

Gepner point and strong Bogomolov-Gieseker inequality for quintic 3-folds

Yukinobu Toda

Abstract.

We propose a conjectural stronger version of Bogomolov-Gieseker inequality for stable sheaves on quintic 3-folds. Our conjecture is derived from an attempt to construct a Bridgeland stability condition on graded matrix factorizations, which should correspond to the Gepner point via mirror symmetry and Orlov equivalence. We prove our conjecture in the rank two case.

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§1. Introduction

1.1. Bogomolov-Gieseker (BG) inequality

First of all, let us recall the following classical result by Bogomolov and Gieseker:

Theorem 1. ([Bog78], [Gie79]) *Let X be a smooth projective complex variety and H an ample divisor in X . For any torsion free H -slope stable sheaf E on X , we have*

$$\Delta(E) \cdot H^{\dim X - 2} \geq 0.$$

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Here $\Delta(E)$ is the discriminant

$$\Delta(E) := \text{ch}_1(E)^2 - 2 \text{ch}_0(E) \text{ch}_2(E).$$

It has been an interesting problem to improve the BG inequality for higher rank stable sheaves (cf. [Jar07], [Nak07]). So far such an improvement is only known for some particular surfaces, e.g. K3 surfaces or Del Pezzo surfaces, which easily follows from Riemann-Roch theorem and Serre duality (cf. Lemma 16, [DRY, Appendix A]). In the 3-fold case, such an improvement is only known for rank two stable sheaves on \mathbb{P}^3 by Hartshorne [Har78]. In a case of other 3-fold, even a conjectural improvement is not known. The purpose of this note is to propose a conjectural improvement of BG inequality for stable sheaves on quintic 3-folds, motivated by an idea from mirror symmetry and matrix factorizations. We first state the resulting conjecture:

Conjecture 1.1. *Let $X \subset \mathbb{P}_{\mathbb{C}}^4$ be a smooth quintic 3-fold and $H := c_1(\mathcal{O}_X(1))$. Then for any torsion free H -slope stable sheaf on X with $c_1(E)/\text{rank}(E) = -H/2$, we have the following inequality:*

$$(1) \quad \frac{\Delta(E) \cdot H}{\text{rank}(E)^2} > 1.5139 \dots$$

The RHS of (1) is a certain irrational real number contained in $\mathbb{Q}(e^{2\pi\sqrt{-1}/5})$, and the detail will be discussed in Conjecture 3.2. Our conjecture is derived from an attempt to construct a Bridgeland stability condition on $D^b \text{Coh}(X)$ corresponding to the Gepner point in the stringy Kähler moduli space of X (cf. Subsection 3.1). The RHS of (1) is related to the coefficient of the corresponding central charge. It seems that Conjecture 1.1 does not appear in literatures even in the rank two case, which we will give a proof in this note:

Proposition 1.1. *Conjecture 1.1 is true if $\text{rank}(E) = 2$.*

The above result will be proved in Subsection 3.7. Based on a similar idea, we also propose a conjectural Clifford type bound for stable coherent systems on quintic surfaces (cf. Section 4). Below we discuss background of the derivation of the above conjecture.

1.2. Background

The notion of stability conditions on triangulated categories introduced by Bridgeland [Bri07] has turned out to be an important mathematical object to study. However it has been a problem for more than ten years to construct Bridgeland stability conditions on the derived categories of coherent sheaves on quintic 3-folds. From a picture of the

mirror symmetry, the space of stability conditions on a quintic 3-fold is expected to be related to its stringy Kähler moduli space, which is described in Figure 1. In Figure 1, we see three special points, large volume limit, conifold point and Gepner point. A conjectural construction of a Bridgeland stability condition near the large volume limit was proposed by Bayer, Macri and the author [BMT], and we reduced the problem to showing a BG type inequality evaluating $\text{ch}_3(*)$ for certain two term complexes. The main conjecture in [BMT] is not yet proved except in the \mathbb{P}^3 case [Mac], and we face our lack of knowledge on the set of Chern characters of stable objects.

In this note, we focus on the Gepner point. A corresponding stability condition is presumably constructed as a Gepner type stability condition [Toda] with respect to the pair

$$\left(\text{ST}_{\mathcal{O}_X} \circ \otimes \mathcal{O}_X(1), \frac{2}{5} \right)$$

where $\text{ST}_{\mathcal{O}_X}$ is the Seidel-Thomas twist [ST01] associated to \mathcal{O}_X . Combined with Orlov's result [Orl09], as discussed in [Wal], such a stability condition is expected to give a natural stability condition on graded matrix factorizations of the defining polynomial of the quintic 3-fold. One may expect that constructing a Gepner type stability condition also requires such a conjectural inequality. It seems worth formulating a conjectural BG type inequality which arises in an attempt to construct a Gepner point, so that making it clear what we should know on Chern characters of stable sheaves. Our Conjecture 1.1 is the resulting output. The inequality (1) itself is interesting since there have been several attempts to improve the classical BG inequality. Assuming Conjecture 1, we construct data which presumably give a Bridgeland stability condition corresponding to the Gepner point.

Compared to the lower degree cases studied in [Toda], constructing Gepner type stability conditions is much harder in the quintic case, and most of the attempts are still conjectural. This is the reason we have separated the arguments for the quintic case from the previous paper [Toda]. We hope that the arguments in this note lead to future developments of the study of Chern characters of stable objects on 3-folds.

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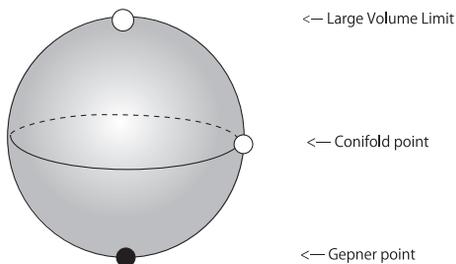


Fig. 1. Stringy Kähler moduli space of a quintic 3-fold

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1.4. Notation and convention

All the varieties or polynomials are defined over complex numbers. For a smooth projective variety X of dimension n and $E \in \text{Coh}(X)$, we write its Chern character as a vector

$$\text{ch}(E) = (\text{ch}_0(E), \text{ch}_1(E), \dots, \text{ch}_n(E))$$

for $\text{ch}_i(E) \in H^{2i}(X)$. For a triangulated category \mathcal{D} and a set of objects \mathcal{S} in \mathcal{D} , we denote by $\langle \mathcal{S} \rangle_{\text{ex}}$ the smallest extension closed subcategory in \mathcal{D} which contains \mathcal{S} .

§2. Background

2.1. Bridgeland stability condition

Let \mathcal{D} be a triangulated category and $K(\mathcal{D})$ its Grothendieck group. We first recall Bridgeland’s definition of stability conditions on it.

Definition 2.1. ([Bri07]) *A stability condition σ on \mathcal{D} consists of a pair $(Z, \{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}})$*

$$(2) \quad Z: K(\mathcal{D}) \rightarrow \mathbb{C}, \quad \mathcal{P}(\phi) \subset \mathcal{D}$$

where Z is a group homomorphism (called central charge) and $\mathcal{P}(\phi)$ is a full subcategory (called σ -semistable objects with phase ϕ) satisfying the following conditions:

- For $0 \neq E \in \mathcal{P}(\phi)$, we have $Z(E) \in \mathbb{R}_{>0} \exp(\sqrt{-1}\pi\phi)$.
- For all $\phi \in \mathbb{R}$, we have $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$.
- For $\phi_1 > \phi_2$ and $E_i \in \mathcal{P}(\phi_i)$, we have $\text{Hom}(E_1, E_2) = 0$.
- For each $0 \neq E \in \mathcal{D}$, there is a collection of distinguished triangles

$$E_{i-1} \rightarrow E_i \rightarrow F_i \rightarrow E_{i-1}[1], \quad E_N = E, \quad E_0 = 0$$

with $F_i \in \mathcal{P}(\phi_i)$ and $\phi_1 > \phi_2 > \dots > \phi_N$.

The full subcategory $\mathcal{P}(\phi) \subset \mathcal{D}$ is shown to be an abelian category, and its simple objects are called σ -stable. In [Bri07], Bridgeland shows that there is a natural topology on the set of ‘good’ stability conditions $\text{Stab}(\mathcal{D})$, and its each connected component has a structure of a complex manifold. (The above ‘good’ property means that Z factors through a fixed group homomorphism $K(\mathcal{D}) \rightarrow \Gamma$ for a fixed finitely generated abelian group Γ , together with a technical condition called support property introduced in [KS].) Let $\text{Aut}(\mathcal{D})$ be the group of autoequivalences on \mathcal{D} . There is a left $\text{Aut}(\mathcal{D})$ -action on the set of stability conditions on \mathcal{D} . For $\Phi \in \text{Aut}(\mathcal{D})$, it acts on the pair (2) as follows:

$$\Phi_*(Z, \{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}) = (Z \circ \Phi^{-1}, \{\Phi(\mathcal{P}(\phi))\}_{\phi \in \mathbb{R}}).$$

There is also a right \mathbb{C} -action on the set of stability conditions on \mathcal{D} . For $\lambda \in \mathbb{C}$, it acts on the pair (2) as follows:

$$(Z, \{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}) \cdot (\lambda) = (e^{-\sqrt{-1}\pi\lambda}Z, \{\mathcal{P}(\phi + \text{Re } \lambda)\}_{\phi \in \mathbb{R}}).$$

The notion of Gepner type stability conditions is defined as follows:

Definition 2.2. ([Toda]) *A stability condition σ on \mathcal{D} is called Gepner type with respect to $(\Phi, \lambda) \in \text{Aut}(\mathcal{D}) \times \mathbb{C}$ if the following condition holds:*

$$\Phi_*\sigma = \sigma \cdot (\lambda).$$

2.2. Gepner type stability conditions on graded matrix factorizations

Let W be a homogeneous element

$$(3) \quad W \in A := \mathbb{C}[x_1, x_2, \dots, x_n]$$

of degree d such that $(W = 0) \subset \mathbb{C}^n$ has an isolated singularity at the origin. For a graded A -module P , we denote by P_i its degree i -part, and $P(k)$ the graded A -module whose grade is shifted by k , i.e. $P(k)_i = P_{i+k}$.

Definition 2.3. *A graded matrix factorization of W is data*

$$(4) \quad P^0 \xrightarrow{p^0} P^1 \xrightarrow{p^1} P^0(d)$$

where P^i are graded free A -modules of finite rank, p^i are homomorphisms of graded A -modules, satisfying the following conditions:

$$p^1 \circ p^0 = \cdot W, \quad p^0(d) \circ p^1 = \cdot W.$$

The category $\text{HMF}^{\text{gr}}(W)$ is defined to be the triangulated category whose objects consist of graded matrix factorizations of W (cf. [Orl09]). The grade shift functor $P^\bullet \mapsto P^\bullet(1)$ induces the autoequivalence τ of $\text{HMF}^{\text{gr}}(W)$, which satisfies the following identity:

$$(5) \quad \tau^{\times d} = [2].$$

The following is the main conjecture in [Toda]:

Conjecture 2.1. *There is a Gepner type stability condition*

$$\sigma_G = (Z_G, \{\mathcal{P}_G(\phi)\}_{\phi \in \mathbb{R}}) \in \text{Stab}(\text{HMF}^{\text{gr}}(W))$$

with respect to $(\tau, 2/d)$, whose central charge Z_G is given by

$$(6) \quad Z_G(P^\bullet) = \text{str}(e^{2\pi\sqrt{-1}/d}: P^\bullet \rightarrow P^\bullet).$$

The definition of the central charge Z_G first appeared in [Wal]. It is more precisely written as follows: since P^i are free A -modules of finite rank, they are written as

$$P^i \cong \bigoplus_{j=1}^m A(n_j^i), \quad n_j^i \in \mathbb{Z}.$$

Then (6) is written as

$$Z_G(P^\bullet) = \sum_{j=1}^m \left(e^{2n_j^0 \pi \sqrt{-1}/d} - e^{2n_j^1 \pi \sqrt{-1}/d} \right).$$

So far Conjecture 2.1 is proved when $n = 1$ [Tak], $d < n = 3$ [KST07], and $n \leq d \leq 4$ [Toda]. The most important unproven case is when $n = d = 5$, in which the variety X is a quintic Calabi-Yau 3-fold.

2.3. Orlov’s theorem

We recall Orlov’s theorem [Orl09] relating the triangulated category $\text{HMF}^{\text{gr}}(W)$ with the derived category of coherent sheaves on the smooth projective variety

$$(7) \quad X := (W = 0) \subset \mathbb{P}^{n-1}.$$

We only use the results for $d = n$ case, i.e. X is a Calabi-Yau manifold, and $d = n + 1$ case, i.e. X is general type.

Theorem 2. ([Orl09, Theorem 2.5], [BFK12, Proposition 5.8]) *If $d = n$, there is an equivalence of triangulated categories*

$$\Psi: D^b \text{Coh}(X) \xrightarrow{\sim} \text{HMF}^{\text{gr}}(W)$$

such that the following diagram commutes:

$$\begin{array}{ccc} D^b \text{Coh}(X) & \xrightarrow{\Psi} & \text{HMF}^{\text{gr}}(W) \\ F \downarrow & & \downarrow \tau \\ D^b \text{Coh}(X) & \xrightarrow{\Psi} & \text{HMF}^{\text{gr}}(W). \end{array}$$

Here F is the autoequivalence given by $F = \text{ST}_{\mathcal{O}_X} \circ \otimes_{\mathcal{O}_X}(1)$.

Recall that $\text{ST}_{\mathcal{O}_X}$ is the Seidel-Thomas twist [ST01], given by

$$\text{ST}_{\mathcal{O}_X}(\ast) = \text{Cone}(\mathbf{R} \text{Hom}(\mathcal{O}_X, \ast) \otimes_{\mathcal{O}_X} \rightarrow \ast).$$

Theorem 3. ([Orl09, Theorem 2.5], [Toda, Proposition 3.22]) *If $d = n + 1$, then there is a fully faithful functor*

$$\Psi: D^b \text{Coh}(X) \hookrightarrow \text{HMF}^{\text{gr}}(W)$$

such that we have the semiorthogonal decomposition

$$\text{HMF}^{\text{gr}}(W) = \langle \mathbb{C}(0), \Psi D^b \text{Coh}(X) \rangle$$

where $\mathbb{C}(0)$ is a certain exceptional object. Moreover the subcategory

$$\mathcal{A}_W := \langle \mathbb{C}(0), \Psi \text{Coh}(X) \rangle_{\text{ex}}$$

is the heart of a bounded t -structure on $\text{HMF}^{\text{gr}}(W)$, and there is an equivalence of abelian categories

$$\Theta: \text{Syst}(X) \xrightarrow{\sim} \mathcal{A}_W.$$

Here $\text{Syst}(X)$ is the abelian category of coherent systems on X .

Recall that a coherent system on X consists of data

$$V \otimes \mathcal{O}_X \xrightarrow{s} F$$

where V is a finite dimensional \mathbb{C} -vector space, $F \in \text{Coh}(X)$ and s is a morphism in $\text{Coh}(X)$. The set of morphisms in $\text{Syst}(X)$ is given by the commutative diagrams in $\text{Coh}(X)$

$$\begin{array}{ccc} V \otimes \mathcal{O}_X & \xrightarrow{s} & F \\ \downarrow & & \downarrow \\ V' \otimes \mathcal{O}_X & \xrightarrow{s'} & F'. \end{array}$$

The equivalence Θ sends $(\mathcal{O}_X \rightarrow 0)$ to $\mathbb{C}(0)$ and $(0 \rightarrow F)$ for $F \in \text{Coh}(X)$ to $\Psi(F) \in \mathcal{A}_W$.

§3. Stronger BG inequality for quintic 3-folds

In this section, we take W to be a quintic homogeneous polynomial with five variables

$$(8) \quad W \in \mathbb{C}[x_0, x_1, x_2, x_3, x_4], \quad \deg(W) = 5.$$

The variety

$$X := (W = 0) \subset \mathbb{P}^4$$

is a smooth quintic Calabi-Yau 3-fold. This is the most interesting case in the study of Conjecture 2.1. We have an equivalence by Theorem 2

$$(9) \quad \Psi: D^b \text{Coh}(X) \xrightarrow{\sim} \text{HMF}^{\text{gr}}(W).$$

The goal of this section is to translate Conjecture 2.1 in terms of $D^b \text{Coh}(X)$, and relate it to a stronger version of BG inequality for stable sheaves on X .

3.1. Stringy Kähler moduli space of a quintic 3-fold

Let us first recall a mirror family of a quintic 3-fold X and its stringy Kähler moduli space. The mirror family of X is a simultaneous crepant resolution $\widehat{Y}_\psi \rightarrow Y_\psi$ of the following one parameter family of quotient varieties [CdLOGP91]:

$$Y_\psi := \left\{ \sum_{i=0}^5 y_i^5 - 5\psi \prod_{i=0}^5 y_i = 0 \right\} / G.$$

Here $[y_1 : y_2 : y_3 : y_4 : y_5]$ is the homogeneous coordinate of \mathbb{P}^4 , and $G = (\mathbb{Z}/5\mathbb{Z})^3$ acts on \mathbb{P}^4 by

$$\xi \cdot [y_1 : y_2 : y_3 : y_4 : y_5] = [\xi_1 y_1 : \xi_2 y_2 : \xi_3 y_3 : \xi_1^{-1} \xi_2^{-1} \xi_3^{-1} y_4 : y_5]$$

for $\xi = (\xi_i)_{1 \leq i \leq 3} \in G$. Let α be the root of unity

$$\alpha := e^{2\pi\sqrt{-1}/5}.$$

Note that we have the isomorphism

$$(10) \quad \widehat{Y}_\psi \xrightarrow{\cong} \widehat{Y}_{\alpha\psi}$$

by $y_i \mapsto y_i$ for $1 \leq i \leq 4$ and $y_5 \mapsto \alpha y_5$. Also \widehat{Y}_ψ is a non-singular Calabi-Yau 3-fold if and only if $\psi^5 \neq 1$. Hence the mirror family \widehat{Y}_ψ is parametrized by the following quotient stack (see Figure 1)

$$\mathcal{M}_K := \left[\frac{\{\psi \in \mathbb{C} : \psi^5 \neq 1\}}{\mu_5} \right]$$

where the generator of μ_5 acts on \mathbb{C} by the multiplication of α . The stack \mathcal{M}_K is called the stringy Kähler moduli space of X . We see that there are 3-special points in Figure 1:

- The point $\psi^5 = \infty$, called *Large volume limit*.
- The point $\psi^5 = 1$, called *Conifold point*.
- The point $\psi^5 = 0$, called *Gepner point*.

The mirror variety \widehat{Y}_ψ is non-singular except at the first two special points. It is also non-singular at the Gepner point, but there admits a non-trivial $\mathbb{Z}/5\mathbb{Z}$ -action by the isomorphism (10).

3.2. Relation to Bridgeland stability

We discuss a relationship between the space \mathcal{M}_K and the Bridgeland’s space

$$\text{Stab}(X) := \text{Stab}(D^b \text{Coh}(X))$$

based on the papers [Asp], [Bri09]. Let $\text{Auteq}(X)$ be the group of auto-equivalences of $D^b \text{Coh}(X)$. It is expected that there is an embedding

$$(11) \quad I: \mathcal{M}_K \hookrightarrow [\text{Auteq}(X) \backslash \text{Stab}(X) / \mathbb{C}]$$

such that, if we write

$$I(\psi) = (Z_\psi, \{\mathcal{P}_\psi(\phi)\}_{\phi \in \mathbb{R}})$$

then the central charge $Z_\psi(E)$ for $E \in D^b \text{Coh}(X)$ is a solution of the Picard-Fuchs (PF) equation which the period integrals of the mirror family \widehat{Y}_ψ should satisfy. Using the following notation

$$z := 5^{-5}\psi^{-5}, \quad \theta_z := z \frac{d}{dz}$$

the PF equation is given by

$$(12) \quad \theta_z^4 \Phi - 5z(5\theta_z + 1)(5\theta_z + 2)(5\theta_z + 3)(5\theta_z + 4)\Phi = 0.$$

The solution space of the above PF equation is known to be four dimensional. The basis at $\psi = 0$ is given by (cf. [CdLOGP91])

$$\varpi_j(\psi) := -\frac{1}{5} \sum_{m=1}^{\infty} \frac{\Gamma(m/5)}{\Gamma(m)\Gamma(1-m/5)^4} (5\alpha^{2+j}\psi)^m$$

for $0 \leq j \leq 3$. For an object $E \in D^b \text{Coh}(X)$, the central charge $Z_\psi(E)$ should satisfy the above PF equation, hence is written as

$$Z_\psi(E) = \sum_{i=0}^3 \Phi_i(\psi) \cdot H^{3-i} \text{ch}_i(E)$$

where $H := c_1(\mathcal{O}_X(1))$ and $\Phi_i(\psi)$ is a linear combination of the basis $\{\varpi_j(\psi)\}_{0 \leq j \leq 3}$ which is independent of E . Here we have identified $H^6(X, \mathbb{Q})$ with \mathbb{Q} via the integration map. On the other hand, around the large volume limit and the conifold point, the monodromy transformations induce linear isomorphisms M_L, M_C on the solution space of the PF equation (12). Hence that monodromy transformations act on the central charge $Z_\psi(E)$, which are expected to coincide with the actions of autoequivalences $\otimes_{\mathcal{O}_X}(1), \text{ST}_{\mathcal{O}_X}$ respectively. Namely we should have the following identities:

$$\begin{aligned} Z_\psi(E \otimes \mathcal{O}_X(1)) &= \sum_{i=0}^3 M_L \Phi_i(\psi) \cdot H^{3-i} \text{ch}_i(E) \\ Z_\psi(\text{ST}_{\mathcal{O}_X}(E)) &= \sum_{i=0}^3 M_C \Phi_i(\psi) \cdot H^{3-i} \text{ch}_i(E). \end{aligned}$$

The coefficients of $\Phi_i(\psi)$ are uniquely determined by the above matching property of the monodromy transformations on both sides of (11).

Indeed, the above idea is used to give an embedding similar to (11) when X is the local projective plane in [BM11]. In the quintic 3-fold

case, based on a similar idea as above, the central charges $Z_\psi(E)$ for line bundles $E = \mathcal{O}_X(m)$ are computed by Aspinwall [Asp, Equation (217)]:

$$Z_\psi(\mathcal{O}_X(m)) = \frac{1}{6}(5m^3 + 3m^2 + 16m + 6)\varpi_0(\psi) - \frac{1}{2}(3m^2 + 3m + 2)\varpi_1(\psi) - m^2\varpi_2(\psi) - \frac{1}{2}m(m - 1)\varpi_3(\psi).$$

Since e^{mH} for $m \in \mathbb{Z}$ span $H^{\text{even}}(X, \mathbb{Q})$, the above formula uniquely determines $\Phi_i(\psi)$. A direct computation shows that

$$\begin{aligned} \Phi_0(\psi) &= \frac{1}{5}(\varpi_0(\psi) - \varpi_0(\psi)) \\ \Phi_1(\psi) &= \frac{1}{30}(16\varpi_0(\psi) - 9\varpi_1(\psi) + 3\varpi_3(\psi)) \\ \Phi_2(\psi) &= \frac{1}{5}(\varpi_0(\psi) - 3\varpi_1(\psi) - 2\varpi_2(\psi) - \varpi_3(\psi)) \\ \Phi_3(\psi) &= \varpi_0(\psi). \end{aligned}$$

As a result, $Z_\psi(E)$ is written as

$$\begin{aligned} &(\varpi_0(\psi) - \varpi_1(\psi)) \text{ch}_0(E) + \frac{1}{30}(16\varpi_0(\psi) - 9\varpi_1(\psi) + 3\varpi_3(\psi)) H^2 \text{ch}_1(E) \\ &+ \frac{1}{5}(\varpi_0(\psi) - 3\varpi_1(\psi) - 2\varpi_2(\psi) - \varpi_3(\psi)) H \text{ch}_2(E) + \varpi_0(\psi) \text{ch}_3(E). \end{aligned}$$

3.3. Gepner point and Gepner type stability conditions

Let us consider a conjectural stability condition $\sigma_G \in \text{Stab}(X)$ satisfying

$$[\sigma_G] = I(0) \in [\text{Auteq}(X) \setminus \text{Stab}(X) / \mathbb{C}]$$

where I is an expected embedding (11). Since the point $\psi^5 = 0$ (Gepner point) in \mathcal{M}_K is an orbifold point with stabilizer group $\mathbb{Z}/5\mathbb{Z}$, the stability condition σ_G should also have the stabilizer group $\mathbb{Z}/5\mathbb{Z}$ with respect to the $\text{Auteq}(X) \times \mathbb{C}$ action on $\text{Stab}(X)$. Under a suitable choice of σ_G , the generator of the above stabilizer group should be given by

$$(13) \quad \left(\text{ST}_{\mathcal{O}_X} \circ \otimes \mathcal{O}_X(1), -\frac{2}{5} \right) \in \text{Auteq}(X) \times \mathbb{C}$$

since the action of $\text{ST}_{\mathcal{O}_X} \circ \otimes \mathcal{O}_X(1)$ on $H^{\text{even}}(X, \mathbb{Q})$ corresponds to the composition of monodromy transformations at the large volume limit and the conifold point under the embedding (11), and the five times composition of $\text{ST}_{\mathcal{O}_X} \circ \otimes \mathcal{O}_X(1)$ coincides with [2]. (This is a consequence of Theorem 2 and the identity (5).) The property of σ_G

fixed by (13) is nothing but the Gepner type property with respect to $(\mathrm{ST}_{\mathcal{O}_X} \circ \otimes \mathcal{O}_X(1), 2/5)$. By the above argument and Theorem 2, a stability condition corresponding to the Gepner point gives a desired stability condition in Conjecture 2.1 via Orlov equivalence (9).

As for the central charge at the Gepner point, we consider the normalized central charge Z_G^\dagger so that $Z_G^\dagger(\mathcal{O}_x) = -1$ holds for any $x \in X$. Under this normalization, Z_G^\dagger is given by

$$\begin{aligned} Z_G^\dagger(E) &:= \lim_{\psi \rightarrow 0} -Z_\psi(E)/\varpi_0(\psi) \\ &= -\mathrm{ch}_3(E) + \frac{1}{5}(\alpha^3 + 2\alpha^2 + 3\alpha - 1)H \mathrm{ch}_2(E) \\ &\quad + \frac{1}{30}(-3\alpha^3 + 9\alpha - 16)H^2 \mathrm{ch}_1(E) + (\alpha - 1) \mathrm{ch}_0(E). \end{aligned}$$

Indeed, the coefficients $\alpha_j^\dagger \in \mathbb{C}H^{3-j}$ of $Z_G^\dagger(E)$ at $\mathrm{ch}_j(E)$ are checked to form the unique solution of the linear equation

$$(\alpha_0^\dagger, \dots, \alpha_3^\dagger) \cdot M = \alpha \cdot (\alpha_0^\dagger, \dots, \alpha_3^\dagger), \quad \alpha_3^\dagger = -1$$

where M is given by the composition of matrices (cf. [Toda, Subsection 4.1])

$$M := \begin{pmatrix} 1 & -(\mathrm{td}_X)_2 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ H & 1 & 0 & 0 \\ H^2/2 & H & 1 & 0 \\ H^3/6 & H^2/2 & H & 1 \end{pmatrix}.$$

Here $(\mathrm{td}_X)_2 = 5H^2/6$ is the $H^{2,2}(X)$ -component of td_X . The above matrix M induces the isomorphism on $H^{\mathrm{even}}(X)$, which is identified with the action of $\mathrm{ST}_{\mathcal{O}_X} \circ \otimes \mathcal{O}_X(1)$ on it. By [Toda, Proposition 4.4], the central charge Z_G^\dagger is related to the central charge Z_G on $\mathrm{HMF}^{\mathrm{gr}}(W)$ given by (6) as

$$Z_G(\Psi(E)) = -(1 - \alpha)^4 \cdot Z_G^\dagger(E)$$

for any $E \in D^b \mathrm{Coh}(X)$. Here Ψ is the equivalence (9). By applying \mathbb{C} -action on $\mathrm{Stab}(X)$, Conjecture 2.1 for the polynomial (8) leads to the following conjecture:

Conjecture 3.1. *Let $X \subset \mathbb{P}_{\mathbb{C}}^4$ be a smooth quintic 3-fold, $H := c_1(\mathcal{O}_X(1))$ and $\alpha := e^{2\pi\sqrt{-1}/5}$. Then there is a Gepner type stability condition*

$$(14) \quad (Z_G^\dagger, \{\mathcal{P}_G^\dagger(\phi)\}_{\phi \in \mathbb{R}}) \in \mathrm{Stab}(X)$$

with respect to $(\text{ST}_{\mathcal{O}_X} \circ \otimes \mathcal{O}_X(1), 2/5)$, whose central charge Z_G^\dagger is given by

$$Z_G^\dagger(E) = -\text{ch}_3(E) + \frac{1}{5}(\alpha^3 + 2\alpha^2 + 3\alpha - 1)H \text{ch}_2(E) + \frac{1}{30}(-3\alpha^3 + 9\alpha - 16)H^2 \text{ch}_1(E) + (\alpha - 1) \text{ch}_0(E).$$

3.4. Some observations

Let us try to construct a desired stability condition in Conjecture 3.1. By [Bri07, Proposition 5.3], giving data (14) is equivalent to giving the heart of a bounded t-structure

$$\mathcal{A}_G \subset D^b \text{Coh}(X)$$

satisfying

$$(15) \quad Z_G^\dagger(\mathcal{A}_G \setminus \{0\}) \subset \{r \exp(\sqrt{-1}\pi\phi) : r > 0, \phi \in (0, 1]\}$$

and any object $E \in \mathcal{A}_G$ admits a Harder-Narasimhan filtration with respect to the Z_G^\dagger -stability. We propose that a desired heart \mathcal{A}_G is constructed as a double tilting of $\text{Coh}(X)$, similar to the one in [BMT]. This is motivated by the following observations:

Firstly in [Toda], we constructed a Gepner type stability condition for a quartic K3 surface S via a tilting of $\text{Coh}(S)$. The construction is similar to the one near the large volume limit in [Bri08], [AB13]. A different point is that, although we only need a classical BG inequality to construct a stability condition near the large volume limit, a construction at the Gepner point requires a stronger version of BG inequality given as follows:

Lemma 3.1. *Let S be a K3 surface and E a torsion free stable sheaf E on S with $\text{rank}(E) \geq 2$. Then we have the following inequality*

$$(16) \quad \frac{\Delta(E)}{\text{rank}(E)^2} \geq 2 - \frac{2}{\text{rank}(E)^2} \geq \frac{3}{2}.$$

The above lemma is an easy consequence of the Riemann-Roch theorem and Serre duality (cf. [Muk87, Corollary 2.5]) and a similar improvement is not known for other surfaces except Del Pezzo surfaces. By the above observation, we expect that a desired Gepner type stability condition on a quintic 3-fold is also constructed in a way similar to the one near the large volume limit, after an improvement of BG inequality.

Secondly we can rewrite the central charge $Z_G^\dagger(E)$ in the following way:

$$(17) \quad -\text{ch}_3^B(E) + aH^2 \text{ch}_1^B(E) + \sqrt{-1} \left(bH \text{ch}_2^B(E) + c \text{ch}_0^B(E) \right).$$

Here $B = -H/2$ and $\text{ch}^B(E)$ is the twisted Chern character

$$\text{ch}^B(E) := e^{-B} \text{ch}(E).$$

In (17), a, b, c are some real numbers in $\mathbb{Q}(\alpha, \sqrt{-1})$, given by

$$\begin{aligned} a &= -\frac{1}{5}\alpha^3 - \frac{1}{5}\alpha^2 - \frac{67}{120} \\ b\sqrt{-1} &= \frac{1}{5}\alpha^3 + \frac{2}{5}\alpha^2 + \frac{3}{5}\alpha + \frac{3}{10} \\ c\sqrt{-1} &= \frac{3}{8}\alpha^3 + \frac{1}{4}\alpha^2 + \frac{5}{8}\alpha + \frac{5}{16}. \end{aligned}$$

They are approximated by

$$a = -0.8819\dots, \quad b = 0.68819\dots, \quad c = 0.52088\dots.$$

The expression (17) is very similar to the central charge near the large volume limit, given by

$$Z_{B,tH}(E) := - \int_X e^{-\sqrt{-1}tH} \text{ch}^B(E)$$

for $t \in \mathbb{R}_{>0}$. The above integration is expanded as

$$(18) \quad -\text{ch}_3^B(E) + \frac{t^2}{2}H^2 \text{ch}_1^B(E) + \sqrt{-1} \left(tH \text{ch}_2^B(E) - \frac{5t^3}{6} \text{ch}_0^B(E) \right).$$

By comparing (17) with (18), although they are in a similar form, we see that some signs of the coefficients are different. In [BMT], we constructed a double tilting of $\text{Coh}(X)$ which, together with the central charge (18), conjecturally gives a Bridgeland stability condition near the large volume limit. We propose to construct the heart \mathcal{A}_G via a double tilting of $\text{Coh}(X)$ in a way similar to [BMT], by taking the difference of the signs of the coefficients into consideration.

3.5. Conjectural stronger Bogomolov-Gieseker inequality

We imitate the argument in [BMT] to construct \mathcal{A}_G . In what follows, we fix $B = -H/2$. Let $\mu_{B,H}$ be the twisted slope function on

$\text{Coh}(X)$ defined by

$$\mu_{B,H}(E) := \frac{H^2 \text{ch}_1^B(E)}{\text{rank}(E)}.$$

Here we set $\mu_{B,H}(E) = \infty$ if E is a torsion sheaf. The above slope function defines the classical slope stability on $\text{Coh}(X)$. We define the pair of full subcategories $(\mathcal{T}_{B,H}, \mathcal{F}_{B,H})$ of $\text{Coh}(X)$ to be

$$\begin{aligned} \mathcal{T}_{B,H} &:= \langle E : \mu_{B,H}\text{-semistable with } \mu_{B,H}(E) > 0 \rangle_{\text{ex}} \\ \mathcal{F}_{B,H} &:= \langle E : \mu_{B,H}\text{-semistable with } \mu_{B,H}(E) \leq 0 \rangle_{\text{ex}}. \end{aligned}$$

The above subcategories form a torsion pair in $\text{Coh}(X)$. The associated tilting $\mathcal{B}_{B,H}$ is defined to be

$$\mathcal{B}_{B,H} := \langle \mathcal{F}_{B,H}[1], \mathcal{T}_{B,H} \rangle_{\text{ex}}.$$

The category $\mathcal{B}_{B,H}$ is the heart of a bounded t-structure on $D^b \text{Coh}(X)$. In [BMT, Lemma 3.2.1], it is observed that the central charge (18) satisfies the following condition: an object $E \in \mathcal{B}_{B,H}$ with $H^2 \text{ch}_1^B(E) = 0$ satisfies $\text{Im } Z_{B,tH}(E) \geq 0$. The classical BG inequality is used to show the above property. We propose that a similar property also holds for the central charge Z_G^\dagger , i.e. an object $E \in \mathcal{B}_{B,H}$ with $H^2 \text{ch}_1^B(E) = 0$ satisfies $\text{Im } Z_G^\dagger(E) \geq 0$. Note that such an object E is contained in the category

$$\langle F[1], \text{Coh}_{\leq 1}(X) : F \text{ is } \mu_{B,H}\text{-stable with } H^2 \text{ch}_1^B(F) = 0 \rangle_{\text{ex}}$$

where $\text{Coh}_{\leq 1}(X)$ is the category of coherent sheaves $T \in \text{Coh}(X)$ with $\dim \text{Supp}(T) \leq 1$. Also noting the equality

$$\Delta(E) = \text{ch}_1^B(E)^2 - 2 \text{ch}_0^B(E) \text{ch}_2^B(E)$$

the above requirement leads to the following conjecture:

Conjecture 3.2. *Let $X \subset \mathbb{P}^4$ be a smooth quintic 3-fold and E a torsion free slope stable sheaf on X with $c_1(E)/\text{rank}(E) = -H/2$. Then we have the following inequality:*

$$(19) \quad \frac{\Delta(E) \cdot H}{\text{rank}(E)^2} > \frac{2c}{b} = 1.5139 \dots$$

The RHS of (19) is irrational, hence the equality is not achieved. Note that the RHS in (19) is very close to the RHS in (16) for the K3 surface case.

Remark 3.1. *A stronger BG inequality similar to (19) is predicted by [DRY] without the condition $c_1(E)/\text{rank}(E) = -H/2$. The prediction in [DRY] is shown to be false in [Jar07], [Nak07]. Conjecture 3.2 does not contradict to the results in [Jar07], [Nak07] since we restrict to the sheaves with fixed slope $c_1(E)/\text{rank}(E) = -H/2$.*

There are few examples of stable sheaves on quintic 3-folds in literatures. The following example is taken in [Jar07]:

Example 3.1. *Let E be the kernel of the morphism $\mathcal{O}_X^{\oplus 6} \rightarrow \mathcal{O}_X(1)^{\oplus 2}$ given by the matrix*

$$\begin{pmatrix} x_0 & x_1 & 0 & x_2 & x_3 & 0 \\ 0 & x_0 & x_1 & 0 & x_2 & x_3 \end{pmatrix}.$$

Here $[x_0 : x_1 : x_2 : x_3 : x_4]$ is the homogeneous coordinates in \mathbb{P}^4 . By [Jar07], E is a stable vector bundle on X with

$$\text{ch}(E) = (4, -2H, -H^2, -H^3/3).$$

Then we have

$$\frac{\Delta(E) \cdot H}{\text{rank}(E)^2} = \frac{15}{4} > 1.5139 \dots$$

The rank two case will be treated in Subsection 3.7.

3.6. Conjectural construction of a Gepner type stability condition

We now give a conjectural construction of a desired \mathcal{A}_G assuming Conjecture 3.2. Similarly to [BMT, Lemma 3.2.1], we have the following lemma:

Lemma 3.2. *Suppose that Conjecture 3.2 is true. Then for any non-zero $E \in \mathcal{B}_{B,H}$, we have the following:*

- *We have $H^2 \text{ch}_1^B(E) \geq 0$.*
- *If $H^2 \text{ch}_1^B(E) = 0$, then we have $\text{Im } Z_G^\dagger(E) \geq 0$.*
- *If $H^2 \text{ch}_1^B(E) = \text{Im } Z_G^\dagger(E) = 0$, then $-\text{Re } Z_G^\dagger(E) > 0$.*

Proof. The same argument of [BMT, Lemma 3.2.1] is applied by using Conjecture 3.2 instead of the classical BG inequality. Q.E.D.

The above lemma shows that the triple

$$(H^2 \text{ch}_1^B(E), \text{Im } Z_G^\dagger(E), -\text{Re } Z_G^\dagger(E))$$

should behave like $(\text{rank}, c_1, \text{ch}_2)$ on coherent sheaves on algebraic surfaces. Similarly to the slope function on coherent sheaves, we consider the following slope function on $\mathcal{B}_{B,H}$

$$\nu_G(E) := \frac{\text{Im } Z_G^\dagger(E)}{H^2 \text{ch}_1^B(E)}.$$

Here we set $\mu_G(E) = \infty$ if $H^2 \text{ch}_1^B(E) = 0$. If we assume Conjecture 3.2, then Lemma 3.2 shows that the slope function ν_G satisfies the weak see-saw property.

Definition 3.1. *An object $E \in \mathcal{B}_{B,H}$ is ν_G -(semi)stable if, for any non-zero proper subobject $F \subset E$ in $\mathcal{B}_{B,H}$, we have the inequality*

$$\nu_{B,H}(F) < (\leq) \nu_{B,H}(E/F).$$

We have the following lemma:

Lemma 3.3. *Suppose that Conjecture 3.2 is true. Then the ν_G -stability on $\mathcal{B}_{B,H}$ satisfies the Harder-Narasimhan property.*

Proof. Although the central charge $Z_G^\dagger(*)$ is irrational, the values $H^2 \text{ch}_1^B(*)$ are contained in $\frac{1}{2} + \mathbb{Z}$, hence they are discrete. This is enough to apply the same argument of [BMT, Lemma 3.2.4], [Bri08, Proposition 7.1] to show the existence of Harder-Narasimhan filtrations with respect to ν_G -stability. Q.E.D.

Assuming Conjecture 3.2, we define the full subcategories in $\mathcal{B}_{B,H}$

$$\begin{aligned} \mathcal{T}_G &:= \langle E : \nu_G\text{-semistable with } \nu_G(E) > 0 \rangle_{\text{ex}} \\ \mathcal{F}_G &:= \langle E : \nu_G\text{-semistable with } \nu_G(E) \leq 0 \rangle_{\text{ex}}. \end{aligned}$$

As before, the pair $(\mathcal{T}_G, \mathcal{F}_G)$ forms a torsion pair on $\mathcal{B}_{B,H}$. By taking the tilting, we obtain the heart of a bounded t-structure

$$\mathcal{A}_G := \langle \mathcal{F}_G[1], \mathcal{T}_G \rangle_{\text{ex}}.$$

We propose the following conjecture:

Conjecture 3.3. *Let $X \subset \mathbb{P}^4$ be a smooth quintic 3-fold and assume that Conjecture 3.2 is true. Then the pair*

$$(20) \quad (Z_G^\dagger, \mathcal{A}_G)$$

determines a Gepner type stability condition on $D^b \text{Coh}(X)$ with respect to $(\text{ST}_{\mathcal{O}_X} \circ \otimes_{\mathcal{O}_X}(1), 2/5)$.

Remark 3.2. *By the construction and the irrationality of Z_G^\dagger , the pair (20) satisfies the condition (15). On the other hand, the irrationality of Z_G^\dagger makes it hard to prove the Harder-Narasimhan property of the pair (20).*

3.7. Conjecture 3.2 for the rank two case

We show that Conjecture 3.2 is true in the rank two case.

Proposition 3.1. *Conjecture 3.2 is true when $\text{rank}(E) = 2$.*

Proof. Since we have the inequality

$$\Delta(E^{\vee\vee}) \cdot H \geq \Delta(E) \cdot H$$

we may assume that E is reflexive. Since $\text{rank}(E) = 2$, we have $c_1(E) = -H$ and

$$\Delta(E) \cdot H = -H^3 + 4c_2(E) \cdot H.$$

The classical BG inequality implies that $\Delta(E) \cdot H \geq 0$, i.e. $c_2(E) \cdot H \geq 5/4$. The conjectural inequality (19) is equivalent to that $c_2(E) \cdot H > 2.7639 \dots$. It is enough to exclude the case $c_2(E) \cdot H = 2$, or equivalently $\text{ch}_2(E) \cdot H = 1/2$.

Suppose by contradiction that $\text{ch}_2(E) \cdot H = 1/2$. Let us set $F := E^\vee$, which is also a torsion free slope stable sheaf. Since F is reflexive, we have

$$\mathcal{E}xt^i(F, \mathcal{O}_X) = 0, \quad i \geq 2$$

and $Q := \mathcal{E}xt^1(F, \mathcal{O}_X)$ is a zero dimensional sheaf by [HL97, Proposition 1.1.10]. This implies that there is a distinguished triangle

$$F^\vee \rightarrow \mathbb{D}(F) \rightarrow Q[-1]$$

where $\mathbb{D}(\ast)$ is the derived dual $\mathbf{R}\mathcal{H}om(\ast, \mathcal{O}_X)$. Therefore if we write

$$(21) \quad \text{ch}(F) = (2, H, \text{ch}_2(F), \text{ch}_3(F))$$

then we have $\text{ch}_2(F^\vee) = \text{ch}_2(F) = \text{ch}_2(E)$ and

$$(22) \quad \text{ch}(F^\vee) = (2, -H, \text{ch}_2(E), -\text{ch}_3(F) + |Q|).$$

Here $|Q|$ is the length of the zero dimensional sheaf Q . On the other hand, since F is a rank two reflexive sheaf, we have the isomorphism (cf. [Har80, Proposition 1.10])

$$F \cong F^\vee \otimes \det(F).$$

Noting that $\det(F) = \mathcal{O}_X(H)$, and (21), (22), we have

$$(2, H, \text{ch}_2(E), \text{ch}_3(F)) = e^H \cdot (2, -H, \text{ch}_2(E), -\text{ch}_3(F) + |Q|).$$

The above equality and the assumption $\text{ch}_2(E) \cdot H = 1/2$ imply that

$$(23) \quad \text{ch}_3(F) = -\frac{1}{6} + \frac{|Q|}{2}.$$

Noting that $c_2(X) = 10H^2$, the Riemann-Roch theorem and (23) imply that

$$(24) \quad \begin{aligned} \chi(F) &:= \sum_{i=0}^3 (-1)^i \dim H^i(X, F) \\ &= 4 + \frac{|Q|}{2}. \end{aligned}$$

We divide into two cases:

Case 1. $H^0(X, F) = 0$.

By the Serre duality and stability, we have

$$H^3(X, F) \cong H^0(F, \mathcal{O}_X) \cong 0.$$

Therefore, by the assumption $H^0(X, F) = 0$ and (24), we have

$$(25) \quad \dim \text{Ext}^1(F, \mathcal{O}_X) = \dim H^2(X, F) \geq 4.$$

Let us take the universal extension

$$0 \rightarrow \mathcal{O}_X \otimes \text{Ext}^1(F, \mathcal{O}_X)^\vee \rightarrow \mathcal{U} \rightarrow F \rightarrow 0.$$

Then by [Toddb, Lemma 2.1], the sheaf \mathcal{U} is a torsion free slope stable sheaf. Applying the BG inequality to \mathcal{U} , we obtain the inequality

$$(H^2 - 2 \text{ch}_2(E)(2 + \dim \text{Ext}^1(F, \mathcal{O}_X))) \cdot H \geq 0.$$

The above inequality implies that $\dim \text{Ext}^1(F, \mathcal{O}_X) \leq 3$, which contradicts to (25).

Case 2. $H^0(X, F) \neq 0$.

Let us take a non-zero element $s \in H^0(X, F)$, and an exact sequence

$$(26) \quad 0 \rightarrow \mathcal{O}_X \xrightarrow{s} F \rightarrow M \rightarrow 0.$$

By [Todb, Lemma 2.2], the sheaf M is a torsion free slope stable sheaf. Therefore it is written as

$$M \cong \mathcal{O}_X(H) \otimes I_Z$$

for some subscheme $Z \subset X$ with $\dim Z \leq 1$. We have the equalities of Chern characters

$$\begin{aligned} \text{ch}_2(F) &= \frac{1}{2}H^2 - [Z] \\ \text{ch}_3(F) &= \frac{1}{6}H^3 - H \cdot [Z] - \chi(\mathcal{O}_Z). \end{aligned}$$

Because $\text{ch}_2(F) \cdot H = \text{ch}_2(E) \cdot H = 1/2$, we have $H \cdot [Z] = 2$. Hence we obtain

$$\text{ch}_3(F) = -\frac{7}{6} - \chi(\mathcal{O}_Z).$$

On the other hand, (23) implies that $\text{ch}_3(F) \geq -1/6$, hence we have $\chi(\mathcal{O}_Z) \leq -1$. By taking the generic projection of the one dimensional subscheme $Z \subset \mathbb{P}^4$ to \mathbb{P}^3 , the Castelnuovo inequality implies

$$g(Z) := h^1(\mathcal{O}_Z) \leq \frac{1}{2}(H \cdot [Z] - 1)(H \cdot [Z] - 2).$$

Since $H \cdot [Z] = 2$, we have $h^1(\mathcal{O}_Z) = 0$, which contradicts to $\chi(\mathcal{O}_Z) \leq -1$. Q.E.D.

§4. Clifford type bound for quintic surfaces

In this section, we take W' to be a quintic homogeneous polynomial with four variables

$$W' \in \mathbb{C}[x_0, x_1, x_2, x_3], \quad \deg(W') = 5.$$

We consider Conjecture 2.1 in this case. We relate it with some Clifford type bound for stable coherent systems on the smooth quintic surface

$$S := (W' = 0) \subset \mathbb{P}^3.$$

4.1. Computation of the central charge

The surface S is a hyperplane section ($x_4 = 0$) of a quintic 3-fold $X := (W = 0) \subset \mathbb{P}^4$, where W is defined by

$$W := W' + x_4^5 \in \mathbb{C}[x_0, x_1, x_2, x_3, x_4].$$

By Theorem 3, there is the heart of a bounded t-structure $\mathcal{A}_{W'} \subset \text{HMF}^{\text{gr}}(W')$, and an equivalence

$$\Theta: \text{Syst}(S) \xrightarrow{\sim} \mathcal{A}_{W'}.$$

Below we abbreviate Θ and regard a coherent system $(\mathcal{O}_S^{\oplus R} \rightarrow F)$ as an object in $\mathcal{A}_{W'}$. There is a natural push-forward functor (cf. [Ued])

$$i_*: \text{HMF}^{\text{gr}}(W') \rightarrow \text{HMF}^{\text{gr}}(W)$$

such that by [Toda, Lemma 3.12] and [Toda, Lemma 4.5], we have

$$i_*(\mathcal{O}_S^{\oplus R} \rightarrow F) \cong \Psi(\mathcal{O}_X^{\oplus R} \rightarrow i_*F).$$

Here i_*F is the usual sheaf push-forward for the embedding $i: S \hookrightarrow X$, $\Psi: D^b \text{Coh}(X) \xrightarrow{\sim} \text{HMF}^{\text{gr}}(W)$ an equivalence in Theorem 2 and

$$(\mathcal{O}_X^{\oplus R} \rightarrow i_*F) \in D^b \text{Coh}(X)$$

is an object in the derived category with i_*F located in degree zero. Let us consider the central charge Z_G^\dagger on $\text{HMF}^{\text{gr}}(W')$ defined by

$$Z_G^\dagger(P) := Z_G^\dagger(\Psi^{-1}i_*P), \quad P \in \text{HMF}^{\text{gr}}(W')$$

where Z_G^\dagger is the central charge (17) on $D^b \text{Coh}(X)$ considered in the previous section. By the argument in [Toda, Section 4], the central charge Z_G^\dagger on $\text{HMF}^{\text{gr}}(W')$ differs from (6) only up to a scalar multiplication. For $F \in \text{Coh}(S)$, let us write

$$\text{ch}(F) = (r, l, n) \in H^0(S) \oplus H^2(S) \oplus H^4(S)$$

with $r \in \mathbb{Z}$ and $n \in \frac{1}{2} + \mathbb{Z}$. By setting $H = c_1(\mathcal{O}_X(1))$ and $B = -H/2$, we have

$$\begin{aligned} & \text{ch}^B(\Psi^{-1}i_*(\mathcal{O}_S^{\oplus R} \rightarrow F)) \\ &= \text{ch}^B(\mathcal{O}_X^{\oplus R} \rightarrow i_*F) \\ &= \left(-R, \left(r - \frac{R}{2} \right) H, i_*l - \frac{R}{8}H^2, n + \frac{5}{24}r - \frac{5}{48}R \right). \end{aligned}$$

Applying the computation of Z_G^\dagger in the previous section, we have

$$\begin{aligned} Z_G^\dagger(\mathcal{O}_S^{\oplus R} \rightarrow F) &= -n - \frac{5}{24}r + \frac{5}{48}R + 5a \left(r - \frac{R}{2} \right) \\ &\quad + \sqrt{-1} \left(b \left(h \cdot l - \frac{5}{8}R \right) - cR \right). \end{aligned}$$

Here $h := H|_S$ and a, b, c are irrational numbers given in (17).

4.2. Conjectural Clifford type bound

We expect that a desired Gepner type stability condition in this case is constructed via double tilting of $\mathcal{A}_{W'}$, similarly to the previous section. Let μ' be the slope function on $\mathcal{A}_{W'}$, given by (using the notation in the previous subsection)

$$\begin{aligned} \mu'(\mathcal{O}_S^{\oplus R} \rightarrow F) &:= -\frac{\text{ch}_1^B(i_*F) \cdot H^2}{R} \\ &= 5 \left(\frac{1}{2} - \frac{\text{rank}(F)}{R} \right). \end{aligned}$$

Here we set $\mu'(*) = -\infty$ if $R = 0$. (Also see [Toda, Subsection 5.4].) The above slope function defines the μ' -stability on $\mathcal{A}_{W'}$, which satisfies the Harder-Narasimhan property (cf. [Toda, Lemma 5.14]). Following the same argument in the previous section, we expect that any μ' -stable object $E \in \mathcal{A}_{W'}$ with $\mu'(E) = 0$ satisfies $\text{Im } Z_G^\dagger(E) \geq 0$. It leads to the following conjecture:

Conjecture 4.1. *Let $S \subset \mathbb{P}^3$ be a smooth quintic surface and $h = c_1(\mathcal{O}_S(1))$. For a μ' -stable coherent system $(\mathcal{O}_S^{\oplus R} \rightarrow F)$ on S with $R = 2 \text{rank}(F) > 0$, we have the following inequality*

$$\frac{c_1(F) \cdot h}{R} > \frac{5}{8} + \frac{c}{b} = 1.3818 \dots$$

If we assume the above conjecture, we are able to construct a double tilting \mathcal{A}'_G of $\mathcal{A}_{W'}$, such that the pair $(Z_G^\dagger, \mathcal{A}'_G)$ satisfies

$$Z_G^\dagger(\mathcal{A}'_G \setminus \{0\}) \subset \{r \exp(\sqrt{-1}\pi\phi) : r > 0, \phi \in (0, 1]\}.$$

We conjecture that the pair $(Z_G^\dagger, \mathcal{A}'_G)$ gives a Gepner type stability condition on $\text{HMF}^{\text{gr}}(W')$ with respect to $(\tau, 2/5)$. The construction of \mathcal{A}'_G is similar to \mathcal{A}_G in the previous section, and we leave the readers to give its explicit construction. We just check the easiest case of Conjecture 4.1:

Lemma 4.1. *Conjecture 4.1 is true if $R = 2 \text{rank}(F) = 2$.*

Proof. Let $(\mathcal{O}_S^{\oplus 2} \xrightarrow{s} F)$ be a μ' -stable coherent system on S with $\text{rank}(F) = 1$. The inequality in Conjecture 4.1 is equivalent to $c_1(F) \cdot h > 2.7636 \dots$. It is enough to show that $c_1(F) \cdot h \geq 3$. Let $F \twoheadrightarrow F'$ be a torsion free quotient. There is a surjection in $\mathcal{A}_{W'}$

$$(\mathcal{O}_S^{\oplus 2} \rightarrow F) \twoheadrightarrow (\mathcal{O}_S^{\oplus 2} \rightarrow F')$$

whose kernel is of the form $(0 \rightarrow F'')$ for a torsion sheaf F'' on S . Obviously $(\mathcal{O}_S^{\oplus 2} \rightarrow F')$ is also μ' -stable, and $c_1(F') \cdot h \leq c_1(F) \cdot h$. Hence we may assume that F is torsion free. Also note that $h^0(F) \geq 2$, since otherwise there is an injection in $\mathcal{A}_{W'}$

$$(\mathcal{O}_S \rightarrow 0) \hookrightarrow (\mathcal{O}_S^{\oplus 2} \rightarrow F)$$

satisfying

$$\mu'(\mathcal{O}_S \rightarrow 0) = 5/2 > 0 = \mu'(\mathcal{O}_S^{\oplus 2} \rightarrow F)$$

which contradicts to the μ' -stability of $(\mathcal{O}_S^{\oplus 2} \rightarrow F)$. Let us set $\mathcal{L} := F^{\vee\vee}$, and take a smooth member $C \in |h|$. Note that \mathcal{L} is a line bundle satisfying $h^0(\mathcal{L}) \geq 2$, and C is a smooth quintic curve in \mathbb{P}^2 . Suppose by contradiction that $c_1(F) \cdot h = c_1(\mathcal{L}) \cdot h \leq 2$. We have the exact sequence

$$0 \rightarrow \mathcal{L}(-C) \rightarrow \mathcal{L} \rightarrow \mathcal{L}|_C \rightarrow 0.$$

Since $c_1(\mathcal{L}(-C)) \cdot h = c_1(\mathcal{L}) \cdot h - 5 < 0$ by our assumption, we have $h^0(\mathcal{L}(-C)) = 0$ and $h^0(\mathcal{L}|_C) \geq 2$. On the other hand, Clifford's theorem on C yields (cf. [Har77, Theorem 5.4])

$$h^0(\mathcal{L}|_C) \leq \frac{1}{2} \deg(\mathcal{L}|_C) + 1 \leq 2.$$

Furthermore, the first inequality is strict since $\mathcal{L}|_C \neq 0, K_C$, and C is not hyperelliptic. Therefore we obtain a contradiction. Q.E.D.

References

[AB13] D. Arcara and A. Bertram, *Bridgeland-stable moduli spaces for K-trivial surfaces. With an appendix by Max Lieblich*, J. Eur. Math. Soc. (JEMS) **15** (2013), 1–38.

[Asp] P. Aspinwall, *D-Branes on Calabi-Yau Manifolds*, arXiv:hep-th/0403166.

[BFK12] M. Ballard, D. Favero, and L. Katzarkov, *Orlov spectra: bounds and gaps*, Invent. Math **189** (2012), 359–430.

[BM11] A. Bayer and E. Macri, *The space of stability conditions on the local projective plane*, Duke Math. J. **160** (2011), 263–322.

[BMT] A. Bayer, E. Macri, and Y. Toda, *Bridgeland stability conditions on 3-folds I: Bogomolov-Gieseker type inequalities*, J. Algebraic Geom. (to appear), arXiv:1103.5010.

[Bog78] F. A. Bogomolov, *Holomorphic tensors and vector bundles on projective manifolds*, Izv. Akad. Nauk SSSR Ser. Mat. **42** (1978), 1227–1287.

[Bri07] T. Bridgeland, *Stability conditions on triangulated categories*, Ann. of Math **166** (2007), 317–345.

- [Bri08] T. Bridgeland, *Stability conditions on K3 surfaces*, Duke Math. J. **141** (2008), 241–291.
- [Bri09] T. Bridgeland, *Spaces of stability conditions*, Proc. Sympos. Pure Math. **80** (2009), 1–21, Algebraic Geometry-Seattle 2005.
- [CdLOGP91] P. Candelas, X. de la Ossa, P. Green, and L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal field theory*, Nuclear Physics **B359** (1991), 21–74.
- [DRY] M. R. Douglas, R. Reinbacher, and S. T. Yau, *Branes, bundles and attractors: Bogomolov and beyond*, arXiv:0604597.
- [Gie79] D. Gieseker, *On a theorem of Bogomolov on Chern Classes of Stable Bundles*, Amer. J. Math. **101** (1979), 77–85.
- [Har77] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, Springer-Verlag, 1977.
- [Har78] ———, *Stable vector bundles of rank 2 on \mathbb{P}^3* , Math. Ann. **238** (1978), 229–280.
- [Har80] ———, *Stable Reflexive Sheaves*, Math. Ann. **254** (1980), 121–176.
- [HL97] D. Huybrechts and M. Lehn, *Geometry of moduli spaces of sheaves*, Aspects in Mathematics, vol. E31, Vieweg, 1997.
- [Jar07] M. Jardim, *Stable bundles on 3-fold hypersurfaces*, Bull. Braz. Math. Soc. **38** (2007), 649–659.
- [KST07] H. Kajiura, K. Saito, and A. Takahashi, *Matrix factorizations and representations of quivers II. Type ADE case*, Advances in Math **211** (2007), 327–362.
- [KS] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, preprint, arXiv:0811.2435.
- [Mac] E. Macri, *A generalized Bogomolov-Gieseker inequality for the three-dimensional projective space*, preprint, arXiv:1207.4980.
- [Muk87] S. Mukai, *On the moduli space of bundles on K3 surfaces I*, Vector Bundles on Algebraic Varieties, M. F. Atiyah et al. ,Oxford University Press (1987), 341–413.
- [Nak07] T. Nakashima, *Strong Bogomolov inequality for stable vector bundles*, Journal of Geometry and Physics **57** (2007), 1997–1983.
- [Orl09] D. Orlov, *Derived categories of coherent sheaves and triangulated categories of singularities*, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin, Progr. Math. **270** (2009), 503–531.
- [ST01] P. Seidel and R. P. Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. **108** (2001), 37–107.
- [Tak] A. Takahashi, *Matrix factorizations and representations of quivers I*, preprint, arXiv:0506347.
- [Toda] Y. Toda, *Gepner type stability conditions on graded matrix factorizations*, preprint, arXiv:1302.6293.
- [Todb] ———, *A note on Bogomolov-Gieseker type inequality for Calabi-Yau 3-folds*, Proc. of AMS (to appear), arXiv:1201.4911.

- [Ued] K. Ueda, *Hyperplane sections and stable derived categories*, preprint, arXiv:1207.1167.
- [Wal] J. Walcher, *Stability of Landau-Ginzburg branes*, Journal of Mathematical Physics **46**, 082305 (2005), arXiv:hep-th/0412274.

*Kavli Institute for the Physics and Mathematics of the Universe,
University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, 277-8583, Japan.
E-mail address: yuKinobu.toda@ipmu.jp*