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Effective degree bounds for generalized Gauss map images

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Dedicated to Professor Yujiro Kawamata on the occasion of his sixtieth birthday.

Abstract.

We establish effective uniform degree bounds for the generalized Gauss map images of an embedded projective variety $X \subset \mathbb{P}^N$ in terms of numerical invariants such as $\dim X$, $\deg X$ and N. This can be seen as a generalization of a classical Castelnuovo type bound.

§1. Introduction

The aim of this paper is to give effective uniform degree bounds for the generalized Gauss map images of an embedded projective variety $X \subset \mathbb{P}^N$ in terms of numerical invariants such as $\dim X$, $\deg X$ and N.

We first recall the generalized Gauss maps ([Zak93, I.§2]). We denote by $G(m, N)$ the Grassmann variety of m-planes $V \subset \mathbb{P}^N$, and denote the corresponding points by $[V] \in G(m, N)$. In our convention, $G(m, N)$ is the Grassmannian of all $0 \in \mathbb{C}^{m+1} \subset \mathbb{C}^{N+1}$. For every integer m with $n := \dim X \leq m < N$, we let

$$\Gamma_m = \overline{\{(x, [V]) \in X_{reg} \times G(m, N);\ T_{X,x} \subset V\}} \subset X \times G(m, N),$$

where $T_{X,x}$ is the projectivized tangent n-plane in $\mathbb{P}^N$ and where the overline means the Zariski closure in $X \times G(m, N)$. We let

$$g_m : \Gamma_m \longrightarrow G(m, N)$$

be the projection to the second factor, which we call *the m-th Gauss map* of X, and define

$$X_m^* := g_m(\Gamma_m) \subset G(m, N).$$

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When $m = N-1$, $X^*_{N-1} = X^* \subset (\mathbb{P}^N)^*$ is the so-called dual variety, and when $m = n$, $g_n : \Gamma_n \to G(n,N)$ or the rational map $X \dashrightarrow G(n,N)$ is the (standard) Gauss map. We define the *defect* of the m-th Gauss map to be

$$\operatorname{def}_m X := \dim \Gamma_m - \dim X^*_m.$$

It is immediate that $0 \le \operatorname{def}_m X \le n$, and $\operatorname{def}_m X \le \operatorname{def}_{m+1} X$. For example, $n - \operatorname{def}_n X = \dim X^*_n$, and $\operatorname{def}_m X = n$ for some m if and only if X is a linear subspace. Our main result is the following

Theorem 1.1. *Let $X \subset \mathbb{P}^N$ be an n-dimensional projective variety of degree $d > 1$. Then the degree of the m-th Gauss map image $X^*_m \subset G(m,N)$ with respect to the Plücker embedding of $G(m,N)$ is bounded as follows.*

(1) $$\deg X^*_n \le d(d-1)^{n-\operatorname{def}_n X} \le d(d-1)^n.$$

Moreover, $\deg X^*_n = d(d-1)^n$ holds if and only if X is smooth and contained in a linear subspace $\mathbb{P}^{n+1}$.

(2) $$\deg X^*_m \le \deg F(n,m;N) \binom{n + \dim G(m,N)}{n} \deg G(m,N) \deg X^*_n$$

for m with $n < m < N$, where $\binom{a}{b}$ is a binomial coefficient.

Here, $\deg G(m,N)$ is the Plücker degree, and $F(n,m;N) \subset G(n,N) \times G(m,N)$ is a flag manifold whose degree is measured by the Plücker embeddings of $G(n,N)$ and $G(m,N)$. We note that the integer $C = \deg F(n,m;N) \binom{n+\dim G(m,N)}{n} \deg G(m,N)$ is independent of $d = \deg X$. In fact, this integer is explicit and can be estimated by $C < \big(\ell + (m+1)(m-n)\big)!(\ell+n)!/(n!)$ with $\ell = \dim G(m,N) = (m+1)(N-m)$ for example. The bound (1) can be improved by taking the codimension of X into account as well as a Castelnuovo type bound for the genus of projective curves. The bound in (2) can also be improved by taking the defect into account. For the sake of readability, we did not include these sharpenings in Theorem 1.1; instead we refer the reader to Theorem 2.1 and Corollary 5.2. Section 5 is devoted to dealing with the situation of positive defect. Actually, it is only with that same Corollary 5.2 that Theorem 1.1 is completely established.

This type of topic is undoubtedly classical, and hence there exist a lot of works related to this paper, especially when $m = N-1$ or $m = n$. We do not attempt to present the history here; instead we refer to the monograph [Zak93], the article [Zak12] and the references contained in these. As a matter of fact, it was the article [Zak12] and the results of

Castelnuovo type in it, especially [Zak12, Theorems 1.18, 1.21], which originally inspired us to investigate generalized Gauss maps. However, regarding the issue of effectivity, we are not aware of any previous results establishing degree bounds for generalized Gauss maps, or even just for standard Gauss maps.

It came as a surprise to us that we were able to prove the bound in Theorem 1.1(1), since it is exactly of the same form as the bound in [Zak12, Theorem 1.18] for the dual variety, which reads

$$\deg X_{N-1}^* \leq d(d-1)^{n-\mathrm{def}_{N-1}X} \leq d(d-1)^n$$

(proven there under the assumption that $X \subset \mathbb{P}^N$ is linearly non-degenerate). This certainly raises the question of the existence of further relations among the values $\deg X_m^*$ with $n \leq m < N$ and the underlying reasons for them, such as a certain kind of symmetry or duality.

The possible existence of such relations is furthermore suggested by Example 3.7 (see Remark 3.8(1)), which is the case of the Veronese curves. Treating this example is not entirely elementary, and the discussion in Example 3.7 is basically due to Kaji [Kaj15]. Most importantly, we will find that it is not the case that $\deg X_m^* \leq d(d-1)^n$ for general m. More concretely, for the Veronese embedding $X \subset \mathbb{P}^d$ of $\mathbb{P}^1$ of degree d, we will see $\deg X_1^* = 2(d-1), \deg X_2^* = 2(d-1)(d-2), \deg X_{d-1}^* = 2(d-1)$ and

$$\deg X_m^* = 2(d-m)\big((m-1)(d-m)+1\big)\deg G(m-2, d-2)$$

for $2 \leq m \leq d-1$. We interpret this formula in two ways. One way is to note for example that $\deg X_3^* = 2(d-3)(2d-5)\frac{(2(d-3))!}{(d-2)!(d-3)!} > 2^d$ for all $d \geq 5$ and also $\deg X_{d-2}^* = 4(2d-5)\frac{(2(d-3))!}{(d-2)!(d-3)!} > 2^d$ for all $d \geq 5$, which are already exponential in d. A second way to understand this formula is to consider the final formula established in Example 3.7(3), which, after setting $n = 1$ for $\dim X$ and $N = d$ for the dimension of the ambient projective space, results in

$$\deg X_m^* = \binom{n+\dim G}{n} \cdot \deg G \cdot \big((n+1)(d-1) - 2(m-n)\big),$$

where $G = G(m-n-1, N-n-1)$. This is linear in d and should also be compared with Theorem 1.1(2), which is polynomial in d (at most of degree $n+1$) multiplied by the degrees of Grassmann and flag varieties and a binomial coefficient. We readily admit that the bound in Theorem 1.1(2) is likely not sharp, but on the other hand, Example 3.7 shows that it is certainly not too far off.

A significant subtlety in our work is that X can be singular. If X were smooth, one could make use of various techniques to study this subject and obtain at least some effective bounds rather easily. For example, one could use the canonical bundle together with the adjunction formula and a ramification formula for finite morphisms, as well as Katz' degree formula [Kat73, Proposition 5.7.2], the Lefschetz theorem for hyperplane sections, and the theory of bundles of principal parts or jet bundles (see [Pie77, §2, §6], or Example 3.7 for a glimpse of these techniques). However, these techniques are not applicable in many key steps in this paper due to the possible presence of singularities.

Let us discuss the more technical part of our approach. Experience shows that the most fundamental case is the case of the standard Gauss map case without defect, i.e., $g_n : \Gamma_n \to X_n^*$ (and $X \dashrightarrow X_n^*$) is birational. The degree of the standard Gauss map image $X_n^* \subset G(n, N)$ is the intersection number of X_n^* and hyperplanes under the Plücker embedding of $G(n, N)$. By an application of a Hodge index theorem type inequality, bounding this degree can be reduced to bounding the intersection number of a general hyperplane section curve on X and an effective Weil divisor on X which is a strict transform, via the Gauss map $g_n : X \dashrightarrow X_n^*$, of a hyperplane section of $X_n^* \subset G(n, N)$ by the Plücker embedding. This type of effective divisor on X corresponds to the ramification divisor of a general linear projection of $\pi : X \to \mathbb{P}^n$ in $\mathbb{P}^N$. We study carefully these standard geometric processes, i.e., hyperplane cuts and linear projections. We will use the Kleiman-Bertini theorem and another refinement of Bertini's theorem to study codimension 1 points in X (this amounts to saying that, if X is a curve, we study the behavior of tangent directions around the singular points) and estimate the number of intersecting points by hand with the aid of a Castelnuovo type bound [Har82, 3.7]. This line of argument is given in [Zak12, Example 1.4] in the case X is smooth, where the canonical bundle is used at some point. This will be discussed in Section 2.

Theorem 1.1(2) is a statement of reduction from generalized Gauss maps to the standard Gauss map. This reduction will be done by using an incident variety technique in Section 3. Strictly speaking, this is completed only after some other reduction steps in Section 4 and Section 5 (see Corollary 5.2). In our context, subvarieties $X \subset \mathbb{P}^N$ can be degenerate in two manners. The first is linear degeneracy and the second is non-birationality of Gauss maps. A subvariety $X \subset \mathbb{P}^N$ is said to be *linearly non-degenerate* if X is not contained in a lower dimensional linear subspace of $\mathbb{P}^N$. This kind of degeneracy is handled by way of Lemma 4.3. For the case when $\mathrm{def}_m X > 0$ also, we obtain a natural reduction to the case of zero defect by general hyperplane cuts in Proposition 5.1 thanks to the tangency theorem of Zak [Zak93, I.2.3].

We work over the field of complex numbers $\mathbb{C}$. Our projective space is the space of all complex lines passing through the origin in a complex vector space. By a variety, we mean a reduced and irreducible scheme of finite type over $\mathbb{C}$. Our argument works without any changes for varieties over an algebraically closed field of characteristic zero.

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§2. Bound on the birational standard Gauss map image

We shall devote this section to proving the following version of Theorem 1.1(1) which represents its most fundamental form.

Theorem 2.1. *Let $X \subset \mathbb{P}^N$ be an n-dimensional projective variety of degree d. We denote by N_X the dimension of the smallest linear subspace $\langle X\rangle (= \mathbb{P}^{N_X}) \subset \mathbb{P}^N$ containing X. Let $a := N_X - n$, and let ε be the unique integer with $\varepsilon \equiv d \pmod a$ and $1 \le \varepsilon \le a$. Let $\gamma : X \dashrightarrow G(n,N)$ be the Gauss map defined by $x \in X_{reg} \mapsto [T_{X,x}] \in G(n,N)$, and denote by $Y = \overline{\gamma(X_{reg})}$ the Zariski closure, i.e., $Y = X^*_n$. Suppose that the map $\gamma : X \dashrightarrow Y$ is birational. Then the degree of Y with respect to the Plücker embedding of $G(n,N)$ is bounded by*

$$\deg Y \le \frac{1}{d^{n-1}}\left(\frac{1}{a}(d-\varepsilon)(d-a+\varepsilon-2)+2d-2\right)^n \le d(d-1)^n.$$

Moreover, $\deg Y = d(d-1)^n$ holds if and only if X is smooth and contained in a linear subspace $\mathbb{P}^{n+1}$, i.e., $N_X = n+1$.

Remark 2.2. (1) To obtain a bound of $\deg Y$ without using ε, one can weaken the above bound to

$$\deg Y \leq \frac{1}{d^{n-1}}\left(\frac{1}{a}(d-1)(d-2)+2d-2\right)^n = \frac{d^{n+1}}{a^n}+O(d^n).$$

(2) It is not hard to understand when $\deg Y = \frac{1}{d^{n-1}}\left(\frac{1}{a}(d-\varepsilon)(d-a+\varepsilon-2)+2d-2\right)^n$ holds. Namely, by the proof of Theorem 2.1 (see Step (5) there), this happens only when, letting $C = X \cap \mathbb{P}^{n-1}$ be a curve obtained by general hyperplane cuts, the embedded curve $C \subset \langle X \rangle \cap \mathbb{P}^{n-1} = \mathbb{P}^{N_X-n+1}$ of $\deg C = d$ and $\operatorname{codim} C = a$ in $\mathbb{P}^{N_X-n+1}$ satisfies the equality $g(C) = \frac{1}{2a}(d-\varepsilon)(d-a+\varepsilon-2)$ in the Castelnuovo type bound, where $g(C)$ is the arithmetic genus of C. Thus, a characterization of X with $\deg Y = \frac{1}{d^{n-1}}\left(\frac{1}{a}(d-\varepsilon)(d-a+\varepsilon-2)+2d-2\right)^n$ will be reduced to that of hyperplane cuts C with $g(C) = \frac{1}{2a}(d-\varepsilon)(d-a+\varepsilon-2)$, which has been studied classically (we refer to [Har82, Ch. 3] for a modern treatment). $\square$

Let us start a discussion towards the proof of Theorem 2.1. We set

$$\Gamma = \overline{\{(x,[T_{X,x}]) \in X_{reg} \times G(n,N)\}} \subset X \times G(n,N),$$

and let $p : \Gamma \to X$ and $q : \Gamma \to G(n,N)$ be the projections:

$$\begin{array}{ccc} \Gamma & \xrightarrow{q} & Y \subset G(n,N) \\ {\scriptstyle p}\downarrow & & \\ X \subset \mathbb{P}^N & & \end{array} .$$

By an abuse of terminologies, we call the projection $q : \Gamma \to G(n,N)$, as well as the regular map $\gamma : X_{reg} \to G(n,N)$ and also the rational map $\gamma : X \dashrightarrow G(n,N)$, the (standard) Gauss map. For every subvariety $V \subset X$ not contained in X_{sing}, we set $\gamma(V) := \overline{\gamma(V \cap X_{reg})}$.

Recall that a brief outline of the proof is given in the Introduction. We will estimate an intersection number of a general curve on X and a divisor on X which comes from a Schubert subvariety in $G(n,N)$. We count such an intersection number by hand, by using the geometry of Grassmannians. Let us start to prepare for the proof of Theorem 2.1, which will be given in the final part of this section.

We suppose $n > 1$ for the remainder. In the case $n = 1$, Theorem 2.1 is classically known, and it can also be proved by the argument in this section with many trivial modifications. The birationality assumption on $\gamma : X \dashrightarrow Y$ trivially excludes the case $d = 1$. The following are some additional noteworthy remarks.

Remark 2.3. (1) We recall (or describe) the rational map $\gamma : X \dashrightarrow G(n, N)$ on a general point of X_{sing} outside of an $(n-2)$-dimensional Zariski closed subset. Let

$$X_0 \subset X$$

be a Zariski open subset such that $p : \Gamma \to X$ is finite over X_0. Since p is birational, we see $\dim(X \setminus X_0) \leq n-2$. For every $x \in X_0$, we define the positive integer $J(x)$ to be the number of points in $\operatorname{Supp} p^{-1}(x) \subset \Gamma$. We have $J(x) = 1$ and $p^{-1}(x) = (x, [T_{X,x}])$ for $x \in X_{reg}$ for example. We let in general

$$p^{-1}(x) = \{(x, [T_{X,x,j}]) \text{ with } [T_{X,x,j}] \in G(n, N), j = 1, \ldots, J(x)\}$$

for $x \in X_0$ and refer to $T_{X,x,j}$ as a tangent plane at x (these $[T_{X,x,j}]$ are defined by $p^{-1}(x)$). Then $q(p^{-1}(x)) = \{[T_{X,x,j}], j = 1, \ldots, J(x)\} \subset Y$. For every integer $k \geq 1$, the set $\{x \in X_0;\ J(x) \leq k\}$ is Zariski open.

We now proceed to define a certain subset $Z \subset X$. In case there is no $(n-1)$-dimensional part of X_{sing}, set $Z = \emptyset$. Otherwise, let $\sum_\lambda Z_\lambda$ be the irreducible decomposition of the $(n-1)$-dimensional part of X_{sing}. For every Z_λ, there exists an integer $k \geq 1$ such that $\{x \in Z_\lambda \cap X_0;\ J(x) \leq k\}$ is non-empty and Zariski open in Z_λ. We take k_λ to be the smallest integer such that

$$Z_{\lambda 0} := \{x \in Z_\lambda \cap X_0;\ X_{sing} \text{ is smooth at } x \text{ and } J(x) = k_\lambda\}$$

is non-empty and Zariski open in Z_λ. Then we set

$$Z = \bigcup\nolimits_\lambda Z_{\lambda 0} \subset X_0.$$

(2) Let $M \subset \mathbb{P}^N$ be a general $(N-n+1)$-plane so that the intersections $X_{reg} \cap M$ and $Z \cap M$ are transverse (recall $Z \subset X_{sing}$ from (1) above). Then

$$C := X \cap M$$

is an irreducible curve of degree $d > 1$, and $C_{reg} = X_{reg} \cap M$ by Bertini's theorem ([Har95, Theorem 17.16]). Codimension 2 (or higher) points in X are irrelevant for C due to M being general. Thus, we can suppose $C \subset X_0$ and $X_{sing} \cap M = Z \cap M$. $\square$

In the situation in Remark 2.3(2), we can further suppose that $T_{X,x,j} \cap M$ is a line for every $x \in C$ and $j = 1, \ldots, J(x)$ by the following Bertini-type lemma. (If $x \in C_{reg}$, then $T_{X,x} \cap M = T_{C,x}$ and it is certainly a line.)

Lemma 2.4 (Bertini-type)**.** *There exists a non-empty Zariski open subset $U \subset G(N-n+1, N)$ such that, for every $[M] \in U$, $C = X \cap M \subset X_0$ has the properties in Remark 2.3(2) and $T_{X,x,j} \cap M$ is a line for every tangent plane $T_{X,x,j}$ of X at $x \in C$.*

Proof. We follow the arguments in [Har77, II.8.18] for the proof of the usual Bertini theorem. We set $G := G(N-n+1, N)$. The conditions in Remark 2.3(2) pose only a Zariski open condition on G. For every $\xi = (x, [T]) \in \Gamma \subset X \times G(n, N)$, we consider

$$B_\xi = \{[M] \in G;\ x \in M, \dim(M \cap T) \geq 2\}.$$

It is always the case that $M \cap T$ is a linear subspace of dimension ≥ 1. This B_ξ is a Schubert variety of partition type $(n-1, 1, 1)$ and hence of codimension $n+1$ in G (see Remark 2.5(1) below). Thus, $\dim B_\xi = \dim G - (n+1)$.

We consider $B \subset \Gamma \times G$ consisting of all pairs $(\xi, [M])$ such that $[M] \in B_\xi$. The fiber of the first projection $p_1 : B \to \Gamma$ over $\xi \in \Gamma$ is nothing but B_ξ. The subset B is a kind of incident variety over Γ (see Remark 2.5(2)) and we have $\dim B = \dim \Gamma + \dim B_\xi = \dim G - 1$. The second projection $p_2 : B(\subset \Gamma \times G) \to G$ cannot be surjective simply because of the dimensions. If we take an element $[M] \in G \backslash p_2(B)$, which is non-empty Zariski open, then $[M] \notin B_\xi$ for any $\xi \in \Gamma \cap p^{-1}(X_0)$. This means $\dim(M \cap T_{X,x,j}) = 1$ for any $x \in M \cap X_0$ and any tangent plane $T_{X,x,j}$ of X at x. Q.E.D.

Remark 2.5. The following are mostly purely general remarks on Grassmannians.

(1) Let $x \in \mathbb{P}^N$, and let $T \subset \mathbb{P}^N$ be an n-plane containing x. We then observe that $\sigma_{x,T} := \{[M] \in G(N-n+1, N);\ x \in M, \dim(M \cap T) \geq 2\}$ is a Schubert variety of the partition type $(n-1, 1, 1)$ in the convention of [GH94, Ch. 1, §5]. We set $k' = N - n + 2, n' = N + 1$ and denote by $G_A(k', n')$ the Grassmannian of all $0 \in \mathbb{C}^{k'} \subset \mathbb{C}^{n'}$ ("A" stands for "affine"). By convention $G_A(k', n') = G(N - n + 1, N)$ in a natural way, which is given by the projectivization $[\Lambda] \in G_A(k', n') \mapsto [\mathbb{P}(\Lambda)] \in G(N - n + 1, N)$.

We take a flag: $0 \in V_1 \subset V_2 \subset \ldots \subset V_{n'-1} \subset V_{n'}$ in $\mathbb{C}^{n'}$ (i.e., each V_i is an i-dimensional linear subspace) so that $\mathbb{P}(V_1) = x$ and $\mathbb{P}(V_{n+1}) = T$. For a k'-plane $\Lambda \subset \mathbb{C}^{n'}$, $[\mathbb{P}(\Lambda)] \in \sigma_{x,T}$ if and only if $\dim(\Lambda \cap V_1) \geq 1, \dim(\Lambda \cap V_n) \geq 2$ and $\dim(\Lambda \cap V_{n+1}) \geq 3$. Note that $\dim(\Lambda \cap V_n) \geq 2$ is a trivial necessary condition for $\dim(\Lambda \cap V_{n+1}) \geq 3$. Namely, letting $(a_1, a_2, a_3) = (n-1, 1, 1)$, $\sigma_{x,T}$ can be identified with $\{[\Lambda] \in G_A(k', n');\ \dim(\Lambda \cap V_{n'-k'+i-a_i}) \geq i \text{ for all } i = 1, 2, 3\}$. The latter subset in $G_A(k', n')$ is a Schubert variety of the partition type $(n-1,1,1)$, which is commonly denoted by $\sigma_{n-1,1,1}$ as in [GH94, p. 196].

(2) Let $\Omega \subset \mathbb{P}^N \times G(n,N)$ be the universal family of n-planes, defined by $\Omega = \{(x,[T]) \in \mathbb{P}^N \times G(n,N);\ x \in T\}$. We consider another incident variety $\Sigma \subset \Omega \times G(N-n+1,N)$ defined by $\Sigma = \{((x,[T]),[M]) \in \Omega \times G(N-n+1,N);\ x \in M, \dim(M \cap T) \geq 2\}$. The fiber of the projection $\Sigma \to \Omega$ is a Schubert variety of the partition type $(n-1,1,1)$ as we saw in (1).

Our variety Γ sits in Ω via the inclusions $\Gamma \subset X \times G(n,N) \subset \mathbb{P}^N \times G(n,N)$. By restricting this family $\Sigma \to \Omega$ to Γ, we have a family $\Sigma_\Gamma \to \Gamma$, which we denote by $p_1 : B \to \Gamma$. □

We next study a special type of divisor on X coming from a Schubert subvariety in $G(n,N)$.

Remark 2.6. (1) Let $L \subset \mathbb{P}^N$ be an $(N-n-1)$-plane. We set

$$D_L = \{[\Lambda] \in G(n,N);\ \Lambda \cap L \neq \emptyset\},$$

which is a special hyperplane section of $G(n,N)$ with respect to the Plücker embedding. Although D_L is defined as a set, there is a natural scheme structure as a restriction of a hyperplane. By the Kleiman-Bertini theorem ([Kle74, Remark 7, Corollary 8], [Har77, III.10.8]), if L is general, the hyperplane section D_L is reduced and irreducible on $G(n,N)$, and moreover Y_{reg} and $(D_L)_{reg}$ intersect transversally with expected dimension for our Y.

(2) Let $L \subset \mathbb{P}^N$ be an $(N-n-1)$-plane. We denote by $\pi_L : \mathbb{P}^N \dashrightarrow \mathbb{P}^n_L$ the linear projection from L (we prefer to denote the target $\mathbb{P}^n$ by $\mathbb{P}^n_L$ to avoid any potential for confusion). If L is general, the map π_L induces a finite morphism $\pi_L : X \to \mathbb{P}^n_L$. We then set

$$R_L = \overline{\{x \in X_{reg};\ T_{X,x} \cap L \neq \emptyset\}},$$

which is a codimension 1 subset of X, and give it the reduced structure. This R_L is the locus where the rank of the differential of $\pi_L : X \to \mathbb{P}^n_L$ drops. We can put a natural scheme structure Ram_L on the set R_L as follows. Letting $R_L = \sum R_{Li}$ be the irreducible decomposition and e_i be the ramification index of $\pi_L : X \to \mathbb{P}^n_L$ along the generic point of R_{Li}, we set

$$\mathrm{Ram}_L = \sum (e_i - 1) R_{Li}.$$

(Cf. [Zak12, Example 1.4]. The notation is slightly different, and the smoothness of X is assumed at some point there.) We set $B_L = \pi_L(R_L) \subset \mathbb{P}^n_L$.

(3) We will take a resolution of singularities $\mu : \widetilde{\Gamma} \to \Gamma$, and let $\widetilde{p} = p \circ \mu$, $\widetilde{q} = q \circ \mu$ be the induced morphisms:

$$\begin{array}{ccc} \widetilde{\Gamma} & \xrightarrow{\widetilde{q}} & Y \subset G(n,N) \\ {\scriptstyle \widetilde{p}}\Big\downarrow & & \\ X \subset \mathbb{P}^N & & \end{array} .$$

We will also take a Zariski closed subset $S_X \subset X$ (resp. $S_Y \subset Y$) such that $\widetilde{p} : \widetilde{\Gamma} \setminus \widetilde{p}^{-1}(S_X) \to X \setminus S_X$ (resp. $\widetilde{q} : \widetilde{\Gamma} \setminus \widetilde{q}^{-1}(S_Y) \to Y \setminus S_Y$) is isomorphic, and set

$$S := S_X \cup p(q^{-1}(S_Y)) \subset X.$$

In particular, we note that $S \supset X_{sing}$ and $q(p^{-1}(S)) \supset Y_{sing}$, $\widetilde{p}$ is isomorphic over $X \setminus S$, and $\widetilde{q}$ is isomorphic over $Y \setminus q(p^{-1}(S))$, which is $Y \setminus \widetilde{q}(\widetilde{p}^{-1}(S))$. Observe that trivially $p^{-1}(S) \supset q^{-1}(S_Y)$. We will be able to take $E = S \subset X$ and $F = q(p^{-1}(S)) \subset Y$ in Lemma 2.7 below.

For every $x \in R_L \cap X_{reg}$, we have $[T_{X,x}] \in D_L \cap Y$ by the definitions. Via this correspondence, we can identify $R_L|_{X \setminus S}$, $\widetilde{p}^*(R_L|_{X \setminus S})$, $\widetilde{q}^*(D_L|_{Y \setminus q(p^{-1}(S))})$ and $D_L|_{Y \setminus q(p^{-1}(S))}$, under the isomorphisms $X \setminus S \cong \widetilde{\Gamma} \setminus \widetilde{p}^{-1}(S) \cong Y \setminus q(p^{-1}(S)) = Y \setminus \widetilde{q}(\widetilde{p}^{-1}(S))$. □

The next lemma is a kind of base point freeness statement. A slight subtlety is the constraint $L \subset M$. Without $L \subset M$, it would be much easier and entirely straight forward.

Lemma 2.7. *Let $E \subset X$ (resp. $F \subset Y$) be a Zariski closed subset satisfying $X_{sing} \subset E \neq X$ (resp. $Y_{sing} \subset F \neq Y$). Then there exist a (general) $(N-n-1)$-plane $L \subset \mathbb{P}^N$ and a (general) $(N-n+1)$-plane $M \subset \mathbb{P}^N$ with $L \subset M$ such that (i) $C = X \cap M$ has the properties in Remark 2.3(2), (ii)*

$$C \cap R_L \subset X \setminus E \subset X_{reg} \quad \textit{and} \quad \gamma(C) \cap D_L \subset Y \setminus F \subset Y_{reg},$$

(iii) C and R_L intersect transversally where they are smooth, and (iv) D_L is reduced and irreducible on $G(n,N)$, and Y_{reg} and $(D_L)_{reg}$ intersect transversally.

Proof. (1) We consider $A = E \cup p(q^{-1}(F))$. We take a general $(N-n+1)$-plane M_0 as in Lemma 2.4 as an auxiliary object and set $C_0 = X \cap M_0$. We can further suppose that M_0 contains an $(N-n-1)$-plane L_0 which is general in view of the Kleiman-Bertini theorem in Remark 2.6(1). If the Kleiman-Bertini theorem holds for one $L_0 \subset M_0$, it holds for general $L \subset M_0$. Since $p : \Gamma \to X$ and $q : \Gamma \to Y$ are birational, these are finite morphisms in codimension 1 over the targets. We may further assume that p is finite around C_0 and hence $p^{-1}(C_0 \cap A)$ consists of a finite number of points. The number of points in $C_0 \cap A$ is just the degree d_A in $\mathbb{P}^N$ of the $(n-1)$-dimensional components in A. Then $q(p^{-1}(C_0 \cap A))$ corresponds to a finite number of tangent n-planes $\{T_{X,x_i,j}; 1 \leq i \leq d_A, 1 \leq j \leq J(x_i)\}$ of X. By Lemma 2.4, we know that these $T_{X,x_i,j} \cap M_0, 1 \leq i \leq d_A, 1 \leq j \leq J(x_i)$, form a finite number of lines in M_0. We then take a general $(N-n-1)$-plane $L \subset M_0$ such that L does not intersect any of these lines $T_{X,x_i,j} \cap M_0$ and L satisfies the genericity condition in Remark 2.6(1). Then $D_L \cap q(p^{-1}(C_0 \cap A)) = \emptyset$ by definition of D_L. We fix this L for the rest of the argument.

(2) We shall then take an $M(\supset L)$ which is close to M_0 in $G(N-n+1, N)$ so that the intersection $(X \cap M) \cap R_L$ becomes transverse. We first note that the set $F := \{[M] \in G(N-n+1, N);\ M \supset L\}$ and the set $F' := \{\text{lines in } \mathbb{P}^n_L\} \cong G(1, n)$ can be identified in a natural way via $F \ni [M] \mapsto [M \cap \mathbb{P}^n_L] \in F'$. For the Zariski open $U \subset G(N-n+1, N)$ in Lemma 2.4, the set $U \cap F$ is Zariski open in F and non-empty because of $[M_0] \in F$. We take a general point $[\ell] \in F'$ so that (i) the corresponding $[M] \in F$ is still in $U \subset G(N-n+1, N)$, (ii) ℓ and $B_L = \pi_L(R_L)$ intersect transversally where B_L is smooth and where the morphism $R_L \to B_L$ is unramified over a Zariski open subset containing $\ell \cap B_L$, and (iii) $D_L \cap q(p^{-1}(C \cap A)) = \emptyset$, where $C = X \cap M$. The last property (iii) follows since $D_L \cap q(p^{-1}(C_0 \cap A)) = \emptyset$ for M_0 and a small perturbation of M_0 will not change the fact that the intersection is empty. We shall show that these L and M are what we are looking for.

(3) We shall prove that $\gamma(C) \cap D_L \cap F = \emptyset$. We take a point $y \in \gamma(C) \cap F$. The condition $y \in \gamma(C)$ means that y is represented by a tangent plane at some point $x \in C$, i.e., $y = [T_{X,x,j}]$ for some $1 \leq j \leq J(x)$. Then by $y \in F$, we have $x \in p(q^{-1}(y))$, in fact $(x, [T_{X,x,j}]) \in q^{-1}(y)$ and $p((x, [T_{X,x,j}])) = x$. Hence $x \in C \cap p(q^{-1}(y)) \subset C \cap A$. However, by our choice of L and M, $D_L \cap q(p^{-1}(C \cap A)) = \emptyset$ holds. It follows that $y = [T_{X,x,j}] \notin D_L$.

(4) We shall prove that $C \cap R_L \cap E = \emptyset$. We take a point $x \in C \cap R_L$. We first note that $T_{X,x,j} \cap L \neq \emptyset$ (not only as a limit) for some tangent plane at x. In the case $x \in X_{reg}$, this follows from the definition of R_L. If $x \in X_{sing}$, $x \in X_{sing} \cap C$ must be in general position in X_{sing} as in Remark 2.3. Then by definition of R_L, there exists a sequence of points $\{x_k\}$ such that $x_k \in X_{reg}$, $T_{X,x_k} \cap L \neq \emptyset$ and $\lim_{k\to\infty} x_k = x$. By passing to a subsequence, we may assume that $p^{-1}(x_k) = (x_k, [T_{X,x_k}]) \in \Gamma$ converges to a point in $p^{-1}(x) = \{(x, [T_{X,x,j}])\}_{1 \leq j \leq J(x)} \subset \Gamma$, namely $[T_{X,x,j}] = \lim_{k\to\infty}[T_{X,x_k}]$ in $G(n, N)$ for some $1 \leq j \leq J(x)$. Hence $T_{X,x,j} \cap L \neq \emptyset$ too.

On the other hand, if $x \in E$, we have $x \in C \cap E \subset C \cap A$. However, then $T_{X,x,j} \cap L = \emptyset$ by our choice of L and M. Thus, $C \cap R_L \cap E = \emptyset$.

(5) We shall prove that C and R_L intersect transversally where they are smooth. We take a point $x \in C \cap R_L$, which is contained in $X \setminus E \subset X_{reg}$ as we have just seen. The property (ii) in (2) implies that R_L is smooth at x. Since $x \in C = X \cap M$ and $x \in C_{reg}$, we see $T_{C,x} = T_{X,x} \cap M$ and then $T_{C,x} \cap L = T_{X,x} \cap (M \cap L) = T_{X,x} \cap L \neq \emptyset$ as $x \in R_L \cap X_{reg}$. If C and R_L are tangent at the point x, we have $T_{C,x} \subset T_{R_L,x}$. Thus, we see $T_{R_L,x} \cap L \supset T_{C,x} \cap L \neq \emptyset$, which implies that the morphism $R_L \to B_L$ is ramified at $x \in R_L$. This cannot happen by the condition (ii) in (2).

We add a remark for a later purpose. Let us take a point $x \in C \cap R_L$ above. Once we know that C and R_L intersect transversally, the ramification index e of $\pi_L : X \to \mathbb{P}^n_L$ along the component of R_L containing x and that of $\pi_L|_C : C \to \mathbb{P}^1$ coincide. Q.E.D.

We are ready to give a proof of Theorem 2.1.

Proof of Theorem 2.1. We take a resolution of singularities $\mu : \widetilde{\Gamma} \to \Gamma$, and let $\widetilde{p} = p \circ \mu, \widetilde{q} = q \circ \mu$ be the induced morphisms as in Remark 2.6.

(1) Let H be the hyperplane section class on $\mathbb{P}^N$, and let D be the hyperplane section class on $G(n, N)$ with respect to the Plücker embedding. We let $\widetilde{H} = \widetilde{p}^* H$ and $\widetilde{D} = \widetilde{q}^* D$, which are nef and big classes and give base point free linear systems on $\widetilde{\Gamma}$. Let $r_i = \widetilde{D}^i \cdot \widetilde{H}^{n-i}$ for $i = 0, 1, \ldots, n$ (which satisfy $r_i > 0$ for all i). In particular, $r_0 = H^n \cdot X = \deg X = d$, $r_n = D^n \cdot Y = \deg Y$. By the Khovanskii-Teissier inequality ([Laz04, Example 1.6.4]), we have the Hodge index theorem type inequalities as in [Zak12, Theorem 1.1]; in particular, we have $r_n \leq r_1^n / r_0^{n-1}$, i.e., $\deg Y \leq r_1^n / d^{n-1}$. For our purpose, it is enough to show that

$$r_1 = \widetilde{D} \cdot \widetilde{H}^{n-1} \leq \frac{1}{a}(d - \varepsilon)(d - a + \varepsilon - 2) + 2d - 2 \leq d(d-1).$$

(2) Here we explain a slightly more general situation. For every choice of general members $\widetilde{H}_1, \ldots, \widetilde{H}_{n-1} \in \widetilde{p}^*|H|$ (i.e., $\widetilde{H}_i = \widetilde{p}^* H_i$ for a general $H_i \in |H|$), $\widetilde{C} = \widetilde{H}_1 \cap \ldots \cap \widetilde{H}_{n-1}$ is a smooth (because of the base point freeness of $\widetilde{p}^*|H|$) irreducible curve on $\widetilde{\Gamma}$ and $C = H_1 \cap \ldots \cap H_{n-1} \cap X$ is a reduced and irreducible curve on X. Then $C \subset \langle X \rangle \cap M (= \mathbb{P}^{N_X - n+1})$ is linearly non-degenerate. The induced morphism $\nu := \widetilde{p}|_{\widetilde{C}} : \widetilde{C} \to C$ is in fact the normalization. Every general $(N - n + 1)$-plane $M \subset \mathbb{P}^N$ can be written as such an intersection $M = H_1 \cap \ldots \cap H_{n-1}$.

Let $L \subset \mathbb{P}^N$ be a general $(N-n-1)$-plane with $L \subset M$ so that the linear projection $\pi_L : \mathbb{P}^N \dashrightarrow \mathbb{P}^n_L$ from L induces finite morphisms $\pi_L : X \to \mathbb{P}^n_L$ and $\pi_L : C \to \mathbb{P}^1$, where $\mathbb{P}^1$ is a line in the target $\mathbb{P}^n_L$. Let

$$f_L = \pi_L \circ \nu : \widetilde{C} \to \mathbb{P}^1$$

be the induced d-sheeted covering, and let $Q_L \subset \widetilde{C}$ be the ramification divisor of f_L. Then by Hurwitz' formula, $2g(\widetilde{C}) - 2 = -2d + \deg Q_L$, where $g(\widetilde{C})$ is the genus of $\widetilde{C}$ (and $g(C)$ will denote the arithmetic genus of C). By a Castelnuovo type bound [Har82, 3.7] (the "genus" there means the arithmetic genus, see [Har82, p. 2]), we have

$$g(\widetilde{C}) \leq g(C) \leq \frac{1}{2a}(d-\varepsilon)(d-a+\varepsilon-2).$$

We note that $\frac{1}{2a}(d-\varepsilon)(d-a+\varepsilon-2) \leq \frac{1}{2}(d-1)(d-2)$ holds, and the equality holds only when $a=1$, i.e., $\langle X \rangle = \mathbb{P}^{n+1}$. Thus,

$$\deg Q_L \leq \frac{1}{a}(d-\varepsilon)(d-a+\varepsilon-2) + 2d - 2 \leq d(d-1).$$

The integer r_1 in (1) can further be written as $\widetilde{D} \cdot \widetilde{H}^{n-1} = (\widetilde{q}^* D) \cdot \widetilde{p}^{-1}(C) = D \cdot \gamma(C) = \deg \gamma(C)$. Since $D_L \in |D|$, our object of interest is $\gamma(C) \cap D_L$.

(3) Using Lemma 2.7 for $E := S \subset X$ and $F := q(p^{-1}(S)) \subset Y$ as mentioned in Remark 2.6(3), we choose an $(N-n-1)$-plane $L \subset \mathbb{P}^N$ and an $(N-n+1)$-plane $M \subset \mathbb{P}^N$ with $L \subset M$ as in Lemma 2.7 such that

$$C \cap R_L \subset X \setminus S \quad \text{and} \quad \gamma(C) \cap D_L \subset Y \setminus q(p^{-1}(S))$$

plus other conditions stated there, where $C = X \cap M$. We note that the birational morphisms $\widetilde{p}$ and $\widetilde{q}$ (and also the Gauss map $\gamma : X \dashrightarrow Y$) induce isomorphisms $X \setminus S \cong \widetilde{\Gamma} \setminus \widetilde{p}^{-1}(S) \cong Y \setminus q(p^{-1}(S)) = Y \setminus \widetilde{q}(\widetilde{p}^{-1}(S))$. Under these isomorphisms, we can identify $C \setminus S$, $\widetilde{C} \setminus \widetilde{p}^{-1}(S)$ and $\gamma(C) \setminus q(p^{-1}(S))$, as well as $R_L|_{X \setminus S}$, $\widetilde{p}^*(R_L|_{X \setminus S})$, $\widetilde{q}^*(D_L|_{Y \setminus q(p^{-1}(S))})$ and $D_L|_{Y \setminus q(p^{-1}(S))}$ (as in Remark 2.6(3)). Thanks to these identifications, and noting the inclusions $\gamma(C) \cap D_L \subset Y \setminus q(p^{-1}(S))$ and $C \cap R_L \subset X \setminus S$, we have

$$\nu^*(R_L|_C) = (\widetilde{q}^* D_L)|_{\widetilde{C}}$$

for the normalization $\nu = \widetilde{p}|_{\widetilde{C}} : \widetilde{C} \to C$. It is true that the effective Cartier divisor $D_L|_Y$ and $\gamma(C)$ intersect transversally where they are smooth, since R_L and C do so. Thus, the Plücker degree of $\gamma(C) \subset G(n, N)$ is

$$\deg \gamma(C) = \#(D_L \cap \gamma(C)) = \#(R_L \cap C)$$

just by counting the number of intersection points (without multiplicities).

Recall that, for $x \in C_{reg}$, $\pi_L : X \to \mathbb{P}^n_L$ (resp. $\pi_L : C \to \mathbb{P}^1$) is ramified at x if and only if $T_{X,x} \cap L \neq \emptyset$ (resp. $T_{C,x} \cap L \neq \emptyset$). Since $T_{X,x} \cap L = T_{C,x} \cap L$ for $x \in C_{reg}$, we obtain the equivalence that $\pi_L : X \to \mathbb{P}^n_L$ is ramified at x if and only if $\pi_L : C \to \mathbb{P}^1$ is ramified at x. Thus, we have

$$Q_L|_{\nu^{-1}(C_{reg})} = (\nu|_{\nu^{-1}(C_{reg})})^*(\mathrm{Ram}_L|_{C_{reg}})$$

on $\nu^{-1}(C_{reg}) \subset \widetilde{C}$.

(4) Adapting the construction and notations in (2) for these L and M in (3), we see

$$Q_L \succeq \nu^*(R_L|_C),$$

i.e., Q_L is more effective than $\nu^*(R_L|_C)$. This is because of the fact that (i) $Q_L|_{\nu^{-1}(C_{reg})} = (\nu|_{\nu^{-1}(C_{reg})})^*(\mathrm{Ram}_L|_{C_{reg}})$ on $\nu^{-1}(C_{reg}) \subset \widetilde{C}$ as we have seen in (3), (ii) $\mathrm{Ram}_L|_C \succeq R_L|_C$, and (iii) R_L has no support on C_{sing} as a consequence of $C \cap R_L \subset X \setminus S$ (while Q_L may have a support on $\nu^{-1}(C_{sing})$).

Thus, noting $\widetilde{D} \cdot \widetilde{H}^{n-1} = \deg(\widetilde{q}^* D_L)|_{\widetilde{C}}$, we have

$$\begin{aligned}\widetilde{D} \cdot \widetilde{H}^{n-1} &= \deg \nu^*(R_L|_C) \leq \deg Q_L \\ &\leq \frac{1}{a}(d-\varepsilon)(d-a+\varepsilon-2) + 2d - 2 \leq d(d-1).\end{aligned}$$

This is what we wanted to prove in (1).

(5) Suppose now that $\deg Y = d(d-1)^n$ holds. Then it has to be that $\deg Q_L = d(d-1)$ in the preceding argument, and then $a = 1$, i.e., $X \subset \langle X \rangle = \mathbb{P}^{n+1}$ in (2). By Lemma 4.3 (an independent general result), we have $\deg Y = \deg \widehat{X}^*_n$, where $\widehat{g}_n : \widehat{\Gamma}_n \to \widehat{X}^*_n \subset G(n, \langle X \rangle)$ is the n-th Gauss map for $X \subset \langle X \rangle = \mathbb{P}^{n+1}$. We then have $\deg \widehat{X}^*_n = d(d-1)^n$, which can happen only when X is smooth due to [Zak12, Theorem 1.18]. It is also known due to [Zak12, Theorem 1.18] that, if X is a smooth hypersurface in $\mathbb{P}^{n+1}$, then $\deg X^*_n = d(d-1)^n$ holds (the Gauss map $\gamma = g_n : X \to X^*_n$ is birational as soon as $d > 1$ in this setting). Q.E.D.

§3. Reduction to the standard Gauss map

We shall prove Theorem 1.1(2) in the case when $g_m : \Gamma_m \to X_m^*$ is birational. In the case when g_m is not birational, the proof of Theorem 1.1(2) gets completed with Corollary 5.2.

We temporarily take three positive integers n, m, N satisfying $n < m < N$ until we reach Proposition 3.4. When we consider polarizations and degrees of Grassmannians $G(n, N)$, $G(m, N)$, $G(n, N) \times G(m, N), \ldots$, and of any subvarieties of those spaces, it is always with respect to the Plücker embeddings.

3.1. (1) We let $F(n, m; N) \subset G(n, N) \times G(m, N)$ be a flag manifold defined by

$$F(n, m; N) = \{([V], [W]) \in G(n, N) \times G(m, N);\ V \subset W\},$$

and let

$$\begin{array}{ccc} F(n,m;N) & \xrightarrow{\pi_m} & G(m,N) \\ {\scriptstyle \pi_n}\downarrow & & \\ G(n,N) & & \end{array}$$

be the projections. This $F(n, m; N)$ is an incident variety fibered over $G(n, N)$ with fibers isomorphic to $G(m-n-1, N-n-1)$; in particular, $\dim F(n, m; N) = (n + 1)(N - n) + (m - n)(N - m)$. Various incident varieties play important roles in this paper (some of them have already appeared in Section 2). There is an explicit (but somewhat involved) formula for $\deg F(n, m; N)$ by representation theory (see Remark 3.5 below). Here we employ the simpler estimate

$$\deg F(n, m; N) \le (\dim F(n, m; N))!.$$

(2) This $F := F(n, m; N)$ connects our X_n^* with X_m^* in the following way, where $X \subset \mathbb{P}^N$ is as in Theorem 1.1. We pull back the bundle structure $F \to G(n, N)$ by the inclusion $X_n^* \subset G(n, N)$ and also the standard Gauss map $g_n : \Gamma_n \to G(n, N)$. We then obtain an induced diagram as follows:

$$\begin{array}{ccccccc} g_n^* F & \longrightarrow & F_{X_n^*} & \xrightarrow{incl.} & F & \longrightarrow & G(m,N) \\ \downarrow & & \downarrow & & \downarrow & & \\ \Gamma_n & \xrightarrow{g_n} & X_n^* & \xrightarrow{incl.} & G(n,N) & & \end{array}.$$

By definition $g_n^* F = \{(x, [V]) \times [W] \in \Gamma_n \times G(m, N);\ V \subset W\}$. We have a natural morphism $\beta : g_n^* F \to X \times G(m, N)$ given by $(x, [V]) \times [W] \mapsto$

$(x, [W])$. If we restrict everything to X_{reg}, then by the definitions of Γ_m and g_n^*F, β gives a natural identification of g_n^*F and Γ_m. Since Γ_m and g_n^*F (as fiber bundles over Γ_n) are irreducible, we have $\beta(g_n^*F) = \Gamma_m$. This construction also shows that g_n^*F is the fiber product of the projections $p_n : \Gamma_n \to X$ and $p_m : \Gamma_m \to X$ over X. As a result, we have the following commutative diagram:

$$\begin{tikzcd}
X & \Gamma_m \arrow[l, "p_m"'] \arrow[r, "g_m"] & G(m,N) \\
\Gamma_n \arrow[u, "p_n", "birat."'] \arrow[d, "g_n"'] & g_n^*F \arrow[l] \arrow[r] \arrow[u, "\beta", "birat."'] \arrow[d] & G(m,N) \arrow[u, equal] \arrow[d, equal] \\
G(n,N) & F \arrow[l, "\pi_n"'] \arrow[r, "\pi_m"] & G(m,N)
\end{tikzcd} .$$

□

The main reduction step towards Proposition 3.4 is the following.

Lemma 3.2. *Let $Y \subset G(n, N)$ be a closed subvariety. Consider*

$$\begin{aligned} F_Y &= \{([V], [W]) \in G(n,N) \times G(m,N);\ V \subset W, [V] \in Y\} \\ &= F(n,m;N) \cap (Y \times G(m,N)), \\ Y_m &= \pi_m(F_Y) \subset G(m,N) \end{aligned}$$

with reduced structures. This F_Y can be seen as a $G(m-n-1, N-n-1)$-bundle over Y. Suppose that the induced morphism $\pi_m : F_Y \to Y_m$ is birational. Then

$$\deg Y_m \le \deg F(n,m;N) \binom{\dim Y + \dim G(m,N)}{\dim Y} \deg G(m,N) \deg Y.$$

Proof. Let H_n (resp. H_m) be a hyperplane section under the Plücker embedding of $G(n, N)$ (resp. $G(m, N)$). We set $k = \dim Y_m$. Since $\pi_m : F_Y \to Y_m$ is birational, we have $\deg Y_m = Y_m H_m^k = F_Y \pi_m^* H_m^k$. Combining this with $F_Y \pi_m^* H_m^k \le F_Y(\pi_n^* H_n + \pi_m^* H_m)^k = \deg F_Y$, we obtain $\deg Y_m \le \deg F_Y$. We set $F = F(n, m; N)$ and $G = G(m, N)$ for simplicity. Since $F_Y = F \cap (Y \times G)$, we have $\deg F_Y \le \deg F \cdot \deg(Y \times G)$ by a Bézout type inequality. We also have $\deg(Y \times G) = (Y \times G)(\pi_n^* H_n + \pi_m^* H_m)^{\dim Y + \dim G} = \binom{\dim Y + \dim G}{\dim Y} \deg Y \cdot \deg G$. Thus, our claim is proven. Q.E.D.

Remark 3.3. Our estimate is not optimal due to the inequality $F_Y \pi_m^* H_m^k \le F_Y(\pi_n^* H_n + \pi_m^* H_m)^k$ in the argument above. □

Proposition 3.4. *Let $X \subset \mathbb{P}^N$ be an n-dimensional projective variety and let m be an integer with $n < m < N$. Then Theorem 1.1(2) holds if the m-th Gauss map $g_m : \Gamma_m \to X_m^*$ is birational.*

Proof. We take $Y = X_n^* \subset G(n,N)$ as the standard Gauss map image of X in the setting of Lemma 3.2. In the second commutative diagram in 3.1(2), if $g_m : \Gamma_m \to X_m^*$ is birational, it follows that $g_n^* F \to g_m(\Gamma_m) = X_m^* \subset G(m,N)$ is birational too. The latter implies $F_{X_n^*} \to \pi_m(F_{X_n^*})$, i.e., $F_Y \to \pi_m(F_Y)$, is birational. Since the birationality of $g_m : \Gamma_m \to X_m^*$ implies the birationality of $g_{m'} : \Gamma_{m'} \to X_{m'}^*$ for any m' with $n \leq m' \leq m$, we have $\dim X_n^* = \dim \Gamma_n = n$ (the projection $\Gamma_n \to X$ is always birational). Thus, by Lemma 3.2, we have

$$\deg X_m^* \leq C \deg X_n^* \quad \text{with}$$
$$C = \deg F(n,m;N) \binom{n + \dim G(m,N)}{n} \deg G(m,N).$$

Let us again write $F = F(n,m;N)$ and $G = G(m,N)$. The following rough bounds give the bound $C < (\ell + (m+1)(m-n))!(\ell+n)!/(n!)$ with $\ell = (m+1)(N-m) = \dim G$. By Remark 3.5, we have $\deg F \leq (\dim F)!$ and $\deg G \leq (\dim G)!$. We recall $\dim F = (n+1)(N-n) + (m-n)(N-m) = \ell + (m+1)(m-n)$. We then have $\deg F \leq (\ell + (m+1)(m-n))!$, and $\binom{n+\dim G}{n} \deg G \leq \frac{(n+\ell)!}{n!\ell!} \ell! = \frac{(\ell+n)!}{n!}$. Q.E.D.

Remark 3.5. We make some comments on a degree formula of homogeneous varieties. We refer to the nice paper [GW11] for an explanation of the following type of calculation. Let F be a homogeneous variety with an ample line bundle H_λ. Let λ be the dominant weight corresponding to H_λ. Let ρ be one half times the sum of the positive roots. Then the degree of F with respect to H_λ is given, due to Borel-Hirzebruch (see [GW11, Introduction]), by

$$\deg_\lambda F = (\dim F)! \prod_\alpha \frac{\langle \lambda, \alpha^* \rangle}{\langle \rho, \alpha^* \rangle},$$

where the product is taken over positive roots α with $\langle \lambda, \alpha^* \rangle \neq 0$. In general, these products are quite involved. For example, the Plücker degree of the Grassmannian is

$$\deg G(m,N) = (\dim G(m,N))! \frac{0!}{(N-m)!} \frac{1!}{(N-m+1)!} \cdots \frac{(m-1)!}{(N-1)!} \frac{m!}{N!}$$

with $0! = 1$ ([Har95, p. 247]). By a somewhat similar reasoning, we have $\deg F \leq (\dim F)!$ for the Plücker degree of flag manifolds. Thus, we can use $(\dim F)!$ as a rough bound, which is only achieved for the full flag manifold. $\square$

Remark 3.6. We take this opportunity to establish two inductive relations of degrees which appear in Theorem 1.1(2) and Corollary 5.2: $\deg G(m-1,N-1) \leq \deg G(m,N)$ and $\deg F(n-1,m-1;N-1) \leq \deg F(n,m;N)$. The first relationship follows from the formula in Remark 3.5, but we prefer to give an independent self-contained proof in (1) below for the proof of (2). The condition $n \leq m < N$ plays no role here, so we will replace m by the unencumbered variable n below. Note that the first relationship immediately implies $\binom{n-1+\dim G(m-1,N-1)}{n-1} \leq \binom{n+\dim G(m,N)}{n}$.

(1) We shall prove $\deg G(n,N) \leq \deg G(n+1,N+1)$ for $0 \leq n \leq N$. Here we work under the convention $G(n,N) = \{(0 \in)\mathbb{C}^n \subset \mathbb{C}^N\}$ for a technical reason. We write $\mathbb{C}^{N+1} = \mathbb{C}^N \oplus \mathbb{C}e_0$ for a non-zero vector $e_0 \in \mathbb{C}^{N+1}$. Let $G = G(n,N)$, $g = \dim G = n(N-n)$, $\widetilde{G} = G(n+1,N+1)$, $\widetilde{g} = \dim \widetilde{G} = (n+1)(N-n)$. We take an embedding

$$\alpha : G \to \widetilde{G} \text{ given by } V(=\mathbb{C}^n) \mapsto \widetilde{V} = V \oplus \mathbb{C}e_0.$$

For $[W] \in \widetilde{G}$, $[W] \in \alpha(G)$ if and only if $W \supset \mathbb{C}e_0$. We see $\operatorname{codim}(\alpha(G) \subset \widetilde{G}) = \widetilde{g} - g = N - n$. If we take Plücker embeddings $G \to \mathbb{P}_G$ and $\widetilde{G} \to \mathbb{P}_{\widetilde{G}}$, there is an embedding $\mathbb{P}_G \to \mathbb{P}_{\widetilde{G}}$ as a linear subspace which makes the following diagram commutative:

$$\begin{array}{ccc} G & \longrightarrow & \mathbb{P}_G \\ {\scriptstyle\alpha}\downarrow & & \downarrow \\ \widetilde{G} & \longrightarrow & \mathbb{P}_{\widetilde{G}} \end{array}.$$

For a full flag $0 \in V_1 \subset V_2 \subset \ldots \subset \mathbb{C}^{N+1}$ of $\mathbb{C}^{N+1}$ starting with $V_1 = \mathbb{C}e_0$, we consider a special Schubert cycle $\sigma_{N-n}(=\sigma_{N-n,0,0,\ldots}) = \{W = \mathbb{C}^{n+1} \subset \mathbb{C}^{N+1};\ W \supset V_1\}$ on $\widetilde{G}$. This is nothing but $\alpha(G)$, i.e., $\alpha(G) = \sigma_{N-n}$. We refer to [GH94, Ch. 1, §5] for Schubert cycles. We recall in particular that the Schubert cycle σ_1 of codimension 1 is a hyperplane cut of $\widetilde{G}$ under the Plücker embedding. By Pieri's formula ([GH94, p. 203]), we have $\sigma_1 \cdot \sigma_b = \sigma_{b+1} + \sigma_{b,1}$ for every positive integer b $(< \widetilde{g})$, and thus $\sigma_1^b = \sigma_b + R_b$ inductively with some effective (perhaps equal to zero) codimension b cycle R_b on $\widetilde{G}$ (R_b is a sum of Schubert cycles with non-negative coefficients). In particular,

$$\sigma_1^{N-n} = \alpha(G) + R_{N-n}.$$

Noting that $\widetilde{g} - g = N - n$, we have

$$\begin{aligned}\deg \widetilde{G} = \sigma_1^{\widetilde{g}} = \sigma_1^{\widetilde{g}-g} \cdot \sigma_1^g = (\alpha(G) + R_{N-n}) \cdot \sigma_1^g \\ \geq \alpha(G) \cdot \sigma_1^g = \alpha^*(\sigma_1)^g = \deg G.\end{aligned}$$

(2) Next, we prove $\deg F(n,m;N) \leq \deg F(n{+}1, m{+}1; N{+}1)$. Here we work under the convention $F(n,m;N) = \{(0 \in)\mathbb{C}^n \subset \mathbb{C}^m \subset \mathbb{C}^N\}$. Again, let $F = F(n,m;N), f = \dim F = n(N-n) + (m-n)(N-m)$, $\widetilde{F} = F(n{+}1, m{+}1; N{+}1)$, $\widetilde{f} = \dim \widetilde{F} = (n{+}1)(N{-}n){+}(m{-}n)(N{-}m)$. We let $G = G(n,N)$, $H = G(m,N)$, $\widetilde{G} = G(n+1, N+1)$, $\widetilde{H} = G(m+1, N+1)$. We then have $F \subset G \times H, \widetilde{F} \subset \widetilde{G} \times \widetilde{H}$ and projections:

$$\begin{array}{ccc} F & \xrightarrow{q} & H \\ {\scriptstyle p}\downarrow & & \\ G & & \end{array} \quad ; \quad \begin{array}{ccc} \widetilde{F} & \xrightarrow{\widetilde{q}} & \widetilde{H} \\ {\scriptstyle \widetilde{p}}\downarrow & & \\ \widetilde{G} & & \end{array}$$

as in 3.1. Let $\alpha_G := \alpha : G \to \widetilde{G}$ be the embedding in (1), and let $\alpha_H : H \to \widetilde{H}$ be the one given by $W \mapsto W \oplus \mathbb{C}e_0$. We also consider an embedding $\alpha_F : F \to \widetilde{F}$ given by $[V \subset W] \mapsto [V \oplus \mathbb{C}e_0 \subset W \oplus \mathbb{C}e_0]$. We note that α_F is not only an embedding of F, but also, if we pull-back the $G(m-n, N-n)$-bundle structure $\widetilde{p} : \widetilde{F} \to \widetilde{G}$ to G via $\alpha_G : G \to \widetilde{G}$, it is exactly $p : F \to G$. In fact, if $[\widetilde{V} \subset \widetilde{W}] \in \widetilde{F}$ with $\widetilde{V} \in \alpha_G(G)$, then $\mathbb{C}e_0 \subset \widetilde{V} \subset \widetilde{W}$ and $[V := \widetilde{V}/\mathbb{C}e_0 \subset W := \widetilde{W}/\mathbb{C}e_0]$ lies in F over $[V] \in G$ (it should be $V = pr(\widetilde{V})$ under the projection $pr : \mathbb{C}^{N+1} = \mathbb{C}^N \oplus \mathbb{C}e_0 \to \mathbb{C}^N$). We have commutative diagrams:

$$\begin{array}{ccc} F = \alpha_G^* \widetilde{F} & \xrightarrow{\alpha_F} & \widetilde{F} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \widetilde{p}} \\ G & \xrightarrow{\alpha_G} & \widetilde{G} \end{array} ; \quad \begin{array}{ccccc} F & \longrightarrow & G \times H & \longrightarrow & \mathbb{P}_G \times \mathbb{P}_H \\ {\scriptstyle \alpha_F}\downarrow & & \downarrow{\scriptstyle \alpha_G \times \alpha_H} & & \downarrow \\ \widetilde{F} & \longrightarrow & \widetilde{G} \times \widetilde{H} & \longrightarrow & \mathbb{P}_{\widetilde{G}} \times \mathbb{P}_{\widetilde{H}} \end{array} .$$

Let σ_1 (resp. τ_1) be a Schubert cycle which is a hyperplane cut of $\widetilde{G}$ (resp. $\widetilde{H}$) under the Plücker embedding. We consider the ample divisor $\widetilde{\sigma}_1 + \widetilde{\tau}_1$ on $\widetilde{G} \times \widetilde{H}$, where $\widetilde{\sigma}_1 = \widetilde{p}^* \sigma_1$ on $\widetilde{G}$ and $\widetilde{\tau}_1 = \widetilde{q}^* \tau_1$ on $\widetilde{H}$. Then

$$\deg \widetilde{F} = \widetilde{F} \cdot (\widetilde{\sigma}_1 + \widetilde{\tau}_1)^{\widetilde{f}} = \widetilde{F} \cdot (\widetilde{\sigma}_1 + \widetilde{\tau}_1)^{\widetilde{f}-f} \cdot (\widetilde{\sigma}_1 + \widetilde{\tau}_1)^f.$$

Noting $\widetilde{f} - f = N - n$, we have $\widetilde{F} \cdot (\widetilde{\sigma}_1 + \widetilde{\tau}_1)^{\widetilde{f}-f} = \widetilde{F} \cdot \widetilde{\sigma}_1^{N-n} + \widetilde{F} \cdot R'$, where $R' = \sum_{k=1}^{\widetilde{f}-f} \binom{\widetilde{f}-f}{k} \widetilde{\sigma}_1^{\widetilde{f}-f-k} \widetilde{\tau}_1^k$, which is a sum of intersections of semi-ample divisors with non-negative coefficients, and $\widetilde{F} \cdot \widetilde{\sigma}_1^{N-n} = \alpha_F(F) +$

$\widetilde{p}^* R_{N-n}$ thanks to the relation $\sigma_1^{N-n} = \alpha_G(G) + R_{N-n}$ in (1). Thus,

$$\begin{aligned}\deg \widetilde{F} &= (\alpha_F(F) + \widetilde{p}^* R_{N-n} + \widetilde{F} \cdot R') \cdot (\widetilde{\sigma}_1 + \widetilde{\tau}_1)^f \\ &\geq \alpha_F(F) \cdot (\widetilde{\sigma}_1 + \widetilde{\tau}_1)^f = \deg F.\end{aligned}$$

□

We close this section by giving a fundamental example, which is due to Kaji. He, in [Kaj15], treats these types of examples with methods due to him, including the case of positive characteristics. Here we quote his argument in a slightly modified manner in view of the connection with our Theorem 1.1.

Example 3.7 (Kaji [Kaj15]). (1) Let $v_d : \mathbb{P}^1 \to \mathbb{P}^d$ be the d-Veronese embedding, i.e., $[s,t] \mapsto [s^d, s^{d-1}t, \ldots, st^{d-1}, t^d]$ in homogeneous coordinates, and denote by $C = v_d(\mathbb{P}^1)$ the image. We suppose $d \geq 2$. It is classically known that $\deg C_1^* = 2(d-1)$ ([Har95, Exercise 19.12]). Here we shall establish the formulas

$$\deg C_m^* = 2(d-m)\big((m-1)(d-m)+1\big) \deg G(m-2, d-2)$$

for every $2 \leq m \leq d-1$ (this also holds for $m = 1$ under a suitable convention). Here the Plücker degree of the Grassmannian is

$$\deg G(m-2, d-2) = \frac{(m-2)!(m-3)!\ldots 1!\,0!}{(d-2)!(d-3)!\ldots(d-m)!} \big((m-1)(d-m)\big)!.$$

Thus, for example,

$\deg C_2^* = 2(d-2)(d-1)$,

$\deg C_3^* = 2(d-3)(2d-5) \deg G(1, d-2) = 2(d-3)(2d-5)\frac{(2(d-3))!}{(d-2)!(d-3)!}$,

$\ldots$

$\deg C_{d-2}^* = 2 \cdot 2(2d-5) \deg G(d-4, d-2) = 4(2d-5)\frac{(2(d-3))!}{(d-2)!(d-3)!}$,

$\deg C_{d-1}^* = 2 \cdot 1(d-1)$.

(2) Let us begin a general discussion to show the formula in (1). Let $X \subset \mathbb{P}^N$ be an n-dimensional smooth projective variety. When X is smooth in 3.1(2), the projection $p_n : \Gamma_n \to X$, as well as $\beta : g_n^* F \to \Gamma_m$ $(n < m < N)$ are isomorphic. We will identify $p_m : \Gamma_m \to X$ and $g_n^* F \to \Gamma_n$, and in particular we regard $p_m : \Gamma_m \to X$ as a $G(m-n-1, N-n-1)$-bundle. We then shall build the universal bundle on it and recall a bundle theoretic construction of Gauss maps.

Let $0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{\oplus(N+1)} \to T_{\mathbb{P}^N}|_X \to 0$ be the Euler exact sequence restricted to X, where $\mathcal{O}_X(1) := \mathcal{O}_{\mathbb{P}^N}(1)|_X$, and let $0 \to T_X \to T_{\mathbb{P}^N}|_X \to N_{X/\mathbb{P}^N} \to 0$ be the normal bundle sequence of X

in $\mathbb{P}^N$. We pull-back (i.e., restrict in this setting) the Euler exact sequence by the bundle injection $T_X \to T_{\mathbb{P}^N}|_X$, and have an extension $0 \to \mathcal{O}_X \to E \to T_X \to 0$ (i.e., we take the Yoneda pairing of these in $\mathrm{Hom}\,(T_X, T_{\mathbb{P}^N}|_X) \times \mathrm{Ext}^1(T_{\mathbb{P}^N}|_X, \mathcal{O}_X) \to \mathrm{Ext}^1(T_X, \mathcal{O}_X)$). We then have the following commutative diagram, which is exact in rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_X & = & \mathcal{O}_X & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & E & \to & \mathcal{O}_X(1)^{\oplus(N+1)} & \to & N_{X/\mathbb{P}^N} & \to & 0\,. \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & T_X & \to & T_{\mathbb{P}^N}|_X & \to & N_{X/\mathbb{P}^N} & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

We take a tensor product with $\mathcal{O}_X(-1)$ in the middle row of the diagram and obtain an exact sequence

$$(*) \quad 0 \to \mathcal{P}^{1*} \to \mathcal{O}_X^{\oplus(N+1)} \to N_{X/\mathbb{P}^N}(-1) \to 0,$$

where $N_{X/\mathbb{P}^N}(-1) := N_{X/\mathbb{P}^N} \otimes \mathcal{O}_X(-1)$, and where $\mathcal{P}^1 := (E \otimes \mathcal{O}_X(-1))^*$ is the so-called bundle (of rank $n+1$) of principal parts of $\mathcal{O}_X(1)$ of first order on X (cf. [Pie77, §2, §6]). We note $\det \mathcal{P}^1 = K_X \otimes \mathcal{O}_X(n+1)$ (as is well-known, see [Zak93, p. 25]), which can be computed by $\det \mathcal{P}^1 = \det N_{X/\mathbb{P}^N}(-1)$ and $K^*_{\mathbb{P}^N}|_X = K^*_X \otimes \det N_{X/\mathbb{P}^N}$ (hence $\det N_{X/\mathbb{P}^N} = K_X \otimes \mathcal{O}_X(N+1)$).

The sub-bundle $\mathcal{P}^{1*} \subset \mathcal{O}_X^{\oplus(N+1)}$ defines the standard Gauss map $g_n : X \to G(n, N)$ (cf. [Zak93, p. 25]; in fact this is often taken as a definition of the standard Gauss map). Then $\deg X^*_n = (\det \mathcal{P}^1)^n = (K_X \otimes \mathcal{O}_X(n+1))^n$ if g_n is birational onto its image. More generally for $n < m < N$, at each $x \in X$, every m-plane $W (= \mathbb{P}^m)$ containing $T_{X,x} (= \mathbb{P}^n)$ corresponds to an $(m+1)$-dimensional vector subspace in $\mathcal{O}_{X,x}^{\oplus(N+1)}$ containing ${\mathcal{P}^1}^*_x$, i.e., corresponds to an $(m-n)$-dimensional vector subspace S of $N_{X/\mathbb{P}^N}(-1)_x$ in view of the exact sequence $(*)$. Then the $G(m-n-1, N-n-1)$-bundle structure of $p_m : \Gamma_m \to X$ (i.e., $g_n^* F \to \Gamma_n$) can be written as

$$\Gamma_m = G(m-n-1, \mathbb{P}(N_{X/\mathbb{P}^N}(-1))) := \coprod_{x \in X} G(m-n-1, \mathbb{P}(N_{X/\mathbb{P}^N}(-1)_x))$$

with $G(m-n-1, \mathbb{P}(N_{X/\mathbb{P}^N}(-1)_x)) = \{\mathbb{P}^{m-n-1} \text{ in } \mathbb{P}(N_{X/\mathbb{P}^N}(-1)_x)\} = \{(0 \in) S = \mathbb{C}^{m-n} \text{ in } N_{X/\mathbb{P}^N}(-1)_x = \mathbb{C}^{N-n}\}$. We let $\mathcal{S} \subset p_m^*(N_{X/\mathbb{P}^N}(-1))$

be the universal sub-bundle on Γ_m of rank $m-n$ of this $G(m-n-1, N-m-1)$-bundle structure. By pulling back (i.e., restricting) the exact sequence $0 \to p_m^*\mathcal{P}^{1*} \to \mathcal{O}_{\Gamma_m}^{\oplus(N+1)} \to p_m^*(N_{X/\mathbb{P}^N}(-1)) \to 0$ on Γ_m by $\mathcal{S} \subset p_m^*(N_{X/\mathbb{P}^N}(-1))$, we have an extension $\mathcal{W}$ of $\mathcal{S}$ to $p_m^*\mathcal{P}^{1*}$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & p_m^*\mathcal{P}^{1*} & \to & \mathcal{W} & \to & \mathcal{S} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & p_m^*\mathcal{P}^{1*} & \to & \mathcal{O}_{\Gamma_m}^{\oplus(N+1)} & \to & p_m^*(N_{X/\mathbb{P}^N}(-1)) & \to & 0\,, \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{Q} & = & \mathcal{Q} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $\mathcal{Q} = p_m^*(N_{X/\mathbb{P}^N}(-1))/\mathcal{S}$ is a vector bundle of rank $N-n$. The subbundle $\mathcal{W} \subset \mathcal{O}_{\Gamma_m}^{\oplus(N+1)}$ of rank $m+1$ is the collection of all m-planes W in $\mathcal{O}_{X,x}^{\oplus(N+1)}$ containing $T_{X,x}$ as we indicated before, and $\mathcal{W} \subset \mathcal{O}_{\Gamma_m}^{\oplus(N+1)}$ gives the morphism $\Gamma_m \to G(m,N)$, which is nothing but the m-th Gauss map g_m. (This $\mathcal{W}$ corresponds to the flag construction in 3.1. We do not want to use the symbol F here, because of the potential for confusion.) Then

$$g_m^*\mathcal{O}_{G(m,N)}(1) = \det \mathcal{W}^* \ = \det p_m^*\mathcal{P}^1 \otimes \det \mathcal{S}^*$$

on Γ_m. We denote (in general) by $\mathcal{O}_{\Gamma_m}(1) := \det \mathcal{S}^*$ the determinant of the dual of the universal sub-bundle.

If $N_{X/\mathbb{P}^N}(-1)$ is of the form $(\mathcal{O}_X^{\oplus(N-n)}) \otimes L$ for a line bundle L on X, then we have an isomorphism

$$i : G(m-n-1, \mathbb{P}(\mathcal{O}_X^{\oplus(N-n)})) \to \Gamma_m,$$

which is defined by $[V \subset \mathcal{O}_{X,x}^{\oplus(N-n)}] \mapsto [V \otimes L \subset (\mathcal{O}_X^{\oplus(N-n)} \otimes L)_x]$. We note that $G(m-n-1, \mathbb{P}(\mathcal{O}_X^{\oplus(N-n)})) = G(m-n-1, N-n-1) \times X$ and $\mathcal{O}_{G(m-n-1,\mathbb{P}(\mathcal{O}_X^{\oplus(N-n)}))}(1) = pr_1^*\mathcal{O}_{G(m-n-1,N-n-1)}(1)$, where pr_1 is the first projection. Since $\det(V \otimes L)^* = (\det V)^* \otimes (L^*)^{\otimes(m-n)}$, it follows that

$$i^*\mathcal{O}_{\Gamma_m}(1) = pr_1^*\mathcal{O}_{G(m-n-1,N-n-1)}(1) \otimes (pr_2^*L^*)^{\otimes(m-n)}.$$

We note that the second projection $pr_2 : G(m-n-1, \mathbb{P}(\mathcal{O}_X^{\oplus(N-n)})) \to X$ is compatible with the projection $p_m : \Gamma_m \to X$, i.e., $p_m \circ i = pr_2$. Up to this point, our discussion was of a general nature.

(3) We now suppose that $X(=C) = v_d(\mathbb{P}^1) \subset \mathbb{P}^d$ is the Veronese curve of degree d. We still keep $n(=1)$ and $N(=d)$. We understand well T_X and $N_{X/\mathbb{P}^N}$; in particular, $\det \mathcal{P}^1 = K_X \otimes \mathcal{O}_X(n+1) = \mathcal{O}_{\mathbb{P}^1}((n+1)(d-1))$ and $N_{X/\mathbb{P}^N} = \mathcal{O}_{\mathbb{P}^1}(d+2)^{\oplus(N-n)}$ by [Kaj85, Example 3.5]. Moreover, $N_{X/\mathbb{P}^N}(-1) = (\mathcal{O}_{\mathbb{P}^1}^{\oplus(N-n)}) \otimes L$ with $L = \mathcal{O}_{\mathbb{P}^1}(2)$. For $2 \leq m \leq N-1$, there is an isomorphism $i : G \times \mathbb{P}^1 \to \Gamma_m$ and $i^*\mathcal{O}_{\Gamma_m}(1) = pr_1^*\mathcal{O}_G(1) \otimes pr_2^*\mathcal{O}_{\mathbb{P}^1}(-2(m-n))$, where we set $G = G(m-n-1, N-n-1)$ for short.

By an abuse of notations, we still denote by $pr_1 : \Gamma_m \to G$ and $pr_2 : \Gamma_m \to \mathbb{P}^1$ the induced projections. Then, as $\det \mathcal{W}^* = \det pr_2^*\mathcal{P}^1 \otimes \det \mathcal{S}^*$, we have

$$\det \mathcal{W}^* = pr_1^*\mathcal{O}_G(1) \otimes pr_2^*\mathcal{O}_{\mathbb{P}^1}((n+1)(d-1) - 2(m-n))$$

on Γ_m. Finally, since g_m is birational onto its image by [Zak93, I.2.3.c], we have

$$\begin{aligned}\deg X_m^* &= (\det \mathcal{W}^*)^{\dim X_m^*} \\ &= \binom{n+\dim G}{n} \cdot \deg G \cdot \big((n+1)(d-1) - 2(m-n)\big).\end{aligned}$$

If we write $N_{X/\mathbb{P}^N} = \mathcal{O}_{\mathbb{P}^1}(d+n+1)^{\oplus(N-n)}$ and $L = \mathcal{O}_{\mathbb{P}^1}(n+1)$, then $\deg X_m^* = \binom{n+\dim G}{n} \cdot \deg G \cdot \big((n+1)(d-m+n-1)\big)$. In particular, setting $n=1$ and $N=d$, we obtain our initial formula. $\square$

Remark 3.8. The following are some comments on Example 3.7.

(1) We would like to emphasize the identity $\deg C_1^* = \deg C_{d-1}^*$ and the following "symmetry:"

$$(m-1)\deg C_m^* = (m'-1)\deg C_{m'}^*$$

for every pair $2 \leq m \leq m' \leq d-1$ with $m+m' = d+1$. In fact, letting $G_A(k,N) = \{0 \in \mathbb{C}^k \subset \mathbb{C}^N\} = G(k-1, N-1)$ in our convention, we observe $(m-1)(d-m) = \dim G_A(m-1, d-1) = \dim G_A(m'-1, d-1) = (m'-1)(d-m')$ and $(d-m)\deg G_A(m-1, d-1) = (m'-1)\deg G_A(d-m', d-1)$ (the roles of $d-m$ and $m-1$ are switched), which establishes the "symmetry." There may be a reasonably nice symmetric bound for $\deg X_{n+k}^*$ (or $\deg X_{n+k+1}^*$) and $\deg X_{N-1-k}^*$ for $X^n \subset \mathbb{P}^N$ in general.

(2) We can also observe that the argument in Example 3.7 is somewhat close to our general approach to proving the reduction step Proposition 3.4. The method in Example 3.7(2) is advantageous especially when we know well T_X and $N_{X/\mathbb{P}^N}$ under a smoothness assumption

on the variety X. Moreover, in the example, we used the particular facts that $C \subset \mathbb{P}^N$ satisfies $N = \deg C$ and that $N_{C/\mathbb{P}^d}$ splits as $\mathcal{O}_{\mathbb{P}^1}(d+2)^{\oplus(d-1)}$.

(3) If we apply Theorem 2.1 (the result only) to the Veronese curve $C \subset \mathbb{P}^d$, noting $a = d-1$ and $\varepsilon = 1$, we have $\deg C_1^* \leq 2(d-1)$. $\square$

§4. Reduction to the linearly non-degenerate case

In the remaining two sections, we will treat the case where $X \subset \mathbb{P}^N$ has positive defect, i.e., the (generalized) Gauss map is not birational. In this case, we need to be concerned about another possible degeneracy, which is the linear degeneracy of $X \subset \mathbb{P}^N$. The present section is rather independent from other parts of the paper, and will reduce Theorem 1.1 to the linearly non-degenerate case which we already used in Section 2.

We start with some general remarks.

Remark 4.1. Let V be a $\mathbb{C}$-vector space of $\dim V = N+1$, and let $W \subset V$ be a linear subspace of $\dim W = N$. We suppose that $\mathbb{P}(V)$ is our ambient space $\mathbb{P}^N$ and $\mathbb{P}(W)$ is a hyperplane H. We take a vector $v_0 \in V \setminus W$. There is then a direct sum decomposition $V = W \oplus \mathbb{C}v_0$ and

$$\wedge^{m+1} V = \wedge^{m+1} W \oplus ((\wedge^m W) \wedge v_0)$$

for every $m \geq 0$ in general. Let $1 \leq m \leq N-1$. The decomposition $\wedge^{m+1} V = \wedge^{m+1} W \oplus ((\wedge^m W) \wedge v_0)$ induces a linear projection

$$h(= \pi_W^m) : \mathbb{P}(\wedge^{m+1} V) \dashrightarrow \mathbb{P}(\wedge^m W)$$

from $\mathbb{P}(\wedge^{m+1} W) \subset \mathbb{P}(\wedge^{m+1} V)$ ([Har95, Exercise 3.8]).

We then restrict this projection h to the Grassmannian $G(m,N) \subset \mathbb{P}(\wedge^{m+1} V)$ via the Plücker embedding. The indeterminacy set $G(m,N) \cap \mathbb{P}(\wedge^{m+1} W)$ is nothing but $G(m,H) := \{[\Lambda] \in G(m,N);\ \Lambda \subset H\}$. If $[\Lambda] \in G(m,N) \setminus G(m,H)$, then $h([\Lambda]) \in \mathbb{P}(\wedge^m W)$ is contained in the Grassmannian $G(m-1,H) \subset \mathbb{P}(\wedge^m W)$ (of $(m-1)$-planes in H). In fact, $h([\Lambda])$ is represented by an $(m-1)$-dimensional linear subspace $\Lambda|_H \subset H$. Thus, the linear projection induces a morphism

$$h : G(m,N) \setminus G(m,H) \longrightarrow G(m-1,H) \text{ by } [\Lambda] \mapsto [\Lambda|_H].$$

This map will be the cornerstone of our subsequent reduction arguments. $\square$

Lemma 4.2. *Let $X \subset \mathbb{P}^N$ be a projective variety of* $\dim X = n$. *Let $[H] \in (\mathbb{P}^N)^* \setminus X_{N-1}^*$, where $(\mathbb{P}^N)^*$ is the dual projective space (recall* $\dim X_{N-1}^* \leq N-1$ *in general). Then*

(1) $X_m^* \cap G(m, H) = \emptyset$ *in* $G(m, N)$ *for any* m *with* $n \le m < N$.

(2) The restricted projection morphism $h : G(m, N) \setminus G(m, H) \to G(m-1, H)$ *induces a finite morphism* $h : X_m^* \to h(X_m^*)$.

Proof. (1) We let $X^* = X_{N-1}^*$. Let $[H] \in (\mathbb{P}^N)^*$, and suppose that there exists $[W] \in X_k^* \cap G(k, H)$ for some k $(n \le k < N)$. Then we shall show $[H] \in X^*$.

We consider the incident variety $I = \{([V], [L]) \in G(k, N) \times G(N-1, N);\ [V] \in X_k^*, V \subset L\}$, and the naturally induced diagram:

$$\begin{array}{ccc} I \subset G(k, N) \times G(N-1, N) & \xrightarrow{\pi_{N-1}} & G(N-1, N) = (\mathbb{P}^N)^* \\ {\scriptstyle \pi_k}\downarrow & & \\ X_k^* \subset G(k, N) & & \end{array} .$$

We note that $\pi_{N-1}(I) = X^*$. This can be checked directly and also by the method in 3.1(2). Furthermore, $([W], [H]) \in I$ holds. Thus, $[H] = \pi_{N-1}(([W], [H])) \in \pi_{N-1}(I) = X^*$.

(2) This is rather a general fact. We again look at the linear projection $\mathbb{P}(\wedge^{m+1} V) \dashrightarrow \mathbb{P}(\wedge^m W)$ from $\mathbb{P}(\wedge^{m+1} W)$ associated to H as in Remark 4.1. We saw $X_m^* \cap \mathbb{P}(\wedge^{m+1} W) = X_m^* \cap G(m, H) = \emptyset$ in (1). We take an arbitrary point $y \in X_m^*$. Then $h(y)$ is given by the unique intersection point $\langle \mathbb{P}(\wedge^{m+1} W), y \rangle \cap \mathbb{P}(\wedge^m W)$ in $\mathbb{P}(\wedge^{m+1} V)$, where $\langle \mathbb{P}(\wedge^{m+1} W), y \rangle$ is the smallest linear subspace containing $\mathbb{P}(\wedge^{m+1} W)$ and y. We set $S := h^{-1}(h(y))(\subset X_m^*)$. We would like to show that S consists of isolated points. By definition of the projection, we have $S \subset \langle \mathbb{P}(\wedge^{m+1} W), y \rangle$. If $\dim S > 0$, we have $S \cap \mathbb{P}(\wedge^{m+1} W) \neq \emptyset$, since $\mathbb{P}(\wedge^{m+1} W)$ is a hyperplane in $\langle \mathbb{P}(\wedge^{m+1} W), y \rangle$. This implies $X_m^* \cap \mathbb{P}(\wedge^{m+1} W) \neq \emptyset$, which is a contradiction to (1). Q.E.D.

Lemma 4.3. *Let* $X \subset \mathbb{P}^N$ *be a projective variety of* $\dim X = n$. *Let* $L = \mathbb{P}^M \subset \mathbb{P}^N$ $(n \le M \le N)$ *be the smallest linear subspace containing* X. *In particular,* $X \subset L$ *is linearly non-degenerate in* L. *Let the integer* m *satisfy* $n \le m < N$, *and let* $k = \max\{n, m - (N - M) = M - (N - m)\}$ $(n \le k < M$, *e.g.,* $k = n$ *if* $m = n$, $k = M - 1$ *if* $m = N - 1)$. *Then*

(1) $\deg X_m^* = \deg \widehat{X}_k^*$, *where* $\widehat{g}_k : \widehat{\Gamma}_k \to \widehat{X}_k^* \subset G(k, L)$ *is the* k*-th Gauss map for* $X \subset L$.

(2) There exists a non-empty Zariski open subset $U_m^* \subset X_m^*$ *such that, for every* $y \in U_m^*$, $X_y := p_m(g_m^{-1}(y)) \subset X$ *is a linear subspace in* $\mathbb{P}^N$, *where* $p_m : \Gamma_m \to X$ *is the projection.*

We note that (2) is very similar to [Zak93, I.2.3.c]. We provide a proof here because in [Zak93] the proof is given under the blanket assumption that X is linearly non-degenerate. The proof below shows how to obtain the general case from that case. Alternatively, a close reading of the proof in [Zak93] shows that the blanket assumption of linear non-degeneracy is not actually used in it, so the new justification given below is strictly speaking unnecessary.

Proof. (o) Suppose $M = N$. Then $L = \mathbb{P}^N$, $k = m$, and then (1) is trivial and (2) is [Zak93, I.2.3.c]. We suppose $M \le N-1$ for the remainder. If $m = n$, then we have $k = n$ and $X_n^* = \widehat{X}_n^*$. Here, by definition of Γ_n and $\Gamma_{H,n}$, $\widehat{X}_n^*(\subset G(n,H))$ can be identified with $X_n^*(\subset G(n,N))$ via the natural inclusion $G(n,H) \subset G(n,N)$. This is the meaning of $X_n^* = \widehat{X}_n^*$. Then (1) is clear, and (2) for X_n^* follows from that for $\widehat{X}_n^*$, for which [Zak93, I.2.3.c] can be applied. We may suppose $n \le M \le N-1$ and $n+1 \le m < N$ for the remainder.

The following are some preliminary considerations. We take a hyperplane $H \subset \mathbb{P}^N$ containing L. We consider the $(m-1)$-th Gauss map $g_{H,m-1} : \Gamma_{H,m-1} \to X_{H,m-1}^* \subset G(m-1,H)$ for $X \subset H$. Let us denote by $\mathbb{C}_H^N \subset \mathbb{C}^{N+1}$ the linear subspace corresponding to $H \subset \mathbb{P}^N$. If we take a vector $v_0 \in \mathbb{C}^{N+1} \setminus \mathbb{C}_H^N$, we have an inclusion $G(m-1,H) \subset G(m,N)$ by $[W] \mapsto [\langle W, v_0\rangle]$, where $\langle W, v_0\rangle$ is the linear subspace spanned by W and v_0 (what we mean is the smallest linear subspace in $\mathbb{P}^N$ containing W and the point in $\mathbb{P}^N$ corresponding to v_0). Then we can see that $X_m^* \subset G(m,N)$ is a cone over $X_{H,m-1}^* \subset G(m-1,H)$ with "the vertex" $G(m,H)$. This is due to (a) if $[V] \in X_m^* \setminus G(m,H)$, then $V \cap H \in X_{H,m-1}^*$, and (b) for every given $[W] \in X_{H,m-1}^*$, if $[V] \in G(m,N) \setminus G(m,H)$ and if $V \cap H = W$, then $[V] \in X_m^*$. In other words, after a choice of $v_0 \in \mathbb{C}^{N+1} \setminus \mathbb{C}_H^N$, we have a linear projection

$$h : \mathbb{P}(\wedge^{m+1}\mathbb{C}^{N+1}) \dashrightarrow \mathbb{P}(\wedge^m \mathbb{C}_H^N)$$

from $\mathbb{P}(\wedge^{m+1}\mathbb{C}_H^N) \subset \mathbb{P}(\wedge^{m+1}\mathbb{C}^{N+1})$ (see Remark 4.1). The map h induces a morphism

$$h : G(m,N) \setminus G(m,H) \longrightarrow G(m-1,H) \text{ by } [V] \mapsto [V|_H].$$

Now, we have $X_m^* = \overline{h^{-1}(X_{H,m-1}^*)}$, where the Zariski closure is taken in $G(m,N)$. (Recall that $X \subset H$ and $\dim X \le m-1 < \dim H$.) Since h is a restriction of a linear projection and X_m^* is a cone over $X_{H,m-1}^*$, we have

$$\deg X_m^* = \deg X_{H,m-1}^*.$$

We also note the following. Let $z = [W] \in X_{H,m-1}^*$. Then

$$X_y = X_{H,z}$$

holds for any $y = [V_y] \in X^*_m \setminus G(m, H)$ such that $V_y \cap H = W$, where $X_{H,z} := p_{H,m-1}(g^{-1}_{H,m-1}(z)) \subset X$ with the projection $p_{H,m-1} : \Gamma_{H,m-1} \to X$. To see this, we first note that, for $x \in X$ and $[V] \in G(m, N)$, $(x, [V]) \in \Gamma_m$ if and only if there exists $[T] \in g_n(p_n^{-1}(x)) \subset X^*_n$ such that $T \subset V$, where $p_n : \Gamma_n \to X$ and $g_n : \Gamma_n \to X^*_n$ are the projections. This can be concluded from 3.1(2). Then $x \in X_y$ if and only if there exists $[T] \in g_n(p_n^{-1}(x)) \subset X^*_n$ such that $T \subset V_y$ (as $X \subset H$, we have $T \subset H$). This is equivalent to the existence of $[T] \in g_{H,n}(p^{-1}_{H,n}(x)) \subset X^*_{H,n}$ such that $T \subset V_y \cap H = W$, namely $x \in X_{H,z}$. In particular, we have shown that our assertions (1) and (2) are reduced to those of $X^*_{H,m-1}$. We shall proceed by induction, taking special care to keep m, M, n and N in balance.

(i) If $M = N-1$, then we can only take $H = L$ and $k = \max\{n, m-1\} = m-1$. Noting that $\widehat{X}^*_{m-1} = X^*_{H,m-1}$, we have (1) and (2) by the reduction above. We suppose $M \leq N-2$ for the remainder. If $m = n+1$, then we have $k = n$ and $X^*_n = \widehat{X}^*_n$, and we have (1) and (2) as in (o). We may suppose $n \leq M \leq N-2$ and $n+2 \leq m < N$ for the remainder.

We take a linear subspace $L_2 = \mathbb{P}^{N-2}$ so that $L \subset L_2 \subset L_1 := H$ (H is the one taken above). We have a morphism

$$h_2 : G(m-1, L_1) \setminus G(m-1, L_2) \longrightarrow G(m-2, L_2) \text{ by } [V] \mapsto [V|_{L_2}].$$

(Recall that $X \subset L_2$ and $\dim X \leq m-2 < \dim L_2$.) We can see that $X^*_{L_1,m-1} \subset G(m-1, L_1)$ is a cone over $X^*_{L_2,m-2} \subset G(m-2, L_2)$ with "the vertex" $G(m-1, L_2)$. We then have $\deg X^*_m = \deg X^*_{L_1,m-1} = \deg X^*_{L_2,m-2}$, and (2) is also reduced to that of $X^*_{L_2,m-2}$.

(ii) If $M = N-2$, then $L_2 = L$ and $k = \max\{n, m-2\} = m-2$. Noting that $\widehat{X}^*_{m-2} = X^*_{L_2,m-2}$, we have (1) and (2) as before. We suppose $M \leq N-3$ for the remainder. If $m = n+2$, then we have $k = n$ and $X^*_n = \widehat{X}^*_n$, and have (1) and (2) as before.

(iii) We can continue this process at most $N-M$ times: $L \subset L_i \subset \ldots \subset L_2 \subset L_1$ with $L_j = \mathbb{P}^{N-j}$. After $(i =)N-M$ steps, we have in fact $L_{N-M} = \mathbb{P}^M = L$ and $k = \max\{n, m-(N-M)\} = m-(N-M)$. The rest is similar. Q.E.D.

Remark 4.4. If a subvariety $X \subset \mathbb{P}^N$ is linearly non-degenerate, then $X \cap H$ is linearly non-degenerate in $H = \mathbb{P}^{N-1}$ for a general hyperplane H ([CGN98, Proposition 1.1] for example). In that sense, we do not have to be concerned about linear (non-)degeneracy in further steps. □

§5. Reduction to the birational generalized Gauss map case

Here we consider the case when the (generalized) Gauss map is not birational, i.e., the case when the defect is positive. In fact, the birationality of the m-th Gauss map $g_m : \Gamma_m \to X_m^*$ is equivalent to $\dim \Gamma_m = \dim X_m^*$ (cf. Lemma 4.3(2), which is essentially [Zak93, I.2.3.c]). Proposition 5.1 below reduces Theorem 1.1 to the cases of zero defect: Theorem 2.1 and Proposition 3.4.

Proposition 5.1. *Let $X \subset \mathbb{P}^N$ be a projective variety of $\dim X = n$ and $\deg X = d > 1$. Let the integer m satisfy $n \le m < N$ and suppose that the m-th Gauss map $g_m : \Gamma_m \to X_m^* \subset G(m,N)$ is* **not** *birational (then it has to hold true that $n > 1$ and $1 \le \mathrm{def}_m X \le n-1$). Then*

$$\deg X_m^* \text{ in } G(m,N) \text{ is equal to } \deg(X \cap H)_{m-1}^* \text{ in } G(m-1,H)$$

for a general hyperplane $H \subset \mathbb{P}^N$.

In particular, letting $n' = n - \mathrm{def}_m X$, $m' = m - \mathrm{def}_m X$ and $N' = N - \mathrm{def}_m X$, for a general linear subspace $L = \mathbb{P}^{N'} \subset \mathbb{P}^N$, the m'-th Gauss map $\widehat{g}_{m'} : \widehat{\Gamma}_{m'} \to (X \cap L)_{m'}^* \subset G(m',L)$ for the subvariety $(X \cap L) \subset L$ is birational, and

$$\deg X_m^* \text{ in } G(m,N) = \deg(X \cap L)_{m'}^* \text{ in } G(m',L).$$

Furthermore, $\widehat{g}_{n'} : \widehat{\Gamma}_{n'} \to (X \cap L)_{n'}^* \subset G(n',L)$, which is the standard Gauss map of $(X \cap L) \subset L$, is birational.

Let us now state a slightly more precise version of Theorem 1.1 which we obtain as a corollary of Theorem 2.1, Proposition 3.4 and Proposition 5.1.

Corollary 5.2. *Let $X \subset \mathbb{P}^N$ be a projective variety of $\dim X = n$ and $\deg X = d > 1$.*

(1) Let N_X be the dimension of the smallest linear subspace $\langle X \rangle (= \mathbb{P}^{N_X}) \subset \mathbb{P}^N$ containing X, let $a := N_X - n$, and let ε be an integer with $\varepsilon \equiv d \pmod{a}$ and $1 \le \varepsilon \le a$. Then

$$\deg X_n^* \le \frac{1}{d^{n'-1}} \left(\frac{1}{a}(d-\varepsilon)(d-a+\varepsilon-2) + 2d - 2 \right)^{n'} \le d(d-1)^{n'},$$

where $n' := n - \mathrm{def}_n X$.

(1') Suppose that $\deg X_n^* = d(d-1)^{n'}$ holds in (1). Then for a general linear subspace $\mathbb{P}^{N'} \subset \mathbb{P}^N$ where $N' = N - \mathrm{def}_n X$, $X' := X \cap \mathbb{P}^{N'}$ satisfies $\dim X' = n'$ and $\deg X' = d$ and is smooth and contained in a

linear subspace $\mathbb{P}^{n'+1} \subset \mathbb{P}^{N'}$*. In particular, X is contained in a linear subspace* $\mathbb{P}^{n+1} \subset \mathbb{P}^N$ *by Remark 4.4.*

(2) Let m be an integer with $n < m < N$. Then

$$\deg X_m^* \leq C'' \deg X_n^* \quad \textit{with}$$
$$C'' = \deg F(n'', m''; N'') \binom{n'' + \dim G(m'', N'')}{n''} \deg G(m'', N''),$$

where $n'' = n - \mathrm{def}_m X, m'' = m - \mathrm{def}_m X$ *and* $N'' = N - \mathrm{def}_m X$*.*

Eventually, the classification of the varieties in (1') above is reduced to that of hypersurfaces X in $\mathbb{P}^{n+1}$ with possibly degenerate Gauss map $\gamma : X \dashrightarrow X^* \subset (\mathbb{P}^{n+1})^*$. This is also a classical subject (see [Zak93] for example).

5.3 (A reduction of Proposition 5.1)**.** Here we make a reduction step in Proposition 5.1 and prepare some notations for the later arguments.

(1) We shall slightly simplify the notations as follows: let

$$\Gamma = \overline{\{(x, [V]) \in X_{reg} \times G(m, N);\ T_{X,x} \subset V\}} \subset X \times G(m, N),$$

$p : \Gamma \to X$ and $q : \Gamma \to G(m, N)$ be the projections, and set $Y := X_m^* = q(\Gamma)$ to obtain a commutative diagram as follows:

$$\begin{array}{ccc} \Gamma & \xrightarrow{q} & Y = X_m^* \subset G(m, N) \\ {\scriptstyle p}\downarrow & & \\ X \subset \mathbb{P}^N & & \end{array}.$$

We set $X_y = p(q^{-1}(y)) \subset X$ for every $y \in Y$. By [Zak93, I.2.3.c], there exists a non-empty Zariski open subset $Y_0 \subset Y$ such that, for every $y \in Y_0$, the fiber $q^{-1}(y) \subset \Gamma$, viewed as a subset of $X \times \{y\} \subset \mathbb{P}^N \times \{y\} \cong \mathbb{P}^N$ (or after identifying $q^{-1}(y)$ and X_y by the projection p) is a linear subspace $\mathbb{P}^r$ in $\mathbb{P}^N$ with $r = \dim \Gamma - \dim Y = \mathrm{def}_m(X)$. Since X is not linear, we have $r < n$, and since $q : \Gamma \to Y$ is not birational (i.e., $\dim \Gamma > \dim Y$), we have $r > 0$. Thus, $0 < r < n$.

(2) Let $H \subset \mathbb{P}^N$ be a general hyperplane and let $X_H = X \cap H \subset \mathbb{P}^{N-1}$. By Bertini's theorem, we may suppose $X_{H,reg} = X_{reg} \cap H$. We let $q_H : \Gamma_H \to G(m-1, H)$ be the $(m-1)$-th Gauss map of $X_H \subset H = \mathbb{P}^{N-1}$, which comes with the following maps:

$$\begin{array}{ccc} \Gamma_H & \xrightarrow{q_H} & Y_H = (X_H)_{m-1}^* \subset G(m-1, H) \\ {\scriptstyle p_H}\downarrow & & \\ X_H \subset H & & \end{array}.$$

(3) By Remark 4.1, we have a morphism

$$h : G(m,N) \setminus G(m,H) \longrightarrow G(m-1,H) \text{ by } [V] \mapsto [V|_H],$$

which is a restriction of a linear projection in a larger projective space via the Plücker embedding of these Grassmannians. By Lemma 4.2, for every $[H] \in (\mathbb{P}^N)^* \setminus X^*_{N-1}$, h is regular around Y and gives a finite morphism $h : Y \to h(Y)$. We then let

$$U_X = \{[H] \in (\mathbb{P}^N)^*;\ X_{H,reg} = X_{reg} \cap H, [H] \in (\mathbb{P}^N)^* \setminus X^*_{N-1}\},$$

which is non-empty and Zariski open. We will establish in Corollary 5.7 that, for every $[H] \in U_X$, the projection h gives a birational morphism $h : Y \to Y_H$. We then have $\deg Y = \deg Y_H$, since h is a restriction of a linear projection in a larger projective space. That is nothing but our assertion $\deg X^*_m = \deg(X \cap H)^*_{m-1}$. Hence Proposition 5.1 is reduced to Corollary 5.7. □

We shall use the setup in 5.3 for the rest of this section. Our aim is to show that $h(Y) = Y_H$ and $h : Y \to Y_H$ is birational in 5.3(3).

Lemma 5.4. *Let $[H] \in U_X$. Then (1) $\dim Y_H > 0$, and (2) $h(Y) = Y_H$ in $G(m-1,H)$; in particular, h is well-defined as a morphism $h : Y \to Y_H$.*

Proof. (1) Suppose $\dim Y_H = 0$. Then $m = n$, X_H is an $(n-1)$-plane and $\deg(X \cap H) = 1$. That means X is linear, which is excluded from the beginning.

(2) We first show that $Y_H \subset h(Y)$ (without using the fact that $q : \Gamma \to Y$ has positive dimensional fibers). In any case, we have $T_{X_H,x'} = T_{X,x'} \cap H$ for any $x' \in X_{H,reg}$. It is enough to show that there exists a non-empty Zariski open $Y_{H,0} \subset Y_H$ such that $Y_{H,0} \subset h(Y)$. If $y' = [V'] \in Y_H$ is general, there exists $(x',[V']) \in \Gamma_H$ for some $x' \in X_{H,reg}$, i.e., $T_{X_H,x'} \subset V' \subset H$. By Lemma 4.2, $T_{X,x'} \not\subset H$ (otherwise $[T_{X,x'}] \in X^*_n \cap G(n,H)$). Thus, we can take $v \in T_{X,x'} \setminus H$. We set $V = \langle V', v\rangle = \mathbb{P}^m$. Then we see $(x',[V]) \in \Gamma(\subset X \times G(m,N))$, $y := [V] \in Y$, and $V|_H = V'$, i.e., $h(y) = y'$. Thus, $y' \in h(Y)$.

We next show that $h(Y) \subset Y_H$. If $y = [V] \in Y$ is general, $X_y = p(q^{-1}(y)) \subset X$ is a linear subspace $\mathbb{P}^r$ with $0 < r < n$. We can suppose, if $y \in Y$ is general, that $X_y \cap X_{reg} \neq \emptyset$ and $X_y \cap X_{reg} \cap H \neq \emptyset$. For any $x \in X_y \cap X_{reg}$, we have $(x,[V]) \in \Gamma$, i.e., $T_{X,x} \subset V$ and $q((x,[V])) = y$. For any $x' \in X_{H,reg}$, we have $T_{X_H,x'} = T_{X,x'} \cap H$. Then for $x' \in X_y \cap X_{H,reg} = X_y \cap X_{reg} \cap H$, we have $T_{X_H,x'} = T_{X,x'} \cap H \subset V \cap H$. Thus, $(x',[V \cap H]) \in \Gamma_H$ and $q_H(x',[V \cap H]) = [V \cap H] = h(y)$. Thus, $h(y) \in Y_H$. Q.E.D.

Remark 5.5. Let $[H] \in U_X$. We set $\Gamma_0 = p^{-1}(X_{reg})$ and $\Gamma_{H,0} = p^{-1}(X_{H,reg})$. We have a natural inclusion $\Gamma_{H,0} \to \Gamma_0$ which makes the following diagram commutative:

$$\begin{array}{ccccccc} X_{reg} & \xleftarrow{p} & \Gamma_0 & \xrightarrow{q} & Y & \subset G(m,N) \\ {\scriptstyle incl.}\uparrow & & {\scriptstyle incl.}\uparrow & & \downarrow{\scriptstyle h} & & . \\ X_{H,reg} & \xleftarrow{p_H} & \Gamma_{H,0} & \xrightarrow{q_H} & Y_H & \subset G(m-1,H) \end{array}$$

For every $x \in X_{H,reg}(= X_{reg} \cap H)$, we have $p^{-1}(x) \cong \{[V] \in G(m,N);\ T_{X,x} \subset V\}$ and $p_H^{-1}(x) \cong \{[W] \in G(m-1,H);\ T_{X_H,x} \subset W(\subset H)\}$. We have a morphism $p^{-1}(x) \to p_H^{-1}(x)$ by $[V] \mapsto [V|_H]$, and the converse $p_H^{-1}(x) \to p^{-1}(x)$ as follows by using the idea which appeared in the proof of Lemma 5.4. By Lemma 4.2, we can take $v \in T_{X,x} \setminus H$. We set $V = \langle W, v\rangle = \mathbb{P}^m$, which is the linear subspace spanned by W and v. We can see that $\langle W, v\rangle$ does not depend on the choice of $v \in T_{X,x} \setminus H$. Then we also see that $(x,[V]) \in \Gamma_0$, $p((x,[V])) = x = p_H((x,[W]))$, and $h \circ q((x,[V])) = h([V]) = [V \cap H] = [W] = q_H((x,[W]))$. Thus, the inclusion $\Gamma_{H,0} \to \Gamma$ is given by $(x,[W]) \mapsto (x,[V])$. $\square$

Lemma 5.6. *Let $[H] \in U_X$. Then the surjection $h : Y \to Y_H$ in Lemma 5.4 has connected general fibers.*

Proof. Suppose that general fibers of $h : Y \to Y_H$ are disconnected. Then for a general $y' \in Y_H$, $h^{-1}(y')$ consists of a finite number of connected components $F_1, \dots, F_k \subset Y$ with $k > 1$. We may suppose that (i) every F_i is irreducible of $\dim F_i = s$, where $s := \dim Y - \dim Y_H$, (ii) $q^{-1}(F_i) \cap \Gamma_0 \neq \emptyset$ for any i, where $\Gamma_0 := p^{-1}(X_{reg})$, and (iii) $X_{H,y'} := p_H({q_H}^{-1}(y')) \subset X_H$ is a linear subspace ($\mathbb{P}^{r+s-1}$) by [Zak93, I.2.3c] and $X_{H,y'} \cap X_{H,reg} \neq \emptyset$ (in particular, $X_{H,y'}$ is irreducible). Needless to say, we have $q^{-1}(F_i) \cap q^{-1}(F_j) = \emptyset$ in Γ if $i \neq j$.

We set $A_i = q^{-1}(F_i) \subset \Gamma$. We first prove that $(A_i \cap \Gamma_0) \cap \Gamma_{H,0} (= A_i \cap \Gamma_{H,0}) \neq \emptyset$ for every i. If $y_i \in F_i$ is general, we have $X_{y_i} \cap X_{H,reg} \neq \emptyset$ (as we saw in the proof of Lemma 5.4). This yields $p(q^{-1}(y_i)) \cap X_{H,reg} \neq \emptyset$, and implies $q^{-1}(y_i) \cap p^{-1}(X_{H,reg}) \neq \emptyset$. Since $\Gamma_{H,0} = p^{-1}(X_{H,reg})$ (here we understand $p_H = p$ on $\Gamma_{H,0}$ in view of the left hand square in the commutative diagram in Remark 5.5), we have $q^{-1}(y_i) \cap \Gamma_{H,0} \neq \emptyset$. Since $q^{-1}(y_i) \cap \Gamma_{H,0} \subset A_i \cap \Gamma_{H,0}$, our assertion holds.

We see clearly that $\amalg_i A_i$ is a disjoint union in Γ. Thus, $\amalg_i A_i|_{\Gamma_{H,0}}$ is a disjoint union too (note $A|_{\Gamma_{H,0}} \neq \emptyset$ for any i by the previous argument) and has at least d irreducible components. By the commutativity of the diagram in Remark 5.5, we have $q_H^{-1}(y') \cap \Gamma_{H,0} = \amalg_i A_i|_{\Gamma_{H,0}}$.

However, recalling that $q_H^{-1}(y') \cap \Gamma_{H,0}$ is irreducible, we have obtained a contradiction. Q.E.D.

The above Lemma 5.6 now immediately yields the following corollary, which concludes the proof of Proposition 5.1.

Corollary 5.7. *Let $[H] \in U_X$. Then the morphism $h : Y \to Y_H$ in Lemma 5.4 is birational.*

Proof. By Lemma 4.2(2), the map $h : Y \to h(Y)$ is finite. By Lemma 5.4(2), $h(Y) = Y_H$, and by Lemma 5.6, h has connected general fibers. This proves the corollary. Q.E.D.

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