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Automorphisms of Calabi-Yau threefolds with Picard number three

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Dedicated to Professor Yujiro Kawamata on the occasion of his 60th birthday

Abstract.

We prove that the automorphism group of a Calabi-Yau threefold with Picard number three is either finite, or isomorphic to the infinite cyclic group up to finite kernel and cokernel.

§1. Introduction

In this paper we are interested in the automorphism group of a Calabi-Yau threefold with small Picard number. Here, a Calabi-Yau threefold is a smooth complex projective threefold X with trivial canonical bundle K_X such that $h^1(X, \mathcal{O}_X) = 0$.

It is a classical fact that the group of birational automorphisms $\mathrm{Bir}(X)$ and the automorphism group $\mathrm{Aut}(X)$ are finite groups and coincide when X is a Calabi-Yau threefold with $\rho(X) = 1$. It is, however, unknown which finite groups really occur as automorphism groups, even for smooth quintic threefolds. When $\rho(X) = 2$, the automorphism group is also finite by [Ogu14, Theorem 1.2] (see also [LP13]), while there is an example of a Calabi-Yau threefold with $\rho(X) = 2$ and with infinite $\mathrm{Bir}(X)$ [Ogu14, Proposition 1.4].

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In contrast, Borcea [Bor91] gave an example of a Calabi-Yau threefold with $\rho(X) = 4$ having infinite automorphism group, and the same phenomenon is expected for any Picard number $\rho(X) \geq 4$; for examples with large Picard numbers, see [GM93, OT13].

Thus far, the case of Picard number 3 remained unexplored. Perhaps surprisingly, we show that the automorphism groups of such threefolds are relatively small:

Theorem 1.1. *Let X be a Calabi-Yau threefold with $\rho(X) = 3$.*

Then the automorphism group $\mathrm{Aut}(X)$ *is either finite, or it is an almost abelian group of rank* 1*, i.e. it is isomorphic to* $\mathbb{Z}$ *up to finite kernel and cokernel.*

We investigate automorphisms g of infinite order and distinguish the cases when g has an eigenvalue different than 1, and when g only has eigenvalue 1. Theorem 1.1 then follows from Corollary 3.3 and Proposition 4.3 below.

At the moment, we do not have an example where $\mathrm{Aut}(X)$ is an infinite group. Existence of such an example would show that 3 is the smallest possible Picard number of a Calabi-Yau threefold with infinite automorphism group. However, finiteness of the automorphism group is known when the fundamental group of X is infinite: when X is a Calabi-Yau threefold of Type A, i.e. X is an étale quotient of a torus, then $\mathrm{Aut}(X)$ is finite by [OS01, Theorem (0.1)(IV)]. The case when X is of Type K, i.e. X is an étale quotient of a product of an elliptic curve and a K3 surface, is studied in [HK14].

It is our honour to dedicate this paper to Professor Yujiro Kawamata on the occasion of his sixtieth birthday. This article and our previous papers [Ogu14, LP13] are inspired by his beautiful paper [Kaw97].

§2. Preliminaries

We first fix some notation. Let X be a Calabi-Yau threefold with Picard number $\rho(X) = 3$. The automorphism group of X is denoted by $\mathrm{Aut}(X)$ and $N^1(X)$ is the Néron-Severi group of X generated by the numerical classes of line bundles on X. Note that $N^1(X)$ is a free $\mathbb{Z}$-module of rank 3. There is a natural homomorphism

$$r\colon \mathrm{Aut}(X) \to \mathrm{GL}(N^1(X)),$$

and we set $\mathcal{A}(X) = r(\mathrm{Aut}(X))$. Note that the kernel of r is finite [Ogu14, Proposition 2.4], hence $\mathrm{Aut}(X)$ is finite if and only if $\mathcal{A}(X)$ is finite. We furthermore let $N^1(X)_{\mathbb{R}} := N^1(X) \otimes \mathbb{R}$ be the vector space generated by $N^1(X)$.

Proposition 2.1. *Let ℓ_1 and ℓ_2 be two distinct lines in $\mathbb{R}^2$ through the origin, and let G be a subgroup of $\mathrm{GL}(2,\mathbb{Z})$ which acts on $\ell_1 \cup \ell_2$.*

If G is infinite, then it is an almost abelian group of rank 1, i.e. G contains a rank 1 abelian subgroup of finite index.

Proof. The proof follows from that of [LP13, Theorem 3.9], and we recall the argument for the convenience of the reader. Fix nonzero points $x_1 \in \ell_1$ and $x_2 \in \ell_2$. Then for any $g \in G$ there exist a permutation (i_1, i_2) of the set $\{1,2\}$ and real numbers α_1 and α_2 such that $gx_1 = \alpha_1 x_{i_1}$ and $gx_2 = \alpha_2 x_{i_2}$. It follows that there are positive numbers β_1 and β_2 such that $g^4 x_i = \beta_i x_i$. Hence, passing to a finite index subgroup, we may assume that G acts on $\mathbb{R}_+ x_1$ and $\mathbb{R}_+ x_2$.

For every $g \in G$, let α_g be the positive number such that $gx_1 = \alpha_g x_1$, and set $\mathcal{S} = \{\alpha_g \mid g \in G\}$. Then $\mathcal{S}$ is a multiplicative subgroup of $\mathbb{R}^*$ and the map

$$G \to \mathcal{S}, \quad g \mapsto \alpha_g$$

is an isomorphism of groups. It therefore suffices to show that $\mathcal{S}$ is an infinite cyclic group. By [For81, 21.1], it is enough to prove that $\mathcal{S}$ is discrete. Otherwise, we can pick a sequence (g_i) in G such that (α_{g_i}) converges to 1. Fix two linearly independent points $h_1, h_2 \in \mathbb{Z}^2$. Then $g_i h_1 \to h_1$ and $g_i h_2 \to h_2$ when $i \to \infty$. Since $g_i h_1, g_i h_2 \in \mathbb{Z}^2$, this implies that $g_i h_1 = h_1$ and $g_i h_2 = h_2$ for $i \gg 0$, and hence $g_i = \mathrm{id}$ for $i \gg 0$. Q.E.D.

In order to prove our main result, Theorem 1.1, we first show that the cubic form on our Calabi-Yau threefold X always splits in a special way, and this almost immediately has strong consequences on the structure of the automorphism group.

In this paper, when L is a linear, quadratic or cubic form on $N^1(X)_{\mathbb{R}}$, we do not distinguish between L and the corresponding locus $(L = 0) \subseteq N^1(X)_{\mathbb{R}}$.

We start with the following lemma.

Lemma 2.2. *Let X be a Calabi-Yau threefold with Picard number 3. Assume that $\mathrm{Aut}(X)$ is infinite.*

Then there exists $g \in \mathcal{A}(X)$ with $\det g = 1$ such that $\langle g \rangle \simeq \mathbb{Z}$.

Proof. By possibly replacing the group $\mathcal{A}(X)$ by the subgroup $\mathcal{A}(X) \cap SL(N^1(X))$ of index at most 2, we may assume that all elements of $\mathcal{A}(X)$ have determinant 1. Assume that all elements of $\mathcal{A}(X)$

have finite order, and fix an element

$$h \in \mathcal{A}(X) \subseteq \mathrm{GL}(N^1(X))$$

of order n_h. Since $\rho(X) = 3$, the characteristic polynomial $\Phi_h(t) \in \mathbb{Z}[t]$ of h is of degree 3. If ξ is an eigenvalue of h, then $\xi^{n_h} = 1$, and hence $\varphi(n_h) \leq 3$, where φ is Euler's function. An easy calculation shows that then $n_h \leq 6$, and therefore $\mathcal{A}(X)$ is a finite group by Burnside's theorem on matrix groups, a contradiction. Q.E.D.

If $c_2(X) = 0$ in $H^4(X, \mathbb{R})$, then $\mathrm{Aut}(X)$ is finite by [OS01, Theorem (0.1)(IV)]. Combining this with Lemma 2.2, we may assume the following:

Assumption 2.3. *Let X be a Calabi-Yau threefold with Picard number 3. We assume that $c_2(X) \neq 0$ and that $\mathrm{Aut}(X)$ is infinite, and we fix an element $g \in \mathcal{A}(X)$ of infinite order as given in Lemma 2.2. We denote by C the cubic form on $N^1(X)_\mathbb{R}$ given by the intersection product.*

Proposition 2.4. *Let $h \in \mathcal{A}(X)$.*

(i) If h is of infinite order, then there exist a real number $\alpha \geq 1$ and (when $\alpha = 1$ not necessarily distinct) nonzero elements $u, v, w \in N^1(X)_\mathbb{R}$ such that w is integral, v is nef, and

$$hu = \frac{1}{\alpha}u, \quad hv = \alpha v, \quad hw = w.$$

Moreover, if $\alpha = 1$, then α is the unique eigenvalue of (the complexified) h.

(ii) If $h \neq \mathrm{id}$ has finite order, then (the complexified) h has eigenvalues 1, λ, $\bar{\lambda}$, where $\lambda \in \{\pm i, \pm(\frac{1}{2} \pm i\frac{\sqrt{3}}{2})\}$.

Proof. Let h^* denote the dual action of h on $H^4(X, \mathbb{Z})$. Since h^* preserves the second Chern class $c_2(X) \in H^4(X, \mathbb{Z})$, one of its eigenvalues is 1, and therefore h also has an eigenvector w with eigenvalue 1. Since h acts on the nef cone $\mathrm{Nef}(X)$, by the Birkhoff-Frobenius-Perron theorem [Bir67] there exist $\alpha \geq 1$ and $v \in \mathrm{Nef}(X) \setminus \{0\}$ such that $hv = \alpha v$. As $\det h = 1$, if $\alpha > 1$, then the remaining eigenvalue of h is $1/\alpha$.

Assume that $\alpha = 1$. Then by the Birkhoff-Frobenius-Perron theorem, all eigenvalues of h have absolute value 1. Thus the characteristic polynomial of h reads

$$\Phi_h(t) = (t-1)(t-\lambda)(t-\bar{\lambda})$$

with $|\lambda| = 1$. Since Φ_h has integer coefficients, a direct calculation gives $\lambda \in \{1, \pm i, \pm(\frac{1}{2} \pm i\frac{\sqrt{3}}{2})\}$. When $\lambda \neq 1$, it is easily checked that h has finite order.

Finally, if $\lambda = 1$, then the Jordan form of h is

$$\text{either} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In both cases it is clear that h has infinite order. Q.E.D.

In the following two sections, we fix an element of infinite order as in Lemma 2.2 and analyse separately the cases $\alpha > 1$ and $\alpha = 1$ as in Proposition 2.4(i).

§3. The case $\alpha > 1$

Proposition 3.1. *Under Assumption 2.3 and in the notation from Proposition 2.4 for $h = g$, assume that $\alpha > 1$. Then u and v are nef and irrational, we have*

$$(1) \qquad u^3 = v^3 = u^2v = uv^2 = u^2w = uw^2 = v^2w = vw^2 = 0,$$

and the plane $\mathbb{R}u+\mathbb{R}v$ is in the kernel of the linear form given by $c_2(X) \in H^4(X, \mathbb{Z})$.

Proof. We first need to show that the eigenspace of $1/\alpha$ intersects $\mathrm{Nef}(X)$. Pick $u \neq 0$ such that $gu = \frac{1}{\alpha}u$, and note that u, v and w form a basis of $N^1(X)_{\mathbb{R}}$. Take a general ample class

$$H = xv + yu + zw,$$

and observe that $y \neq 0$ by the general choice of H. Then $g^{-n}H$ is ample for every positive integer n, hence the divisor

$$\lim_{n\to\infty} \frac{1}{\alpha^n|y|} g^{-n}H = \lim_{n\to\infty} \left(\frac{x}{\alpha^{2n}|y|}v + \frac{y}{|y|}u + \frac{z}{\alpha^n|y|}w \right) = \frac{y}{|y|}u$$

is nef. Now replace u by $yu/|y|$ if necessary to achieve the nefness of u.

Furthermore, since $v^3 = (gv)^3 = \alpha^3v^3$, we obtain $v^3 = 0$; other relations in (1) are proved similarly. Also,

$$v \cdot c_2(X) = gv \cdot gc_2(X) = \alpha v \cdot c_2(X),$$

hence $v \cdot c_2(X) = 0$, and analogously $u \cdot c_2(X) = 0$.

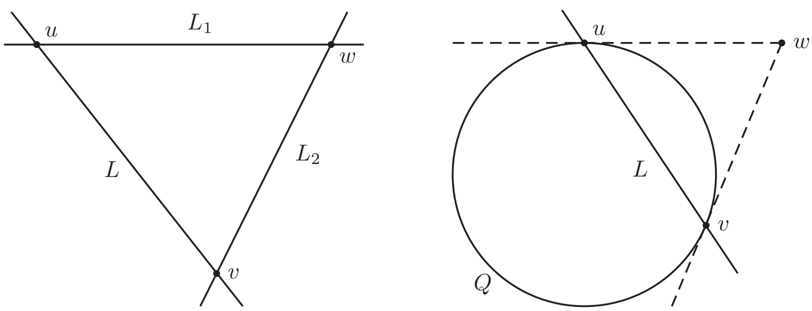
Finally, assume that v is rational. By replacing v by a rational multiple, we may assume that v is a primitive element of $N^1(X)$. But the eigenspace associated to α is 1-dimensional, and since gv is also primitive, we must have $gv = v$, a contradiction. Irrationality of u is proved in the same way. Q.E.D.

Proposition 3.2. *Under Assumption 2.3 and in the notation of Proposition 2.4 for $h = g$, assume that $\alpha > 1$. Let L be the linear form on $N^1(X)_{\mathbb{R}}$ given by $c_2(X)$.*

Then one of the following holds:

(i) *$C = L_1 L_2 L$, where L_1 and L_2 are irrational linear forms such that*

$$L_1 \cap L_2 = \mathbb{R}w, \quad L_1 \cap L = \mathbb{R}u, \quad L_2 \cap L = \mathbb{R}v;$$



(ii) *$C = QL$, where Q is an irreducible quadratic form. Then*

$$Q \cap L = \mathbb{R}u \cup \mathbb{R}v,$$

and the planes $\mathbb{R}u + \mathbb{R}w$ and $\mathbb{R}v + \mathbb{R}w$ are tangent to Q at u and v respectively.

Proof. Denote $A = w^3$ and $B = uvw$. We first claim that $B \neq 0$. In fact, suppose that $B = 0$ and let H be any ample class. Then the relations (1) imply $uv = 0$, hence $0 = (Huv)^2 = (H^2u) \cdot (v^2u)$, and the Hodge index theorem [BS95, Corollary 2.5.4] yields that H and v are proportional, which is a contradiction since $v^3 = 0$. This proves the claim.

Therefore, for any real variables x, y, z we have

$$(xu + yv + zw)^3 = z(Az^2 + 6Bxy),$$

and thus in the basis (u, v, w) we have $C = QL$, where $Q = Az^2 + 6Bxy$. We consider two cases.

Assume first that $A = 0$. Then $C = 6Bxyz$, and we set $L_2 = x$ and $L_1 = 6By$. This gives (i).

If $A \neq 0$, then

$$Q = Az^2 + 6Bxy = \begin{pmatrix} x \\ y \\ z \end{pmatrix}^t \begin{pmatrix} 0 & 3B & 0 \\ 3B & 0 & 0 \\ 0 & 0 & A \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

and the signature of Q is $(2, 1)$. Therefore, Q is a non-empty smooth quadric. It is now easy to see that the tangent plane to Q at u is $(y = 0)$, and the tangent plane to Q at v is $(x = 0)$. This proves the proposition. Q.E.D.

Corollary 3.3. *Under Assumption 2.3 and in the notation of Proposition 2.4 for $h = g$, assume that $\alpha > 1$. Then $\mathcal{A}(X)$ is an almost abelian group of rank* 1*.*

Proof. First note that every element $h \in \mathcal{A}(X)$ fixes the cubic C and the plane $L = c_2(X)^{\perp}$. Further, the singular locus $\mathrm{Sing}(C)$ of C is h-invariant. In the case (i) of Proposition 3.2, $\mathrm{Sing}(C) = \mathbb{R}u \cup \mathbb{R}v \cup \mathbb{R}w$. This implies that the set

$$\mathbb{R}u \cup \mathbb{R}v \subseteq L$$

is h-invariant, and hence so is $\mathbb{R}w$. In particular, the sets $\mathbb{R}u$, $\mathbb{R}v$, $\mathbb{R}w$ are each h^2-invariant. Then Proposition 2.4 immediately shows that $hw = w$, and hence the map

$$\mathcal{A}(X) \to \mathrm{GL}(2, \mathbb{Z}), \quad h \mapsto h|_L$$

is injective. Now the claim follows from Proposition 2.1.

In the case (ii) of Proposition 3.2, we have $\mathrm{Sing}(C) = \mathbb{R}u \cup \mathbb{R}v \subseteq L$, and $\mathbb{R}w$ is h-invariant as it is the intersection of tangent planes to Q at u and v. Now we conclude similarly as above. Q.E.D.

§4. The case $\alpha = 1$

Lemma 4.1. *Under Assumption 2.3 and in the notation of Proposition 2.4 for $h = g$, assume that $\alpha = 1$. Then the Jordan form of g is*

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

In particular, the eigenspace of g associated to the eigenvalue 1 *has dimension* 1*.*

Proof. By Proposition 2.4, $\alpha = 1$ is the unique eigenvalue of g. Therefore the Jordan form of g is either of the form (2) or of the form

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

Assume that the Jordan form is of the form (3); in other words, there is a basis (u_1, u_2, u_3) of $N^1(X)_{\mathbb{R}}$ such that

$$gu_1 = u_1, \quad gu_2 = u_1 + u_2, \quad gu_3 = u_3.$$

Clearly,

$$g^n u_2 = u_2 + nu_1$$

for every integer n, and furthermore,

$$u_2^3 = (g^n u_2)^3 = u_2^3 + 3nu_2^2 u_1 + 3n^2 u_2 u_1^2 + n^3 u_1^3.$$

This gives

$$u_2^2 u_1 = u_2 u_1^2 = u_1^3 = 0. \tag{4}$$

Similarly, from the equations

$$u_2^2 u_3 = (g^n u_2)^2 g^n u_3 \quad \text{and} \quad u_2 u_3^2 = g^n u_2 (g^n u_3^2)^2$$

we get

$$u_1^2 u_3 = u_1 u_3^2 = u_1 u_2 u_3 = 0. \tag{5}$$

For any smooth very ample divisor H on X, (4) and (5) give $u_1^2 \cdot H = u_1 \cdot H^2 = 0$, thus $(u_1|_H)^2 = 0$ and $u_1|_H \cdot H|_H = 0$, and hence $u_1|_H = 0$, applying the Hodge index theorem on H. This implies $u_1 = 0$ by the Lefschetz hyperplane section theorem, a contradiction. Thus the Jordan form cannot be of type (3), and the assertion is proved. Q.E.D.

Proposition 4.2. *Under Assumption 2.3 and in the notation of Proposition 2.4 for $h = g$, assume that $\alpha = 1$. Then, possibly by rescaling w, there exist $w_1, w_2 \in N^1(X)$ such that (w, w_1, w_2) is a basis of $N^1(X)_{\mathbb{R}}$ with respect to the Jordan form* (2)*, and we have*

$$w \cdot c_2(X) = w_1 \cdot c_2(X) = w^2 = w_1^3 = ww_1^2 = ww_1w_2 = 0 \tag{6}$$

and

$$ww_2^2 = 2w_1w_2^2 = -2w_1^2w_2 \neq 0. \tag{7}$$

Proof. Pick any $w_2 \in N^1(X)$ such that $w_1 := (g - \mathrm{id})w_2 \neq 0$ and $u := (g - \mathrm{id})^2 w_2 \neq 0$, which is possible by Lemma 4.1. Then

$$gu = u, \quad gw_1 = u + w_1, \quad gw_2 = w_1 + w_2,$$

and it is easy to check that (u, w_1, w_2) is a basis of $N^1(X)_{\mathbb{R}}$. Since the eigenspace associated to the eigenvalue 1 of g is 1-dimensional by Lemma 4.1, by Proposition 2.4 we may assume that $u = w$. We first observe that

$$g^n w_1 = w_1 + nw \quad \text{and} \quad g^n w_2 = w_2 + nw_1 + \frac{n(n-1)}{2} w$$

for any integer n. Then the equations

$$w_1 \cdot c_2(X) = g^n w_1 \cdot c_2(X) \quad \text{and} \quad w_2 \cdot c_2(X) = g^n w_2 \cdot c_2(X)$$

give

$$w \cdot c_2(X) = w_1 \cdot c_2(X) = 0.$$

Similarly, from $w_1^3 = (g^n w_1)^3$ and $ww_2^2 = (g^n w)(g^n w_2)^2$ we get

$$ww_1^2 = ww_1w_2 = w^2 = 0,$$

and $w_1^2 w_2 = (g^n w_1)^2 (g^n w_2)$ yields

$$w_1^3 = 0.$$

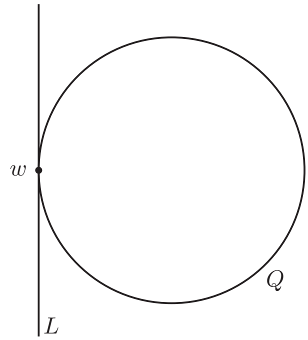
Finally, from $w_2^3 = (g^n w_2)^3$ we obtain (7), up to the non-vanishing statement. Assume that $ww_2^2 = 0$. Since w, w_1, w_2 generate $N^1(X)_{\mathbb{R}}$, this implies that for any two smooth very ample line bundles H_1 and H_2 on X we have $w \cdot H_1 \cdot H_2 = 0$, and in particular $w|_{H_1} = 0$. But then $w = 0$ by the Lefschetz hyperplane section theorem, a contradiction. Q.E.D.

Proposition 4.3. *Under Assumption 2.3 and in the notation of Proposition 2.4 and Proposition 4.2 for $h = g$, assume that $\alpha = 1$.*

- *(i) Let L be the linear form on $N^1(X)_{\mathbb{R}}$ given by $c_2(X)$. Then $C = QL$, where Q is an irreducible quadratic form, and L is tangent to Q at w.*
- *(ii) The automorphism group* $\mathrm{Aut}(X)$ *is an almost abelian group of rank* 1.

Proof. Set $E = 3ww_2^2/2$ and $F = w_2^3$. Then, using (6) and (7), for all real variables x, y, z we obtain the equation

$$(xw + yw_1 + zw_2)^3 = z(Fz^2 + 2Exz - Ey^2 + Eyz).$$



Since $L = z$ by (6), we have $C = QL$, where $Q = Fz^2 + 2Exz - Ey^2 + Eyz$. Noticing that $E \neq 0$ by Proposition 4.2, the tangent plane to Q at w is $(z = 0)$. This shows (i).

For (ii), consider any $h \in \mathcal{A}(X)$. We may assume $\det h = 1$, possibly replacing $\mathcal{A}(X)$ by $\mathcal{A}(X) \cap \mathrm{SL}(N^1(X))$.

The singular locus of C is $\mathbb{R}w$, hence $\mathbb{R}w$ is h-invariant and therefore defined over $\mathbb{Q}$. By the shape of the cubic and by Proposition 3.2, and since the element g in Assumption 2.3 is chosen arbitrarily, h has a unique real eigenvalue $\alpha = 1$. By Proposition 2.4 and by Lemma 4.1, $\mathbb{R}w$ is the only eigenspace of h, thus $hw = w$.

The plane $L = c_2(X)^\perp$ is h-invariant, and note that L is spanned by w and w_1 by (6). In the basis (w, w_1), the restriction $h|_L$ has the form

$$\begin{pmatrix} 1 & a_h \\ 0 & b_h \end{pmatrix},$$

and $\det(h|_L) = \pm 1$. By possibly replacing $\mathcal{A}(X)$ by the preimage of $\mathcal{A}(X)|_L \cap \mathrm{SL}(L)$ under the restriction map $\mathcal{A}(X) \to \mathcal{A}(X)|_L$, which has index at most 2, we may assume that $\det(h|_L) = 1$, and thus $b_h = 1$. Hence, the matrix of h in the basis (w, w_1, w_2) is

$$\mathcal{H} = \begin{pmatrix} 1 & a_h & d_h \\ 0 & 1 & c_h \\ 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

This implies, in particular, that h cannot be of finite order. The quadric Q is given in this basis by the matrix

$$\mathcal{Q} = \begin{pmatrix} 0 & 0 & E \\ 0 & -E & \frac{1}{2}E \\ E & \frac{1}{2}E & F \end{pmatrix}.$$

We now view Q as a quadric over $\mathbb{C}$. Since Q is h-invariant, by the Nullstellensatz there exists $\lambda \in \mathbb{Q}$ such that $hQ = \lambda Q$, i.e. $\mathcal{H}^t \mathcal{Q} \mathcal{H} = \lambda \mathcal{Q}$. By taking determinants, we conclude that $\lambda^3 = 1$, hence $\lambda = 1$. Putting the explicit matrices into the formula, we obtain

$$a_h = c_h \quad \text{and} \quad d_h = \frac{a_h(a_h - 1)}{2}. \tag{9}$$

Since $w \in N^1(X)$, there is a primitive element $\overline{w} \in N^1(X)$ and a positive integer p such that $w = p\overline{w}$. We have $a_h p\overline{w} = a_h w = hw_1 - w_1 \in N^1(X)$, hence the number $a_h p$ must be an integer. Consider the group homomorphism

$$\tau \colon \mathcal{A}(X) \to \mathbb{Z}, \qquad h \mapsto pa_h.$$

By (9), τ is injective, and therefore $\mathcal{A}(X) \simeq \mathbb{Z}$. Thus $\mathcal{A}(X)$ is abelian of rank 1. Q.E.D.

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