# Chapter 6

# Bases

## 6.1 The Aomoto Complex

In this section we study the cohomology groups $H^p(\mathsf{A}^{\cdot}(\mathcal{A}), a_\lambda\wedge)$. We do this by considering a universal complex whose specialization is the complex $(\mathsf{A}^{\cdot}(\mathcal{A}), a_\lambda\wedge)$.

**Definition 6.1.1 ([A1]).** *Let* $\mathbf{y} = \{y_H \mid H \in \mathcal{A}\}$ *be a system of indeterminates in one-to-one correspondence with the hyperplanes of* $\mathcal{A}$. *Let* $\mathbb{C}[\mathbf{y}]$ *be the polynomial ring in* $\mathbf{y}$. *Define a graded* $\mathbb{C}[\mathbf{y}]$*-algebra:*

$$\mathsf{A}^{\cdot}_{\mathbf{y}} = \mathsf{A}^{\cdot}_{\mathbf{y}}(\mathcal{A}) = \mathbb{C}[\mathbf{y}] \otimes_{\mathbb{C}} \mathsf{A}^{\cdot}(\mathcal{A}).$$

*Let* $a_{\mathbf{y}} = \sum_{H\in\mathcal{A}} y_H \otimes a_H \in \mathsf{A}^1_{\mathbf{y}}$. *The complex* $(\mathsf{A}^{\cdot}_{\mathbf{y}}(\mathcal{A}), a_{\mathbf{y}}\wedge)$

$$(1) \qquad 0 \to \mathsf{A}^0_{\mathbf{y}}(\mathcal{A}) \xrightarrow{a_{\mathbf{y}}\wedge} \mathsf{A}^1_{\mathbf{y}}(\mathcal{A}) \xrightarrow{a_{\mathbf{y}}\wedge} \dots \xrightarrow{a_{\mathbf{y}}\wedge} \mathsf{A}^r_{\mathbf{y}}(\mathcal{A}) \to 0$$

*is called the* **Aomoto complex**.

Let $\mathsf{S}$ be a multiplicative closed subset of $\mathbb{C}[\mathbf{y}]$. Consider the Aomoto complex of quotients by $\mathsf{S}$

$$(2) \qquad 0 \to \mathsf{A}^0_{\mathsf{S}}(\mathcal{A}) \xrightarrow{a_{\mathbf{y}}\wedge} \mathsf{A}^1_{\mathsf{S}}(\mathcal{A}) \xrightarrow{a_{\mathbf{y}}\wedge} \dots \xrightarrow{a_{\mathbf{y}}\wedge} \mathsf{A}^r_{\mathsf{S}}(\mathcal{A}) \to 0,$$

where $\mathsf{A}^{\cdot}_{\mathsf{S}} = \mathsf{A}^{\cdot}_{\mathsf{S}}(\mathcal{A}) = \mathbb{C}[\mathbf{y}]_{\mathsf{S}} \otimes_{\mathbb{C}[\mathbf{y}]} \mathsf{A}^{\cdot}_{\mathbf{y}}(\mathcal{A})$.

**Lemma 6.1.2.** *If* $\mathcal{C}$ *is a nonempty central arrangement and* $\mathsf{Y}$ *is the multiplicative closed subset of* $\mathbb{C}[\mathbf{y}]$ *generated by* $\sum_{H\in\mathcal{C}} y_H$*;* $\mathsf{Y} = \{(\sum_{H\in\mathcal{C}} y_H)^m \mid m \geq 0\}$*, then the complex* $(\mathsf{A}^{\cdot}_{\mathsf{Y}}(\mathcal{C}), a_{\mathbf{y}}\wedge)$ *is acyclic.*

*Proof.* Let $\theta_E = \sum_{j=1}^{\ell} u_j(\partial/\partial u_j)$ be the Euler derivation. Denote the interior product by angle brackets. For $\eta \in \mathsf{A}^q_{\mathsf{Y}}(\mathcal{C})$, a standard formula [OT1, 4.73] gives $\langle\theta_E, a_{\mathbf{y}} \wedge \eta\rangle = \langle\theta_E, a_{\mathbf{y}}\rangle\eta - a_{\mathbf{y}}\langle\theta_E, \eta\rangle$, where $\langle\theta_E, a_{\mathbf{y}}\rangle = \sum_{H\in\mathcal{C}} y_H \otimes 1$. Thus if $a_{\mathbf{y}} \wedge \eta = 0$, the hypothesis gives $\eta = a_{\mathbf{y}} \wedge (\sum_{H\in\mathcal{C}} y_H)^{-1}\langle\theta_E, \eta\rangle$. $\square$

**Lemma 6.1.3.** *Let $\lambda = (\lambda_H)_{H\in\mathcal{A}}$ be a system of weights. Suppose that $\mathsf{S}$ is a multiplicative closed subset of $\mathbb{C}[\mathbf{y}]$ satisfying $f(\lambda) \neq 0$ whenever $f \in \mathsf{S}$. Denote the evaluation map by $ev_\lambda : \mathsf{A}^p_{\mathsf{S}} \to \mathsf{A}^p$ which evaluates at $y_H = \lambda_H$ $(H \in \mathcal{A})$. The evaluation map induces a homomorphism*

$$ev_\lambda : H^{\cdot}(\mathsf{A}^{\cdot}_{\mathsf{S}}(\mathcal{A}), a_{\mathbf{y}}\wedge) \to H^{\cdot}(\mathsf{A}^{\cdot}(\mathcal{A}), a_\lambda\wedge).$$

*(1) The evaluation map $ev_\lambda : \mathsf{A}^p_{\mathsf{S}} \to \mathsf{A}^p$ is surjective.*
*(2) The evaluation map*

$$ev_\lambda : H^r(\mathsf{A}^{\cdot}_{\mathsf{S}}(\mathcal{A}), a_{\mathbf{y}}\wedge) \to H^r(\mathsf{A}^{\cdot}(\mathcal{A}), a_\lambda\wedge)$$

*is surjective.*

*Proof.* (1) is obvious. We obtain (2) from (1) because the map involves the top cohomology groups. □

## 6.2 The Isomorphism

In this section we will prove that the Aomoto complex $(\mathsf{A}^{\cdot}_{\mathsf{D}}, a_{\mathbf{y}}\wedge)$ of quotients by a suitable multiplicative closed subset $\mathsf{D}$ of $\mathbb{C}[\mathbf{y}]$ is isomorphic to the cochain complex of the simplicial complex $\mathsf{NBC}$. The subset $\mathsf{D}$ is defined as follows: recall the projective closure $\mathcal{A}_\infty$ in Section 3.1 and the set of dense edges $\mathsf{D}(\mathcal{A}_\infty)$ in Section 3.2. Define $y_{H_\infty} = -\sum_{H\in\mathcal{A}} y_H$. Then the multiplicative subset $\mathsf{D}$ of $\mathbb{C}[\mathbf{y}]$ is generated by

$$\{\sum_{H\in(\mathcal{A}_\infty)_Z} y_H \mid Z \in \mathsf{D}(\mathcal{A}_\infty)\}.$$

For a flag $P = (Y_1 > Y_2 > \cdots > Y_q)$ in $\hat{L}$, define

$$\Xi_{\mathbf{y}}(P) = \bigwedge_{p=1}^{q} a_{\mathbf{y}}(Y_p) \in \mathsf{A}^q_{\mathbf{y}},$$

where $a_{\mathbf{y}}(X) = \sum_{H\in\mathcal{A}_X} y_H \otimes a_H$ for $X \in \hat{L}$. For $S = \{H_{i_1}, \ldots, H_{i_q}\} \in \mathbf{nbc}$, recall that $\xi(S) = (X_1 > \cdots > X_q)$, where $X_p = \bigcap_{k=p}^{q} H_{i_k}$ for $1 \leq p \leq q$ as in Section 5.2. Let $C^{\cdot}(\mathsf{NBC}, \mathbb{C}[\mathbf{y}])$ be the cochain complex of $\mathsf{NBC}$ over $\mathbb{C}[\mathbf{y}]$. Note that $C^{-1}(\mathsf{NBC}, \mathbb{C}[\mathbf{y}])$ is a rank-one free $\mathbb{C}[\mathbf{y}]$-module whose basis is the cochain $\emptyset^*$, dual to the $(-1)$-simplex $\emptyset$. We define

$$\Theta^q : C^{q-1}(\mathsf{NBC}, \mathbb{C}[\mathbf{y}]) \longrightarrow \mathsf{A}^q_{\mathbf{y}} \quad (1 \leq q \leq r)$$

by

$$\Theta^q(\alpha) = \sum_{\substack{S\in\mathbf{nbc}\\ |S|=q}} \alpha(S)\theta_{\mathbf{y}}(S) \in \mathsf{A}^q_{\mathbf{y}},$$

where $\alpha \in C^{q-1}(\mathsf{NBC}, \mathbb{C}[\mathbf{y}])$ and $\theta_{\mathbf{y}}(S) = \Xi_{\mathbf{y}}(\xi(S))$. For $q = 0$, define

$$\Theta^0 : C^{-1}(\mathsf{NBC}, \mathbb{C}[\mathbf{y}]) \longrightarrow \mathsf{A}^0_{\mathbf{y}}$$

by $\Theta^0(\alpha) = \alpha(\emptyset) \in \mathsf{A}^0_{\mathbf{y}} = \mathbb{C}[\mathbf{y}]$. The important result below is due to Schechtman-Varchenko [SV1] and Brylawski-Varchenko [BV]. We state it in a slightly different way using $\mathsf{NBC}$ and present a new and elementary proof.

**Theorem 6.2.1.** *The maps $\{\Theta^q\}_{0 \le q \le r}$ give a morphism from the (augmented) cochain complex $C^{\cdot -1}(\mathsf{NBC}, \mathbb{C}[\mathbf{y}])$ to the Aomoto complex $(\mathsf{A}^{\cdot}_{\mathbf{y}}, a_{\mathbf{y}}\wedge)$. This morphism induces an isomorphism over the ring of quotients $\mathbb{C}[\mathbf{y}]_{\mathsf{D}}$.*

*Proof. Step 1.* For the first half, it is sufficient to show that the diagram

$$\begin{tikzcd} C^{q-1}(\mathsf{NBC}, \mathbb{C}[\mathbf{y}]) \arrow[r, "\delta"] \arrow[d, "\Theta^q"'] & C^q(\mathsf{NBC}, \mathbb{C}[\mathbf{y}]) \arrow[d, "\Theta^{q+1}"] \\ \mathsf{A}^q_{\mathbf{y}} \arrow[r, "a_{\mathbf{y}}\wedge"'] & \mathsf{A}^{q+1}_{\mathbf{y}} \end{tikzcd}$$

is commutative, where $\delta$ denotes the coboundary map. It is easy to see that the diagram is commutative when $q = 0$. Let $\emptyset^*$ be the $(-1)$-cochain dual to the $(-1)$-simplex $\emptyset$. Then

$$\Theta^1 \circ \delta(\emptyset^*) = \Theta^1\Big(\sum_{H \in \mathcal{A}} \{H\}^*\Big) = \sum_{H \in \mathcal{A}} y_H \otimes a_H = a_{\mathbf{y}} = a_{\mathbf{y}} \wedge \Theta^0(\emptyset^*).$$

Suppose $q > 0$. Let $S = \{H_{i_1}, \dots, H_{i_q}\}$ be a $(q-1)$-simplex in $\mathsf{NBC}$. Let $S^*$ denote the $(q-1)$-cochain dual to $S$:

$$\langle S^*, S' \rangle = \begin{cases} 1 & \text{if } S' = S \\ 0 & \text{otherwise.} \end{cases}$$

It suffices to show that

$$\Theta^{q+1} \circ \delta(S^*) = a_{\mathbf{y}} \wedge \Theta^q(S^*)$$

by induction on $q$. Note that

$$\delta(S^*) = \sum_{k=0}^{q} (-1)^k \sum_{I_k} \{H_{i_1}, \dots, H_{i_k}, H, H_{i_{k+1}}, \dots, H_{i_q}\}^*,$$

where the second summation is over the set

$$I_k = \{H \in \mathcal{A} \mid \{H_{i_1}, \dots, H_{i_k}, H, H_{i_{k+1}}, \dots, H_{i_q}\} \in \mathbf{nbc}\}.$$

Let $\xi(S) = (X_1 > \cdots > X_q)$. It follows from Lemmas 5.2.2 and 5.2.3 that the maps $\xi$ and $\nu$ provide a bijection between $I_0$ and

$$J_0 = \{(Z > X_1 > \dots X_q) \mid \nu(Z) \prec H_{i_1},\, r(Z) = q+1\}$$

and for $1 \le k \le q$ between $I_k$ and

$$\begin{aligned} J_k = \{(Y_1 > \dots Y_k > Z > X_{k+1} > \cdots > X_q) \mid H_{i_k} \prec \nu(Z) \prec H_{i_{k+1}},\\ r(Z) = q-k+1,\, r(Y_j) = q-j+2,\, \nu(Y_j) = H_{i_j}\ (1 \le j \le k)\}. \end{aligned}$$

Fix $Y_1$ with $\nu(Y_1) = H_{i_1}$, $r(Y_1) = q+1$, and $Y_1 > X_1$. Define

$$J_1(Y_1) = \{(Z > X_2 > \dots X_q) \mid \nu(Z) \prec H_{i_2},\, r(Z) = q,\, Y_1 > Z\}$$

and

$$\begin{aligned} J_k(Y_1) = \{(Y_2 > \dots Y_k > Z > X_{k+1} > \cdots > X_q) \mid H_{i_k} \prec \nu(Z) \prec H_{i_{k+1}},\\ r(Z) = q-k+1,\, r(Y_j) = q-j+2,\, \nu(Y_j) = H_{i_j}\ (2 \le j \le k)\} \end{aligned}$$

for $2 \le k \le q$. Then

$$\begin{aligned} &\sum_{k=1}^{q} (-1)^{k-1} \sum_{P \in J_k} \Xi_{\mathbf{y}}(P) \\ &= \sum_{\substack{\nu(Y_1)=H_{i_1} \\ r(Y_1)=q+1 \\ Y_1 > X_1}} a_{\mathbf{y}}(Y_1) \left[ \sum_{k=1}^{q} (-1)^{k-1} \sum_{P \in J_k(Y_1)} \Xi_{\mathbf{y}}(P) - a_{\mathbf{y}}(X_1) a_{\mathbf{y}}(X_2) \dots a_{\mathbf{y}}(X_q) \right]. \end{aligned}$$

By the induction assumption for $\{H_{i_2}, \dots, H_{i_q}\} \in \mathbf{nbc}(\mathcal{A}_{Y_1})$, we have

$$\begin{aligned} \sum_{k=1}^{q} (-1)^{k-1} \sum_{P \in J_k(Y_1)} \Xi_{\mathbf{y}}(P) &= \Theta^q \circ \delta(\{H_{i_2}, \dots, H_{i_q}\}^*) \\ &= a_{\mathbf{y}}(Y_1) \wedge \Theta^{q-1}(\{H_{i_2}, \dots, H_{i_q}\}^*) \\ &= a_{\mathbf{y}}(Y_1) a_{\mathbf{y}}(X_2) \dots a_{\mathbf{y}}(X_q). \end{aligned}$$

Thus

$$\Theta^{q+1} \circ \delta(S^*) = \sum_{k=0}^{q} (-1)^k \sum_{P \in J_k} \Xi_{\mathbf{y}}(P) = \sum_{P \in J_0} \Xi_{\mathbf{y}}(P) - \sum_{k=1}^{q} (-1)^{k-1} \sum_{P \in J_k} \Xi_{\mathbf{y}}(P)$$

$$= \sum_{\substack{\nu(Z) \prec H_{i_1} \\ r(Z)=q+1 \\ Z > X_1}} a_{\mathbf{y}}(Z) a_{\mathbf{y}}(X_1) \ldots a_{\mathbf{y}}(X_q) - \sum_{\substack{\nu(Y_1) = H_{i_1} \\ r(Y_1)=q+1 \\ Y_1 > X_1}} a_{\mathbf{y}}(Y_1)$$

$$\times \left[ \sum_{k=1}^{q} (-1)^{k-1} \sum_{P \in J_k(Y_1)} \Xi_{\mathbf{y}}(P) - \sum_{\substack{\nu(Z) = H_{i_1} \\ r(Z)=q \\ Y_1 > Z > X_2}} a_{\mathbf{y}}(Z) a_{\mathbf{y}}(X_2) \ldots a_{\mathbf{y}}(X_q) \right]$$

$$= \sum_{\substack{\nu(Z) \prec H_{i_1} \\ r(Z)=q+1 \\ Z > X_1}} a_{\mathbf{y}}(Z) a_{\mathbf{y}}(X_1) \ldots a_{\mathbf{y}}(X_q)$$

$$- \sum_{\substack{\nu(Y_1) = H_{i_1} \\ r(Y_1)=q+1 \\ Y_1 > X_1}} a_{\mathbf{y}}(Y_1) \left[ a_{\mathbf{y}}(Y_1) a_{\mathbf{y}}(X_2) \ldots a_{\mathbf{y}}(X_q) - a_{\mathbf{y}}(X_1) \ldots a_{\mathbf{y}}(X_q) \right]$$

$$= \sum_{\substack{\nu(Z) \prec H_{i_1} \\ r(Z)=q+1 \\ Z > X_1}} a_{\mathbf{y}}(Z) a_{\mathbf{y}}(X_1) \ldots a_{\mathbf{y}}(X_q) + \sum_{\substack{\nu(Y_1) = H_{i_1} \\ r(Y_1)=q+1 \\ Y_1 > X_1}} a_{\mathbf{y}}(Y_1) a_{\mathbf{y}}(X_1) \ldots a_{\mathbf{y}}(X_q)$$

$$= \sum_{\substack{r(Y)=q+1 \\ Y > X_1}} a_{\mathbf{y}}(Y) a_{\mathbf{y}}(X_1) \ldots a_{\mathbf{y}}(X_q) = a_{\mathbf{y}} \wedge \Theta^q(S^*).$$

*Step 2.* By Theorem 5.1.2, there are decompositions

$$C^{q-1}(\mathsf{NBC}, \mathbb{C}[\mathbf{y}]) = \bigoplus_{\substack{X \in L \\ r(X)=q}} C^{q-1}(\mathsf{NBC}(\mathcal{A}_X), \mathbb{C}[\mathbf{y}])$$

and

$$\mathsf{A}^q_{\mathbf{y}} = \bigoplus_{\substack{X \in L \\ r(X)=q}} \mathsf{A}^q_{\mathbf{y}}(\mathcal{A}_X).$$

Note that the map $\Theta^q$ is compatible with these decompositions. In other words, $\Theta^q$ induces

$$\Theta^q_X : C^{q-1}(\mathsf{NBC}(\mathcal{A}_X), \mathbb{C}[\mathbf{y}]) \longrightarrow \mathsf{A}^q_{\mathbf{y}}(\mathcal{A}_X)$$

for each $X \in L$ with $r(X) = q$. Since $\mathsf{D}(\mathcal{A}_X) \subseteq \mathsf{D}(\mathcal{A})$ by Lemma 3.2.6, we may assume that $\mathcal{A}$ is nonempty central when we prove the last half of the theorem.

*Step 3.* Suppose that $\mathcal{A}$ is nonempty central. Let $r = r(\mathcal{A})$. Let $0 \le q \le r$. We prove that the induced map

$$\Theta^q_{\mathsf{D}} : C^{q-1}(\mathsf{NBC}, \mathbb{C}[\mathbf{y}]_{\mathsf{D}}) \longrightarrow \mathsf{A}^q_{\mathsf{D}}$$

is an isomorphism by induction on $r \ge 1$. When $r = 1$, $\Theta^q_{\mathsf{D}}$ $(q = 0, 1)$ are isomorphisms because each $y_H$ is invertible in $\mathbb{C}[\mathbf{y}]_{\mathsf{D}}$. By Step 2, we have

$$C^{q-1}(\mathsf{NBC}, \mathbb{C}[\mathbf{y}]_{\mathsf{D}}) = \bigoplus_{\substack{X \in L \\ r(X)=q}} C^{q-1}(\mathsf{NBC}(\mathcal{A}_X), \mathbb{C}[\mathbf{y}]_{\mathsf{D}})$$

and

$$\mathsf{A}^q_{\mathsf{D}} = \bigoplus_{\substack{X \in L \\ r(X)=q}} \mathsf{A}^q_{\mathsf{D}}(\mathcal{A}_X).$$

By the induction assumption, we may assume that the theorem holds true for $\mathcal{A}_X$ when $r(X) < r$. Thus $\Theta^q_{\mathsf{D}}$ is an isomorphism for $0 \le q \le r-1$. Decompose $\mathcal{A}$ into indecomposable subarrangements:

$$\mathcal{A} = \mathcal{A}_1 \uplus \cdots \uplus \mathcal{A}_m$$

as in Lemma 3.2.7. Then it is not difficult to see that

$$(\mathsf{A}^{\cdot}_{\mathsf{D}}, a_{\mathbf{y}}\wedge) \simeq \bigotimes_{i=1}^{m} \left( \mathsf{A}^{\cdot}_{\mathsf{D}}(\mathcal{A}_i), \left( \sum_{H \in \mathcal{A}_i} y_H \otimes a_H \right) \wedge \right).$$

Note $\mathsf{D}(\mathcal{A}_i) \subseteq \mathsf{D}(\mathcal{A})$ for $1 \le i \le m$ by Lemma 3.2.7(2). Thus the complex $(\mathsf{A}^{\cdot}_{\mathsf{D}}, a_{\mathbf{y}}\wedge)$ is acyclic by Lemma 6.1.2. Since $\mathsf{NBC}$ is contractible by Theorem 5.2.10, the (augmented) cochain complex $C^{\cdot-1}(\mathsf{NBC}, \mathbb{C}[\mathbf{y}]_{\mathsf{D}})$ is also acyclic. Thus we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C^{-1}(\mathsf{NBC}, \mathbb{C}[\mathbf{y}]_{\mathsf{D}}) \arrow[r, "\delta"] \arrow[d, "\Theta^0_{\mathsf{D}}"'] & \cdots \arrow[r, "\delta"] & C^{r-1}(\mathsf{NBC}, \mathbb{C}[\mathbf{y}]_{\mathsf{D}}) \arrow[r] \arrow[d, "\Theta^r_{\mathsf{D}}"'] & 0 \\
0 \arrow[r] & \mathsf{A}^0_{\mathsf{D}} \arrow[r, "a_{\mathbf{y}}\wedge"'] & \cdots \arrow[r, "a_{\mathbf{y}}\wedge"'] & \mathsf{A}^r_{\mathsf{D}} \arrow[r] & 0,
\end{tikzcd}$$

whose rows are exact and the vertical maps $\Theta^q_{\mathsf{D}} (0 \le q \le r-1)$ are isomorphisms. Therefore the rightmost vertical map $\Theta^r_{\mathsf{D}}$ is also an isomorphism. This completes the induction step. ☐

**Corollary 6.2.2.** *Let $\mathcal{A}$ be an affine arrangement of rank $r$ with projective closure $\mathcal{A}_\infty$. Assume that $\lambda_X \neq 0$ for every $X \in \mathsf{D}(\mathcal{A}_\infty)$. Then*

$$H^p(\mathsf{A}^{\cdot}(\mathcal{A}), a_\lambda\wedge) = 0 \text{ for } p \neq r, \quad \dim H^r(\mathsf{A}^{\cdot}(\mathcal{A}), a_\lambda\wedge) = \beta(\mathcal{A}).$$

*Proof.* We have a commutative diagram

$$\begin{tikzcd} C^{q-1}(\mathsf{NBC}, \mathbb{C}[\mathbf{y}]_{\mathsf{D}}) \arrow[d, "\Theta^q_{\mathsf{D}}"] & C^{q-1}(\mathsf{NBC}, \mathbb{C}) \arrow[l, hook', "j"'] \arrow[d, "ev_\lambda \circ \Theta^q_{\mathsf{D}} \circ j"] \\ \mathsf{A}^q_{\mathsf{D}} \arrow[r, "ev_\lambda"'] & \mathsf{A}^q. \end{tikzcd}$$

Here $ev_\lambda$ is defined in Lemma 6.1.3 and $j$ is the natural map induced by an extension of the coefficient ring. By Theorem 6.2.1, $\Theta^q_{\mathsf{D}}$ is an $\mathbb{C}[\mathbf{y}]_{\mathsf{D}}$-isomorphism. Therefore the composed map $ev_\lambda \circ \Theta^q_{\mathsf{D}} \circ j$, which is the evaluation of $\Theta^q_{\mathsf{D}}$ at $y_H = \lambda_H (H \in \mathcal{A})$, gives a $\mathbb{C}$-isomorphism: $C^{q-1}(\mathsf{NBC}, \mathbb{C}) \xrightarrow{\sim} \mathsf{A}^q$. Since each map in the diagram commutes with the coboundary maps,

$$H^q(\mathsf{A}^{\cdot}(\mathcal{A}), a_\lambda) \simeq \tilde{H}^{q-1}(\mathsf{NBC}, \mathbb{C}),$$

where $\tilde{H}$ stands for the reduced cohomology. Theorem 5.2.10 completes the proof. ☐

We combine these results to generalize similar theorems of Aomoto [A2] and Kohno [Ko1].

**Theorem 6.2.3.** *Let $\mathcal{A}$ be an affine arrangement of rank $r$ with projective closure $\mathcal{A}_\infty$. Assume that $\lambda_X \notin \mathbb{Z}_{\geq 0}$ for every $X \in \mathsf{D}(\mathcal{A}_\infty)$. Then*
*(1) $H^p(M(\mathcal{A}), \mathcal{L}_\lambda) = 0 = H_p(M(\mathcal{A}), \mathcal{L}^\vee_\lambda)$ for $p \neq r$,*
*(2) $\dim H^r(M(\mathcal{A}), \mathcal{L}_\lambda) = \beta(\mathcal{A}) = \dim H_r(M(\mathcal{A}), \mathcal{L}^\vee_\lambda)$.*

*Proof.* The assertions follow from Theorem 4.2.6, Theorem 5.1.3, and Corollary 6.2.2. ☐

## 6.3 The $\beta$nbc Cohomology Basis

Let $\lambda = (\lambda_H)_{H \in \mathcal{A}}$ be a system of weights. Recall the map $\Theta^r$ from Theorem 6.2.1, the evaluation map $ev_\lambda$ from Lemma 6.1.3, and the isomorphism $\iota$ from Theorem 5.1.3:

$$C^{r-1}(\mathsf{NBC}, \mathbb{C}[\mathbf{y}]) \xrightarrow{\Theta^r} \mathsf{A}^r_{\mathbf{y}} \xrightarrow{ev_\lambda} \mathsf{A}^r \xrightarrow{\iota} \mathsf{B}^r.$$

**Definition 6.3.1.** *Define $\zeta : \beta\mathbf{nbc} \to \mathsf{B}^r$ by $\zeta(B) = \iota \circ ev_\lambda \circ \Theta^r(B^*)$. Explicitly, if $B = \{H_{i_1}, \ldots, H_{i_r}\}$ is a $\beta$**nbc** frame and $\xi(B) = (X_1 > \cdots > X_r)$ where $X_p = \bigcap_{k=p}^r H_{i_k}$ for $1 \leq p \leq r$, then $\zeta(B) = \wedge_{p=1}^r \omega_\lambda(X_p)$.*

**Theorem 6.3.2.** *Let $\mathcal{A}$ be an affine arrangement of rank $r$ with projective closure $\mathcal{A}_\infty$. Assume that $\lambda_X \notin \mathbb{Z}_{\geq 0}$ for every $X \in \mathsf{D}(\mathcal{A}_\infty)$. Then the set*

$$\{\zeta(B) \in H^r(M, \mathcal{L}) \mid B \in \beta\mathbf{nbc}\}$$

*is a basis for the only nonzero local system cohomology group, $H^r(M, \mathcal{L}_\lambda)$.*

*Proof.* We combine the results of Theorems 5.3.3, 6.2.1, 5.1.3, 4.2.6 and Lemma 6.1.3 (2). $\square$

**Example 6.3.3.** *Recall the Selberg arrangement of Examples 3.1.1, 3.2.2, 5.2.4, 5.2.6, and 5.3.4. Assume that the weights of the dense edges satisfy the conditions of Theorem 6.3.2. Then*

$$\begin{aligned}\zeta(\{2,4\}) &= (\lambda_2\omega_2 + \lambda_4\omega_4 + \lambda_5\omega_5)\lambda_4\omega_4 = \lambda_2\lambda_4\omega_{24} - \lambda_4\lambda_5\omega_{45},\\ \zeta(\{2,5\}) &= (\lambda_2\omega_2 + \lambda_4\omega_4 + \lambda_5\omega_5)\lambda_5\omega_5 = \lambda_2\lambda_5\omega_{25} + \lambda_4\lambda_5\omega_{45}\end{aligned}$$

*provide a basis for* $H^2(M, \mathcal{L}_\lambda)$.

The next two statements will be used in Chapter 7.

**Lemma 6.3.4.** *If* $B \in \beta\mathbf{nbc}(\mathcal{A}')$, *then* $\zeta(B)|_{\lambda_n=0} = \zeta'(B)$.

*Proof.* Let $X \in L(\mathcal{A})$. If $X \not\subset H_n$, then $\omega_\lambda(X, \mathcal{A}) = \omega_\lambda(X, \mathcal{A}')$. If $X \subset H_n$, then $\omega_\lambda(X, \mathcal{A}) = \omega_\lambda(X, \mathcal{A}') + \eta \wedge \lambda_n\omega_n$ for some form $\eta$. $\square$

**Lemma 6.3.5.** *Let* $B'' \in \beta\mathbf{nbc}(\mathcal{A}'')$ *and let* $B = \{\nu B'', H_n\} \in \overline{\beta\mathbf{nbc}}(\mathcal{A}'')$. *Then the residue of* $\zeta(B)$ *along* $H_n$ *is equal to* $\zeta''(B'')$.

*Proof.* The last factor of $\zeta(B)$ is $\lambda_n\omega_n$. Since the product is exterior, it follows that $\lambda_n\omega_n$ may be removed as a summand from all the other factors of the product without changing its value. Taking residue of this rewritten product removes the factor $\lambda_n\omega_n$ and restricts the remaining terms to $H_n$. The residue is now just $\zeta''(B'')$. $\square$

## 6.4 The $\beta$nbc Homology Basis

Morse theoretic arguments are used in [OSi] to construct a $\beta$**nbc** local system homology basis for arbitrary arrangements with suitable weights. Here we present a special case used in Chapter 7. If $\Delta$ is a bounded chamber in $M_{\mathbb{R}}$, then $\Delta \in C_\ell^{lf}(M, \mathcal{L}^\vee)$ is a cycle. Let $[\Delta]$ denote its locally finite homology class. Recall from Definition 3.3.8 the set of bounded chambers, $\mathsf{bch}(\mathcal{A})$.

**Proposition 6.4.1.** *Let* $\mathcal{A}$ *be an essential complexified real arrangement with projective closure* $\mathcal{A}_\infty$. *Assume that* $\lambda_X \notin \mathbb{Z}_{\geq 0}$ *for every* $X \in \mathsf{D}(\mathcal{A}_\infty)$. *Then* $\{[\Delta] \mid \Delta \in \mathsf{bch}(\mathcal{A})\}$ *forms a basis for* $H_\ell^{lf}(M(\mathcal{A}), \mathcal{L}^\vee)$.

*Proof.* This result was proved with more restrictions in [Ko1] and [AK, 4.1.1]. Write $M = M(\mathcal{A})$ and let $B = \cup_{\Delta \in \mathsf{bch}(\mathcal{A})}\Delta$. In order to apply the argument in [Ko1], it is sufficient to show that $H^q(M - B, \mathcal{L}^\vee) = 0$ for all $q$. Let $W$ be a small tubular neighborhood in $\mathbb{CP}^\ell$ of the hyperplane at infinity, $H_\infty$. Note that the inclusion map $W \cap M \hookrightarrow M - B$ is a homotopy equivalence.

Let $j : W \cap M \hookrightarrow W$. Let $x \in W$. There are two cases to consider. If $x \in W \cap M$, then there is no hyperplane in $\mathcal{A}_\infty$ going through $x$, so we have $(R^q j_* \mathcal{L}^\vee)_x = 0$ for $q \neq 0$ and $(R^0 j_* \mathcal{L}^\vee)_x \simeq \mathbb{C}$. If $x \in W - M$, then $W - M$ is locally a central arrangement near $x$. In this case the local euler characteristic of $M$ intersected with a small open ball centered at $x$ is zero, so we have $(R^q j_* \mathcal{L}^\vee)_x = 0$ for all $q$ by Theorem 6.2.3. Therefore we have $H^q(W \cap M, \mathcal{L}^\vee) \simeq H^q(W, j_* \mathcal{L}^\vee)$. For each $x \in H_\infty$, there exists a small neighborhood $W_x$ of $x$ in $W$ such that

$$H^q(W_{x_1} \cap \cdots \cap W_{x_k}, j_* \mathcal{L}^\vee) = 0$$

as long as $W_{x_1} \cap \cdots \cap W_{x_k} \neq \emptyset$. Since $H_\infty \simeq \mathbb{CP}^{\ell-1}$ is compact, we may choose $W_{x_1}, \ldots, W_{x_m}$ which cover $H_\infty$. Let $W_0 = W_{x_1} \cup \cdots \cup W_{x_m}$. By applying the Mayer-Vietoris theorem repeatedly, we have $H^q(W_0, j_* \mathcal{L}^\vee) = 0$ for all $q$. By Poincaré duality, we have

$$H^q(M - B, \mathcal{L}^\vee) \simeq H^q(W_0 \cap M, \mathcal{L}^\vee) \simeq H^q(W_0, j_* \mathcal{L}^\vee) = 0.$$

□

Next we establish a bijection between $\mathsf{bch}(\mathcal{A})$ and $\beta\mathbf{nbc}(\mathcal{A})$. This is done recursively by deletion and restriction.

**Definition 6.4.2.** *Let $\rho = (X_1 > \cdots > X_\ell)$ be a flag of affine subspaces $X_i \in L(\mathcal{A})$ with $\dim_{\mathbb{C}} X_i = i - 1$ $(i = 1, \ldots, \ell)$. Let $\Delta \in \mathsf{bch}(\mathcal{A})$ and let $\bar{\Delta}$ be its closure in $\mathbb{R}^\ell$. We say that $\rho$ is* **adjacent** *to $\Delta$ if $\dim_{\mathbb{R}}(X_i \cap \bar{\Delta}) = i - 1$ for $i = 1, \ldots, \ell$.*

**Proposition 6.4.3.** *There exists a unique bijection*

$$\tau : \mathsf{bch}(\mathcal{A}) \longrightarrow \beta\mathbf{nbc}(\mathcal{A})$$

*with the property that $\xi(\tau(\Delta))$ is adjacent to $\Delta$.*

*Proof.* If $\mathsf{bch}(\mathcal{A}) = \emptyset$, then $\beta\mathbf{nbc}(\mathcal{A}) = \emptyset$. Suppose $\mathsf{bch}(\mathcal{A}) \neq \emptyset$. We argue by induction on $|\mathcal{A}|$. Assume that the maps $\tau'$ and $\tau''$ already exist for $\mathcal{A}'$ and $\mathcal{A}''$. There are the following four kinds of bounded chambers of $\mathcal{A}$:

(1) $\Delta \in \mathsf{bch}(\mathcal{A})$ is called *undivided* if $\Delta \in \mathsf{bch}(\mathcal{A}')$. Thus $\Delta$ does not intersect $H_n$. Define $\tau(\Delta) = \tau'(\Delta)$. The adjacency is clear.

(2) $\Delta \in \mathsf{bch}(\mathcal{A})$ is called *newborn* if there exists an unbounded chamber of $\mathcal{A}'$ which contains $\Delta$. In this case $\bar{\Delta} \cap H_n$ is the closure of a bounded chamber $\Delta'' \in \mathsf{bch}(\mathcal{A}'')$. Let $\tau''(\Delta'') = B''$. Define $\tau(\Delta) = \{\nu B'', H_n\}$. Since $\xi(B'')$ is adjacent to $\Delta''$, $\xi(\tau(\Delta))$ is adjacent to $\Delta$.

(3) Suppose that a bounded chamber $\Delta' \in \mathsf{bch}(\mathcal{A}')$ is divided in two by $H_n$. Denote the two chambers of $\mathcal{A}$ inside $\Delta'$ by $\Delta^+$ and $\Delta^-$. Let $\tau'(\Delta') = B' \in \beta\mathbf{nbc}(\mathcal{A}')$. Then $\xi(B')$ is adjacent to $\Delta'$ by the induction hypothesis. Clearly, $\xi(B')$ is adjacent to exactly one of these two chambers, say to $\Delta^+$. The chamber $\Delta^+ \in \mathsf{bch}(\mathcal{A})$ is called the *heir* of $\Delta'$ and we define $\tau(\Delta^+) = B'$. The adjacency is clear.

(4) The chamber $\Delta^- \in \mathsf{bch}(\mathcal{A})$ is called the *cutoff* of $\Delta'$. Here $\bar{\Delta}^- \cap H_n$ is the closure of a bounded chamber $\Delta'' \in \mathsf{bch}(\mathcal{A}'')$. Let $\tau''(\Delta'') = B''$. Define $\tau(\Delta^-) = \{\nu B'', H_n\}$. Since $\xi(B'')$ is adjacent to $\Delta''$, $\xi(\tau(\Delta^-))$ is adjacent to $\Delta^-$.

By construction, $\tau$ is bijective and unique. □

Propositions 6.4.1 and 6.4.3 imply:

**Theorem 6.4.4.** *Let $\mathcal{A}$ be an essential complexified real arrangement. Assume that $\lambda_X \notin \mathbb{Z}_{\geq 0}$ for every $X \in \mathsf{D}(\mathcal{A}_\infty)$. Then the set*

$$\{\tau^{-1}(B) \in H_\ell^{lf}(M, \mathcal{L}^\vee) \mid B \in \beta\mathbf{nbc}(\mathcal{A})\}$$

*is a basis for the only nonzero local coefficient homology group, $H_\ell^{lf}(M, \mathcal{L}^\vee)$.*

Orient each $\Delta \in \mathsf{bch}(\mathcal{A})$ using the adjacent flag $\xi(\tau(\Delta)) = (X_1 > \cdots > X_\ell)$ by choice of an orthonormal frame $\{e_1, \ldots, e_\ell\}$ so that $e_i$ is a unit vector originating from the point $X_1$ in the direction of $X_{i+1} \cap \bar{\Delta}$ for $1 \leq i \leq \ell - 1$ and $e_\ell$ is a unit vector originating from the point $X_1$ in the direction of $V \cap \bar{\Delta}$. Call this the $\beta$**nbc-orientation** of $\Delta$.

**Example 6.4.5.** *Recall the Selberg arrangement of Figure 3.1. Let the closure of $\Delta_1$ have vertices $(1,4), (1,3,5), (2,4,5)$ and the closure of $\Delta_2$ have vertices $(2,3), (1,3,5), (2,4,5)$. Then $\tau(\Delta_1) = (2,4)$ and $\tau(\Delta_2) = (2,5)$. Note that $\Delta_1$ is oriented clockwise, while $\Delta_2$ is oriented counterclockwise.*

**Definition 6.4.6.** *Given the linear order $\prec$, introduce a linear order in $\beta\mathbf{nbc}(\mathcal{A})$ using the lexicographic order on the hyperplanes read from right to left. Write the ordered set $\beta\mathbf{nbc}(\mathcal{A}) = \{B_j\}_{j=1}^{\beta}$. We use Theorem 6.3.2 and write $\psi_j = \zeta(B_j)$ to get the associated linearly ordered basis of global holomorphic forms for $H^\ell(M, \mathcal{L})$, $\Psi(\mathcal{A}) = \{\psi_j\}_{j=1}^{\beta}$. We use Theorem 6.4.4 and write $\Delta_j = \tau^{-1}(B_j)$ to get the associated linearly ordered basis of oriented bounded chambers for $H_\ell^{lf}(M, \mathcal{L}^\vee)$, $\mathsf{bch}(\mathcal{A}) = \{\Delta_j\}_{j=1}^{\beta}$. Call these the $\beta$**nbc**-bases in cohomology and locally finite homology.*

**Example 6.4.7.** *For the Selberg arrangement*

$$\begin{aligned}
\beta\mathbf{nbc} &= \{\{2,4\}, \{2,5\}\}, \\
\Psi &= \{\lambda_2\lambda_4\omega_{24} - \lambda_4\lambda_5\omega_{45},\ \lambda_2\lambda_5\omega_{25} + \lambda_4\lambda_5\omega_{45}\}, \\
\mathsf{bch} &= \{\Delta_1, \Delta_2\}.
\end{aligned}$$

## 6.5 Resonance

It follows from Sard's Theorem that there exists a maximal dense open subset of the space of weights where the local system cohomology groups are independent of the weights.

**Definition 6.5.1.** *Let* $\mathbf{U}(\mathcal{A}) \subset \mathbb{C}^n$ *be the maximal dense open set where the cohomology groups* $H^*(M(\mathcal{A}), \mathcal{L}_\lambda)$ *are independent of* $\lambda \in \mathbf{U}(\mathcal{A})$. *We call weights* $\lambda \in \mathbf{U}(\mathcal{A})$ **nonresonant**. *Let*

$$\mathbf{V}(\mathcal{A}) = \{\lambda \in \mathbb{C}^n \mid \lambda_X \notin \mathbb{Z}_{>0},\ X \in \mathsf{D}(\mathcal{A}_\infty)\},$$
$$\mathbf{W}(\mathcal{A}) = \{\lambda \in \mathbb{C}^n \mid \lambda_X \notin \mathbb{Z}_{\geq 0},\ X \in \mathsf{D}(\mathcal{A}_\infty)\}.$$

It follows from Theorem 6.2.3 that $\mathbf{W}(\mathcal{A}) \subset \mathbf{U}(\mathcal{A})$. Thus nonresonant weights give local systems with vanishing cohomology in all but the top dimension.

**Example 6.5.2.** *In this example we show that* $\mathbf{V}(\mathcal{A}) \not\subset \mathbf{U}(\mathcal{A})$. *Let* $\mathcal{A}$ *consist of three lines through the origin in the plane. The origin is a dense edge. If the weights are in* $\mathbf{U}(\mathcal{A})$, *then the local system cohomology vanishes in all dimensions because* $\beta(\mathcal{A}) = 0$. *If we assign* $1/3$ *to two of the lines and* $-2/3$ *to the third, then* $\lambda_\infty = 0 = \lambda_0$. *We show that these weights are not in* $\mathbf{U}(\mathcal{A})$ *so they are resonant.*

Since these weights are in $\mathbf{V}(\mathcal{A})$, it follows from Theorem 4.2.6 and Theorem 5.1.3 that we may use the complex $(\mathsf{A}^{\cdot}(\mathcal{A}), a_\lambda\wedge)$ to calculate $H^p(M(\mathcal{A}), \mathcal{L}_\lambda)$. We choose the following **nbc** bases: $\{1\}$ in $\mathsf{A}^0$, $\{a_1, a_2, a_3\}$ in $\mathsf{A}^1$ and $\{a_1a_2, a_1a_3\}$ in $\mathsf{A}^2$. Since $a_\lambda = 1/3a_1 + 1/3a_2 - 2/3a_3$, the map $a_\lambda\wedge : \mathsf{A}^0 \to \mathsf{A}^1$ is injective. Thus $H^0(M(\mathcal{A}), \mathcal{L}_\lambda) = 0$. In order to describe the map $a_\lambda\wedge : \mathsf{A}^1 \to \mathsf{A}^2$ we use the relation $a_2a_3 = -a_1a_2 + a_1a_3$:

$$a_\lambda \wedge a_1 = a_\lambda \wedge a_2 = a_\lambda \wedge a_3 = -\frac{1}{3}a_1a_2 + \frac{2}{3}a_1a_3.$$

Thus $\dim H^1(M(\mathcal{A}), \mathcal{L}_\lambda) = 1 = \dim H^2(M(\mathcal{A}), \mathcal{L}_\lambda)$ in this example. Hence these weights are resonant.

**Example 6.5.3.** *It follows from Proposition 2.1.3 that given weights* $\lambda_i$, *an equivalent local system is obtained by the integer modification* $\lambda_i + k_i$ *with* $k_i \in \mathbb{Z}$. *In this example we show that there are resonant weights with no integer modification in* $\mathbf{V}(\mathcal{A})$. *We defined the projective 2-arrangement Ceva(3) in Example 3.3.6. Assign weight* $\lambda_i = 1/3$ *to the eight affine lines so* $\lambda_\infty = -8/3$.

Eight triple points have weight 1 and four triple points have weight $-2$. Thus $\lambda \notin \mathbf{V}(\mathcal{A})$. We show next that no integer modification of $\lambda$ is in $\mathbf{V}(\mathcal{A})$. The sum of the weights of the triple points is zero. If the modified weights are in $\mathbf{V}(\mathcal{A})$, then each triple point must have weight 0. There is a unique solution to the corresponding system of twelve equations in the nine unknowns: $k_i = -1/3$, $k_\infty = 8/3$. In particular, there is no integer solution.

Resonant weights occur in some interesting problems. Cohen and Suciu [CS1] showed that calculation of the cohomology groups with constant coefficients of the Milnor fiber of a central arrangement leads to local coefficient cohomology groups of the complement of the decone. Many of the resulting weight systems are resonant. Examples 6.5.2 and 6.5.3 show that there are two possibilities. If some integer modification of $\lambda$ is in $\mathbf{V}(\mathcal{A})$, then we may use the complex $(\mathsf{A}^{\cdot}(\mathcal{A}), a_\lambda\wedge)$ to calculate $H^p(M(\mathcal{A}), \mathcal{L}_\lambda)$. This is an effective algorithm. If no integer modification of $\lambda$ is in $\mathbf{V}(\mathcal{A})$, then there is no effective method of calculation at present. The best results are the following inequalities:

$$\dim H^p(\mathsf{A}^{\cdot}(\mathcal{A}), a_\lambda\wedge) \leq \dim H^p(M(\mathcal{A}), \mathcal{L}_\lambda) \leq \dim \mathsf{A}^p(\mathcal{A}).$$

The upper bound was obtained by Cohen [Co2], the lower bound by Libgober and Yuzvinsky [LY]. In the special case of 2-arrangements, the local system cohomology groups are computable for all weights using Fox calculus by methods of Cohen and Suciu [CS1]. These calculations reveal that there are examples where the inequalities are strict on both sides. There are several important situations where the weights are all rational, the Milnor fiber is an example. In this case, sharper upper bounds were obtained in [CO].