

The dual complex of singularities

Tommaso de Fernex, János Kollár and Chenyang Xu

*Dedicated to Yujiro Kawamata on the occasion of his
60th birthday*

Abstract.

The dual complex of a singularity is defined, up-to homotopy, using resolutions of singularities. In many cases, for instance for isolated singularities, we identify and study a “minimal” representative of the homotopy class that is well defined up-to piecewise linear homeomorphism. This is derived from a more global result concerning dual complexes of dlt pairs. As an application, we also show that the dual complex of a log terminal singularity as well as the one of a simple normal crossing degeneration of a family of rationally connected manifolds are contractible.

§1. Introduction

To every simple normal crossing variety E one can associate a cell complex $\mathcal{D}(E)$, called *the dual complex* of E , that describes how the irreducible components of E intersect; see Definition 8 for details. Over the complex numbers, one can think of $\mathcal{D}(E)$ as the *combinatorial part* of the topology of E .

Let $(0 \in X)$ be a (not necessarily isolated) singularity and $f: Y \rightarrow X$ a resolution such that $E := \text{Supp}(f^{-1}(0))$ is a simple normal crossing divisor. The corresponding dual complex $\mathcal{D}(E)$ depends on the choice of Y but, as proved in increasingly general forms in the papers [48, 53, 49, 4, 43], the different $\mathcal{D}(E)$ are all homotopy equivalent, even simple-homotopy equivalent; see Definition 18. Their simple-homotopy equivalence class is denoted by $\mathcal{DR}(0 \in X)$.

The main result of this paper is that in many cases, for instance for isolated singularities, one can do even better and select a “minimal”

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representative $\mathcal{DMR}(0 \in X)$ that is well defined up-to piecewise linear homeomorphism.

For surfaces, this representative is given by $\mathcal{D}(E)$ of any resolution $f : Y \rightarrow X$ such that $E := \text{Supp}(f^{-1}(0))$ is a nodal curve with smooth irreducible components and $K_Y + E$ is f -nef, that is, $((K_Y + E) \cdot C) \geq 0$ for every f -exceptional curve C .

In higher dimensions such resolutions usually do not exist but the Minimal Model Program tells us that such objects do exist if we allow Y and E to be mildly singular. The relevant notion is *divisorial log terminal* or *dlt*, see Definition 6. Every simple normal crossing pair (Y, E) is also dlt and for general dlt pairs (Y, E) one can define the dual complex $\mathcal{D}(E)$ by simply ignoring the singularities, see Definition 8. Furthermore, for every normal variety X for which K_X is \mathbb{Q} -Cartier, there are (usually infinitely many) projective, birational morphisms $p : Y \rightarrow X$ such that (Y, E) is dlt and $K_Y + E$ is p -nef, where E is the divisorial part of the exceptional set $\text{Ex}(p)$; see [40]. These are called the *dlt modifications* of X , see Definition 13. (Conjecturally, dlt modifications exist for any X .)

Our main theorem is the following. Here we state it for isolated singularities; the general form is given in Section 4.

Theorem 1. *Let $(0 \in X)$ be an isolated singularity over \mathbb{C} such that K_X is \mathbb{Q} -Cartier. Let $p : Y \rightarrow X$ be a proper, birational morphism and $E \subset Y$ the divisorial part of $\text{Supp } p^{-1}(0)$. Assume that (Y, E) is dlt and $E \neq \emptyset$.*

- (1) *If p is a dlt modification, that is if $K_Y + E$ is p -nef, then the dual complex $\mathcal{D}(E)$ is independent of p , up-to PL homeomorphism. This defines a PL homeomorphism class associated to $(0 \in X)$; we denote it by $\mathcal{DMR}(0 \in X)$.*
- (2) *If $E = \text{Supp } p^{-1}(0)$ then the dual complex $\mathcal{D}(E)$ is simple-homotopy equivalent to $\mathcal{DMR}(0 \in X)$.*
- (3) *If X is \mathbb{Q} -factorial, p is projective and p is an isomorphism over $X \setminus \{0\}$ then $\mathcal{D}(E)$ collapses to $\mathcal{DMR}(0 \in X)$.*

Here \mathcal{DMR} stands for the “Dual complex of the Minimal divisorial log terminal partial Resolution” and a *collapse* is a particularly simple type of homotopy equivalence; see Definition 18.

The assumptions in (1.1–2) are most likely optimal.

If the minimal model conjecture holds for dlt pairs then $\mathcal{DMR}(0 \in X)$ is defined even if K_X is not \mathbb{Q} -Cartier, but it need not be homotopy equivalent to $\mathcal{DR}(0 \in X)$. On the other hand, even in these cases, we can apply the more general Theorem 28 to a pair (X, Δ) where $\Delta \sim_{\mathbb{Q}} -K_X$ is a general \mathbb{Q} -divisor.

Even for hypersurface singularities the various $\mathcal{D}(E)$ are not homeomorphic to each other, even the number of connected components of $\mathcal{D}(E)$ can change.

By contrast, the necessity of the assumptions in (1.3) is not clear and we do not have any examples where $\mathcal{D}(E)$ does not collapse to $\mathcal{DMR}(0 \in X)$. Note also that many singularities are \mathbb{Q} -factorial (for instance all isolated complete intersection singularities of dimension ≥ 4) and many can be made \mathbb{Q} -factorial by choosing a suitable algebraic model [42].

If X is log terminal then its dlt modification is the identity map $X \cong X$, thus $E = \emptyset$. Nonetheless, in this case we define $\mathcal{DMR}(0 \in X)$ to be a point, rather than the empty set. By a simple trick of introducing an auxiliary divisor, we can apply the general version of Theorem 1 to this case and prove the following.

Theorem 2. *Let $0 \in X$ be a point on a dlt pair (X, Δ) over \mathbb{C} . Then $\mathcal{DR}(0 \in X)$ is contractible.*

Although the connection is not immediate, we derive Theorems 1–2 from the following more global result which is the main technical theorem of the paper. It asserts that for a given dlt pair (X, Δ) , one should focus on the sum of those divisors that appear in Δ with coefficient 1. This divisor, denoted by $\Delta^{=1}$, is also the union of all log canonical centers of (X, Δ) .

Theorem 3. *Let (X, Δ) be a quasi projective, dlt pair over \mathbb{C} and $g: Y \rightarrow X$ a projective resolution such that $E := \text{Supp } g^{-1}(\Delta^{=1})$ is a simple normal crossing divisor. Then*

- (1) *The dual complex $\mathcal{D}(E)$ is simple-homotopy equivalent to $\mathcal{D}(\Delta^{=1})$.*
- (2) *If E is \mathbb{Q} -Cartier then $\mathcal{D}(E)$ collapses to $\mathcal{D}(\Delta^{=1})$.*

For a simple normal crossing divisor $E \subset X$ the intersections of the various $E_i \subset E$ are also the log canonical centers of the pair (X, E) , thus one can also view the dual complex as describing the combinatorics of log canonical centers.

For a general log canonical pair (X, Δ) these two approaches give different objects and it is of interest to study the cases when they are the same. This leads to the concept of quotient-dlt pairs, see Definition 35.

Quotient-dlt pairs constitute a useful subclass of log canonical pairs which is preserved by Fano contractions. In Section 5 this leads to a short proof of the following, which extends earlier results of [29, 22].

Theorem 4. *Let $f: X \rightarrow (0 \in C)$ be a flat projective morphism to a smooth, pointed curve over \mathbb{C} . Assume that the general fibers X_t are*

smooth, rationally connected and $F_0 := \text{red } f^{-1}(0)$ is a simple normal crossing divisor. Then $\mathcal{D}(F_0)$ is contractible.

5 (Topological remarks). One should think of elementary collapses and their inverses (see Definition 18) as the obvious homotopy equivalences. A sequence of them gives simple-homotopy equivalences.

By a theorem of Whitehead, every homotopy equivalence between simply connected simplicial complexes is homotopic to a simple-homotopy equivalence, but this can fail when $\pi_1 \neq 1$; see [6].

Being collapsible is much stronger than being contractible. For instance, the *dunce hat* or *Bing's house with two rooms* are contractible, even simple-homotopy equivalent to a point, yet they are not collapsible; cf. [54].

We do not have any examples as in Theorems 2 or 4 where the dual complex is not collapsible. There are some indications, for example Theorems 32–33, that dual complexes coming from algebraic geometry are somewhat special.

History. The dual graph of the exceptional curves of resolutions of surface singularities has been studied for a long time. To the best of our knowledge, higher dimensional versions first appeared in [38] for degenerations of K3 surfaces. The members of the seminar [10] essentially knew Theorem 2 for 3-folds and Theorem 4 for surfaces.

The connectedness conjecture of [44], proved in [27, Sec.17], was the first indication that exceptional divisors with discrepancy ≤ -1 play a special role. The role of the dual complex of these divisors is studied in [14, 16], especially for lc singularities. The effect of the MMP on the dual complex is studied in [12] which essentially leads to the proof of Theorem 1 for log canonical X .

Much of the early work on the abundance conjecture involves understanding how a dual complex changes under birational maps. In retrospect, versions of Proposition 11 are contained in [27, 24, 13].

Theorem 2 is obvious if $\dim X = 2$ and the simple connectedness of $\mathcal{DR}(0 \in X)$ follows from [28, 52]. Much of the 3-dimensional case is proved in [49]. For quotient singularities in arbitrary dimensions Theorem 2 is proved in [26].

It is interesting to connect algebraic properties of singularities with topological properties of the dual complex. Using [47] one can show that if $(0 \in X)$ is a rational singularity then $\mathcal{DR}(0 \in X)$ is \mathbb{Q} -acyclic, that is, $H^i(\mathcal{DR}(0 \in X), \mathbb{Q}) = 0$ for $i > 0$. If $(0 \in S)$ is a rational surface singularity then $\mathcal{DR}(0 \in S)$ is contractible. [48] asked if $\mathcal{DR}(0 \in X)$ is contractible for higher dimensional isolated rational singularities as well, however the opposite turned out to be true. A finite simplicial complex

C occurs as a dual complex of a rational singularity iff C is \mathbb{Q} -acyclic; see [23, Thm.42], [33, Thm.8] and [35, Thm.3].

By [35, Thm.2] every finite simplicial complex occurs as a dual complex of some isolated singularity.

After this, the dlt case seemed a natural candidate. It was explicitly asked in [33, Question 69] and in [8]. Our results also answer [33, Question 68] and solve [33, Conjecture 70].

While we prove contractibility of various dual complexes, the contraction and even the end result depend on auxiliary choices. Nonetheless, in many examples there seems to be a canonical choice and it would be useful to understand this situation better.

We were led to conjecturing Theorems 2–4 partly through arithmetic considerations. If $(0 \in X)$ and the resolution $Y \rightarrow X$ are defined over a field k then the Galois group $\text{Gal}(\bar{k}/k)$ acts on the dual complex $\mathcal{DR}(0 \in X_{\bar{k}})$ over an algebraic closure \bar{k} . The key results of the papers [29, 22, 39] say that this action has a fixed point. This suggests that in these settings the dual complex has a natural retraction to a particular point that is Galois-invariant. See Theorems 32–33 for the corresponding generalizations.

Our methods rely on the minimal model program or MMP. In general, the property of a pair to have simple normal crossings is not preserved during MMP but being dlt is preserved. Our main technical result studies how the dual complex $\mathcal{D}(\Delta^{=1})$ changes as we run MMP on a dlt pair (X, Δ) . In retrospect, the connectedness theorems [37, Thm.5.48] and [34, Sec.4.4] appear as the simplest special cases of Theorem 3.

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6 (Notation and Conventions). We follow the notation and definitions of the books [37, 34]. We work over an algebraically closed field of characteristic 0.

A *pair* (X, Δ) consists of a normal variety X and a \mathbb{Q} -divisor Δ on it. If all coefficients are in $[0, 1]$ (resp. $(-\infty, 1]$), then we say that Δ is a *boundary* (resp. *sub-boundary*).

Let $f: Y \rightarrow X$ be a birational morphism. If $K_X + \Delta$ is \mathbb{Q} -Cartier then the formulas

$$K_Y + \Delta_Y \sim_{\mathbb{Q}} f^*(K_X + \Delta) \quad \text{and} \quad f_*\Delta_Y = \Delta$$

define the \mathbb{Q} -divisor Δ_Y , called the *log pull-back* of Δ . (See [37, Sec.2.3] or [34, 2.6] for details.) For a divisor $E \subset Y$ its *discrepancy*, denoted by $a(E, X, \Delta)$, is the negative of its coefficient in Δ_Y .

A pair (X, Δ) is *log canonical*, abbreviated as *lc*, if $a(E, X, \Delta) \geq -1$ for every divisor E and every birational morphism $f: Y \rightarrow X$.

Let (X, Δ) be an lc pair. An irreducible subvariety $Z \subset X$ is an *lc center* iff there is a birational morphism $f: Y \rightarrow X$ and a divisor $E \subset Y$ such that $a(E, X, \Delta) = -1$ and $f(E) = Z$.

We say that (X, Δ) is a *simple normal crossing pair* if X is smooth and Δ has simple normal crossing support. Given any pair (X, Δ) , there is a largest open set $X^{\text{snc}} \subset X$, called the *simple normal crossing locus* or *snc locus* such that $(X^{\text{snc}}, \Delta|_{X^{\text{snc}}})$ is a simple normal crossing pair.

A log canonical pair (X, Δ) is *divisorial log terminal*, abbreviated as *dlt*, if none of the lc centers of (X, Δ) is contained in $X \setminus X^{\text{snc}}$.

Two pairs (X_i, Δ_i) are called *crepant birational equivalent* if there are proper, birational morphisms $f_i: Y \rightarrow X_i$ such that the log pull back of Δ_1 on Y equals the log pull back of Δ_2 on Y .

§2. The dual complex of a dlt pair

Definition 7. For cell complexes, we follow the terminology of [21], see especially Section 2.1 and the Appendix.

The notion of an (unordered) Δ -*complex* is defined inductively. A 0-dimensional Δ -complex is a collection of points called *vertices*. If k -dimensional Δ -complexes and their attaching maps are already defined, we obtain $(k+1)$ -dimensional Δ -complexes as follows.

Let C_k be a k -dimensional Δ -complex and $\{S_i : i \in I_{k+1}\}$ a collection of $(k+1)$ -dimensional simplices. The boundary ∂S_i is a k -dimensional cell complex. An attaching or characteristic map is a map $\tau_i: \partial S_i \rightarrow C_k$. Identifying the points of ∂S_i with their image in C_k for every i gives a $(k+1)$ -dimensional Δ -complex C_{k+1} . The $\leq k$ -dimensional cells of C_{k+1} are the cells in C_k and the images of the S_i give the $(k+1)$ -dimensional cells. For $j \leq k$ the cell complex C_j is called the *j -skeleton* of C_{k+1} .

If all the attaching maps are embeddings, then the resulting object is a *regular cell complex*.

A regular cell complex is called a *simplicial complex* if the intersection of any 2 simplices S_1, S_2 is a single (possibly empty) simplex

S_3 which is a facet in both of them. In Figure 1 below we have 3 Δ -complexes of dimension 1. The first is a Δ -complex that is not regular, the second is regular but not simplicial and the third is simplicial.

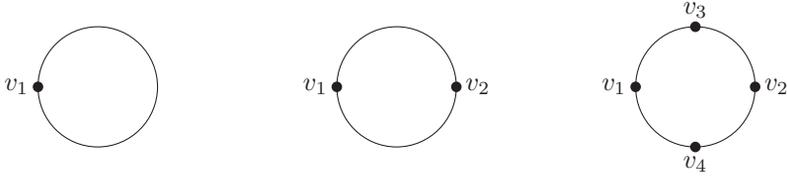


Fig. 1

Let D be a regular cell complex. For a simplex $v \subset D$ let $\text{St}(v)$ be the (open) *star* of v , that is, the union of the interiors of all cells whose closure contains v . Its closure, denoted by $\overline{\text{St}}(v)$, is the *closed star*.

If D is a simplicial complex then their difference $\text{Lk}(v) = \overline{\text{St}}(v) \setminus \text{St}(v)$ is the *link* of v . Note that the stellar subdivision of the closed star $\overline{\text{St}}(v)$ is a cone over the link $\text{Lk}(v)$. (This fails if D is not simplicial.)

Definition 8. Let $Z = \bigcup_{i \in I} Z_i$ be a pure dimensional scheme with irreducible components Z_i . Assume that

- (1) each Z_i is normal and
- (2) for every $J \subset I$, if $\bigcap_{i \in J} Z_i$ is nonempty, then every connected component of $\bigcap_{i \in J} Z_i$ is irreducible and has codimension $|J| - 1$ in Z .

Note that assumption (2) implies the following.

- (3) For every $j \in J$, every irreducible component of $\bigcap_{i \in J} Z_i$ is contained in a unique irreducible component of $\bigcap_{i \in J \setminus \{j\}} Z_i$.

The *dual complex* $\mathcal{D}(Z)$ of Z is the regular cell complex obtained as follows. The vertices are the irreducible components of Z and to each irreducible component of $W \subset \bigcap_{i \in J} Z_i$ we associate a cell of dimension $|J| - 1$. This cell is usually denoted by v_W .

The attaching map is given by condition (3). Note that $\mathcal{D}(Z)$ is a simplicial complex iff $\bigcap_{i \in J} Z_i$ is irreducible (or empty) for every $J \subset I$.

Let X be a variety and E a divisor on X . If $\text{Supp}(E)$ satisfies the conditions (1-2), then $\mathcal{D}(E) := \mathcal{D}(\text{Supp}(E))$ is called the *dual complex* of E .

Note that conditions (1-2) are satisfied in three important cases:

- (4) X is a smooth and E is a simple normal crossing divisor.

- (5) (X, Δ) is a dlt pair and $E := \Delta^{=1}$ is the set of divisors whose coefficient in Δ equals 1.
- (6) We introduce quotient-dlt pairs in Definition 35 with conditions (1–2) in mind.

Here the claim (4) is clear and (5) is proved in [15, Sec.3.9]; see also [34, Thm.4.16]. In the dlt case, let $X^{\text{snrc}} \subset X$ be the simple normal crossing locus. Then $\mathcal{D}(\Delta^{=1}) = \mathcal{D}((\Delta|_{X^{\text{snrc}}})^{=1})$, thus the dual complex is insensitive to the singularities of (X, Δ) .

In the above cases (4–6) the cells of $\mathcal{D}(E) = \mathcal{D}(\Delta^{=1})$ are identified with the log canonical centers of (X, Δ) . Frequently these are also called the *strata* of E or of $\Delta^{=1}$.

Even if X is smooth and $E = \bigcup_{i \in I} E_i$ is a simple normal crossing divisor, the dual complex $\mathcal{D}(E)$ need not be a simplicial complex, but this can be achieved after some blow-ups as in Remark 10.

It is possible to define the dual complex even if Z does not satisfy the conditions (1–2). However, there seem to be several ways to do it and we do not know which variant is the most useful. In this paper we only use the cases arising from (4–6).

9 (Blowing-up and the dual complex). Let X be a smooth variety and $E = \bigcup_{i \in I} E_i$ a simple normal crossing divisor. Let $Z \subset X$ be a smooth, irreducible subvariety that has only simple normal crossing with E ; see [30, 3.24].

Let $\pi : B_Z X \rightarrow X$ denote the blow up of Z with exceptional divisor E'_0 (assuming $0 \notin I$). Let $E'_i := \pi_*^{-1}(E_i)$ denote the birational transform of E_i and $E' := \pi_*^{-1}(E) = \bigcup_{i \in I} E'_i$. Then $E'_0 \cup E'$ is a simple normal crossing divisor. By a direct computation we see the following.

- (1) If Z is a stratum of E then $\mathcal{D}(E'_0 \cup E')$ is obtained from $\mathcal{D}(E)$ by the *stellar subdivision* of $\mathcal{D}(E)$ corresponding to an interior point of the cell v_Z .
- (2) If Z is not a stratum of E then $\mathcal{D}(E') = \mathcal{D}(E)$.
- (3) If Z is not a stratum of E but $Z \subset E$ then $\mathcal{D}(E'_0 \cup E')$ is obtained from $\mathcal{D}(E)$ as follows.

Let E_Z be the smallest stratum that contains Z and v_Z the corresponding simplex. Note that E_Z is an irreducible component of some intersection $\bigcap_{i \in J} E_i$. Let $\mathcal{D}(Z)$ denote the dual complex of $\sum_{i \in I \setminus J} E_i|_Z$. Then the dual complex $\mathcal{D}(E'|_{E'_0})$ is the join $v_Z * \mathcal{D}(Z)$ (see e.g. [21, Page 9]). There is a natural map $\tau_L : \mathcal{D}(Z) \rightarrow \text{Lk}(v_Z)$ hence we get a map $\tau_S : v_Z * \mathcal{D}(Z) \rightarrow \overline{\text{St}}(v_Z)$. We can identify $v_Z * \mathcal{D}(Z)$ with a

subcomplex of the cone $\text{Cone}(v_Z * \mathcal{D}(Z))$ and attach the latter to $\mathcal{D}(E)$ using τ_S to get $\mathcal{D}(E'_0 \cup E')$.

Since the cone over the join retracts (even collapses) to the join, we see that $\mathcal{D}(E'_0 \cup E')$ retracts (even collapses) to $\mathcal{D}(E)$. (However, they need not be PL homeomorphic; even their dimension can be different.)

Note on terminology. Let D be a regular cell complex and $p \in D$ a point. The *stellar subdivision* of D with center p is obtained as follows.

- i) The closed cells not containing p are unchanged.
- ii) If v is a closed cell containing p , we replace it by all the cells that are spans $\langle p, w \rangle$ where $w \subset v$ is any face non containing p .

(Some authors seem to call this a barycentric subdivision, but this goes against standard usage in PL topology [21, 46].)

Let (X, Δ) be an snc pair where Δ is a sub-boundary and apply the above observations to $E := \Delta^=1$. Write

$$K_{B_Z X} + \Delta_Z \sim_{\mathbb{Q}} \pi^*(K_X + \Delta)$$

and set $E_Z := (\Delta_Z)^=1$. Note that $E_Z = E'$ if Z is not a stratum of E and $E_Z = E'_0 \cup E'$ if Z is a stratum of E . Thus we conclude the following.

- (4) If $(K_{B_Z X}, \Delta_Z)$ is obtained from (K_X, Δ) by blowing up a sub-variety $Z \subset X$ that has simple normal crossing with Δ then $\mathcal{D}(\Delta_Z^=1)$ is PL homeomorphic to $\mathcal{D}(\Delta^=1)$.

Remark 10. The barycentric subdivision of any dual complex is simplicial. For the dual complex of a simple normal crossing divisor, it can be realized by blow-ups as follows.

Let X be a smooth variety and $E = \bigcup_{i \in I} E_i$ a simple normal crossing divisor. Let $Z_i \subset X$ denote the union of all i -dimensional strata of E . Consider a sequence of blow-ups

$$\Pi : \tilde{X} := X_{n-1} \xrightarrow{\pi_{n-2}} X_{n-2} \rightarrow \cdots \rightarrow X_1 \xrightarrow{\pi_0} X_0 := X$$

where $\pi_i : X_{i+1} \rightarrow X_i$ denotes the blow-up of the birational transform $(\pi_0 \circ \cdots \circ \pi_{i-1})_*^{-1} Z_i \subset X_i$.

Each blow-up center is smooth and $\Pi^{-1} E$ is a simple normal crossing divisor whose dual complex $\mathcal{D}(\Pi^{-1} E)$ is the stellar subdivision of $\mathcal{D}(E)$.

Finally, we know that a barycentric subdivision can be written as a sequence of stellar subdivisions.

The following invariance result for the dual complexes of crepant-birational dlt pairs is a considerable strengthening of [34, 4.35].

Proposition 11. *Let $f_i : (X_i, \Delta_i) \rightarrow S$ be proper morphisms. Assume that the (X_i, Δ_i) are dlt and crepant-birational to each other over S . Then the dual complexes $\mathcal{D}(\Delta_i^{\neq 1})$ are PL homeomorphic to each other.*

Proof. By [51], every dlt pair (X, Δ_X) has a log resolution $g : (Y, \Delta_Y) \rightarrow (X, \Delta_X)$ such that $\mathcal{D}(\Delta_Y^{\neq 1})$ is identified with $\mathcal{D}(\Delta_X^{\neq 1})$. Thus we may assume that the (X_i, Δ_i) are snc pairs but now the Δ_i are only sub-boundaries.

Next we use the factorization theorem of [1] which says that there is a sequence of smooth blow-ups as in Paragraph 9 and their inverses

$$X_1 = Y_0 \xrightarrow{\pi_0} Y_1 \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_{m-1}} Y_m \xrightarrow{\pi_m} Y_{m+1} \xrightarrow{\pi_{m+1}} \cdots \xrightarrow{\pi_{r-1}} Y_r = X_2.$$

Moreover, we may assume that the induced maps $\pi_0^{-1} \circ \cdots \circ \pi_{i-1}^{-1} : Y_i \dashrightarrow X_1$ are morphisms for $i \leq m$ and the induced maps $\pi_{r-1} \circ \cdots \circ \pi_i : Y_i \dashrightarrow X_2$ are morphisms for $i \geq m$.

Let Θ_i be the log pull-back of Δ_1 for $i \leq m$ and the log pull-back of Δ_2 for $i \geq m$ by the above morphisms. Note that the two definitions of Θ_m agree since the (X_i, Δ_i) are crepant-birational to each other over S .

We use (9.4) to show that at each step $\mathcal{D}(\Theta_i^{\neq 1})$ changes either by a stellar subdivision or its inverse.

Thus the $\mathcal{D}(\Delta_i^{\neq 1})$ are obtained from each other by repeatedly adding and removing stellar subdivisions. Q.E.D.

Complement 12. Using the above notation, let $Z \subset S$ be a closed subset. Let $\Theta_{i,Z}^{\neq 1} \subset \Theta_i^{\neq 1}$ be the union of those divisors that are contained in $f_i^{-1}(Z)$. Assume further that every lc center of $\Theta_i^{\neq 1}$ contained in $f_i^{-1}(Z)$ is also an lc center of $\Theta_{i,Z}^{\neq 1}$.

Then the above proof shows that the dual complexes $\mathcal{D}(\Delta_{i,Z}^{\neq 1})$ are PL homeomorphic to each other.

Although we do not need this, it is worth remarking that if $X_1 \dashrightarrow X_2$ is an isomorphism in codimension 1 then we can go between the X_i by a series of *flips*. The topological analogs of these are the *Pachner moves* or *bistellar flips*, see [41].

We will need several types of partial resolutions of a pair (X, Δ) .

Definition 13. Let X be a normal variety and Δ a boundary on X . Let $g' : (X', \Delta') \rightarrow (X, \Delta)$ be a proper, birational morphism where

$\Delta' = E + (g')_*^{-1}\Delta$ and E is the sum of all divisors in $\text{Ex}(g')$. We say that

$$g' : (X', \Delta') \rightarrow (X, \Delta) \text{ is a } \left\{ \begin{array}{l} \text{dlt} \\ \text{qdlt} \\ \text{lc} \end{array} \right\} \text{ modification}$$

if

$$(X', \Delta') \text{ is } \left\{ \begin{array}{l} \text{dlt} \\ \text{qdlt} \\ \text{lc} \end{array} \right\} \text{ and } K_{X'} + \Delta' \text{ is } \left\{ \begin{array}{l} f\text{-nef.} \\ f\text{-nef.} \\ f\text{-ample.} \end{array} \right.$$

We frequently denote a dlt modification by $g^{\text{dlt}} : (X^{\text{dlt}}, \Delta^{\text{dlt}}) \rightarrow (X, \Delta)$ and an lc modification by $g^{\text{lc}} : (X^{\text{lc}}, \Delta^{\text{lc}}) \rightarrow (X, \Delta)$. For qdlt, see Definition 35.

Every pair (X, Δ) such that $K_X + \Delta$ is \mathbb{Q} -Cartier has a unique log canonical modification and (usually non-unique) dlt modifications by [40]. (Conjecturally, both should exist for any (X, Δ) .) We also remark that because we require that $K_X + \Delta$ is nef, this is indeed stronger than the existence of dlt blow up proved by Hacon (see [17, 10.4]).

Note that $\text{Ex}(g^{\text{lc}})$ has pure codimension 1, but this need not hold for $\text{Ex}(g^{\text{dlt}})$. However, we prove in Lemma 29 that there is a dlt modification such that $\text{Ex}(g^{\text{dlt}})$ has pure codimension 1.

We can now define the most important dual complexes associated to a pair (X, Δ) .

Definition 14. Let X be a normal variety and $W \subset X$ a closed subvariety. Let $p : Y \rightarrow X$ be a resolution of singularities such that $\text{Supp } p^{-1}(W)$ is a simple normal crossing divisor. By [48, 53, 49, 4, 43], the simple-homotopy class of the dual complex $\mathcal{D}(\text{Supp } p^{-1}(W))$ is independent of the choice of p . We denote it by $\mathcal{DR}(W \subset X)$.

Definition 15. The *non-klt locus* of (X, Δ) is the unique smallest subscheme $W \subset X$ such that $(X \setminus W, \Delta|_{X \setminus W})$ is klt. It is frequently denoted by $\text{non-klt}(X, \Delta)$. It can be written as $g^{\text{dlt}}((\Delta^{\text{dlt}})^{=1})$ for any dlt modification.

Any two dlt modifications are crepant-birational to each other over X . (Although not explicitly stated, this is what the proof of [37, 3.52] gives.) Hence, by Proposition 11, the dual complex

$$\mathcal{DMR}(X, \Delta) := \mathcal{D}((\Delta^{\text{dlt}})^{=1})$$

is independent of the choice of $g^{\text{dlt}} : X^{\text{dlt}} \rightarrow X$, up-to PL homeomorphism.

We stress that $\mathcal{DMR}(X, \Delta)$ is a PL homeomorphism equivalence class but $\mathcal{DR}(W \subset X)$ is a simple-homotopy equivalence class.

Note that $\mathcal{DMR}(X, \Delta)$ does depend on Δ , not just on the pair $(W \subset X)$. This will be quite useful to us. In some cases we will be able to compute $\mathcal{DR}(W \subset X)$ by choosing a suitable Δ and then computing $\mathcal{DMR}(X, \Delta)$.

§3. Running MMP

Let (X, Δ) be an lc pair and $f: X \dashrightarrow Y$ a step (a divisorial contraction or a flip) in an (X, Δ) -MMP. Set $\Delta_Y := f_*\Delta$; then (Y, Δ_Y) is also an lc pair. Furthermore, if (X, Δ) is dlt then so is (Y, Δ_Y) by [37, 3.44].

Once we introduce quotient-dlt pairs in Section 5, we see that all the proofs and results in this section hold for quotient-dlt pairs.

Let $Z \subset X$ be an lc center of (X, Δ) . We say that f contracts Z if $Z \subset \text{Ex}(f)$. If $Z \not\subset \text{Ex}(f)$ then $f_*(Z)$ is an lc center of (Y, Δ_Y) . Since discrepancies strictly increase for every divisor whose center is contained in $\text{Ex}(f)$, we see that every lc center of (Y, Δ_Y) is obtained this way from a non-contracted lc center of (X, Δ) . Using this observation repeatedly, we conclude the following.

Lemma 16. *Let (X, Δ) be a dlt pair and $f: X \dashrightarrow Y$ a birational map obtained by running an (X, Δ) -MMP. Set $\Delta_Y := f_*\Delta$. Then the dual complex $\mathcal{D}(\Delta_Y^{-1})$ is naturally a subcomplex of $\mathcal{D}(\Delta^{-1})$. Q.E.D.*

In general, the inclusion $\mathcal{D}(\Delta_Y^{-1}) \subset \mathcal{D}(\Delta^{-1})$ is not a homotopy equivalence, but this happens in many interesting cases. The following examples illustrate the main possibilities.

Example 17. Let $Z = (x_1x_2 - x_3x_4 = 0) \subset \mathbb{C}^4$ be the quadric cone. Consider the planes $A_1 := (x_1 = x_3 = 0), A_2 := (x_2 = x_4 = 0), B_1 := (x_1 = x_4 = 0), B_2 := (x_2 = x_3 = 0)$. The two small resolutions are $Y' := B_{A_1}X = B_{A_2}X$ and $Y'' := B_{B_1}X = B_{B_2}X$. By explicit computation we see the following.

(1) $f: Y' \dashrightarrow Y''$ is a flip for $(Y', A'_1 + A'_2)$. The corresponding dual complexes are as in Figure 2.



Fig. 2

This is not a homotopy equivalence.

(2) $f : Y' \dashrightarrow Y''$ is a flip for $(Y', A'_1 + A'_2 + B'_1)$. The corresponding dual complexes are as in Figure 3.

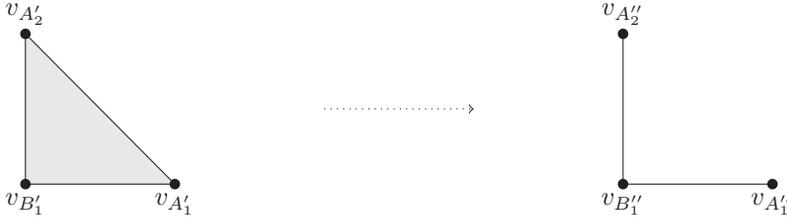


Fig. 3

This is a homotopy equivalence, even a collapse.

(3) $f : Y' \dashrightarrow Y''$ is a flop for $(Y', A'_1 + A'_2 + B'_1 + B'_2)$. The corresponding dual complexes are as in Figure 4.

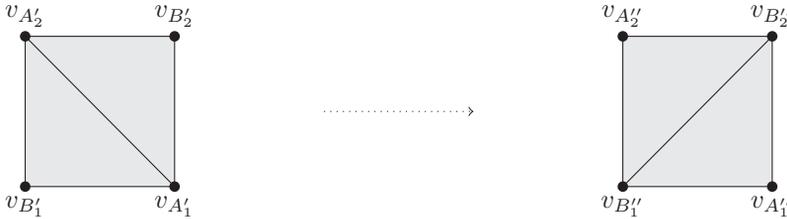


Fig. 4

This is a PL homeomorphism, a composite of a stellar subdivision (whose center is the center of the square) and its inverse.

In order to describe the general case precisely, we need some concepts from simplicial complex theory, especially the notion of collapse.

Definition 18. Let D be a regular cell complex. Let v be a cell in D and w a face of v . We say that (v, w) is a *free pair* if w is not the face of any other cell in D . The *elementary collapse* of (D, v, w) is the regular complex obtained from D by removing the interiors of the cells v and w . This is clearly a homotopy equivalence. A sequence of elementary collapses is called a *collapse*. A regular complex D is *collapsible* if it collapses to a point.

A map $g : C_1 \rightarrow C_2$ of regular complexes is a *simple-homotopy equivalence* if it is homotopic to a map obtained by a sequence of elementary collapses and their inverses. If the C_i are simply connected, then every homotopy equivalence is a simple-homotopy equivalence. However, in general not every homotopy equivalence is a simple-homotopy equivalence; the difference is measured by the *Whitehead torsion*. For details and for proofs see [6].

The following is our key result relating extremal contractions to collapses of the dual complex.

Theorem 19. *Let (X, Δ) be dlt and $f : X \dashrightarrow Y$ a divisorial contraction or flip corresponding to a $(K_X + \Delta)$ -negative extremal ray R . Set $\Delta_Y := f_*\Delta$. Assume that there is a prime divisor $D_0 \subset \Delta^=1$ such that $(D_0 \cdot R) > 0$.*

Then $\mathcal{D}(\Delta^=1)$ collapses to $\mathcal{D}(\Delta_Y^=1)$.

Proof. We distinguish two types of contracted lc centers.

Case 1. Let Z be a contracted lc center such that $Z \subset D_0$. Then there is a subset $0 \notin J \subset I$ such that Z is an irreducible component of $D_0 \cap \bigcap_{i \in J} D_i$. Thus there is a unique irreducible component Z^+ of $\bigcap_{i \in J} D_i$ such that Z is an irreducible component of $D_0 \cap Z^+$. We claim that $Z^+ \subset \text{Ex}(f)$.

Indeed, if $Z^+ \not\subset \text{Ex}(f)$ then $f_*(Z^+) \subset Y$ is an lc center which has nonempty intersection with $\text{Ex}(f^{-1})$. Since $(D_0 \cdot R) > 0$, we see that $f_*(D_0)$ contains $\text{Ex}(f^{-1})$. Denote by $h : X \rightarrow X_1$ the divisorial or flip contraction morphism. Recall that Z is also a connected component of $D_0 \cap Z^+$. Therefore, Proposition 25 implies that if there is another component of $D_0 \cap Z^+$ whose image intersects $h(Z)$, then itself must intersect with Z which is absurd. Hence we conclude that Z is the same as $D_0 \cap Z^+$ over a neighborhood of $h(Z)$. Thus over the neighborhood of $h(Z)$, $f_*(Z^+) \cap f_*(D_0)$ is a nonempty subset of $\text{Ex}(f^{-1})$. By [2], $f_*(Z^+) \cap f_*(D_0)$ is a union of lc centers. However, $\text{Ex}(f^{-1})$ does not contain any lc centers, a contradiction.

Case 2. Let W be a contracted lc center such that $W \not\subset D_0$. We claim that $W^- := W \cap D_0$ is a (nonempty, irreducible) lc center. Thus W^- is also contracted and the operations $Z \mapsto Z^+$ and $W \mapsto W^-$ are inverses.

To see this, let $h : X \rightarrow X_1$ denote the contraction. First apply Proposition 25 to the generic point of the Stein factorization of the morphism $(W, W \cap D_0) \rightarrow h(W)$. It shows that $W \cap D_0$ has a unique irreducible component W^- that dominates $h(W)$. If $W \cap D_0$ has any other irreducible component W^* , this would give a contradiction using Proposition 25 over the generic point of $h(W^*)$.

Let $M \subset \mathcal{D}(\Delta^=1)$ be the union of the interiors of all cells v_Z where $Z \subset \text{Ex}(f)$ is a contracted lc center. Thus

$$\mathcal{D}(\Delta_{\bar{Y}}^=1) = \mathcal{D}(\Delta^=1) \setminus M.$$

We show that all the cells in M can be collapsed, starting with the largest dimensional ones.

By the above considerations, the cells in M naturally come in pairs

$$(\langle v_{D_0}, v_W \rangle, v_W)$$

where $W \subset X$ is a contracted lc center not contained in D_0 . The other cell $\langle v_{D_0}, v_W \rangle$, spanned by v_{D_0} and v_W , is the same as $v_{D_0 \cap W}$. We call v_W *link-type* (since $v_W \subset \text{Lk}(v_{D_0})$) and $\langle v_{D_0}, v_W \rangle$ *star type* (since $\langle v_{D_0}, v_W \rangle \subset \overline{\text{St}}(v_{D_0})$).

Note further that if an lc center W is contracted, then so is every lc center contained in W . Dually, if $v_W \in M$ is a facet of a cell v_V then also $v_V \subset M$.

Let $(\langle v_{D_0}, v_W \rangle, v_W)$ be a maximal dimensional pair in M . If v_W is a face of a cell v_V then v_V is also in M . By maximality of dimension, v_V is of star-type, thus $v_V = \langle v_{D_0}, v_W \rangle$. Thus $(\langle v_{D_0}, v_W \rangle, v_W)$ is a free pair and it can be collapsed.

Set $m := \dim \mathcal{D}(\Delta_{\bar{Y}}^=1)$. Note that these collapses remove all star-type cells of dimension m and all link-type cells of dimension $m - 1$.

Next we look at an $(m - 1)$ -dimensional pair $(\langle v_{D_0}, v_W \rangle, v_W)$. Assume that v_W is a face of a cell v_V . It can not be of link-type since we have already removed all link-type cells of dimension $m - 1$. Thus v_V is of star-type and so $v_V = \langle v_{D_0}, v_W \rangle$. Hence $(\langle v_{D_0}, v_W \rangle, v_W)$ is again a free pair and can be collapsed.

Iterating this completes the proof.

Q.E.D.

Remark 20. It is interesting to see what happens in Theorem 19 if $(D_i \cdot R) < 0$ for every $D_i \subset \Delta^=1$.

Let Z be a contracted lc center that is an irreducible component of $\bigcap_{i \in J} D_i$. Then $\text{Ex}(f) \subset D_i$ for every i , thus $\text{Ex}(f) = Z$. Hence $\mathcal{D}(\Delta_{\bar{Y}}^=1)$ is obtained from $\mathcal{D}(\Delta^=1)$ by removing the cell v_Z . This is not a homotopy equivalence, even the Euler characteristic changes by 1.

The general case when $(D_i \cdot R) \leq 0$ for every $D_i \subset \Delta^=1$ is more complicated.

In order to use Theorem 19 we need to find conditions that ensure the existence of such a divisor D_0 at each step of an MMP.

Lemma 21. *Let (X, Δ) be dlt and $g : X \rightarrow S$ a morphism. Assume that there is a numerically g -trivial effective divisor A whose support*

equals $\Delta^=1$. Let $f: X \dashrightarrow Y$ be a divisorial contraction or flip corresponding to a $K_X + \Delta$ -negative extremal ray R over S . Then

- (1) either $\text{Ex}(f)$ does not contain any lc centers
- (2) or there is a prime divisor $D_0 \subset \Delta^=1$ such that $(D_0 \cdot R) > 0$.

Proof. Let Z be a contracted lc center that is an irreducible component of $\cap_{i \in J} D_i$. We are done if $(D_i \cdot R) > 0$ or some i .

Otherwise $(D_i \cdot R) \leq 0$ for every $D_i \subset \Delta^=1$ and so $(A \cdot R) \leq 0$ with equality holding only if $(D_i \cdot R) = 0$ for every D_i . In particular $(D_i \cdot R) = 0$ for every $i \in J$.

If f contracts a divisor E_f then $(E_f \cdot R) < 0$. Thus E_f is not one of the D_i . Set $Z_Y := f(Z)$. If $f = (h^+)^{-1} \circ h$ is a flip, set $Z_Y := (h^+)^{-1}(h(Z))$. In both cases, as $D_i \cdot R = 0$, we know that $Z_Y \subset \cap_{i \in J} f_*(D_i)$ and $Z_Y \subset \text{Ex}(f^{-1})$. Thus $\cap_{i \in J} f_*(D_i)$ is nonempty and it is not the f_* -image of a non-contracted lc center. However, $\cap_{i \in J} f_*(D_i)$ is a union of lc centers by [2], a contradiction. Q.E.D.

Using Theorem 19 and Lemma 21 for every step of an MMP gives the following.

Corollary 22. *Let (X, Δ) be dlt and $g: X \rightarrow Z$ a morphism. Let $f: X \dashrightarrow Y$ be a birational map obtained by running an (X, Δ) -MMP over Z . Set $\Delta_Y := f_*\Delta$. Assume that there is a numerically g -trivial effective divisor whose support equals $\Delta^=1$. Then $\mathcal{D}(\Delta^=1)$ collapses to $\mathcal{D}(\Delta_Y^=1)$.* Q.E.D.

The next lemma gives other examples where the assumptions of Theorem 19 are satisfied.

Lemma 23. *Let X, Y be normal, \mathbb{Q} -factorial varieties and $p: Y \rightarrow X$ a projective, birational morphism. Let Γ be a boundary on Y such that (Y, Γ) is lc. Let $f: Y \dashrightarrow Y_1$ be a divisorial contraction or flip corresponding to a $(K_Y + \Gamma)$ -negative extremal ray R over X . Then*

- (3) either there is an $E_R \subset \text{Ex}(p)$ such that $(E_R \cdot R) > 0$
- (4) or f contracts a divisor $E_f \subset \text{Ex}(p)$ and $Y_1 \rightarrow X$ is a local isomorphism at the generic point of $f(E_f)$.

Proof. Let $p_1: Y_1 \rightarrow X$ be the natural morphism. Since X is \mathbb{Q} -factorial, $\text{Ex}(p_1)$ has pure codimension 1.

Assume first that f is a divisorial contraction of a divisor E_f . We are done if (4) holds. Otherwise $f(E_f) \subset \text{Ex}(p_1)$ hence there is an irreducible divisor $E_1 \subset \text{Ex}(p_1)$ that contains $f(E_f)$. Hence its birational transform $E_R := f_*^{-1}E_1$ has positive intersection with R .

Next assume that $f = (h^+)^{-1} \circ h$ is a flip. Then $\text{Ex}(h^+) \subset \text{Ex}(p_1)$. Since X is \mathbb{Q} -factorial, there is an effective anti-ample divisor supported

on $\text{Ex}(p_1)$ [37, 2.62]. So there is an irreducible divisor $E_1 \subset \text{Ex}(p_1)$ that has negative intersection with a flipped curve. Thus its birational transform $E_R := f_*^{-1}E_1$ has positive intersection with R . Q.E.D.

As a consequence, we get the following generalization of (1.3).

Corollary 24. *Let (X, Δ) be a \mathbb{Q} -factorial pair and $(0 \in X)$ a point such that $(X \setminus \{0\}, \Delta|_{X \setminus \{0\}})$ is klt. Let $p: Y \rightarrow X$ be a projective, birational morphism that is an isomorphism over $X \setminus \{0\}$. Set $E := \text{Ex}(p)$ and assume that $(Y, E + p_*^{-1}\Delta)$ is dlt. Then*

- (1) $\mathcal{D}(E)$ collapses to $\mathcal{DMR}(0 \in X)$ and
- (2) if (X, Δ) is klt then $\mathcal{D}(E)$ is collapsible.

Proof. As we note in Definition 26, we may assume that Y is \mathbb{Q} -factorial. We run the $(Y, E + p_*^{-1}\Delta)$ -MMP over X to get

$$(Y, E) = (Y_0, E_0) \xrightarrow{f_0} (Y_1, E_1) \xrightarrow{f_1} \cdots \xrightarrow{f_{r-1}} (Y_r, E_r) \xrightarrow{q} X.$$

If (X, Δ) is not klt then (Y_r, E_r) is a dlt modification. If (X, Δ) is klt then we let $q: (Y_r, E_r) \rightarrow X$ denote the last extremal contraction.

By Lemma 23, we can use Theorem 19 to conclude that $\mathcal{D}(E_i)$ collapses to $\mathcal{D}(E_{i+1})$ for $i < r$, proving the first claim.

If (X, Δ) is klt then q contracts a single divisor E_r to a point. Thus $\mathcal{D}(E_r)$ is a point which shows (2). Q.E.D.

During the proof of Theorem 19 we have used the following variant of the connectedness theorem. It is a special case of [2, 6.6]. We include a proof here for reader's convenience.

Proposition 25. *Let (X, Δ) be a dlt pair and $f: X \rightarrow Y$ a projective morphism such that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $-(K_X + \Delta)$ is f -ample. Then for any $y \in Y$ there is at most one minimal lc center Z of (X, Δ) that intersects $f^{-1}(y)$.*

Proof. This follows from [31, Theorem 10] by adding a general ample \mathbb{Q} -divisor $H \sim_{\mathbb{Q}, f} -(K_X + \Delta)$ with small coefficients.

A more direct proof is the following. By the connectedness theorem [37, Thm.5.48], we know that if Z_1 and Z_2 are two lc centers that are minimal among all lc centers intersecting $f^{-1}(y)$, then we can find a chain W_1, \dots, W_k of divisors in $\Delta^=1$ such that $Z_1 \subset W_1$, $Z_2 \subset W_k$, and $W_i \cap W_{i+1} \cap f^{-1}(y) \neq \emptyset$ for all $1 \leq i \leq k-1$. Applying induction to $f: (W_1, \text{Diff}_{W_1}\Delta) \rightarrow f(W_1)$, we conclude that $Z_1 \subset W_1 \cap W_2$, which implies that $Z_1 \subset W_2$. Repeating the above argument for $f: (W_i, \text{Diff}_{W_i}\Delta) \rightarrow f(W_i)$, we eventually obtain that $Z_1 \subset W_k$. Then again by induction, we conclude that $Z_1 = Z_2$. Q.E.D.

§4. Proofs of the main theorems

We prove Theorem 3 and its consequences in this section. The strategy is to run a suitable MMP terminating with a dlt modification of (X, Δ) and show that each step of the program corresponds to a collapse of the dual complex. First we need to deal with divisors that are not \mathbb{Q} -Cartier.

Definition 26. Let X be a normal variety. A *small, \mathbb{Q} -factorial modification* is a proper, birational morphism $\pi : X^{\text{qf}} \rightarrow X$ such that π is small, that is, there are no exceptional divisors, and X^{qf} is \mathbb{Q} -factorial.

For a divisor Δ on X set $\Delta^{\text{qf}} := \pi_*^{-1}\Delta$. Note that if (X, Δ) is dlt then so is $(X^{\text{qf}}, \Delta^{\text{qf}})$ and $\mathcal{D}(\Delta^{\text{qf}}) = \mathcal{D}((\Delta^{\text{qf}})^{\text{=1}})$.

A dlt pair (X, Δ) always has projective, small, \mathbb{Q} -factorial modifications but not all lc pairs have them; see [34, 1.37].

Let $\pi_1 : (X_1, \Delta_1) \rightarrow (X, \Delta)$ be a projective, small, \mathbb{Q} -factorial modification of a dlt pair (X, Δ) . Next run an $(X_1, \Delta_1 - \Delta_1^{\text{=1}})$ -MMP over X to get $\pi_2 : (X_2, \Delta_2) \rightarrow (X, \Delta)$ such that

$$-\Delta_2^{\text{=1}} \sim_{\mathbb{Q}, \pi_2} K_{X_2} + \Delta_2 - \Delta_2^{\text{=1}}$$

is π_2 -nef. Thus $\pi_2 : (X_2, \Delta_2) \rightarrow (X, \Delta)$ is a projective, small, dlt, \mathbb{Q} -factorial modification such that $\text{Supp } \Delta_2^{\text{=1}} = \pi_2^{-1}(\text{Supp } \Delta^{\text{=1}})$.

27 (Proof of Theorem 3). Let $\pi_2 : (X_2, \Delta_2) \rightarrow (X, \Delta)$ be as above.

After a suitable sequence of blow-ups we get $Y_2 \rightarrow Y$ such that the induced rational map $g_2 : Y_2 \dashrightarrow X_2$ is a morphism. As noted in Paragraph 9, $\mathcal{D}(\text{Supp } g_2^{-1}(\Delta_2^{\text{=1}}))$ is simple-homotopy equivalent to $\mathcal{D}(E) = \mathcal{D}(\text{Supp } g^{-1}(\Delta^{\text{=1}}))$. By replacing X by X_2 and Y by Y_2 , we may assume from now on that X is \mathbb{Q} -factorial.

Consider the boundary

$$\Gamma := g_*^{-1}\Delta^{<1} + E + \sum b_i F_i$$

where the sum on the right hand side is taken over all g -exceptional divisors F_i that are not contained in the support of E and

$$b_i := \max \left\{ \frac{1 - a(F_i, X, \Delta)}{2}, \frac{1}{2} \right\}.$$

We run a (K_Y, Γ) -MMP over X as in [5].

By construction, the \mathbb{Q} -divisor

$$G := K_Y + \Gamma - g^*(K_X + \Delta)$$

is effective and its support consists precisely of all the g -exceptional divisors with positive log discrepancy with respect to (X, Δ) . As the push forward of G to \tilde{X} is effective, exceptional and semiample over X , it must be trivial by negativity. This means that the map $Y \dashrightarrow \tilde{X}$ produced by the MMP contracts all components of G . It follows therefore that if we denote by $\tilde{\Delta}$ the birational transform of Γ on \tilde{X} , then $(\tilde{X}, \tilde{\Delta})$ is a dlt modification of (X, Δ) . Thus $\mathcal{D}(\Delta^{\leq 1})$ is PL homeomorphic to $\mathcal{D}(\tilde{\Delta}^{\leq 1})$ by Proposition 11.

Furthermore, since $E = \text{Supp } g^*(\Delta^{\leq 1})$ and $g^*(\Delta^{\leq 1})$ is numerically g -trivial, Corollary 22 implies that $\mathcal{D}(E)$ collapses to $\mathcal{D}(\tilde{\Delta}^{\leq 1})$. Q.E.D.

We can now state and prove the general form of Theorem 1

Theorem 28. *Let X be a normal variety and Δ an effective boundary on X such that $K_X + \Delta$ is \mathbb{Q} -Cartier. Let $W \neq \emptyset$ be the non-klt locus of (X, Δ) . Let $p : Y \rightarrow X$ be a proper, birational morphism and E the divisorial part of $\text{Supp } p^{-1}(W)$. Assume that $(Y, E + p_*^{-1}\Delta^{\leq 1})$ is dlt.*

- (1) *If the divisor $K_Y + E + p_*^{-1}\Delta^{\leq 1}$ is p -nef then the dual complex $\mathcal{DMR}(X, \Delta) := \mathcal{D}(E)$ is independent of p , up-to PL homeomorphism.*
- (2) *If $E = \text{Supp } p^{-1}(W)$ then the dual complex $\mathcal{D}(E)$ is simple-homotopy equivalent to $\mathcal{DMR}(X, \Delta)$.*
- (3) *If X is \mathbb{Q} -factorial and p is an isomorphism over $X \setminus W$ then $\mathcal{D}(E)$ collapses to $\mathcal{DMR}(X, \Delta)$.*

Proof. Part (1) is noted in Definition 15.

In order to see (2) we use Lemma 29 to obtain a \mathbb{Q} -factorial dlt modification $q : (X^{\text{dlt}}, \Delta^{\text{dlt}}) \rightarrow (X, \Delta)$ such that $(\Delta^{\text{dlt}})^{\leq 1} = \text{Supp } q^{-1}(W)$. After some blow-ups we get $\pi : Y_1 \rightarrow Y$ such that $q^{-1} \circ p \circ \pi : Y_1 \rightarrow X^{\text{dlt}}$ is a morphism and $\mathcal{D}(\text{Supp}(p \circ \pi)^{-1}(W))$ is simple-homotopy equivalent to $\mathcal{D}(E)$.

By Theorem 3, $\mathcal{D}(\text{Supp}(p \circ \pi)^{-1}(W))$ collapses to $\mathcal{D}((\Delta^{\text{dlt}})^{\leq 1}) = \mathcal{DMR}(X, \Delta)$.

Finally assume that X is \mathbb{Q} -factorial and p is an isomorphism over $X \setminus W$. Let $f_i : Y_i \rightarrow Y_{i+1}$ be an extremal contraction of a divisor $E_i \subset Y_i$ in our MMP and $p_{i+1} : Y_{i+1} \rightarrow X$ the projection. Note that $p_{i+1}(f_i(E_i)) \subset W$, thus p_{i+1} is not a local isomorphism at the generic point of $f_i(E_i)$. Thus the first alternative of Lemma 23 applies at each step of the MMP, hence Theorem 19 implies that $\mathcal{D}(E)$ collapses to $\mathcal{DMR}(X, \Delta)$. Q.E.D.

The following argument is similar to [19, 3.3.1.4].

Lemma 29. *Let (X, Δ) be a pair such that $K_X + \Delta$ is \mathbb{Q} -Cartier. Let W be the non-klt locus of (X, Δ) . Then there exists a dlt modification $p : Y \rightarrow X$ such that $\text{Supp}(p^{-1}W)$ is a divisor.*

Proof. Let $p^{\text{lc}} : (X^{\text{lc}}, \Delta^{\text{lc}}) \rightarrow (X, \Delta)$ be the log canonical modification as in Definition 13. As we noted, $\text{Ex}(p^{\text{lc}})$ has pure codimension 1 and it is contained in $\text{Supp}((\Delta^{\text{lc}})^=1)$. Thus W^{lc} , the non-klt locus of $(X^{\text{lc}}, \Delta^{\text{lc}})$, is equal to the preimage of W .

Since any dlt modification of $(X^{\text{lc}}, \Delta^{\text{lc}})$ is also a dlt modification of (X, Δ) , it suffices to show that there is a dlt modification of $q : Y \rightarrow (X^{\text{lc}}, \Delta^{\text{lc}})$ such that $\text{Supp}(q^{-1}(W^{\text{lc}}))$ is of pure codimension 1.

Consider an arbitrary dlt modification $q_1 : Y_1 \rightarrow (X^{\text{lc}}, \Delta^{\text{lc}})$ and write

$$q_1^*(K_{X^{\text{lc}}} + \Delta^{\text{lc}}) \sim_{\mathbb{Q}} K_{Y_1} + \Gamma_1$$

then $W^{\text{lc}} = q_1(\Gamma_1^=1)$. It follows from [5] that there is a minimal model $p_2 : Y_2 \rightarrow X^{\text{lc}}$ of $(Y_1, \Gamma_1^<1)$ over X^{lc} . Denote the push forward of Γ_1 on Y_2 to be Γ_2 . Thus $-\Gamma_2^=1 \sim_{\mathbb{Q}, X^{\text{lc}}} K_{Y_2} + \Gamma_2^<1$ is nef, which implies that

$$\text{Supp}(p_2^{-1}(W^{\text{lc}})) = \text{Supp}(p_2^{-1}p_2(\Gamma_2^=1)) = \text{Supp}(\Gamma_2^=1)$$

has the same support as $\Gamma_2^=1$.

Though (Y_2, Γ_2) is only log canonical, we can take any dlt modification $r : Y \rightarrow (Y_2, \Gamma_2)$ and set $q = p_2 \circ r$. Then $\text{Supp}(q^{-1}W^{\text{lc}}) = \text{Supp}(r^*\Gamma_2^=1)$ is of pure codimension 1 since $\Gamma_2^=1$ is \mathbb{Q} -Cartier. Q.E.D.

30 (Proof of Theorem 2). Since a dlt pair is always the limit of klt pairs (see [37, 2.43]), we can assume that $0 \in (X, \Delta)$ is a point on a klt pair. As in [18, Lem.2.5] there exists a \mathbb{Q} -divisor H such that $(X, \Delta + H)$ is klt on $X \setminus \{0\}$ and there is precisely one divisor E with discrepancy -1 over 0 . Thus $\mathcal{DMR}(0 \in X, \Delta + H)$ is a single point.

By (28.2) $\mathcal{DR}(0 \in X)$ is simple-homotopy equivalent to $\mathcal{DMR}(0 \in X, \Delta + H)$ hence to a point. Q.E.D.

31 (Versions over nonclosed fields). Let k be a field of characteristic 0, X a normal variety defined over k and $0 \in X$ a k -point. Let $f : X' \rightarrow X$ be a resolution defined over k such that $E' := \text{Supp} f^{-1}(0)$ is a simple normal crossing divisor defined over k . (See [34, Defn.1.7] for the correct definition of a simple normal crossing divisor.) Set $\mathcal{D}(E') := \mathcal{D}(E'_k)$ where $\bar{k} \supset k$ is an algebraic closure. The Galois group $\text{Gal}(\bar{k}/k)$ acts on $\mathcal{D}(E')$ and the usual arguments show that $\mathcal{DR}(0 \in X)$ is well defined up-to $\text{Gal}(\bar{k}/k)$ -equivariant simple-homotopy equivalence.

One needs to check that the collapses in Theorem 19 can be done equivariantly. Let D_0 be a divisor defined over k such that $(D_0 \cdot R) > 0$.

Over \bar{k} , it can break up into a collection of disjoint divisors D_0^i . The contraction $h : X \rightarrow X_1$ of the ray R becomes the contraction of a face F over \bar{k} , but D_0 is strictly negative on $F \setminus \{0\}$. The proof of Theorem 19 now shows that each contracted lc center over \bar{k} intersects exactly one of the D_0^i . Thus the collapses prescribed by the different D_0^i occur in disjoint sets so they can be performed simultaneously.

The proofs of Theorems 2–4 apply without changes to yield the following generalizations.

Theorem 32. *Let $0 \in X$ be a k -point on a dlt pair (X, Δ) defined over k . Then $\mathcal{DR}(0 \in X)$ is $\text{Gal}(\bar{k}/k)$ -equivariantly contractible. Q.E.D.*

Theorem 33. *Let $0 \in C$ be a k -point on a smooth curve and $f : X \rightarrow (0 \in C)$ a flat, projective morphism. Assume that the general geometric fibers X_t are smooth, rationally connected and $F_0 := \text{red } f^{-1}(0)$ is a simple normal crossing divisor. Then $\mathcal{D}(F_0)$ is $\text{Gal}(\bar{k}/k)$ -equivariantly contractible. Q.E.D.*

Note that a contractible space with a finite group action need not be equivariantly contractible (cf. [11]), thus, in this respect, the dual complexes coming from algebraic geometry behave better than arbitrary simplicial complexes.

§5. Quotients of dlt pairs

Proposition 34. *Let $(x \in X)$ be the spectrum of a d -dimensional local ring and Δ an effective divisor such that (X, Δ) is lc and x is an lc center of (X, Δ) . The following are equivalent.*

- (1) *There are \mathbb{Q} -Cartier divisors $D_1, \dots, D_d \subset \Delta^{\neq 1}$ such that $x \in D_i$.*
- (2) *There is a semi-local, snc pair $(x' \in X', D'_1 + \dots + D'_d)$ and an Abelian group G acting on it (preserving each of the D'_i) such that*

$$(x \in X, \Delta) = (x \in X, D_1 + \dots + D_d) \cong (x' \in X', D'_1 + \dots + D'_d)/G.$$

Proof. The implication (2) \Rightarrow (1) is well known; cf. [37, Prop.5.20].

To converse is outlined in [27, Sec.18]. We construct $\pi : X' \rightarrow X$ as follows. By assumption, for every i there is an $m_i > 0$ such that $m_i D_i \sim 0$. These give degree m_i cyclic covers $X'_i \rightarrow X$; let $\pi : X' \rightarrow X$ be their composite. Then $X' \rightarrow X$ is Galois with group $\prod_i \mathbb{Z}/m_i$ and it branches only along the D_i . Set $D'_i := \text{red } \pi^{-1}(D_i)$. Then $(X', D'_1 + \dots + D'_d)$ is lc by [37, Prop.5.20].

In general $x' := \pi^{-1}(x)$ may consist of several points. At each of them, the D'_i are Cartier. We claim that in fact X' and the D'_i are smooth. This is proved by induction on the dimension. The $d = 1$ case is clear.

By adjunction, $(D'_d, D'_1|_{D'_d} + \cdots + D'_{d-1}|_{D'_d})$ is lc, thus D'_d is smooth by induction. Since D'_d is a Cartier divisor, this implies that X' is smooth. Q.E.D.

Definition 35. A log canonical pair (X, Δ) is called *quotient-dlt*, abbreviated as *qdlt*, if for every lc center $Z \subset X$ the local scheme

$$(\mathrm{Spec} \mathcal{O}_{Z,X}, \Delta|_{\mathrm{Spec} \mathcal{O}_{Z,X}})$$

satisfies the equivalent conditions of Proposition 34.

It is clear that being qdlt is preserved by any (X, Δ) -MMP. It follows from Definition 8, that we can define $\mathcal{D}(\Delta^=1)$ for a qdlt pair. It seems to us that, for dual-complex problems, qdlt pairs form the most general class where the main results hold. We need the following lemma on extending qdlt modifications.

Lemma 36. *Let (X, Δ) be a quasi-projective, lc pair. Let $U \subset X$ be an open subset such that no lc center is contained in $X \setminus U$. Let $g_U : U' \rightarrow U$ be a quasi-projective, qdlt modification. Then there is a quasi-projective, qdlt modification $g_X : X' \rightarrow X$ extending g_U .*

Note that since no lc center is contained in $X \setminus U$, every exceptional divisor of g_X intersects U' .

Proof. Let \bar{X} be a normal compactification of U' that is projective over X . Let Y be a projective resolution of \bar{X} such that the preimage of $\mathrm{Supp} \Delta_{U'} \cup (\bar{X} \setminus U')$ is a simple normal crossing divisor.

Denote by $p : Y \rightarrow X$ the composite morphism. For p -exceptional divisors E_i let $b_i = \max\{\frac{1}{2}, \frac{1-a(E_i, X, \Delta)}{2}\}$ and set $\Delta_Y = p_*^{-1}\Delta + \sum_i b_i E_i$. Applying [20], we obtain that (Y, Δ_Y) has a log canonical model X'' over \bar{X} that is a compactification of U' .

Let Δ'' be the push forward of Δ_Y . Then (X'', Δ'') does not have log canonical centers contained in $X'' \setminus U'$ hence (X'', Δ'') is qdlt. Furthermore, $K_{X''} + \Delta'' - h^*(K_X + \Delta)$ is effective and its support is the same as the divisorial part of $X'' \setminus U'$ which we denote by $\sum_{i \in I} E_i$. We conclude that $\{E_i : i \in I\}$ are precisely the divisorial components in the stable base locus

$$\mathbf{B}(X''/X, K_{X''} + \Delta'') = \mathbf{B}_-(X''/X, K_{X''} + \Delta'').$$

That is, if H'' is an ample divisor on X'' there is a $0 < \epsilon \ll 1$ such that all the E_i are also contained in $\mathbf{B}(X''/X, K_{X''} + \Delta'' + \epsilon H'')$.

By the method of [37, 2.43], there is a divisor Δ_ϵ'' on X'' such that $\Delta_\epsilon'' \sim_{\mathbb{Q}} K_{X''} + \Delta'' + \epsilon H''$ and (X'', Δ_ϵ'') is klt.

By [5] a suitable (X'', Δ_ϵ'') -MMP over X terminates with a minimal model X' . Then the rational map $\phi : X'' \dashrightarrow X'$ is an isomorphism on U' and it contracts all the E_i . Set $\Delta' := \phi_* \Delta''$. Then $K_{X'} + \Delta' \sim_{\mathbb{Q}, X} 0$ and $X' \setminus U'$ has codimension ≥ 2 . Therefore $g_X : X' \rightarrow X$ is a quasi-projective, qdlt modification extending g_U . Q.E.D.

This implies that qdlt pairs have a toroidal dlt modification.

Proposition 37. *Let (X, Δ) be a quasi projective, qdlt pair. Let $U \subset X$ be an open set containing all the log canonical centers of (X, Δ) such that $\Delta|_U = \Delta^{\neq 1}|_U$ and $(U, \Delta_U) = (U, \Delta^{\neq 1}|_U)$ is toroidal (such U exists by definition). Then there is a dlt modification $g^{\text{dlt}} : (X^{\text{dlt}}, \Delta^{\text{dlt}}) \rightarrow (X, \Delta)$ such that over U , $(U', \Delta_{U'}) := (g^{-1}(U), \Delta^{\text{dlt}}|_{U'})$ is snc, $g^{\text{dlt}}|_{U'} : (U', \Delta_{U'}) \rightarrow (U, \Delta_U)$ is toroidal and $\mathcal{D}((\Delta^{\text{dlt}})^{\neq 1})$ is a subdivision of $\mathcal{D}(\Delta^{\neq 1})$.*

Proof. Given the toroidal pair (U, Δ_U) by [25] there is a toroidal log resolution $g_U : (U', \Delta_{U'}) \rightarrow (U, \Delta)$. Since it is a toroidal morphism, $g_U^*(K_U + \Delta_U) = K_{U'} + \Delta_{U'}$ and $\mathcal{D}(\Delta_{U'})$ is subdivision of $\mathcal{D}(\Delta_U)$. By Lemma 36, we can extend $g_U : U' \rightarrow U$ to $g_X : X' \rightarrow X$. Q.E.D.

Corollary 38. *Let (X, Δ_X) be a quasi projective, qdlt pair and $g : (Y, \Delta_Y) \rightarrow (X, \Delta_X)$ a dlt modification. Then $\mathcal{D}(\Delta_X^{\neq 1})$ is PL homeomorphic to $\mathcal{D}(\Delta_Y^{\neq 1})$.* Q.E.D.

Another good property of quotient-dlt singularities is that the theorem on extracting one divisor (cf. [34, 1.39]) that fails for dlt singularities does hold for quotient-dlt singularities.

Lemma 39. *Let (X, Δ) be a quasi-projective qdlt pair. Let E be a divisor such that $a(E; X, \Delta) = -1$. Then there exists a model $f : Y \rightarrow X$ such that E is the sole exceptional divisor of f and $(Y, f_*^{-1}\Delta + E)$ is qdlt.*

Proof. We may assume that X is \mathbb{Q} -factorial. We pick U as in Proposition 37 and let $g_U : U' \rightarrow U$ be the toroidal morphism such that $\text{Ex}(g_U) = E$. By Lemma 36, we can extend $g_U : U' \rightarrow U$ to $g_X : X' \rightarrow X$. Q.E.D.

This following result is essentially in [22, Sec.5].

Proposition 40. *Let (X, Δ_X) be a \mathbb{Q} -factorial dlt or qdlt pair. Let $g : (X, \Delta_X) \rightarrow Y$ be a Fano contraction of an extremal ray. Assume that $\Delta_X^{\neq 1}$ is g -vertical.*

Then Y is \mathbb{Q} -factorial and there is a \mathbb{Q} -divisor Δ_Y such that (Y, Δ_Y) is qdlt and there is a natural identification $\mathcal{D}(\Delta_X^{\neq 1}) \cong \mathcal{D}(\Delta_Y^{\neq 1})$.

Proof. It follows from [37, 3.36] that Y is \mathbb{Q} -factorial. Since $\rho(Y/X) = 1$, we know that each g -vertical component is mapped to a divisor on Y and different g -vertical divisors are mapped to different divisors on Y (see [22, 5.1, 5.2]). Write $\Delta_X^{-1} = \sum_{i \in I} E_i$. Then $\Delta_Y = \sum_{i \in I} F_i$ where $F_i := g(E_i)$ and $E_i = g^{-1}(F_i)$. Thus, for every $J \subset I$ we have

$$g^{-1}(\cap_{i \in J} F_i) = \cap_{i \in J} g^{-1}(F_i) = \cap_{i \in J} E_i,$$

which shows that $\mathcal{D}(\sum_{i \in I} E_i) = \mathcal{D}(\sum_{i \in I} F_i)$.

Similar to the argument of [22, 5.5], we see that the log canonical centers of (Y, Δ_Y) are precisely the nonempty intersections $\cap_{i \in J} F_i$ where $J \subset I$. In particular, (Y, Δ_Y) is qdlt. Q.E.D.

We can now prove the following strengthening of Theorem 4.

Theorem 41. *Let $f : X \rightarrow (0 \in C)$ be a flat proper family. Assume that general fibers X_t are rationally connected and $(X, \text{red } f^{-1}(0))$ is qdlt. Then $\mathcal{D}(\text{red } f^{-1}(0))$ is contractible.*

Proof. By Theorem 3 we may assume that X is smooth, f is projective and $\Delta := \text{red } f^{-1}(0)$ is snc.

Then we run a $(K_X + \Delta)$ -MMP over C . This is the same as running a $(K_X + \Delta - \epsilon f^{-1}(0))$ -MMP with for $1 \gg \epsilon > 0$. As X_t is smooth and rationally connected, K_X is not pseudo-effective, thus by [5] a suitable MMP terminates with a Fano contraction $g : (X', \Delta') \rightarrow Y$ over C where $\dim X' > \dim Y$.

Since $f^{-1}(0)$ is numerically f -trivial and its support is $\Delta^{\neq 1} = \Delta$, Corollary 22 guarantees that $\mathcal{D}(\Delta)$ collapses to $\mathcal{D}(\Delta')$.

If $Y = C$ then $\Delta' = \text{red}(f')^{-1}(0)$ is irreducible since $\rho(X'/Y) = 1$. Thus $\mathcal{D}(\Delta')$ is a point and we are done.

If $\dim Y > 1$ then we look at $f_Y : Y \rightarrow C$. By Proposition 40, $(Y, \text{red } f_Y^{-1}(0))$ is again qdlt and $\mathcal{D}(\text{red}(f')^{-1}(0))$ can be identified with $\mathcal{D}(\text{red } f_Y^{-1}(0))$. The latter is collapsible by induction, hence so is $\mathcal{D}(\Delta')$ and hence $\mathcal{D}(\Delta)$. Q.E.D.

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Dept. of Mathematics, University of Utah, Salt Lake City, UT 84112, USA
E-mail address: defefernex@math.utah.edu

Princeton University, Princeton NJ 08544, USA
E-mail address: kollark@math.princeton.edu

Beijing International Center of Mathematics Research, Beijing, 100871, China
E-mail address: cyxu@math.pku.edu.cn