Chapter 1

Introduction

We are interested in the following quasilinear hyperbolic system of balance laws

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = B(u), \tag{1.1}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown vector function standing for the density of physical quantities, $A(u) = (a_{ij}(u))$ is an $n \times n$ matrix with suitably smooth elements $a_{ij}(u)$ $(i, j = 1, \cdots, n)$, it represents the gradient matrix of the flux function, $B(u) = (B_1(u), \cdots, B_n(u))^T$ is a given smooth vector function denoting for the source term. System (1.1) describes many physical phenomena. In particular, important examples occur in gas dynamics, shallow water theory, plasma physics, combustion theory, nonlinear elasticity, acoustics, classical or relativistic fluid dynamics and petroleum reservoir engineering (see [An], [CF], [CM], [LL], [Se], [Ta], etc.). These equations play an important role in both science (such as physics, mechanics, biology, etc.) and technology.

By hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n real eigenvalues $\lambda_1(u), \cdots, \lambda_n(u)$ and a complete system of left (resp. right) eigenvectors. For $i = 1, \cdots, n$, let $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ (resp. $r_i(u) = (r_{i1}(u), \cdots, r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad (\text{resp. } A(u)r_i(u) = \lambda_i(u)r_i(u)). \tag{1.2}$$

We have

$$\det |l_{ij}(u)| \neq 0 \quad (\text{equivalently, } \det |r_{ij}(u)| \neq 0). \tag{1.3}$$

All $\lambda_i(u)$, $l_{ij}(u)$ and $r_{ij}(u)$ $(i, j = 1, \cdots, n)$ are supposed to have the same regularity as $a_{ij}(u)$ $(i, j = 1, \cdots, n)$.

Without loss of generality, we suppose that on the domain under consideration

$$l_i(u) r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \cdots, n) \tag{1.4}$$

and

$$r_i^T(u) r_i(u) \equiv 1 \quad (i = 1, \cdots, n), \tag{1.5}$$

where δ_{ij} stands for the Kronecker's symbol.

Throughout this paper, we always assume that the source term $B(u)$ satisfies

$$B(0) = 0 \quad \text{and} \quad \nabla B(0) = 0. \tag{1.6}$$

This means that $B(u)$ is a nonlinear source term of higher order. In this paper we only consider "small" solutions to system (1.1).

For the following initial data

$$t = 0: \quad u = \phi(x), \tag{1.7}$$

where $\phi(x)$ is a "small" C^1 vector function of x with certain decay properties as $|x| \to +\infty$, we shall investigate the global existence or the blow-up phenomenon of the C^1 solution to the Cauchy problem (1.1) and (1.7).

It is well-known that the Cauchy problem for linear hyperbolic partial differential equations with smooth coefficients always admits a unique global classical solution on the whole domain, provided that the initial data is suitably smooth. However, the situation for *quasi*linear hyperbolic equations is quite different. Generally speaking, in the quasilinear case the classical solutions to the Cauchy problem exist only locally in time and singularities may occur after a finite time, even for the initial data which is sufficiently smooth and small. A systematic theory on the local existence and uniqueness of the classical solution to the Cauchy problem for quasilinear hyperbolic systems has been developed already (see [LY] and

[Ma]). The global theory is presently under investigation and is of great interest. Therefore, it is natural to propose the following three kinds of problems:

(P1) *Under what conditions does the Cauchy problem for quasilinear hyperbolic systems admit a unique global classical solution? Basing on this problem, we can further study the regularity and the global behaviour of the solution, especially the asymptotic behaviour of the solution as* $t \to +\infty$.

(P2) *Under what conditions does the classical solution to the Cauchy problem blow up in a finite time? When and where does the solution blow up? Which quantities blow up? Can we further investigate the behaviour or mechanisms of the blow-up phenomenon?*

Even if the solution blows up in a finite time, physical phenomenon still exists with singularities. Therefore one wants to understand further

(P3) *How do the singularities, in particular, shocks grow out of nothing? What is the structure of the singularities? What about the stability of the singularities?*

These problems are of great importance in both theory and application.

For a single quasilinear equation, these problems have been solved completely by the method of characteristics and the Whitney's theory of singularities of mappings of the plane into the plane (see [Co] and [Na]).

A systematic theory on the global existence and the breakdown of classical solutions to quasilinear reducible hyperbolic systems has been established (see [Am], [Je], [La], [Li], etc.). However, for problem **(P3)**, according to the author's knowledge, a few results have been known. Making use of the method of characteristics, Lebaud [Le] considered the problem **(P3)** for p-system and discussed the formation of shock for the initial data of simple wave.

For general quasilinear hyperbolic systems, the known results can be summarized as follows.

Homogeneous Systems: $B(u) \equiv 0$**.** Suppose that in a neighbourhood of

$u = 0$, $A(u) \in C^2$, system (1.1) is *strictly hyperbolic*:

$$\lambda_1(0) < \cdots < \lambda_n(0) \tag{1.8}$$

and *genuinely nonlinear* in the sense of P.D.Lax:

$$\nabla\lambda_i(u)r_i(u) \neq 0 \quad (i = 1, \cdots, n). \tag{1.9}$$

Suppose furthermore that $\varphi(x)$ is a C^2 function with compact support:

$$\text{supp}\, \varphi(x) \subseteq [\alpha_0, \beta_0] \quad \text{(where } \alpha_0 \text{ and } \beta_0 \text{ are constants).} \tag{1.10}$$

F.John [Jo] proved that if

$$\tilde{\theta} \triangleq (\beta_0 - \alpha_0)^2 \sup_{x \in \boldsymbol{R}} |\varphi''(x)| > 0 \tag{1.11}$$

is small enough, then the first order derivatives of the C^2 solution to the Cauchy problem (1.1) and (1.7) must blow up in a finite time. Although he still adopted the method of characteristics, he derived the formula on the decomposition of waves and reduced a Riccati's differential equation $z' = az^2$ and then proved the result mentioned above.

T.P.Liu [Lu] generalized John's result to the case that one part of characteristics is genuinely nonlinear, while the other part is *linearly degenerate* in the sense of P.D.Lax, i.e., in a neighbourhood of $u = 0$, for the corresponding indices i

$$\nabla\lambda_i(u)r_i(u) \equiv 0. \tag{1.12}$$

In this situation he showed that for a quite large class of initial data, the first order derivatives of the C^2 solution blow up in a finite time. His result can be applied to the system of one-dimensional gas dynamics.

L.Hörmander [Hol] reproved John's result by a self-contained and somewhat simplified exposition of the method. Moreover, by determining the time of blow-up asymptotically, he gave a sharp estimate on the life span of the solution.

Employing the nonlinear geometrical optics, S.Alinhac [Al] reconsidered the result presented in [Hol] and gave a more precise estimate on the life span.

Recently, by introducing the concept of *weak linear degeneracy*, Li Ta-tsien, Zhou Yi and Kong De-xing [LZK2] gave a complete result on the global existence and the life span of the C^1 solution to the Cauchy problem (1.1) and (1.7), where characteristics of system (1.1) might be neither genuinely nonlinear nor linearly degenerate, and the initial data $\varphi(x)$ is small in the following sense: there exists a constant $\mu > 0$ such that

$$\theta \triangleq \sup_{x \in \boldsymbol{R}} \left\{ (1+|x|)^{1+\mu} \left(|\varphi(x)| + |\varphi'(x)| \right) \right\} \tag{1.13}$$

is small. Li Ta-tsien, Kong De-xing and Zhou Yi [LKZ] generalized the result presented in [LZK2] to the case that the quasilinear system might be non-strictly hyperbolic.

Inhomogeneous systems: $B(u) \not\equiv 0$. In this case, system (1.1) is non-conservative. It can be used to describe some physical problems such as the dynamical systems with dissipation of energy; nonlinear three-wave interaction in plasma physics; propagation of waves in optical fibre, etc. (see [BFJ], [KRB], [WW]). When $B(u)$ is *linearly dissipative*, namely, $B(0) = 0$ and the matrix $L(0)\nabla B(0)(L(0))^{-1}$ is *row-diagonally (or column-diagonally) dominant*, the global existence of the C^1 solution to (1.1) has been established (see [HL], [LQ], [K1]), where $L(0) = (l_{ij}(0))$ and $\nabla B(0) = \left(\frac{\partial B_i}{\partial u_j}(0)\right)$. People find out that the linear dissipation can prevent the formation of shock waves in nonlinear hyperbolic waves with small amplitude. W.Kosiński [Ko] studied the gradient catastrophe of the smooth solutions to some special inhomogeneous hyperbolic systems. Kong De-xing [K2] considered a quasilinear reducible hyperbolic system with a dissipative term of higher order. Assuming that the system is strictly hyperbolic and genuinely nonlinear in the sense of P.D.Lax, he proved that if the dissipative effect is not "strong", then the first order derivatives of the solution must blow up in a finite time. This result can be applied to the nonlinear wave equation with dissipative term of higher order.

For general quasilinear hyperbolic systems ($n \geq 3$), the problem **(P3)** is open.

In this paper, we shall systematically study the global existence or the blow-up phenomenon of the C^1 solution to the Cauchy problem for general quasilinear hyperbolic system (1.1) with small and decaying initial data. By means of investigating the *generalized null condition*, making use of the concept of *weak linear degeneracy* and introducing the concept of *matching condition*, we will give a complete result on the global existence and the life span of the C^1 solution to the Cauchy problem (1.1) and (1.7); furthermore, present a description on the large time behaviour of the global classical solution (if it exists) and two mechanisms of the blow-up phenomenon, particularly, the asymptotic behaviour of the life span (if the classical solution blows up in a finite time).

This paper is divided into seven chapters: Introduction, Preliminaries, Quasilinear strictly hyperbolic systems, Quasilinear non-strictly hyperbolic systems, Homogeneous quasilinear hyperbolic systems, Applications, and Appendices. In Chapter 2 we first state two lemmas, due to L.Hörmander [Ho1], on Riccati's differential equations, then derive F.John's formula on the decomposition of waves with some supplements which will play an important role in the sequel, and finally discuss an equivalent definition of the classical solution to system (1.1). Chapter 3 deals with the Cauchy problem for quasilinear strictly hyperbolic systems. In this chapter we give a systematic result on the global existence and the life span of the C^1 solution to the Cauchy problem for quasilinear strictly hyperbolic system (1.1) with small and decaying initial data (1.7). Moreover, we investigate the large time behaviour of the global classical solution in the case that it exists; in the case that the C^1 solution blows up in a finite time, we give the asymptotic behaviour of the life span of the classical solution and illustrate that envelope of characteristics of the same family will appear and singularities just occur at the starting point of the envelope, i.e., the point with minimum t-value on the envelope. The chapter ends with some remarks on *critical* systems.

Chapter 4 aims at generalizing the result presented in Chapter 3 to the case that system (1.1) might be non-strictly hyperbolic. Basing on a detailed investigation on the *generalized null condition* and assuming the existence of the *normalized transformation* or the *normalized coordinates*, we establish some important relations on the decomposition of waves. Through these relations, we obtain a series of results similar to those in Chapter 3.

A special discussion on homogeneous quasilinear hyperbolic systems is carried out in Chapter 5. The whole discussion in Chapter 3 and Chapter 4 is based on the existence of the normalized coordinates. However, in the non-strictly hyperbolic case, in general we do not know if there exist the normalized coordinates, and even if the normalized coordinates exist, it is still very hard to check the hypotheses given in the normalized coordinates. Therefore, a consideration without the normalized coordinates is needed. In this chapter, essentially restricting our system in such a way that each characteristic is either genuinely nonlinear or linearly degenerate in the sense of P.D.Lax, and without using the normalized coordinates, we obtain more results including a limit formula on life span of the C^1 solution to the Cauchy problem (1.1) and (1.7). In particular, by means of *continuous Glimm functionals* for classical solutions to general homogeneous quasilinear hyperbolic systems (see [Sc]), we establish a uniform *a priori* estimate on the C^0 norm of the C^1 solution to the Cauchy problem (1.1) and (1.7), where the initial data $\varphi(x)$ is small in the following sense: there exists a constant $\mu > 0$ such that

$$\overline{\theta} = \sup_{x \in \boldsymbol{R}} \left\{ (1 + |x|)^{1+\mu} \, |\varphi'(x)| \right\} > 0 \tag{1.14}$$

is small. In this chapter, we only need $\overline{\theta}$ small instead of requiring θ small (see (1.13) for the definition of θ). Hence $\varphi(x)$ might be monotone. Thus we can further consider the monotone initial data, and obtain a lower bound on the life span of the C^1 solution to the Cauchy problem for system (1.1) with certain monotone initial data (1.7). Moreover, some examples of quasilinear hyperbolic systems with constant characteristics are constructed to illustrate two mechanisms of breakdown of C^1 solutions to homogeneous quasilinear hyperbolic systems, and the difference between diagonalizable and non-diagonalizable quasilinear hyperbolic systems.

Chapter 6 is devoted to the applications of our general theory to some physical systems and a system related to geometric problems. These systems include the quasilinear canonical system related to the Monge-Ampère equation, the system of nonlinear three-wave interaction in plasma physics, the nonlinear wave equation with higher order dissipation, the system of one-dimensional gas dynamics with higher order damping, the system of motion of an elastic string, the system of plane elastic waves for hyperelastic materials and the nonlinear wave equation with scalar operators of higher order. For these systems, we give a complete result on the global existence or the blow-up phenomenon, particularly, the life span of the C^1 solutions to their Cauchy problems.

We finally give three appendices in Chapter 7. In Appendix 1 we construct a counter example to illustrate that the assumption on the decay rate of initial data is essential in our theory, otherwise our results might be false. In Appendix 2 we consider critical quasilinear hyperbolic systems in diagonal form and give lower and upper bounds of life span of classical solutions, this result improves the corresponding results presented in §5 of [LZK2]. In Appendix 3 we generalize the Schatzman's theorem on continuous Glimm functionals. This result plays an important role in Chapter 5.

The method employed in this paper is the *extension method of local solution.* This method requires us: first, establish the local classical solution theory, then derive some uniform *a priori* estimates on the solution. Using these uniform *a priori* estimates, we can draw the final conclusions. This method can be expressed simply as follows:

Local classical solution theory

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Uniform *a priori* estimates on solution

$\Downarrow$

Final results (Global existence or Breakdown)

Because the local classical solution theory has been established well, for example, see [LY], the key point of this method is how to establish some uniform *a priori* estimates on the solution. In this paper, according to the eigenvalues of $A(u)$, we first divide the upper half plane into different angle domains, then introduce suitable norms of the solution on these angle domains, and then we obtain different estimates on these norms by making use of the formulas on the decomposition of waves. These estimates show that i-wave is stronger in the corresponding i-th characteristic angle domain than outside. Using these estimates, we observe that the i-wave can be modeled simply by a Riccati's equation $Z = a_0 z^2 + a_1(t)z + a_2(t)$, where $a_1(t)$, $a_2(t)$ are continuous functions and a_0 is a constant, in which $a_0 = 0$ if $\lambda_i(u)$ is weakly linearly degenerate; otherwise, $a \neq 0$. Our final conclusions (including the global existence, the blow-up phenomenon, the estimate of the life span, the large time behaviour of the global classical solution, the mechanisms of breakdown of the C^1 solution, etc.) can be drawn by the theory on ordinary differential equations.

Finally, we point out that, in this paper, we only consider "small" initial data with certain decay properties as $|x| \to +\infty$, for instance, the initial data (1.7), where $\varphi(x)$ is small in the sense that θ or $\overline{\theta}$ is small, in which θ and $\overline{\theta}$ are defined by (1.13) and (1.14) respectively. The method used in this paper might be inapplicable to other kinds of initial data such as periodic functions, etc. (see Appendix 1). For periodic initial data, reader may refer to S.Klainerman and A.Majda [KM].