Chapter 7

Appendices

§A1. A counter example

Consider the following Cauchy problem

$$\begin{cases} r_t + (1+rs)r_x = 0, \\ s_t = 0, \end{cases} \tag{A1.1}$$

$$t = 0: \quad r = \varepsilon r_0(x),\ s = \varepsilon s_0(x), \tag{A1.2}$$

where $r_0(x)$ and $s_0(x)$ are C^1 functions with bounded C^1 norm, $\varepsilon > 0$ is a small parameter.

Obviously, in a neighbourhood of $(r, x) = (0, 0)$, (A1.1) is a strictly hyperbolic system with two distinct real eigenvalues

$$\lambda_1(r,s) \triangleq 1 + rs > \lambda_2(r,s) \triangleq 0. \tag{A1.3}$$

On the other hand, by Definition 3.1 it is easy to check that system (A1.1) is weakly linearly degenerate. Therefore, by Theorem 3.1 we have

Theorem A1.1. Under the hypotheses mentioned above, if $r_0(x)$ and $s_0(x)$ satisfy that there is a constant $\mu > 0$ such that

$$\sup_{x \in \boldsymbol{R}} \left\{ (1+x)^{1+\mu} \left(|r_0(x)| + |s_0(x)| + |r_0'(x)| + |s_0'(x)| \right) \right\} < \infty, \tag{A1.4}$$

then there exists $\varepsilon_0 > 0$ so small that for any given $\varepsilon \in (0, \varepsilon_0]$, the Cauchy problem (A1.1)-(A1.2) admits a unique global C^1 solution $u = u(t,x)$ on $t \geq 0$. $\square$

Now we turn to consider the following initial data:

$$t = 0: \quad r = \varepsilon \overline{r}_0(x) \stackrel{\triangle}{=} \varepsilon(1+x^2)^{-1}, \quad s = \varepsilon \overline{s}_0(x), \tag{A1.5}$$

where $\overline{s}_0(x)$ is a C^1 function satisfying

(i) $\overline{s}_0(x) \geq 0, \quad \forall\, x \in R;$

(ii) $\overline{s}_0'(x) \leq 0, \quad \forall\, x \geq 0;$

(iii) $\overline{s}_0'(x) \geq 0, \quad \forall\, x \leq 0;$

(iv) $||\overline{s}_0(x)||_{C^1(\boldsymbol{R})} \leq M_0$ (where M_0 is a positive constant);

(v) $\overline{s}_0(x) = \begin{cases} \frac{1}{1+x}, & \text{as} \quad x \geq 1, \\ 0, & \text{as} \quad x \leq -1. \end{cases}$

Noting (A1.4), we see that the initial data (A1.5) corresponds to the case that $\mu = 0$ in (A1.4). However the conclusion of Theorem A1.1 is false because we have

Theorem A1.2. There exists $\varepsilon_0 > 0$ so small that for any given $\varepsilon \in (0, \varepsilon_0]$, the first order derivatives of the C^1 solution to the Cauchy problem (A1.1) and (A1.5) must blow up in a finite time and there exist two positive constants a and b independent of ε, such that the life span $\tilde{T}(\varepsilon)$ satisfies

$$\exp\left(a\varepsilon^{-2}\right) \leq \tilde{T}(\varepsilon) \leq \exp\left(b\varepsilon^{-2}\right). \tag{A1.6}$$

$\square$

Proof. Noting the second equation in system (A1.1), we have

$$s(t,x) = \varepsilon \overline{s}_0(x), \quad \forall\, (t,x) \in \boldsymbol{R}^+ \times \boldsymbol{R}. \tag{A1.7}$$

Substituting it into the first equation in (A1.1), we observe that the Cauchy problem (A1.1) and (A1.5) simply reduces to the following Cauchy problem for a scalar equation

$$r_t + (1 + \varepsilon\overline{s}_0(x)r)\, r_x = 0, \tag{A1.8}$$

$$t = 0: \quad r = \varepsilon\overline{r}_0(x) = \varepsilon(1 + x^2)^{-1}, \tag{A1.9}$$

where $\overline{s}_0(x)$ satisfies the properties (i)-(v). Therefore, in what follows it suffices to consider the Cauchy problem (A1.8)-(A1.9).

On the existence domain of the C^1 solution to the Cauchy problem (A1.8)-(A1.9), let $x = x(t, \beta)$ be the characteristic passing through a point $(0, \beta)$ on the x-axis and set

$$\lambda(t, x) = 1 + \varepsilon\overline{s}_0(x)r(t, x). \tag{A1.10}$$

By the definition of characteristic curve, $x = x(t, \beta)$ satisfies

$$\begin{cases} \frac{dx}{dt} = \lambda(t, x), \\ t = 0: \quad x = \beta, \end{cases} \tag{A1.11}$$

on which

$$r = \varepsilon\overline{r}_0(\beta) = \varepsilon(1 + \beta^2)^{-1}. \tag{A1.12}$$

Hence, noting (A1.10) and using (A1.12), we may rewrite (A1.11) as

$$\begin{cases} \frac{dx}{dt} = 1 + \varepsilon^2\overline{s}_0(x)(1 + \beta^2)^{-1}, \\ t = 0: \quad x = \beta. \end{cases} \tag{A1.13}$$

It follows from (A1.12) that along the characteristic $x = x(t, \beta)$

$$r_x(t, x(t, \beta)) = -2\varepsilon\beta\left(1 + \beta^2\right)^{-2} / x_\beta(t, \beta). \tag{A1.14}$$

On the other hand, we obtain from (A1.13) that

$$x_\beta(t, \beta) = A(t, \beta) \exp \triangle(t, \beta), \tag{A1.15}$$

where

$$\triangle(t, \beta) = \frac{\varepsilon^2}{1 + \beta^2} \int_0^t \overline{s}_0'(x(\tau, \beta)) d\tau \tag{A1.16}$$

and

$$A(t, \beta) = 1 - \frac{2\varepsilon^2\beta}{(1 + \beta^2)^2} \int_0^t \overline{s}_0(x(\tau, \beta)) \exp\left(- \triangle\left(\tau, \beta\right)\right) d\tau. \tag{A1.17}$$

Now we estimate $\triangle(t,\beta)$.

Let ε_0 be so small that

$$\varepsilon_0^2 M_0 \le \frac{1}{2}, \tag{A1.18}$$

where M_0 is given in property (iv). Hence, noting (A1.10), (A1.12) and property (i), on the existence domain of the C^1 solution to the Cauchy problem (A1.8)-(A1.9) we have

$$1 \le \lambda(t,x) \le \frac{3}{2}. \tag{A1.19}$$

Then noting (A1.19) and the first equation in (A1.13), we obtain from (A1.16) that

$$\begin{aligned}\triangle(t,\beta) &= \tfrac{\varepsilon^2}{1+\beta^2}\int_0^t \overline{s}_0'(x(\tau,\beta))\tfrac{1}{\lambda(\tau,x(\tau,\beta))}\lambda(\tau,x(\tau,\beta))d\tau \\ &= \tfrac{\varepsilon^2}{1+\beta^2}\int_\beta^{x(t,\beta)} \overline{s}_0'(x)\tfrac{1}{1+\varepsilon^2\overline{s}_0(x)(1+\beta^2)^{-1}}dx.\end{aligned} \tag{A1.20}$$

Noting (A1.19) again and using properties (iv)-(v), we get

$$\begin{aligned}|\triangle(t,\beta)| &\le \varepsilon^2\int_{-\infty}^{\infty}|\overline{s}_0'(x)|\,dx \le \varepsilon^2\left\{\int_{-1}^{1}|\overline{s}_0'(x)|\,dx + \int_1^\infty \tfrac{1}{(1+x)^2}dx\right\} \\ &\le \varepsilon^2\left\{2M_0+\tfrac{1}{2}\right\} \le (1+2M_0)\varepsilon^2, \quad \forall\, t\ge 0, \quad \forall\, \beta\in \boldsymbol{R}.\end{aligned} \tag{A1.21}$$

Thus, we obtain

$$|\triangle(t,\beta)| \le C_1\varepsilon^2, \quad \forall\, t\ge 0, \quad \forall\, \beta\in \boldsymbol{R}, \tag{A1.22}$$

where $C_1 = 1+2M_0$ is a positive constant independent of ε.

Moreover, by (A1.11) and (A1.19), we have

$$\beta + t \le x(t,\beta) \le \beta + \frac{3}{2}t, \quad \forall\, t\ge 0, \quad \forall\, \beta\in \boldsymbol{R}. \tag{A1.23}$$

We next estimate $A(t,\beta)$.

For any $\beta\in\boldsymbol{R}$, noting (A1.22), property (i), (A1.19), the first equation in (A1.13) and (A1.23), we obtain from (A1.17) that

$$\begin{aligned}A(t,\beta) &\ge 1-\tfrac{2\varepsilon^2|\beta|}{(1+\beta^2)^2}\exp\left(C_1\varepsilon^2\right)\int_0^t \overline{s}_0(x(\tau,\beta))d\tau \\ &= 1-\tfrac{2\varepsilon^2|\beta|}{(1+\beta^2)^2}\exp\left(C_1\varepsilon^2\right)\int_0^t \overline{s}_0(x(\tau,\beta))\tfrac{\lambda(\tau,x(\tau,\beta))}{\lambda(\tau,x(\tau,\beta))}d\tau \\ &= 1-\tfrac{2\varepsilon^2|\beta|}{(1+\beta^2)^2}\exp\left(C_1\varepsilon^2\right)\int_\beta^{x(t,\beta)} \overline{s}_0(x)\tfrac{1}{1+\varepsilon^2\overline{s}_0(x)(1+\beta^2)^{-1}}dx \\ &\ge 1-\tfrac{2\varepsilon^2|\beta|}{(1+\beta^2)^2}\exp\left(C_1\varepsilon^2\right)\int_{-\infty}^{|\beta|+\frac{3}{2}t} \overline{s}_0(x)dx.\end{aligned} \tag{A1.24}$$

Case I: $|\beta| + \frac{3}{2}t \le 1$.

By properties (iv) and (v), it follows from (A1.24) that

$$\begin{aligned} A(t,\beta) &\ge 1 - \tfrac{2\varepsilon^2|\beta|}{(1+\beta^2)^2} \exp\left(C_1\varepsilon^2\right) \int_{-1}^{1} \overline{s}_0(x)dx \\ &\ge 1 - 4M_0\varepsilon_0^2 \exp\left(C_1\varepsilon_0^2\right) \\ &\ge \tfrac{1}{2}, \quad \forall\, t \in R^+, \quad \forall\, \beta \in \left\{\beta \in R \,\middle|\, |\beta| + \tfrac{3}{2}t \le 1\right\} \end{aligned} \tag{A1.25}$$

provided that $\varepsilon_0 > 0$ is suitably small.

Case II: $|\beta| + \frac{3}{2}t > 1$.

Noting properties (iv)-(v), we obtain from (A1.24) that

$$\begin{aligned} A(t,\beta) &\ge 1 - \tfrac{2\varepsilon^2|\beta|}{(1+\beta^2)^2} \exp\left(C_1\varepsilon^2\right) \left\{ \int_{-1}^{1} \overline{s}_0(x)dx + \int_{1}^{|\beta|+\frac{3}{2}t} \tfrac{1}{1+x}dx \right\} \\ &\ge 1 - C_2\tfrac{|\beta|}{(1+\beta^2)^2}\varepsilon^2 \left\{2M_0 + \ln\left(1 + |\beta| + \tfrac{3}{2}t\right) - \ln 2\right\} \\ &\ge 1 - C_2\tfrac{|\beta|}{(1+\beta^2)^2}\varepsilon^2 \left\{2M_0 + \ln\left(2\left(|\beta| + \tfrac{3}{2}t\right)\right) - \ln 2\right\} \\ &= 1 - C_2\tfrac{|\beta|}{(1+\beta^2)^2}\varepsilon^2 \left\{2M_0 + \ln\left(|\beta| + \tfrac{3}{2}t\right)\right\} \\ &\ge \tfrac{3}{4} - C_2\tfrac{|\beta|}{(1+\beta^2)^2}\varepsilon^2 \ln\left(|\beta| + \tfrac{3}{2}t\right) \\ &\ge \tfrac{3}{4} - C_2\tfrac{\varepsilon^2}{1+\beta^2} \ln\left(|\beta| + \tfrac{3}{2}t\right), \end{aligned} \tag{A1.26}$$

provided that $\varepsilon_0 > 0$ is suitably small, here and hereafter C_j $(j = 2, 3, \cdots)$ stand for positive constants independent of t, β and ε. It is easy to see that when $\varepsilon_0 > 0$ is suitably small, for any fixed $\varepsilon \in (0, \varepsilon_0]$ we have

$$\frac{1}{3}\exp\left(\frac{1}{4C_2\varepsilon^2}\right) \le \frac{2}{3}\left\{\exp\left[\frac{1+\beta^2}{4C_2\varepsilon^2}\right] - |\beta|\right\}, \quad \forall\, \beta \in R.$$

Choosing C_3 to satisfy

$$\exp\left(C_3\varepsilon^2\right) \le \frac{1}{3}\exp\left(\frac{1}{4C_2\varepsilon^2}\right),$$

we have

$$\frac{C_2\varepsilon^2}{1+\beta^2}\ln\left(|\beta| + \frac{3}{2}t\right) \le \frac{1}{4}, \quad \forall\, t \in \left[0, \exp\left(C_3\varepsilon^{-2}\right)\right].$$

Then it follows from (A1.26) that

$$A(t,\beta) \ge \frac{1}{2}, \quad \forall\, t \in \left[0, \exp\left(C_3\varepsilon^{-2}\right)\right], \quad \forall\, \beta \in \left\{\beta \in R \,\middle|\, |\beta| + \frac{3}{2}t > 1\right\}. \tag{A1.27}$$

Thus, combining (A1.25) and (A1.27) gives

$$A(t,\beta) \geq \frac{1}{2}, \quad \forall\, t \in \left[0, \exp\left(C_3\varepsilon^{-2}\right)\right], \quad \forall\, \beta \in \mathbf{R}. \tag{A1.28}$$

Therefore, noting (A1.15), (A1.22) and (A1.28), we obtain from (A1.14) that

$$|r_x(t,\beta)| \leq C_4\varepsilon, \quad \forall\, t \in \left[0, \exp\left(C_3\varepsilon^{-2}\right)\right], \quad \forall\, \beta \in \mathbf{R}. \tag{A1.29}$$

(A1.29) implies that

$$\tilde{T}(\varepsilon) \geq \exp\left(a\varepsilon^{-2}\right), \tag{A1.30}$$

where $a = C_3$ is a positive constant independent of ε.

Similarly, for any given $\beta \geq 1$ we have

$$\begin{aligned} A(t,\beta) &\leq 1 - \tfrac{2\varepsilon^2\beta}{(1+\beta^2)^2} \exp\left(-C_1\varepsilon^2\right) \int_0^t \overline{s}_0(x(\tau,\beta))d\tau \\ &= 1 - \tfrac{2\varepsilon^2\beta}{(1+\beta^2)^2} \exp\left(-C_1\varepsilon^2\right) \int_\beta^{x(t,\beta)} \tfrac{1}{1+x} \tfrac{1}{1+\varepsilon^2\overline{s}_0(x)(1+\beta^2)^{-1}} dx \\ &\leq 1 - \tfrac{4\varepsilon^2\beta}{3(1+\beta^2)^2} \exp\left(-C_1\varepsilon^2\right) \int_\beta^{x(t,\beta)} \tfrac{1}{1+x} dx \\ &= 1 - \tfrac{4\varepsilon^2\beta}{3(1+\beta^2)^2} \exp\left(-C_1\varepsilon^2\right) \left[\ln(1+x(t,\beta)) - \ln(1+\beta)\right], \quad \forall\, t \geq 0. \end{aligned} \tag{A1.31}$$

Particularly, in what follows we consider the case that $\beta = 1$.

Noting (A1.23), from (A1.31) we get

$$A(t,1) \leq 1 - \frac{1}{3}\varepsilon^2 \exp\left(-C_1\varepsilon^2\right) \left[\ln(2+t) - \ln 2\right], \quad \forall\, t \geq 0. \tag{A1.32}$$

Then it follows from (A1.32) that

$$A(t_0, 1) \leq 0, \tag{A1.33}$$

where

$$t_0 = 2\exp\left\{\frac{3}{\varepsilon^2}\exp\left(C_1\varepsilon^2\right)\right\} - 2. \tag{A1.34}$$

Noting (A1.15), (A1.22) and (A1.33), from (A1.14) we observe that the C^1 solution to the Cauchy problem (A1.8)-(A1.9) must blow up at t_0 at the latest. This implies that

$$\tilde{T}(\varepsilon) \leq t_0 \leq \exp\left(b\varepsilon^{-2}\right), \tag{A1.35}$$

where b is a positive constant independent of ε.

Combining (A1.30) and (A1.35) yields (A1.6). The proof is completed. Q.E.D.

Remark A1.1. Theorem A1.2 makes it clear that the condition that $\mu > 0$ is essential in our theory. $\square$

Finally, we consider a kind of periodic initial data

$$t = 0: \quad r = \varepsilon(1 + \sin x), \; s = \varepsilon. \tag{A1.36}$$

In the present situation, the Cauchy problem (A1.1) and (A1.36) simply reduces to the following Cauchy problem for a scalar equation

$$\begin{cases} r_t + (1 + \varepsilon r) r_x = 0, \\ t = 0: \quad r = \varepsilon(1 + \sin x). \end{cases} \tag{A1.37}$$

By the classical method of characteristics, we can easily prove the following.

Theorem A1.3. The C^1 solution to the Cauchy problem (A1.37) must blow up in a finite time and the life span $\tilde{T}(\varepsilon)$ satisfies

$$\tilde{T}(\varepsilon) = \frac{1}{\varepsilon^2}. \tag{A1.38}$$

$\square$

Theorem A1.3 makes it clear that the method used in this paper might be inapplicable to periodic initial data, even if it is small.

Remark A1.2. A detailed discussion on slow decay initial data has been carried out in [K5]. $\square$

§A2. Critical quasilinear hyperbolic systems in diagonal form

Consider the following Cauchy problem

$$\frac{\partial u}{\partial t} + \Lambda(u)\frac{\partial u}{\partial x} = 0, \tag{A2.1}$$

$$t = 0: \ u = \varepsilon\phi(x), \tag{A2.2}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown vector function, $\Lambda(u) = \mathrm{diag}(\lambda_1(u), \cdots, \lambda_n(u))$ is an $n \times n$ diagonal matrix with smooth elements $\lambda_i(u)$ $(i = 1, \cdots, n)$, $\varepsilon > 0$ is a small parameter and $\phi(x) = (\phi_1(x), \cdots, \phi_n(x))^T$ is a nontrivial C^1 vector function with compact support:

$$\mathrm{supp}\ \phi(x) \subseteq [\alpha_0, \ \beta_0],$$

in which α_0 and β_0 are two constants.

Suppose that in a neighbourhood of $u = 0$, system (A2.1) is strictly hyperbolic:

$$\lambda_1(0) < \cdots < \lambda_n(0). \tag{A2.3}$$

Moreover, suppose that system (A2.1) is critical in the sense of Definition 3.4.

Theorem A2.1. Under the assumptions mentioned above, there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$ the first order derivatives of the C^1 solution $u = u(t, x)$ to the Cauchy problem (A2.1)-(A2.2) must blow up in a finite time and the life span $\tilde{T}(\varepsilon)$ of the C^1 solution $u = u(t, x)$ satisfies

$$(1 - \kappa\varepsilon)\,\Xi \le \tilde{T}(\varepsilon) \le (1 + \kappa\varepsilon)\,\Xi, \tag{A2.4}$$

where κ is a positive constant independent of ε and

$$\Xi = \min_{i=1,\cdots,n} \left\{ -\left[\min_{x \in \mathbf{R}} \left(\frac{\partial \lambda_i}{\partial x}(0, \cdots, \varepsilon\phi_i(x), \cdots, 0) \right) \right]^{-1} \right\} > 0. \quad \square$$

Remark A2.1. It follows from (A2.4) that

$$\lim_{\varepsilon \to 0} \left\{ \tilde{T}(\varepsilon) \cdot \max_{i=1,\cdots,n} \sup_{x \in \mathbf{R}} \left\{ -\frac{\partial \lambda_i}{\partial x}(0, \cdots, \varepsilon\phi_i(x), \cdots, 0) \right\} \right\} = 1. \tag{A2.5}$$

$\square$

Proof of Theorem A2.1. On the existence domain of the C^1 solution, let $x = x_i(t,\xi)$ be the i-th characteristic passing through an arbitrary fixed point $(0,\xi)$ on the x-axis. It satisfies

$$\begin{cases} \frac{dx_i}{dt} = \lambda_i(u(t, x_i(t,\xi))), \\ t = 0: \ x_i = \xi, \end{cases} \tag{A2.6}$$

on which

$$u_i\left(t, x_i(t,\xi)\right) = \varepsilon\phi_i(\xi). \tag{A2.7}$$

Let $\varepsilon_0 > 0$ be so small that

$$\varepsilon_0 M_0 \leq \eta_0, \tag{A2.8}$$

where M_0 denotes the C^0 norm of $\phi(x)$ and $\eta_0 > 0$ is a small constant such that

$$\lambda_j(u) - \lambda_i(v) \geq \eta_1, \quad \forall\ |u|,\ |v| \leq \eta_0, \quad \forall\ j > i, \tag{A2.9}$$

where $\eta_1 > 0$ is a constant. Hence, on the existence domain of the C^1 solution we have

$$|u(t,\ x)| \leq \varepsilon||\phi(\xi)||_{C^0} \leq \eta_0. \tag{A2.10}$$

Differentiating (A2.6)-(A2.7) with respect to ξ yields

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial x_i(t,\xi)}{\partial \xi}\right) = \sum_{j=1}^{n} \frac{\partial \lambda_i(u)}{\partial u_j}\left(t, x_i(t,\xi)\right) \frac{\partial u_j(t, x_i(t,\xi))}{\partial \xi}, \\ t = 0: \ \frac{\partial x_i}{\partial \xi} = 1 \end{cases} \tag{A2.11}$$

and

$$\frac{\partial u_i\left(t, x_i\left(t,\xi\right)\right)}{\partial \xi} = \varepsilon\phi_i'(\xi). \tag{A2.12}$$

Noting (A2.12), we can rewrite (A2.11) as

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial x_i(t,\xi)}{\partial \xi}\right) = \sum_{j\neq i} \frac{\partial \lambda_i(u)}{\partial u_j}\frac{\partial u_j}{\partial \xi_j}\frac{\partial \xi_j}{\partial \xi} + \varepsilon\phi_i'(\xi)\frac{\partial \lambda_i(u)}{\partial u_i}, \\ t = 0: \ \frac{\partial x_i}{\partial \xi} = 1, \end{cases} \tag{A2.13}$$

where $\xi_j = \xi_j(t,\xi)$ stands for the x-coordinate of the intersection point of x-axis with the j-th characteristic passing through any fixed point $(t, x_i(t,\xi))$ on the i-th characteristic $x = x_i(t,\xi)$. We have

$$x_i(t,\xi) = x_j\left(t, \xi_j(t,\xi)\right). \tag{A2.14}$$

Lemma A2.1. Adoptting the symbol mentioned above, if $j < i$ (resp. $j > i$), we have

$$\frac{\partial \xi_j(t,\xi)}{\partial t} > 0 \quad (\text{resp.} < 0) \tag{A2.15}$$

and

$$\xi_j(t,\xi) \longrightarrow \infty \quad (\text{resp.} -\infty) \quad \text{as } t \longrightarrow \infty; \tag{A2.16}$$

moreover, for any given ξ with $\beta_0 \geq \xi$ (resp. $\alpha_0 \leq \xi$), there is a constant $\bar{t} > 0$ such that for any fixed $\varepsilon \in (0, \varepsilon_0]$, it follows that

$$\xi_j(t,\xi) \geq \beta_0 \quad (\text{resp.} \leq \alpha_0), \quad \forall\, t \geq \bar{t}, \tag{A2.17}$$

where $\bar{t}$ is only dependent on $\beta_0 - \xi$ (resp. $\xi - \alpha_0$) but independent of ε. □

Proof. We only prove the case that $j < i$. The proof of the case that $j > i$ is similar.

By the uniqueness of the solution of ordinary differential equation, we have

$$\frac{\partial \xi_j(t,\xi)}{\partial t} \geq 0.$$

Differentiating (A2.14) with respect to ξ gives

$$\frac{\partial x_i(t,\xi)}{\partial t} = \frac{\partial x_j(t,\xi_j)}{\partial t} + \frac{\partial x_j(t,\xi_j)}{\partial \xi_j}\frac{\partial \xi_j(t,\xi)}{\partial t}.$$

Hence we have

$$\frac{\partial x_j(t,\xi_j)}{\partial \xi_j}\frac{\partial \xi_j(t,\xi)}{\partial t} = \lambda_i(u) - \lambda_j(u) > 0.$$

Thus we obtain (A2.15) immediately.

Noting (A2.10), from (A2.6) we get

$$x_i(t,\xi) \geq \xi + \min_{|u|\leq\eta_0}\{\lambda_i(u)\}\, t$$

and

$$x_j(t,\xi_j(t,\xi)) \leq \xi_j(t,\xi) + \max_{|u|\leq\eta_0}\{\lambda_j(u)\}\, t.$$

Noting (A2.14) and (A2.9), we have

$$\xi_j(t,\xi) \geq \xi + \left\{\min_{|u|\leq\eta_0}\{\lambda_i(u)\} - \max_{|u|\leq\eta_0}\{\lambda_j(u)\}\right\} t \geq \xi + \eta_1 t.$$

This implies (A2.16).

For any given ξ with $\beta_0 \geq \xi$, taking $\bar{t} = \frac{\beta_0 - \xi}{\eta_1}$, we get

$$\xi_j(\bar{t}, \xi) \geq \xi + \eta_1 \bar{t} = \beta_0. \tag{A2.18}$$

Thus, the combination of (A2.15) and (A2.18) leads to (A2.17). Q.E.D.

Let us go on to prove Theorem A2.1.

Without loss of generality, we may assume that $\phi_n(x) \not\equiv 0$. The proofs for others ($i = 1, \cdots, n-1$) are similar.

By Lemma A2.1, for any given $\xi \in [\alpha_0, \beta_0]$, there is a positive constant t_* independent of ε such that

$$\xi_j(t, \xi) \geq \beta_0 \quad (j = 1, \cdots, n-1), \quad \text{as } t \geq t_*.$$

This indicates that

$$u_j\left(t, \xi_j(t, \xi)\right) \equiv 0 \quad (j = 1, \cdots, n-1), \quad \text{as } t \geq t_*.$$

Hence, the first equation of (A2.13) simply reduces to

$$\frac{d}{dt}\left(\frac{\partial x_n(t,\xi)}{\partial \xi}\right) = \varepsilon \phi_n'(\xi) \frac{\partial \lambda_n}{\partial u_n}\left(0, \cdots, 0, \varepsilon \phi_n(\xi)\right), \quad \text{as } t \geq t_*.$$

When $\varepsilon_0 > 0$ is suitably small, the Cauchy problem (A2.1)-(A2.2) has a unique C^1 solution $u = u(t,x)$ on $[0, t_*]$ and the solution satisfies

$$||u(t,x)||_{C^1([0,t_*]\times \mathbf{R})} \leq \kappa_0 \varepsilon, \tag{A2.19}$$

where $\kappa_0 > 0$ is a constant independent of ε (see [K3]).

Noting (A2.19), from (A2.11) have

$$1 - \kappa_* \varepsilon \leq \frac{\partial x_n(t,\xi)}{\partial \xi} \leq 1 + \kappa_* \varepsilon, \quad \forall\ t \in [0,\, t_*], \tag{A2.20}$$

where $\kappa_* > 0$ is a constant independent of ε.

Consider the following initial data problem

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial x_n}{\partial \xi}\right) = \varepsilon \phi_n'(\xi) \frac{\partial \lambda_n}{\partial u_n}\left(0, \cdots, 0, \varepsilon \phi_n(\xi)\right), & \text{as} \quad t \geq t_*, \\ t = t_* : \ \frac{\partial x_n}{\partial \xi} = \frac{\partial x_n}{\partial \xi}\left(t_*, \xi\right). \end{cases} \tag{A2.21}$$

It follows from (A2.21) that

$$\frac{\partial x_n(t,\xi)}{\partial \xi} = \frac{\partial x_n}{\partial \xi}(t_*,\xi) + \varepsilon\phi_n'(\xi)\frac{\partial \lambda_n}{\partial u_n}(0,\cdots,0,\varepsilon\phi_n(\xi))\,t, \quad \text{as } t \geq t_*. \tag{A2.22}$$

Since system (A2.1) is critical and the initial data $\phi(x)$ has a compact support, there is a point $\xi_0 \in [\alpha_0, \beta_0]$ such that

$$\begin{aligned} \Theta_n(\xi_0) &\triangleq \phi_n'(\xi_0)\tfrac{\partial \lambda_n}{\partial u_n}(0,\cdots,0,\varepsilon\phi_n(\xi_0)) \\ &= \min_{\xi \in R}\left\{\phi_n'(\xi)\tfrac{\partial \lambda_n}{\partial u_n}(0,\cdots,0,\varepsilon\phi_n(\xi))\right\} < 0, \end{aligned}$$

where we have made use of the assumption that $\phi_n(x) \not\equiv 0$. Then, it follows from (A2.22) that

$$\frac{\partial x_n}{\partial \xi}(t,\xi_0) \longrightarrow 0 \tag{A2.23}$$

as

$$t \nearrow t^* \triangleq t_* - \frac{\frac{\partial x_n}{\partial \xi}(t_*,\xi_0)}{\Theta_n(\xi_0)\,\varepsilon}. \tag{A2.24}$$

On the other hand, from (A2.12) (in which we take $i = n$) we have

$$\frac{\partial u_n}{\partial x} = \frac{\varepsilon\phi_n'(\xi)}{\frac{\partial x_n}{\partial \xi}(t,\xi)}. \tag{A2.25}$$

Thus, noting (A2.23)-(A2.25), we get

$$\left|\frac{\partial u_n}{\partial x}(t,x_n(t,\xi_0))\right| \longrightarrow \infty, \quad \text{as } t \nearrow t^*. \tag{A2.26}$$

Moreover, from (A2.20) and (A2.24), we see that there is a positive constant κ_n independent of ε such that

$$(1-\kappa_n\varepsilon)\left\{-\frac{1}{\Theta_n(\xi_0)\,\varepsilon}\right\} \leq t^* \leq (1+\kappa_n\varepsilon)\left\{-\frac{1}{\Theta_n(\xi_0)\,\varepsilon}\right\}. \tag{A2.27}$$

Similarly, we can obtain the same kind of estimates for $i = 1,\cdots,n-1$.

The combination of these estimates gives (A2.4) immediately. This completes the proof of Theorem A2.1. Q.E.D.

Remark A2.2. From (A2.4) we can draw various kinds of estimates on the life span $\tilde{T}(\varepsilon)$; particularly, for those equations discussed in §5 of [LZK2] we can obtain the same estimates given in [LZK2]. In the special case that $n = 1$, (A2.4) can be improved as (see [ZK])

$$\tilde{T}(\varepsilon) = \left[\sup_{x \in R} \left\{ -\frac{\partial \lambda}{\partial x} (\varepsilon \phi(x)) \right\} \right]^{-1}. \quad \square$$

§A3. Continuous Glimm functionals

Adopting an idea of J.Glimm [Gl], M.Schatzman [Sc] introduces the *continuous Glimm functionals* for smooth solutions to quasilinear strictly hyperbolic systems, and proves that it is a decreasing function of t. Here we aim at generalizing this result to the case that systems might be non-strictly hyperbolic so that this result has more applications.

Consider the following quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} = 0, \tag{A3.1}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown vector function of (t, x) and $A(u) = (a_{ij}(u))$ is an $n \times n$ matrix with C^1 elements $a_{ij}(u)$ $(i, j = 1, \cdots, n)$.

By hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n real eigenvalues $\lambda_1(u), \cdots, \lambda_n(u)$ and a complete system of left (resp. right) eigenvectors. For $i = 1, \cdots, n$, let $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ (resp. $r_i(u) = (r_{i1}(u), \cdots, r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$:

$$l_i(u) A(u) = \lambda_i(u) l_i(u) \quad (\text{resp. } A(u) r_i(u) = \lambda_i(u) r_i(u)). \tag{A3.2}$$

We have

$$\det |l_{ij}(u)| \neq 0 \quad (\text{equivalently, } \det |r_{ij}(u)| \neq 0). \tag{A3.3}$$

All $\lambda_i(u)$, $l_{ij}(u)$ and $r_{ij}(u)$ $(i, j = 1, \cdots, n)$ are supposed to be C^1.

Without loss of generality, we suppose that on the domain under consideration

$$l_i(u) r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \cdots, n) \tag{A3.4}$$

and

$$r_i^T(u)\, r_i(u) \equiv 1 \quad (i = 1, \cdots, n), \tag{A3.5}$$

where δ_{ij} stands for the Kronecker's symbol.

Finally, we suppose that on the domain under consideration

$$\lambda_1(u) \leq \lambda_2(u) \leq \cdots \leq \lambda_n(u). \tag{A3.6}$$

Let $u = u(t,x)$ be a C^1 solution to system (A3.1) and introduce (see [Sc])

$$\mathcal{L}(u(t)) = \sum_{i=1}^{n} \mathcal{L}_i(u(t)) \tag{A3.7}$$

and

$$\mathcal{Q}(u(t)) = \sum_{i<j} \mathcal{Q}_{ij}(u(t)), \tag{A3.8}$$

where

$$\mathcal{L}_i(u(t)) = \int_{\mathbf{R}} |w_i(t,x)|\, dx \quad (i = 1, \cdots, n), \tag{A3.9}$$

$$\mathcal{Q}_{ij}(u(t)) = \iint_{x>y} |w_i(t,x)|\, |w_j(t,y)|\, dxdy \quad (i,j = 1, \cdots, n), \tag{A3.10}$$

in which w_i is defined by (2.2.2).

Lemma A3.1. Under the hypotheses mentioned at the beginning, suppose that $u = u(t,x)$ is a C^2 solution to system (A3.1) on the domain $D(T) = \{(t,x) |\ 0 \leq t \leq T,\ |x| < \infty\}$. Suppose furthermore that the C^0 norm of $u(t,x)$ is bounded, $u_x(t,x) \to 0$ as $|x| \to \infty$ for any fixed $t \in [0,T]$, and the integrals appearing in (A3.11) and (A3.12) make sense. Then we have

$$\frac{d\mathcal{L}(u(t))}{dt} \leq c_1 \int_{\mathbf{R}} \Lambda(t,x) dx, \quad \forall\, t \in [0,T] \tag{A3.11}$$

and

$$\frac{d\mathcal{Q}(u(t))}{dt} \leq (c_2 \mathcal{L}(u(t)) - 1) \int_{\mathbf{R}} \Lambda(t,x) dx, \quad \forall\, t \in [0,T], \tag{A3.12}$$

where c_1 and c_2 are two positive constants only dependent of the C^0 norm of $u(t,x)$ but independent of t and T, and

$$\Lambda(t,x) = \sum_{i>j} (\lambda_i(u) - \lambda_j(u))\, |w_i||w_j|. \tag{A3.13}$$

□

Proof. Multiplying sgnw_i on both sides of (5.1.22) yields

$$\frac{\partial|w_i|}{\partial t}+\frac{\partial\left(\lambda_i(u)|w_i|\right)}{\partial x}=\sum_{j,k=1}^{n}\Gamma_{ijk}(u)w_jw_k\mathrm{sgn}w_i \qquad a.e.\ \text{in } (t,x) \quad (i=1,\cdots,n). \tag{A3.14}$$

Integrating (A3.14) from $-\infty$ to ∞ with respect to x and summing up it with respect to i, we get

$$\frac{d\mathcal{L}(u(t))}{dt}\le\int_R\sum_{i=1}^{n}\sum_{j,k=1}^{n}|\Gamma_{ijk}(u)||w_j||w_k|(t,x)dx \tag{A3.15}$$

and then, noting (5.1.19), we obtain (A3.11).

On the other hand, for any given $p,\ q\in\{1,\cdots,n\}$ with $p<q$, by (A3.14) we have

$$\frac{\partial|w_p|}{\partial t}+\frac{\partial\left(\lambda_p(u)|w_p|\right)}{\partial x}=\sum_{j,k=1}^{n}\Gamma_{pjk}(u)w_jw_k\mathrm{sgn}w_p \qquad a.e.\ \text{in } (t,x) \tag{A3.16}$$

and

$$\frac{\partial|w_q|}{\partial t}+\frac{\partial\left(\lambda_q(u)|w_q|\right)}{\partial x}=\sum_{j,k=1}^{n}\Gamma_{qjk}(u)w_jw_k\mathrm{sgn}w_q \qquad a.e.\ \text{in } (t,x). \tag{A3.17}$$

Taking the following procedure:

$$\int_{-\infty}^{\infty}\left[|w_q(t,x)|\int_x^{\infty}(A3.16)d\xi+|w_p(t,x)|\int_{-\infty}^{x}(A3.17)d\eta\right]dx$$

and noting (5.1.19), we get

$$\begin{aligned}
&\frac{d}{dt}\left[\iint\limits_{x>y}|w_p(t,x)|\,|w_q(t,y)|\,dxdy\right]+\int_R\left(\lambda_q(u)-\lambda_p(u)\right)|w_p(t,x)|\,|w_q(t,x)|\,dx\\
&\qquad\le\int_{-\infty}^{\infty}|w_q(t,x)|\,dx\cdot\int_{-\infty}^{\infty}\sum_{j,k=1}^{n}|\Gamma_{pjk}(u)|\,|w_j|\,|w_k|\,d\xi+\\
&\qquad\quad\int_{-\infty}^{\infty}|w_p(t,x)|\,dx\cdot\int_{-\infty}^{\infty}\sum_{j,k=1}^{n}|\Gamma_{qjk}(u)|\,|w_j|\,|w_k|\,d\eta\\
&\qquad\le C_1\mathcal{L}(u(t))\int_{-\infty}^{\infty}\Lambda(t,x)dx,
\end{aligned} \tag{A3.18}$$

henceforth C_i $(i = 1, 2)$ denote positive constants only dependent of the C^0 norm of $u(t, x)$ but independent of t and T. Summing up (A3.18) with respect to p, $q \in \{1, 2, \cdots n\}$ with $p < q$ gives

$$\frac{dQ(u(t))}{dt} + \int_R \Lambda(t, x)dx \leq C_2\mathcal{L}(u(t)) \int_{-\infty}^{\infty} \Lambda(t, x)dx, \qquad (A3.19)$$

which is nothing but the desired (A3.12). Q.E.D.

¿From Lemma A3.1 we get immediately the following.

Lemma A3.2. Under the assumptions of Lemma A3.1, if the total variation on space of u is small enough ($\mathcal{L}(u(t))$ is small enough) for any fixed $t \in [0, T]$, then there exists a positive constant M only dependent of the C^0 norm of $u(t, x)$ but independent of t and T, such that

$$\mathcal{F}(u(t)) \triangleq \mathcal{L}(u(t)) + MQ(u(t)) \qquad (A3.20)$$

is a non-increasing function of t, namely,

$$\frac{d\mathcal{F}(u(t))}{dt} \leq 0, \quad \forall\, t \in [0, T]. \qquad (A3.21)$$

□

$\mathcal{F}(u(t))$ defined by (A3.20) is actually the *continuous Glimm functionals* for smooth solutions (see [Sc]).

Theorem A3.1. Under the hypotheses mentioned at the beginning, suppose that $u = u(t, x)$ is a C^1 solution to system (A3.1) on the domain $D(T) = \{(t, x) |\ 0 \leq t \leq T,\ |x| < \infty\}$. Suppose furthermore that the C^0 norm of $u(t, x)$ is bounded, $u_x(t, x) \to 0$ as $|x| \to \infty$ for any fixed $t \in [0, T]$, and the integrals appearing in (A3.11) and (A3.12) make sense. Suppose finally that the total variation on space of u is small enough ($\mathcal{L}(u(t))$ is small enough) for any fixed $t \in [0, T]$. Then, there exists a positive constant M only dependent of the C^0 norm of $u(t, x)$ but independent of t and T, such that the functional $\mathcal{F}(u(t))$ (see (A3.20)) is a non-increasing function of t. □

Using the technique adopted in the proof of Lemma 2.2 in [Sc], we get Theorem A3.1 from Lemma A3.2 directly.

In what follows, we consider Cauchy problem for system (A3.1) with the following initial data

$$t = 0 : \quad u = \varphi(x), \tag{A3.22}$$

where $\varphi(x)$ is a C^1 vector function satisfying that there exists a constant $\mu > 0$ such that

$$\theta \triangleq \sup_{x \in \boldsymbol{R}} \left\{ (1 + |x|)^{1+\mu} |\varphi'(x)| \right\} < \infty. \tag{A3.23}$$

Lemma A3.3. Under the assumptions mentioned at the beginning, there exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0, \theta_0]$, on any given existence domain $D(T)$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem (A3.1) and (A3.22) there exist positive constants c_3 and c_4 independent of θ and T, such that the following uniform *a priori* estimates hold:

$$\mathcal{L}(u(t)) \le \mathcal{L}(\varphi) + c_3 (\mathcal{L}(\varphi))^2, \quad \forall\, t \in [0, T] \tag{A3.24}$$

and

$$||u(t, \cdot) - \varphi(0)||_{C^0} \triangleq \sup_{x \in \boldsymbol{R}} |u(t, x) - \varphi(0)| \le c_4 \mathcal{L}(\varphi), \quad \forall\, t \in [0, T]. \tag{A3.25}$$

□

Remark A3.1. It is easy to see that

$$\mathcal{L}(\varphi) \le C_0 \theta, \tag{A3.26}$$

where C_0 is a positive constant. Hence, (A3.23) implies the boundedness of u and $\mathcal{L}(u(t))$, provided that θ is small. □

Proof of Lemma A3.3. For the time being it is supposed that on the whole domain $D(T)$ we have

$$|u(t, x) - \varphi(0)| \le \delta, \tag{A3.27}$$

where $\delta > 0$ is a small number.

At the end of the proof we shall explain that this hypothesis is reasonable.

In what follows, C_i $(i = 3, 4, \cdots)$ will denote positive constants independent of $\mathcal{L}(\varphi)$ (or θ) and T.

Noting (A3.23), we have

$$\varphi(x) - \varphi(0) = \int_0^x \varphi'(\xi)d\xi, \tag{A3.28}$$

then

$$|\varphi(x) - \varphi(0)| \leq C_3\mathcal{L}(\varphi). \tag{A3.29}$$

By continuity, it is easy to see that there exists $T_\varrho > 0$ small enough such that

$$||u(t, \cdot) - \varphi(0)||_{C^0} \leq 2C_3\mathcal{L}(\varphi), \quad \forall\, t \in [0, T_\varrho] \tag{A3.30}$$

and

$$||w(t, \cdot)||_{C^0} \leq C_4, \quad \forall\, t \in [0, T_\varrho]\,. \tag{A3.31}$$

By Lemma 3.2 in [LZK2], we observe that there exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0, \theta_0]$, $w = w(t, x)$ is integrable in space for any fixed $t \in [0, T_\varrho]$ (then the integrals appearing in (A3.11) and (A3.12) make sense on the strip $[0, T_\varrho] \times R$), and

$$|w(t, x)| \leq C_\varrho\theta, \quad \forall\, t \in [0, T_\varrho]\,, \quad \forall\, x \in R, \tag{A3.32}$$

where C_ϱ is a positive constant independent of T_ϱ and T_ϱ is suitably small; moreover, on any given existence domain $D(T)$

$$w(t, x) \to 0 \quad \text{as} \quad |x| \to \infty. \tag{A3.33}$$

Integrating (A3.14) from $-\infty$ to ∞ with respect to x and using (A3.27) and (A3.32)-(A3.33), we obtain

$$\frac{d\mathcal{L}(u(t))}{dt} \leq C_5\mathcal{L}(u(t)), \quad \forall\ t \in [0, T_\varrho]\,, \tag{A3.34}$$

then we get

$$\mathcal{L}(u(t)) \leq \mathcal{L}(\varphi)\exp\left(C_5 T_\varrho\right), \quad \forall\, t \in [0, T_\varrho]\,. \tag{A3.35}$$

Hence, we can choose $T_\varrho > 0$ so small that

$$\mathcal{L}(u(t)) \le 2\mathcal{L}(\varphi), \quad \forall\, t \ \in [0, T_\varrho]\,. \tag{A3.36}$$

Thus, when $\theta_0 > 0$ is suitably small, for any fixed $\theta \in [0, \theta_0]$, noting (A3.26) and using Theorem A3.1, we obtain

$$\begin{aligned} \mathcal{L}(u(t)) \ &\le \ \mathcal{L}(u(t)) + MQ(u(t)) \le \mathcal{L}(\varphi) + MQ(\varphi) \\ &\le \ \mathcal{L}(\varphi) + M(\mathcal{L}(\varphi))^2, \quad \forall\, t \in [0, T_\varrho]\,. \end{aligned} \tag{A3.37}$$

Noting (A3.30) and (A3.37), we see that (A3.24) and (A3.25) hold at least for $t \in [0, T_\varrho]$ if we take $c_3 \ge M$ and $c_4 \ge 2C_3$.

We now prove that two positive constants c_3 and c_4 independent of $\mathcal{L}(\varphi)$ and T can be chosen in such a way that (A3.24)-(A3.25) hold on the whole interval $[0, T]$, provided that $\theta_0 > 0$ is small enough. For this purpose, it suffices to show that we can choose c_3 and c_4 such that for any fixed T_0 $(0 < T_0 \le T)$ such that

$$\mathcal{L}(u(t)) \le \mathcal{L}(\varphi) + 2c_3(\mathcal{L}(\varphi))^2, \quad \forall\, t \in [0, T_0] \tag{A3.38}$$

and

$$||u(t,\cdot) - \varphi(0)||_{C^0} \le 2c_4\mathcal{L}(\varphi), \quad \forall\, t \in [0, T_0]\,, \tag{A3.39}$$

we have

$$\mathcal{L}(u(t)) \le \mathcal{L}(\varphi) + c_3(\mathcal{L}(\varphi))^2, \quad \forall\, t \in [0, T_0] \tag{A3.40}$$

and

$$||u(t,\cdot) - \varphi(0)||_{C^0} \le c_4\mathcal{L}(\varphi), \quad \forall\, t \in [0, T_0]\,, \tag{A3.41}$$

provided that $\theta_0 > 0$ is small enough.

In fact, if $\theta_0 > 0$ is suitably small, for any fixed $\theta \in [0, \theta_0]$, noting (A3.38) and (A3.26) we observe that Theorem A3.1 is valid on the domain $D(T_0) = \{(t, x)|\ 0 \le t \le T_0,\ |x| < \infty\}$, and then we obtain

$$\mathcal{L}(u(t)) \le \mathcal{L}(\varphi) + M(\mathcal{L}(\varphi))^2, \quad \forall\, t \in [0, T_0]\,. \tag{A3.42}$$

Since we have taken $c_3 \ge M$, from (A3.42) we get (A3.40) immediately.

On the other hand, similar to (A3.29), when $\theta_0 > 0$ is suitably small we have

$$\begin{aligned} ||u(t,\cdot) - \varphi(0)||_{C^0} &\le C_6\mathcal{L}(u(t)) \le C_6\left(\mathcal{L}(\varphi) + M(\mathcal{L}(\varphi))^2\right) \\ &\le 2C_6\mathcal{L}(\varphi), \quad \forall\, t \in [0, T_0], \end{aligned} \tag{A3.43}$$

then, taking $c_4 \ge 2C_6$ $(\ge 2C_3)$, we get (A3.41) immediately.

Moreover, taking $\theta_0 > 0$ suitably small and noting (A3.26), from (A3.25) we get (A3.27) easily. This implies the validity of hyperthesis (A3.27). The proof is completed. Q.E.D.

In particular, we suppose that on the domain under consideration, each eigenvalue of $A(u)$ has a constant multiplicity. Without loss of generality, we may suppose that

$$\lambda(u) \triangleq \lambda_1(u) \equiv \cdots \equiv \lambda_p(u) < \lambda_{p+1}(u) < \cdots < \lambda_n(u), \tag{A3.44}$$

where $1 \le p \le n$. When $p = 1$, system (A3.1) is strictly hyperbolic; while, when $p > 1$, (A3.1) is a non-strictly hyperbolic system with characteristics with constant multiplicity.

Theorem A3.2. Under the assumptions of Lemma A3.3, if (A3.44) holds, then there exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0, \theta_0]$, on any given existence domain $D(T)$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (A3.1) and (A3.22), the following estimate holds:

$$\int_{\tilde{C}_j} |w_i(t,x)|dt \le c_5\theta, \tag{A3.45}$$

where $\tilde{C}_j$ stands for any given j-th characteristic curve in which $j \in \{p+1, \cdots, n\}$ if $i \in \{1, \cdots, p\}$; while $j \ne i$ if $i \in \{p+1, \cdots, n\}$, and c_5 is a positive constant independent of θ, T and $\tilde{C}_j$. □

Proof. Using the technique adopted in the proof of Lemma 2.2 in [Sc], we observe that it suffices to prove (A3.45) for u of C^2 class.

Taking θ_0 suitably small and noting (A3.44) and (A3.25)-(A3.26), we see that on the existence domain of the C^1 solution to the Cauchy problem (A3.1) and

(A3.22), there exists a positive constant c_0 independent of θ and T such that

$$\begin{aligned} &\lambda_{p+1}(u) - \lambda_i(u) \geq c_0 \quad (i = 1, \cdots, p), \\ &\lambda_{j+1}(u) - \lambda_j(u) \geq c_0 \quad (j = p+1, \cdots, n-1). \end{aligned} \tag{A3.46}$$

Moreover, noting (A3.24) and (A3.26) and using (A3.12) we have

$$\frac{dQ(u(t))}{dt} \leq -\frac{1}{2}\int_{\boldsymbol{R}} \Lambda(t,x)dx, \tag{A3.47}$$

provided that $\theta_0 > 0$ is suitably small. Then, noting that

$$\mathcal{Q}(u(t)) \leq C_7(\mathcal{L}(u(t)))^2, \quad \forall\, t \in [0,T] \tag{A3.48}$$

and using (A3.26) again, we obtain from (A3.47) that

$$\int_0^T \int_{\boldsymbol{R}} \Lambda(t,x)dxdt \leq 2\mathcal{Q}(\varphi) \leq C_8\theta^2. \tag{A3.49}$$

By (5.1.18) we have

$$\int_{\partial D} |w_i|(dx - \lambda_i(u)dt) = \iint\limits_D \operatorname{sgn}\, w_i \sum_{j,k=1}^n \Gamma_{ijk} w_j w_k dtdx, \tag{A3.50}$$

where D is the part of the strip: $[0,T] \times \boldsymbol{R}$ to the left of the j-th characteristic curve $\tilde{C}_j$. On the other hand, we obtain from (A3.46) that on $\tilde{C}_j$

$$\begin{aligned} |dx - \lambda_i dt| = |(\lambda_j(u) - \lambda_i(u))dt| \geq c_0|dt|, \;\; &j \in \{p+1, \cdots, n\}, \text{ if } i \in \{1, \cdots, p\}; \\ &j \neq i, \quad \text{if} \quad i \in \{p+1, \cdots, n\}. \end{aligned} \tag{A3.51}$$

Hence, noting (5.1.19), (A3.13) and (A3.24)-(A3.26), and using (A3.33), (A3.49) and (A3.51), we obtain from (A3.50) that

$$\begin{aligned} c_0 \int_{\tilde{C}_j} |w_i(t,x)|dx \;&\leq\; \mathcal{L}(\varphi) + \mathcal{L}(u(T)) + C_9 \int_0^T \int_{\boldsymbol{R}} \Lambda(t,x)dxdt \\ &\leq\; C_{10}\mathcal{L}(\varphi) + C_{11}\left(\mathcal{L}(\varphi)\right)^2 \\ &\leq\; C_{12}\theta, \end{aligned} \tag{A3.52}$$

provided that $\theta_0 > 0$ is suitably small.

(A3.45) follows from (A3.52) directly. Thus, the proof is completed. Q.E.D.