{\lambda_n=0} = R(\mathcal{A}',\lambda')R(\mathcal{A}'',\lambda'')\,[\prod_{j\in P}\alpha_j^{\lambda_j}|_{H_n}]^{\beta(\mathcal{A}'')}.$$

In order to find the corresponding formula for the determinant, we have to specify the branches of Φ_λ. Let $\hat{\alpha}_p$ be the restriction of α_p to H_n and define

$$\Phi'_\lambda = \prod_{p\in I'}\alpha_p^{\lambda'_p}, \quad \Phi''_\lambda = \prod_{r\in I''}\hat{\alpha}_r^{\lambda''_r}.$$

Choose a branch of Φ'_λ on each bounded chamber of $\mathsf{bch}(\mathcal{A}')$ and a branch of Φ''_λ on each bounded chamber of $\mathsf{bch}(\mathcal{A}'')$. Also choose a branch of $\alpha_n^{\lambda_n}$ on each bounded chamber $\Delta\in\mathsf{bch}(\mathcal{A})$. Let c_n denote one of the complex numbers $\prod_{j\in P}\alpha_j^{\lambda_j}|_{H_n}$. We use the terminology from the proof of Proposition 6.4.3 to define a branch Φ_Δ of Φ_λ on Δ as follows:

(1) If Δ is undivided, then $\Delta\in\mathsf{bch}(\mathcal{A}')$. Define $\Phi_\Delta = (\alpha_n^{\lambda_n}|_\Delta)(\Phi'_\Delta)$.

(2) If Δ is the heir of $\Delta'\in\mathsf{bch}(\mathcal{A}')$, then define $\Phi_\Delta = (\alpha_n^{\lambda_n}|_\Delta)(\Phi'_{\Delta'})|_\Delta$.

(3) If Δ is either a cutoff or a newborn, then it has a unique wall $\bar{\Delta}\cap H_n \supset \Delta''\in\mathsf{bch}(\mathcal{A}'')$. Choose the branch Φ'_Δ of Φ'_λ on Δ such that $\Phi'_\Delta|_{\Delta''} = c_n\Phi''_{\Delta''}$. Let $\Phi_\Delta = (\alpha_n^{\lambda_n}|_\Delta)\Phi'_\Delta$.

Recall that we are using the β**nbc**-orientation for every chamber of $\mathsf{bch}(\mathcal{A}')$, $\mathsf{bch}(\mathcal{A}'')$ and $\mathsf{bch}(\mathcal{A})$. If $\Delta\in\mathsf{bch}(\mathcal{A})$ is undivided or the heir of $\Delta'\in\mathsf{bch}(\mathcal{A}')$, then the corresponding β**nbc**-flags are equal. The orientation of Δ is induced from the orientation of Δ'. If $\Delta\in\mathsf{bch}(\mathcal{A})$ is either a cutoff or newborn, then the orthonormal frame for Δ is given by the orthonormal frame for Δ'' together with the unit vector in the direction of Δ as the last vector of the frame.

Define $\mathsf{PM}(\mathcal{A},\lambda)$, $\mathsf{PM}(\mathcal{A}',\lambda')$, and $\mathsf{PM}(\mathcal{A}'',\lambda'')$ using these branches and orientations. We analytically continue the determinant $\det\mathsf{PM}(\mathcal{A},\lambda)$ onto the hyperplane H_n. Write $I' = I\setminus\{n\}$.

Proposition 7.3.3. *Suppose $\Re\lambda_p > 0$ for all $p\in I'$. Then*

$$\det\mathsf{PM}(\mathcal{A},\lambda)|_{\lambda_n=0} = \det\mathsf{PM}(\mathcal{A}',\lambda')\det\mathsf{PM}(\mathcal{A}'',\lambda'')\,c_n^{\beta(\mathcal{A}'')}.$$

Proof. Let $\Delta'\in\mathsf{bch}(\mathcal{A}')$ be a divided chamber. Let Δ^+ be its heir and let Δ^- be its cutoff. Let Φ^+ and Φ^- be the branches of Φ_λ on Δ^+ and Δ^-. Define the constant $c_{\Delta'}$ by $\Phi^+ = c_{\Delta'}\Phi^-$ on $\Delta'\cap H_n$. Add the column corresponding to the cutoff Δ^- multiplied by $c_{\Delta'}$ to the column corresponding to the heir Δ^+ and set $\lambda_n = 0$. The resulting column has entries $\{(\int_{\Delta'}\Phi'_\lambda\alpha_n^{\lambda_n}\psi_j)|_{\lambda_n=0}\}_{j=1}^{\beta}$. Let M be the matrix obtained from $\mathsf{PM}(\mathcal{A},\lambda)$ by performing this operation for every divided chamber. Then $\det\mathsf{M} = \det\mathsf{PM}(\mathcal{A},\lambda)|_{\lambda_n=0}$. Write

$$\mathsf{M} = \begin{pmatrix} P & Q \\ R & S\end{pmatrix}$$

where P is a square matrix of size $\beta(\mathcal{A}')$ and S is a square matrix of size $\beta(\mathcal{A}'')$. Since the first $\beta(\mathcal{A}')$ columns of M are labeled by $\mathsf{bch}(\mathcal{A}')$, it follows from Lemma 6.3.4 that $P = \mathsf{PM}(\mathcal{A}', \lambda')$.

When computing R and S we may take $\psi = \zeta(B)$ where $B = \{\nu B'', H_n\} \in \overline{\beta\mathbf{nbc}}(\mathcal{A}'')$ with $B'' \in \beta\mathbf{nbc}(\mathcal{A}'')$. Write $\psi = \eta \wedge \lambda_n \omega_n$. Consider R first. Let $\Delta' \in \mathsf{bch}(\mathcal{A}')$. Set $\Delta'_t = \Delta' \cap \{\alpha_n = t\}$ and $F(t) = \int_{\Delta'_t} \Phi'_\lambda \eta$. Define real numbers $a < b$ such that $\Delta'_t \neq \emptyset$ if and only if $a \leq t \leq b$. Using the variable $t = \alpha_n$, Fubini's theorem and integration by parts give

$$\pm \int_{\Delta'} \Phi'_\lambda \alpha_n^{\lambda_n} \psi = \int_a^b \lambda_n t^{\lambda_n - 1} F(t) dt = [t^{\lambda_n} F(t)]_a^b - \int_a^b t^{\lambda_n} F'(t) dt.$$

Taking the limit as $\lambda_n \to 0$, $\Re\lambda_n > 0$, we get

$$\lim \left[[t^{\lambda_n} F(t)]_a^b - \int_a^b t^{\lambda_n} F'(t) dt \right] = \begin{cases} 0 & 0 \notin \{a, b\} \\ F(0) & 0 = a < b \\ -F(0) & a < b = 0. \end{cases}$$

If Δ' is divided, then we apply the first part to get zero. If H_n intersects $\bar{\Delta}'$ in a face of codimension > 1, then $F(0) = 0$. If H_n does not intersect $\bar{\Delta}'$, then the integral is again zero. Thus $\mathsf{M}(\Delta', \psi) = 0$. This shows that $R = 0$.

It remains to compute the entries of S. Let $\Delta \in \mathsf{bch}(\mathcal{A})$ be either a cutoff or a newborn. In this case $H_n \cap \bar{\Delta} \supset \Delta''$. Let $\tau''(\Delta'') = B''$. It follows from Lemma 6.3.5 that $\psi'' = \eta|_{\Delta''} = \zeta''(B'')$. Set $\Delta_t = \Delta \cap \{\alpha_n = t\}$ and $G(t) = \int_{\Delta_t} \Phi'_\Delta \eta$, where Φ'_Δ is the unique branch of Φ'_λ on Δ such that $\Phi'_\Delta|_{\Delta''} = c_n \Phi''_{\Delta''}$. Define real numbers $a < b$ such that $\Delta_t \neq \emptyset$ if and only if $a \leq t \leq b$. Then $0 \in \{a, b\}$. Recall the choice of branch of Φ_λ on Δ and orientation of Δ. By the same calculation as above, using the variable $t = \alpha_n$, Fubini's theorem and integration by parts, we get

$$\mathsf{M}(\Delta, \psi) = \lim \int_\Delta \Phi_\lambda \psi = G(0) = c_n \int_{\Delta''} \Phi''_\lambda \psi'' = c_n \mathsf{M}(\Delta'', \psi'')$$

as $\lambda_n \to 0$, $\Re\lambda_n > 0$. So $S = c_n \mathsf{PM}(\mathcal{A}'', \lambda'')$. Thus we have

$$\begin{aligned} \det \mathsf{PM}(\mathcal{A}, \lambda)|_{\lambda_n = 0} &= \det \mathsf{M} = (\det P)(\det S) \\ &= \det \mathsf{PM}(\mathcal{A}', \lambda') \det \mathsf{PM}(\mathcal{A}'', \lambda'')\, c_n^{\beta(\mathcal{A}'')}. \end{aligned}$$

□

Corollary 7.3.4.

$$\det \mathsf{PM}^*(\mathcal{A}, \lambda)|_{\lambda_n = 0} = \det \mathsf{PM}^*(\mathcal{A}', \lambda') \det \mathsf{PM}^*(\mathcal{A}'', \lambda'')\, c_n^{\beta(\mathcal{A}'')}.$$

Proof. When the real part of λ_p is positive for all $p \in I'$, this functional equality was proved in Proposition 7.3.3. Therefore this equality holds true everywhere. □

7.4 Proof of the Main Theorem

We argue by induction on (ℓ, n). If $\ell = 1$ the theorem is well-known. If $\beta(\mathcal{A}) = 0$, then the theorem asserts $1 = 1$. Note that $\beta(\mathcal{A}) = 0$ whenever $n \le \ell$. From the induction hypothesis we have

$$\det \mathsf{PM}^*(\mathcal{A}', \lambda') = R(\mathcal{A}', \lambda')B(\mathcal{A}', \lambda'), \quad \det \mathsf{PM}^*(\mathcal{A}'', \lambda'') = R(\mathcal{A}'', \lambda'')B(\mathcal{A}'', \lambda'').$$

We determine the product of critical values first. The induction hypothesis together with Corollary 7.3.4 and Corollary 7.3.2 give

$$c_1^{\lambda_1} \dots c_n^{\lambda_n}|_{\lambda_n = 0} = R(\mathcal{A}, \lambda)|_{\lambda_n = 0}.$$

Thus $c_p^{\lambda_p} = \prod_{j=1}^{\beta} \alpha_p^{\lambda_p}(x_{p,j})$ for $p \neq n$. By considering another linear order $\prec_m$ where H_m is the largest hyperplane $m \neq n$, we get

$$c_1^{\lambda_1} \dots c_n^{\lambda_n}|_{\lambda_m = 0} = R(\mathcal{A}, \lambda)|_{\lambda_m = 0}.$$

so we obtain $c_n^{\lambda_n} = \prod_{j=1}^{\beta} \alpha_n^{\lambda_n}(x_{n,j})$.

We have

$$\det \mathsf{PM}^*(\mathcal{A}, \lambda) = R(\mathcal{A}, \lambda)B(\mathcal{A}, \lambda)h(\lambda).$$

It remains to determine the rational function h. Let

$$\mathcal{L} = \{\lambda_X + m \mid X \in \mathsf{D}(\mathcal{A}_\infty),\ m \in \mathbb{Z}\}.$$

Lemma 7.4.1. *(1) Up to sign, h is independent of the linear order.*

(2) The numerator and the denominator of h are (up to sign) products of linear forms belonging to $\mathcal{L}$.

(3) For all $k \in I$, $h(\lambda_1, \cdots, \lambda_n)|_{\lambda_k = 0}$ is equal to either 1 or -1.

Proof. (1) follows from [FT, Proposition 3.10] and the fact that both $B(\mathcal{A}, \lambda)$ and $R(\mathcal{A}, \lambda)$ are independent of the linear order.

As for (2), recall that $\det \mathsf{PM}(\mathcal{A}, \lambda)$ takes a finite nonzero value at each $\lambda \in \mathbf{U}_{\mathbb{Z}}(\mathcal{A})$. Neither $B(\mathcal{A}, \lambda)$ nor $R(\mathcal{A}, \lambda)$ has a zero or pole at $\lambda \in \mathbf{U}_{\mathbb{Z}}(\mathcal{A})$. Therefore h is a rational function which takes a finite nonzero value at every $\lambda \in \mathbf{U}_{\mathbb{Z}}(\mathcal{A})$. Since the complement of $\mathbf{U}_{\mathbb{Z}}(\mathcal{A})$ is the union of a locally finite infinite family of hyperplanes, we have (2).

Lastly, (3) is a consequence of the induction assumption, Corollary 7.3.4 and Corollary 7.3.2. □

Lemma 7.4.2. *h is equal to a constant function which is either 1 or -1 .*

Proof. Suppose that h is not constant. By Lemma 7.4.1(2), we may write h as a fraction whose denominator and numerator are both products of finitely many elements of $\mathcal{L}$. Suppose $\lambda_X + m$ appears in the expression. By Lemma 7.4.1(3), $\lambda_X - \lambda_j + m$ also appears in the expression for each j such that λ_j appears in λ_X.

Moreover, $\lambda_X + \lambda_j + m$ also appears in the expression for each j such that λ_j does not appear in λ_X. Using this observation repeatedly, we conclude that $\sum_{i\in J} \lambda_i + m$ appears for every subset J of I. In particular, $\lambda_1 + \cdots + \lambda_{n-1} + m$ appears in the expression. This implies either (i) $X = H_1 \cap \cdots \cap H_{n-1}$ is dense and $\mathcal{A}_X = \{H_1, \ldots, H_{n-1}\}$, or (ii) $X_\infty = \bar{H}_n \cap H_\infty$ is dense and $(\mathcal{A}_\infty)_{X_\infty} = \{\bar{H}_n, H_\infty\}$. Since (ii) is a contradiction, (i) always occurs. In particular, $H_1, \ldots, H_{n-1}$ are dependent and there exists $j \in \{1, \ldots, n-1\}$ such that $X = H_1 \cap \cdots \cap H_{j-1} \cap H_{j+1} \cap \cdots \cap H_{n-1}$. If $\mathcal{A}$ is central, there is nothing to prove because $\beta(\mathcal{A}) = 0$. We may assume $\emptyset = H_1 \cap \cdots \cap H_n$. Thus

$$\emptyset = H_1 \cap \cdots \cap H_n = X \cap H_n = H_1 \cap \cdots \cap H_{j-1} \cap H_{j+1} \cap \cdots \cap H_n.$$

This implies that $\lambda_1 + \cdots + \lambda_{j-1} + \lambda_{j+1} + \cdots + \lambda_n + m$ does not appear in the expression of h, which is a contradiction. This shows that h is a constant. By Lemma 7.4.1(3), the constant is equal to either 1 or -1. □

It follows from Lemma 7.4.2 that

$$\det \mathsf{PM}^*(\mathcal{A}, \lambda) = \pm R(\mathcal{A}, \lambda) B(\mathcal{A}, \lambda).$$

It remains to determine the sign. It is known that the sign is positive when $\ell = 1$ or $\beta(\mathcal{A}) = 0$. By Corollaries 7.3.4 and 7.3.2, we can show inductively that the sign is always positive:

$$\det \mathsf{PM}^*(\mathcal{A}, \lambda) = R(\mathcal{A}, \lambda) B(\mathcal{A}, \lambda).$$

Since $\mathsf{PM}^* = \mathsf{PM}$, this proves the main theorem.