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Diptych varieties. II: Apolar varieties

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To Yujiro Kawamata in friendship and admiration

Abstract.

This paper constructs all the diptych varieties with $de \leq 4$ (see [BR1], Main Theorem 3.3). Our construction involves several new classes of Gorenstein almost homogeneous spaces for $\mathrm{GL}(2) \times \mathbb{G}_m^r$, in particular two infinite series arising from the algebra of apolarity.

Contents

Diptych varieties and Mori flips

We introduced *diptych varieties* in [BR1], motivated by our attempts to understand Mori's explicit calculations [M] in the Picard group of a 3-fold extremal neighbourhood. Mori's argument associates a 2-step continued fraction expansion $[d, e, d, \dots]$ with an extremal neighbourhood. Roughly, for $C = \mathbb{P}^1 \subset X$ a flipping curve of Type A in a 3-fold X with two terminal singularities $P, Q \in C$ of type $cA_n/\boldsymbol{\mu}_r$ and a pair of divisors transverse to C at P and Q respectively, Mori sets up a 'continued division' algorithm that constructs a sequence of divisors

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$F_{2i} \sim F_{2i-1} - dF_{2i-2}$, $F_{2i+1} \sim F_{2i} - eF_{2i-1}$, and proves that it terminates in the set theoretic equality $C = F_k \cap F_{k+1}$ for some k. This expresses a flipping curve C as the base locus of a pencil of divisors, and hence proves the existence of the flip of $C \subset X$, showing moreover that it can in principle be computed as the normalisation of the pencil. Diptych varieties are *key varieties* for the $\mathbb{G}_m$ cover of these *Type A flips*: flips arise as regular pullbacks from diptychs after some massaging; see [Ki] §11 (especially 11.2) and [BR4] for details of this last step from diptychs to extremal neighbourhoods.

For completeness, we give some details in §2 of what we understand by a diptych variety; in brief, each is an affine 6-fold V_{ABLM} arising as a 4-parameter deformation of a *tent*, a reducible Gorenstein toric surface consisting of a cycle $T = S_0 \cup S_1 \cup S_2 \cup S_3$ of four affine toric components meeting along their 1-dimensional strata; the four deformation parameters smooth the axes of transverse intersections of the cycle. A diptych variety is characterised by three natural numbers d, e, k, or by a 2-step recurrent continued fraction $[d, e, d, \dots]$ to k terms – of course, these correspond to the d, e, k of Mori's continued division algorithm.

Theorem 1.1 of [BR1] asserts that a diptych variety exists for any d, e, k (with the bounds of [BR1], Theorem 3.3, (3.7) on k in the cases $de \le 3$). In the main case $de > 4$ and $d, e \ge 2$, we proved this in [BR1], Section 5. In [BR3] we treat the cases $de > 4$ with d or $e = 1$ using variants of the same methods. This paper constructs diptych varieties in the remaining cases $de \le 4$, fulfilling the promise of [BR1], Theorem 1.1, and providing key varieties for the remaining extremal neighbourhoods of Type A.

Apolar geometry

The diptych varieties with $de = 4$ have a beautiful description in terms of key 5-folds $V_k \subset \mathbb{A}^{k+5}$ that play a principal role in this paper (see §1, and especially 1.3). These are *almost homogeneous spaces* that are easy to describe based on the algebra of apolarity, and we offer several alternative approaches. With a final unprojection argument, any of these descriptions is enough to prove the existence of diptych varieties with $de = 4$.

Geometrically, the V_k are almost homogeneous spaces for the group $G = \mathrm{GL}(2) \times \mathbb{G}_m$: each is the closure of the orbit of an 'apolar' vector in a reducible representation of G, and we refer to them as *apolar varieties*, as yet with no general formal definition, but see 1.3. It would be interesting to know whether apolar varieties such as the V_k and the W_d introduced in 4.1 arise naturally in other parts of geometry and representation theory;

we see similar phenomena in other calculations in codimension ≥ 4, and this type of *apolar geometry* should apply more widely.

From the point of view of equations, we express the V_k using a generalised form of Cramer's rule. This provides all the equations of V_k in closed form, in contrast to the small subset of Pfaffian equations that we get away with in [BR1]. The varieties V_k are serial unprojections, although this does not itself provide all the equations directly.

§4 introduces a second series of apolar varieties, this time almost homogeneous 7-folds $W_d \subset \mathbb{A}^{d+9}$, and applies them as models for diptychs with $k = 2$. With a single additional unprojection, they also provide a format for diptychs with $k = 3$ involving *crazy Pfaffians*, reminiscent of Riemenschneider's 'quasi-determinants' [R]; see 4.2 where we discuss the equations in terms of *floating factors*. §5 handles the few remaining cases with $k = 4, 5$ and $de = 3$, where unprojection methods and pentagrams provide the equations directly. Rather than our apolar varieties V_k and W_d given by serial unprojection, these cases are most naturally described as regular pullbacks from a parallel unprojection key variety, a 10-fold $W \subset \mathbb{A}^{16}$.

Gorenstein rings in high codimension

Gorenstein rings arise naturally in geometry as homogeneous coordinate rings of Fanos, Calabi–Yaus, regular canonical n-folds, and other constructions – and, most notably for our purposes here, of 3-fold extremal neighbourhoods. Thus a supply of model Gorenstein rings, with explicit information about their generators and relations, gradings and so on, is of practical importance. It is hard to construct Gorenstein rings in high codimension in general; there is no practical classification beyond codimension 3 (although see [R2, R3] for a first structure theorem in codimension 4). Grojnowski and Corti and Reid [CR] study weighted homogeneous spaces or closed orbits in highest weight representations of semisimple algebraic groups, in particular for SL(5) and SO(10); Qureshi and Szendrői [QS] generalise these to more classes of examples. The almost homogeneous spaces V_k in §1 (dimension 5, codimension k), W_d in §4 (dimension 7, codimension $d + 2$) and W in §5 (dimension 10, codimension 6) present new Gorenstein rings purpose built to model certain 3-fold flips of Type A.

§1. The apolar variety V_k

The apolar varieties $V_k \subset \mathbb{A}^{k+5}$ introduced here provide an infinite family of affine Gorenstein 5-folds that are almost homogeneous spaces under $\mathrm{GL}(2) \times \mathbb{G}_m$. We treat the V_k as varieties in their own right from several different points of view.

1.1. The definition by equations

We define 5-folds $V_k \subset \mathbb{A}^{k+5}_{\langle x_{0\dots k}, a,b,c,z\rangle}$ for each $k \geq 3$. First set up $2 \times k$ and $k \times (k-2)$ matrixes

$$M = \begin{pmatrix} x_0 & \dots & x_{i-1} & \dots & x_{k-1} \\ x_1 & \dots & x_i & \dots & x_k \end{pmatrix}$$

and

$$N = \begin{pmatrix} a & & & & & \\ b & a & & & & \\ c & b & a & & & \\ & \vdots & & & \vdots & \\ & & & c & b & a \\ & & & & c & b \\ & & & & & c \end{pmatrix}.$$

Our variety $V_k \subset \mathbb{A}^{k+5}_{\langle x_{0\dots k}, z,a,b,c\rangle}$ is defined by two sets of equations:

$$\text{(1.1)} \qquad \text{(I)} \quad MN = 0 \qquad \text{and} \qquad \text{(II)} \quad \bigwedge^2 M = z \cdot \bigwedge^{k-2} N.$$

(I) is a recurrence relation

$$\text{(1.2)} \qquad ax_{i-1} + bx_i + cx_{i+1} = 0 \quad \text{for } i = 1, \dots, k-1.$$

(II) is a $(k-2) \times k$ adaptation of Cramer's rule giving the Plücker coordinates of the space of solutions of (I) up to a scalar factor z. The order and signs of the minors in (II) is not a problem here, as one sees from the guiding cases

$$x_{i-1}x_{i+1} - x_i^2 = a^{i-1}c^{k-i-1}z \text{ and } x_{i-1}x_{i+2} - x_i x_{i+1} = a^{i-1}bc^{k-i-2}z.$$

(However, in subsequent cases, in particular when we work with Pfaffians in 1.2, we need to fix a convention on their order and signs.) Note that the maximal $(k-2) \times (k-2)$ minors of N include a^{k-2} (delete the last two row) and c^{k-2} (delete the first two). More generally, deleting two adjacent rows $i-1, i$ gives $a^{i-1}c^{k-i-1}$ as a minor (only the diagonal contributes), whereas deleting two rows $i-1, i+1$ gives the minor $a^{i-1}bc^{k-i-2}$.

Thus our second set of equations is

$$x_{i-1}x_{j+1} - x_i x_j = z \det N(i-1, j).$$

Relations for $x_ix_j - x_kx_l$ for all $i+j = k+l$ are obtained as combinations of these; for example

$$\begin{aligned} x_{i-1}x_{j+2} - x_{i+1}x_j &= x_{i-1}x_{j+2} - x_ix_{j+1} + x_ix_{j+1} - x_{i+1}x_j \\ &= zN(i-1, j+1) + zN(i,j). \end{aligned}$$

Theorem 1.1. *For $k \geq 3$, (I) and (II) define a reduced irreducible Gorenstein 5-fold*

$$V_k \subset \mathbb{A}^{k+5}_{\langle x_{0\dots k}, a,b,c,z\rangle}.$$

This also holds for $k = 2$, with (II) involving interpreting the 0×0 minors as the single equation $1 \cdot z = x_0x_2 - x_1^2$.

This theorem follows at once from the following lemma.

Lemma 1.2. (i) *z is a regular element for V_k.*

(ii) *The section $z = 0$ of V_k is the quotient of the hypersurface*

$$\widetilde{W} : (g := au^2 + buv + cv^2 = 0) \subset \mathbb{A}^5_{\langle a,b,c,u,v\rangle}$$

by the $\boldsymbol{\mu}_k$ action $\frac{1}{k}(0,0,0,1,1)$. It is Gorenstein because

$$\frac{\mathrm{d}a \wedge \mathrm{d}b \wedge \mathrm{d}c \wedge \mathrm{d}u \wedge \mathrm{d}v}{g} \in \omega_{\mathbb{A}^5}(\widetilde{W}).$$

is $\boldsymbol{\mu}_k$ invariant.

(iii) *Also z, a, c is a regular sequence, and the section $z = a = c = 0$ of V_k is the toric Gorenstein surface (three-sided tent) consisting of $\frac{1}{k}(1,1)$ with coordinates $x_0, \dots, x_k$ and two copies of $\mathbb{A}^2$ with coordinates x_0, b and x_k, b.*

Proof. First, if $c \neq 0$ then a, b, c, x_0, x_1 are free parameters, and the recurrence relation (I) gives $x_2, \dots, x_k$ as rational function of these. One checks that the first equation in (II) gives $z = -\frac{ax_0^2 + bx_0x_1 + cx_1^2}{c^{k-1}}$ and the remainder follow. Similarly if $a \neq 0$.

If $a = c = 0$ and $b \neq 0$ then one checks that x_0, x_k, b are free parameters, $x_i = 0$ for $i = 1, \dots, k-1$ and $z = \frac{x_0x_k}{b^{k-2}}$. Finally, if $a = b = c = 0$ then $x_0, \dots, x_k$ and z obviously parametrise $\frac{1}{k}(1,1) \times \mathbb{A}^1$.

Therefore, no component of V_k is contained in $z = 0$, proving (i).

After we set $z = 0$, the equations (II) become $\bigwedge^2 M = 0$, and define the cyclic quotient singularity $\frac{1}{k}(1,1)$ (the cone over the rational normal curve). Introducing u, v as the roots of $x_0, \dots, x_k$, with $x_i = u^{k-i}v^i$, boils the equations $MN = 0$ down to the single equation $g := au^2 + buv + cv^2 = 0$. This proves (ii). (iii) is easy. Q.E.D.

1.2. The equations as Pfaffians

The equations of V_k fit together as 4×4 Pfaffians of a skew matrix. For this, edit M and N to get two new matrixes,

$$M' = \begin{pmatrix} x_0 & \dots & x_{i-1} & x_i & \dots & x_{k-2} \\ x_1 & \dots & x_i & x_{i+1} & \dots & x_{k-1} \\ x_2 & \dots & x_{i+1} & x_{i+2} & \dots & x_k \end{pmatrix} \tag{1.3}$$

which is $3\times(k-1)$ and N', the $(k-1)\times(k-3)$ matrix with the same display as N (that is, delete the first (or last) row and column of N). Equations (I) can be rewritten $(a,b,c)M' = 0$.

Now all of the equations (1.1) can be written as the 4×4 Pfaffians of the $(k+2)\times(k+2)$ skew matrix

$$\begin{pmatrix} c & -b & & M' & \\ & a & & & \\ & & & & \\ & & & z\bigwedge^{k-3} N' & \end{pmatrix}. \tag{1.4}$$

The Pfaffians $\mathrm{Pf}_{12.3(i+3)}$ give the recurrence relation (1.2), while the remaining Pfaffians give (II). In more detail, the big matrix is

$$\begin{pmatrix} c & -b & x_0 & \dots & x_{i-1} & x_i & \dots & x_{k-2} \\ & a & x_1 & \dots & x_i & x_{i+1} & \dots & x_{k-1} \\ & & x_2 & \dots & x_{i+1} & x_{i+2} & \dots & x_k \\ & & & zc^{k-3} & \dots & & \dots & \dots \\ & & & & zc^{k-i-1}a^{i-2} & -zbc^{k-i-2}a^{i-2} & \dots & \dots \\ & & & & & zc^{k-i-2}a^{i-1} & \dots & \dots \\ & & & & & & \dots & \dots \\ & & & & & & & za^{k-3} \end{pmatrix}$$

with bottom right $(k-1)\times(k-1)$ block equal to the $(k-3)$rd wedge of N' (with signs).

Small values of k. Our family starts with $k\geq 3$; the case $k=2$ would give the hypersurface $ax_0+bx_1+cx_2=0$, with $z := x_0x_2 - x_1^2$.

The first regular case is $k = 3$, which gives the 5×5 skew determinantal

$$\begin{pmatrix} c & -b & x_0 & x_1 \\ & a & x_1 & x_2 \\ & & x_2 & x_3 \\ & & & z \end{pmatrix}$$

a regular section of the affine Grassmannian $\mathrm{aGr}(2, 5)$. The case $k = 4$ is

$$\begin{pmatrix} c & -b & x_0 & x_1 & x_2 \\ & a & x_1 & x_2 & x_3 \\ & & x_2 & x_3 & x_4 \\ & & & zc & -zb \\ & & & & za \end{pmatrix},$$

an easy case of the standard extrasymmetric 6×6 determinantal of Dicks and Reid, [TJ], 9.1, equation (9.4).

The first really new case is $k = 5$, with equations the 4×4 Pfaffians of the 7×7 skew matrix

$$\begin{pmatrix} c & -b & x_0 & x_1 & x_2 & x_3 \\ & a & x_1 & x_2 & x_3 & x_4 \\ & & x_2 & x_3 & x_4 & x_5 \\ & & & zc^2 & -zbc & z(b^2 - ac) \\ & & & & zac & -zab \\ & & & & & za^2 \end{pmatrix}$$

We first arrived at this matrix by guesswork (with the z floated over from the row-columns $4, 5, 6, 7$ to $1, 2, 3$), determining the superdiagonal entries c^2, ac, a^2 and those immediately above $-bc, -ac$ by eliminating variables to smaller cases; the entry $b^2 - ac$ is then fixed so that the bottom 4×4 Pfaffian vanishes identically.

1.3. The variety V_k by apolarity

We can treat V_k as an almost homogeneous space under $\mathrm{GL}(2) \times \mathbb{G}_m$. For this, view $x_0, \ldots, x_k$ as coefficients of a binary form and a, b, c as coefficients of a binary quadratic form in dual variables, so that the equations $MN = 0$ or $(a, b, c)M' = 0$ are the apolarity relations. In general terms, *polarity* can be described as a choice of splitting of maps such as $\operatorname{Sym}^{d-1} U \otimes U \twoheadrightarrow \operatorname{Sym}^d U$ (here $U = \mathbb{C}^2$ is the given representation of GL(2)), or more vaguely as a way of viewing the $2 \times d$ matrix $\left(\begin{smallmatrix} y_0 & \cdots & y_{d-1} \\ y_1 & \cdots & y_d \end{smallmatrix}\right)$ or his bigger cousin (1.3) as a single object in determinantal constructions.

More formally, write

$$q = a\check{u}^2 + 2b\check{u}\check{v} + c\check{v}^2 \in \operatorname{Sym}^2 U^\vee \quad \text{and}$$
$$f = x_0 u^k + kx_1 u^{k-1} v + \tbinom{k}{2} x_2 u^{k-2} v^2 + \cdots + x_k v^k \in \operatorname{Sym}^k U.$$

Including the factor $\binom{k}{i}$ in the coefficient of $u^i v^{k-i}$ is a standard move in this game.

The second polar of f is the polynomial

$$\begin{aligned}
\Phi(u, v, u', v') &= \frac{1}{k(k-1)} \left(\frac{\partial^2 f}{\partial u^2} \otimes u'^2 + 2 \frac{\partial^2 f}{\partial u \partial v} \otimes u'v' + \frac{\partial^2 f}{\partial v^2} \otimes v'^2 \right) \\
&= \sum_{i=0}^{k-2} \tbinom{k-2}{i} x_i u^{k-i-2} v^i \otimes u'^2 \\
&\qquad + 2 \sum_{i=1}^{k-1} \tbinom{k-2}{i-1} x_i u^{k-i-1} v^{i-1} \otimes u'v' \\
&\qquad\qquad + \sum_{i=2}^{k} \tbinom{k-2}{i-2} x_i u^{k-i} v^{i-2} \otimes v'^2 \\
&= \sum_{i=0}^{k-2} \tbinom{k-2}{i} u^{k-2-i} v^i \otimes \left(x_i u'^2 + 2x_{i+1} u'v' + x_{i+2} v'^2 \right) \\
&\qquad \in \operatorname{Sym}^{d-2} U \otimes \operatorname{Sym}^2 U.
\end{aligned}$$

We apply $q \in \operatorname{Sym}^2 U^\vee$ to the second factor and equate to zero to obtain the recurrence relation $(a, b, c)M = 0$. In other words, substitute $u'^2 \mapsto a$, $u'v' \mapsto \frac{1}{2} b$, and $v'^2 \mapsto c$ in Φ.

Moreover, the second set of equations follows from the first by substitution, provided (say) that $c \neq 0$ and we fix the value of $x_0 x_2 - x_1^2$; for example, in

$$x_i x_{i+2} - x_{i+1}^2$$

substituting $x_{i+2} = -\frac{a}{c} x_i - \frac{b}{c} x_{i+1}$ gives

$$x_i \left(-\frac{a}{c} x_i - \frac{b}{c} x_{i+1} \right) - x_{i+1}^2 = -\frac{a}{c} x_i^2 - \left(\frac{b}{c} x_i + x_{i+1} \right) x_{i+1},$$

and we can substitute $-\frac{a}{c} x_{i-1}$ for the bracketed expression, to deduce that

$$x_i x_{i+2} - x_{i+1}^2 = \frac{a}{c} \left(x_{i-1} x_{i+1} - x_i^2 \right), \quad \text{etc.}$$

A normal form for a quadratic form under GL(2) is uv, so that a typical solution to the equations is

$$(a,b,c) = (0,1,0), \quad (x_{0\dots k}) = (1,0,\dots,0,1), \quad z = 1.$$

in the representation $\operatorname{Sym}^2 U^\vee \oplus \operatorname{Sym}^k U \oplus \mathbb{C}^1$ of $\mathrm{GL}(2) \times \mathbb{G}_m$, where the final $\mathbb{G}_m$ acts by homotheties on $U^\vee$, so acts on $q \in \operatorname{Sym}^2 U^\vee$ by $q \mapsto \lambda^2 q$ and on z by $z \mapsto \lambda^2 z$. Then V_k is the closure of the orbit of this typical apolar vector.

§2. Diptych varieties and Mori flips of Type A

The varieties $V_k \subset \mathbb{A}^{k+5}$ form a simple and natural series of Gorenstein 5-folds, each with an action of a large algebraic group and, by Lemma 1.2, a regular sequence z, a, c whose common zero locus is a reducible toric surface composed of a cycle of three affine toric surfaces.

In [BR1], we introduce a rather more complicated series of Gorenstein varieties: these are 6-folds

$$V_{ABLM} \subset \mathbb{A}^{k+l+6}$$

(where l is the number appearing in (2.1)), each admitting a regular sequence A, B, L, M whose common zero locus $T \subset V_{ABLM}$ is a reducible toric surface composed of a cycle of four affine toric surfaces which we call a *tent*. There is more combinatorial structure inside V_{ABLM}: namely $V_{LM} := (A = B = 0)$ and $V_{AB} := (L = M = 0)$ are toric 4-folds inside V_{ABLM} whose intersection equals T. In the language of [AH], V_{ABLM} is an affine T-variety (T for torus, not for tent): it admits an action of a torus $\mathbb{T} = (\mathbb{G}_m^\times)^4$ which restricts to the intrinsic torus action on each of the toric strata described so far.

Each diptych variety depends on a 2-step recurrent continued fraction $[d, e, d, \dots]$ to k terms. Starting from nothing, this data determines the toric configuration $V_{AB} \supset T \subset V_{LM}$, and the *existence of diptych varieties* is then the claim that this configuration arises inside an irreducible 6-fold, the diptych variety, as above; this claim is proved in the case $de > 4$, $d, e \geq 2$ in [BR1].

In §3 we use V_k to prove the existence of diptych varieties in the case $de = 4$. We need some of the definitions and notions of [BR1] for this. Given integers $d, e, k \geq 1$, consider the continued fraction expansion with k terms

$$[d, e, d, \dots] = d - \frac{1}{e - \cdots}.$$

Define $[b_1, \dots, b_{l-1}]$ to be the complementary continued fraction of a truncation as follows. Truncate the expansion $[d, e, d, \dots]$ to $k-1$ terms and reverse it, and then consider the uniquely defined minimal sequence of $b_j \geq 2$ for which

$$[\dots, d, e, d, 1, b_{l-1}, \dots, b_1] = 0. \tag{2.1}$$

For example, starting with $[4, 3, 4]$, one calculates $[3, 4, 1, 2, 2, 3, 2] = 0$, so in this case $[b_1, b_2, b_3, b_4] = [2, 3, 2, 2]$. (This is the Riemenschneider complementary continued fraction, in the sense of [BR1] Proposition 2.1(d).) Set $b_l = 1$.

Now define a toric variety V_{AB} as follows. Start with four variables x_k, y_l, A, B. Define the Laurent monomial $x_{k-1} = Ax_k^d y_l^{-1}$, and then

$$x_{k-2i} = x_{k-2i+1}^e x_{k-2i+2}^{-1} \quad \text{and} \quad x_{k-2i-1} = x_{k-2i}^d x_{k-2i+1}^{-1} \tag{2.2}$$

alternating the exponents d, e until you reach x_0. Similarly define $y_{l-1} = Bx_k^{-1} y_l^{b_l}$, and then

$$y_{j-1} = y_j^{b_j} y_{j+1}^{-1}$$

until you reach y_0. We treat these expressions in two ways: first as monomials in a lattice $\mathbb{M}_{AB} = \mathbb{Z}^4$ based by A, B, x_k, y_l; second as independent variables A, B, $x_{0\dots k}$, $y_{0\dots l}$ on affine space $\mathbb{A}^{k+l+4}$. The cone

$$\sigma_{AB} = \langle A, B, x_0, \dots, x_k, y_0, \dots, y_l \rangle \subset \mathbb{M}_{AB}$$

defines a toric variety $V_{AB} = X_{\sigma_{AB}}$ which embeds naturally as

$$V_{AB} \subset \mathbb{A}^{k+l+4}$$

defined by the relations above (after multiplying up denominators) and others that follow from syzygies. (In other words, the relations above define a union of components, of which V_{AB} is the unique component not contained in a coordinate hyperplane.)

Similarly we define V_{LM} starting from the four variables x_0, y_0, L, M and applying analogous relations for $x_1, x_2, \dots$ and $y_1, y_2, \dots$ but with the terms of the reversed continued fraction: that is, with $[d, e, d, \dots]$ if k is even, and from $[e, d, e, \dots]$ to k terms if k is odd. Again there is a lattice $\mathbb{M}_{LM}$ containing the defining cone σ_{LM}.

We sketch all of this data in a picture, called a *pair of long rectangles*, as in Figure 2.1, in which the bullet points represent $x_0, x_1, \dots, x_k$ up the left-hand side of each long rectangle and $y_0, \dots, y_l$ up the right-hand side, the *tags* d, e and b_j appear next to the corresponding variable on which they appear as an exponent, and the four auxilliary variables, or *annotations*, A, B, L, M positioned near the corners where they appear in the initial defining relations. Influenced by this picture, we refer to

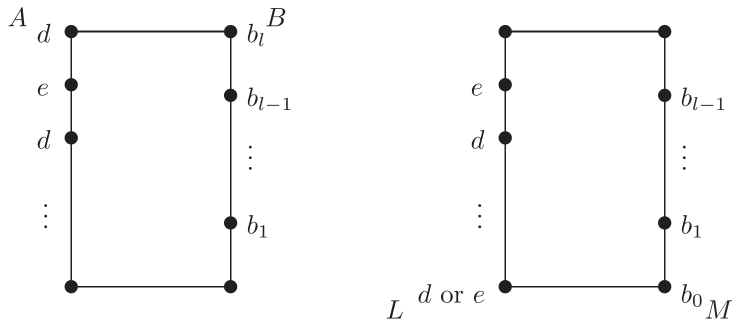


Figure 2.1. The pair of long rectangles for $[d, e, d, \dots]$ to k terms

data associated to x_0, y_0 as the *bottom end* of the long rectangles, and to x_k, y_l as the *top end*.

Notice from the defining relations that the lattices $\mathbb{M}_{AB}$ and $\mathbb{M}_{LM}$ are in fact identical, and so we identify them as $\mathbb{M}$. To avoid prejudice, we use the *impartial basis* L, M, A, B of $\mathbb{M}$. Although these four monomials are only a $\mathbb{Q}$-basis spanning an index de sublattice of $\mathbb{M}$, expressing lattice points in them turn out to express the antagonistic convexity properties of σ_{AB} and σ_{LM} most cleanly.

Although it is not completely obvious, the data assembled so far describes the toric monomial cones of the configuration $V_{AB} \supset T \subset V_{LM}$ for the initial continued fraction expansion $[d, e, d, \dots]$; see [BR1], §3. To show the existence of the corresponding diptych 6-fold, we simply build its equations from the bottom end up. We start by combining the equations of V_{AB} and V_{LM} at the bottom end in a naive way:

$$\begin{aligned} x_1 y_0 &= y_1 A^\alpha B^\beta + x_0^{(d \text{ or } e)} L \\ x_0 y_1 &= A^\gamma B^\delta + y_0 M, \end{aligned} \tag{2.3}$$

where the exponents α, β, λ, μ are determined by the tag relations we started from (and, unsurprisingly, appear in convergents of the continued fraction expansion $[d, e, d, \dots]$). These relations define a Gorenstein 6-fold $V_0 \subset \mathbb{A}^8_{\langle A,B,L,M,x_0,x_1,y_0,y_1 \rangle}$, that contains a divisor

$$D_0 = (x_0 = y_0 = A^\lambda B^\mu = 0) \subset V_0,$$

where $A^\lambda B^\mu = \gcd(A^\alpha B^\beta, A^\gamma B^\delta)$. We now apply the Gorenstein unprojection theorem of [PR] serially to construct a sequence of pairs $D_\nu \subset V_\nu$, adding the remaining variables x_i, y_j one at a time until we reach $V_\nu = V_{ABLM}$.

We demonstrate the first step by use of a magic *pentagram*: we seek to include the variable x_2 and calculate any relations that involve it. Consider the 5×5 antisymmetric matrix (we write only the strict upper triangle), which we also refer to as the *Pfaffian matrix*,

$$M_0 = \begin{pmatrix} x_1 & y_1 A^{\alpha-\lambda} B^{\beta-\mu} & -x_0^{(d \text{ or } e)-1} L & -x_2 \\ & x_0 & A^\lambda B^\mu & -M \\ & & y_0 & A^{\gamma-\lambda} B^{\delta-\mu} \\ & & & y_1 \end{pmatrix}. \tag{2.4}$$

The first and last of the maximal Pfaffians of M give precisely the pair of relations (2.3). The other three maximal Pfaffians involve expressions for $x_2 \cdot I_{D_0}$, where $I_{D_0} = (x_0, y_0, A^\lambda B^\mu)$ is the defining ideal of the unprojection divisor $D_0 \subset V_0$. These five Pfaffians define a Gorenstein variety $V_1 \subset \mathbb{A}^9$ in variables A, B, L, M, x_0, x_1, x_2, y_0, y_1. If $k = 1$, then this is V_{ABLM}, otherwise it contains a divisor

$$D_1 = (x_0 = x_1 = y_0 = A^? B^? = 0) \subset V_1,$$

where the exponents on $A^? B^?$ can be determined from the particular values of d, e, k. One can check that the 4-fold locuses $(A = B = 0)$ and $(L = M = 0)$ and their surface intersection correspond to the toric configuration; this is part of the claim of the existence of diptych varieties. The five equations constructed here have leading terms

$$\begin{gathered} x_0 y_1 = \cdots \qquad x_1 y_0 = \cdots \\ x_2 x_0 = \cdots \qquad x_2 y_0 = \cdots \qquad x_1 y_1 = \cdots\ , \end{gathered}$$

and joining these pairs of variables on Figure 2.1 draws a pentagram – hence the name. (It is magic because it works.)

The order we add the variables is important. We lay a *bar* at the level of variables we have considered so far: we start with the bar x_1 —— y_1, to indicate that we have all variables below these, then raise it to x_2 —— y_1 and so on as we add subsequent variables. Fortunately the precise order required is a technical point that our use of V_k in this paper sidesteps.

As an exercise, one can write an alternative proof of Theorem 1.1 above in the style of [BR1]: start with any of the codimension 2 complete intersections

$$\begin{pmatrix} x_{i-1} x_{i+1} = x_i^2 + a^{i-1} c^{k-i-1} z \\ a x_{i-1} + b x_i + c x_{i+1} = 0 \end{pmatrix} \subset \mathbb{A}^7_{\langle x_{i-1}, x_i, x_{i+1}, a, b, c, z \rangle}$$

and add the remaining variables one at a time as a serial unprojection using magic pentagrams at each step. (Or see [BR1], 1.2, for a fully-worked example of a similar calculation.)

Once set up properly, much of this construction is automatic. Curiously, the hardest part, and the bulk of the subtle machinery developed in [BR1], is to show that the natural unprojection divisor D_ν is a subscheme of V_ν. Again, our use of the V_k here completely sidesteps that point – when we need to make unprojection arguments in §3, the inclusion of the divisor is straightforward.

The contrast between the simple geometric constructions of this paper and the delicate and lengthy methods of [BR1] is striking. The varieties V_k arise naturally from the representation theory of $\mathrm{GL}(2)\times\mathbb{G}_m$, in contrast to any construction we could find in [BR1]. There is still some work to do in Section 3 to go from V_k to the diptych varieties, but it is easy compared to [BR1]. Whether the other diptychs of [BR1] can be modelled on almost homogeneous spaces in a similar way remains a mystery; this point has eluded us for a couple of decades.

§3. Application of V_k to diptych varieties with $de = 4$

Diptych varieties V_{ABLM} depend on three parameters $d, e, k \geq 1$. The solutions of $de = 4$ are $(d, e) = (2, 2)$, $(4, 1)$ and $(1, 4)$, and we allow any $k \geq 1$. In each case, we construct almost all of the coordinate ring of V_{ABLM} by a regular pullback from the key variety V_k of §1. We then adjoin the remaining few variables by an unprojection argument using the ideas of §2. Our proofs here are selfcontained, but we refer to [BR1] in places this clarifies the argument; see especially the worked example [BR1], 1.2.)

3.1. Case $[2, 2]$

We first construct the diptych variety V_{ABLM} with the monomial cones σ_{AB} and σ_{LM} of Figure 3.1. It has variables $x_{0\dots k}$ on the left

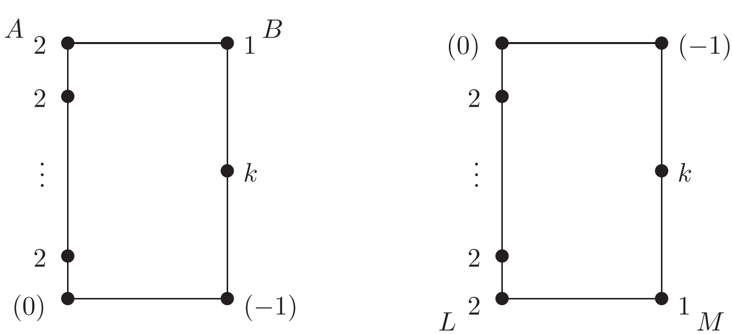


Figure 3.1. Case $[2, 2]$

against $y_{0\dots 2}$ on the right, tagged as in Figure 3.1, together with A,B, L,M. Although we do not yet own V_{ABLM}, we know some of its equations: by (2.3), we find the two bottom equations:

$$x_1y_0 = A^{k-1}B^k + x_0^2L \quad \text{and} \quad x_0y_1 = ABx_1 + y_0M. \tag{3.1}$$

Then, following the model of (2.4), the pentagram y_1, y_0, x_0, x_1, x_2 adjoins x_2, and $x_3, \dots, x_k$ are adjoined by a long rally of *flat* pentagrams $y_1, x_{i-1}, x_i, x_{i+1}, x_{i+2}$ with matrixes

$$\begin{pmatrix} y_1 & x_1 & -M & -x_2 \\ & y_0 & AB & -x_0L \\ & & x_0 & A^{k-2}B^{k-1} \\ & & & x_1 \end{pmatrix} \tag{3.2}$$

and

$$\begin{pmatrix} y_1 & x_{i+1} & -LM & -x_{i+2} \\ & x_{i-1} & AB & -x_i \\ & & x_i & (AB)^{k-i-2}(LM)^{i-1}BM \\ & & & x_{i+1} \end{pmatrix}$$

giving the Pfaffian equations

$$y_1x_i = ABx_{i+1} + LMx_{i-1}, \quad x_{i-1}x_{i+1} = x_i^2 + (AB)^{k-i-1}(LM)^{i-1}BM$$
$$\text{and} \quad x_{i-1}x_{i+2} = x_ix_{i+1} + (AB)^{k-i-2}(LM)^{i-1}BMy_1.$$

We see that these are the equations of V_k after the substitution

$$(a,b,c,z) \mapsto (LM, -y_1, AB, BM). \tag{3.3}$$

Thus to construct our diptych variety, we pull back $V_k \subset \mathbb{A}^{k+5}$ by (3.3), then adjoin the two corners y_0, y_2 as unprojection variables. Adjoining either of these is easy, but adjoining the second then requires a simple application of some of the main ideas of proof in Sections 4–5 of [BR1] which we work out here.

Lemma 3.1. *Define* $W_0 \subset \mathbb{A}^{k+6}_{\langle x_{0\dots k}, y_1, A, B, L, M\rangle}$ *to be the pullback of* V_k *under the morphism* $\mathbb{A}^{k+6} \to \mathbb{A}^{k+5}$ *given by* (3.3).

(i) $W_0 \subset \mathbb{A}^{k+6}$ *is an irreducible* 6*-fold.*

(ii) $D_0 = (x_1 = \cdots = x_k = M = 0)$ *is contained in* W_0 *as a divisor.*

(iii) *The unprojection* $W_1 \subset \mathbb{A}^{k+6} \times \mathbb{A}^1_{\langle y_0\rangle}$ *of* $D_0 \subset W_0$ *with unprojection variable* y_0 *includes the equations* (3.1) *as generators of its defining ideal.*

Proof. (ii) is immediate from the defining equations (1.1) of V_k: setting $x_1 = \cdots = x_k = 0$ leaves only terms divisible by M. It is a divisor because it has the right dimension. (iii) follows from the Pfaffians of the matrix (3.2), that express the unprojection variable y_0 as a rational function in $x_0, x_1, y_1, A, B, L, M$ with a simple pole on D. This includes the equations (3.1). Q.E.D.

Once we own $y_0 \in \mathbb{C}[W_1]$, we have to establish that the unprojection divisor of y_2 is contained in the variety W_1. The detailed statement is Theorem 3.3 below. (This is the same as the key point of the proof of [BR1], but our case here is much easier.) To prove it, we work with the $\mathbb{T}$-weights of each homogeneous polynomial in $x_0, \dots, y_2, A, B, L, M$, written in terms of the impartial basis dual to the monomials L, M, A, B (compare [BR1], Proposition 4.1). These base a slightly smaller lattice, giving some of the impartial coordinates of monomials little denominators d or e. The tag equations of V_{AB} and V_{LM} from Figure 3.1 determine the impartial coordinates, as follows.

Lemma 3.2. *In the impartial basis L, M, A, B, the monomials x_0, ..., y_2 have $\mathbb{T}$-weights:*

$$\begin{array}{rl} & \begin{array}{cccc} L & M & A & B \end{array} \\ x_0 = & \left(\begin{array}{cccc} -\frac{1}{2} & 0 & \frac{k-1}{2} & \frac{k}{2} \end{array} \right) \\ x_1 = & \left(\begin{array}{cccc} 0 & \frac{1}{2} & \frac{k-2}{2} & \frac{k-1}{2} \end{array} \right) \\ x_2 = & \left(\begin{array}{cccc} \frac{1}{2} & 1 & \frac{k-3}{2} & \frac{k-2}{2} \end{array} \right) \\ \vdots & \\ x_i = & \left(\begin{array}{cccc} \frac{i-1}{2} & \frac{i}{2} & \frac{k-i-1}{2} & \frac{k-i}{2} \end{array} \right) \\ \vdots & \\ x_{k-1} = & \left(\begin{array}{cccc} \frac{k-2}{2} & \frac{k-1}{2} & 0 & \frac{1}{2} \end{array} \right) \\ x_k = & \left(\begin{array}{cccc} \frac{k-1}{2} & \frac{k}{2} & -\frac{1}{2} & 0 \end{array} \right) \end{array} \quad \text{and} \quad \begin{array}{rl} & \begin{array}{cccc} L & M & A & B \end{array} \\ y_0 = & \left(\begin{array}{cccc} 0 & -\frac{1}{2} & \frac{k}{2} & \frac{k-1}{2} \end{array} \right) \\ y_1 = & \left(\begin{array}{cccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array} \right) \\ y_2 = & \left(\begin{array}{cccc} \frac{k}{2} & \frac{k+1}{2} & 0 & -\frac{1}{2} \end{array} \right) \end{array}$$

Proof. These vectors satisfy all the tag relations of the pair of long rectangles; or if you prefer, plug in the formulas from [BR1], Proposition 4.1. Q.E.D.

The following statement specifies the unprojection divisor $D_1 \subset W_1$ of y_2, completing our construction.

Theorem 3.3. *In the notation of Lemma 3.1, define*

$$D_1 = (x_0 = \cdots = x_{k-1} = y_0 = B = 0) \subset \mathbb{A}^{k+7}_{\langle x_{0\dots k}, y_0, y_1, A, B, L, M \rangle}.$$

Then $D_1 \subset W_1$, and the unprojection of D_1 in W_1 is the diptych variety V_{ABLM} on the pair of long rectangles of Figure 3.1.

Proof. Most of the generators of I_{W_1} are already in the ideal of I_{W_0}, and so lie in the ideal I_{D_1} by the argument of Lemma 3.1 applied to y_2 rather than y_0. The equation (3.1) of the form $x_0y_1 = \cdots$ is known by Lemma 3.1(iii), and also lies in I_{D_1}.

The remaining generators of I_{W_1} have leading terms x_iy_0 for $i = 1, \dots, k$. To prove that each of these lies in I_{D_1}, we prove a stronger statement: every monomial in any of these generator relations is divisible by one of $x_{0\dots k-1}$, y_0 or B. In fact, we prove some stronger still. As in [BR1], 5.1, rather than working directly with these generators, we work with their $\mathbb{T}$-weights, and we show that any monomial of $\mathbb{T}$-weight equal to that of x_iy_0 (that is, any monomial that could appear in a $\mathbb{T}$-homogeneous equation which included x_iy_0) is divisible by one of $x_{0\dots k-1}$, y_0 or B.

For monomials m, n, write $m \overset{\mathbb{T}}{\sim} n$ if m and n have the same $\mathbb{T}$-weight, or equivalently, the same impartial coordinates. Suppose $m \in \mathbb{C}[W_1]$ is a monomial with $m \overset{\mathbb{T}}{\sim} x_iy_0$ for some $i = 1, \dots, k$. (Any term in the equation having leading term x_iy_0 satisfies this equivalence, so if each such monomial lies in I_{D_1} then certainly the generator itself does.) We may assume that the monomial m is of the form $x_k^{\xi}y_2^{\eta}L^{\lambda}M^{\mu}A^{\alpha}B^{\beta}$, since the other variables already lie in I_{D_1}. We may assume further that $\xi = 0$: otherwise, dividing through by x_i, the $\mathbb{T}$-weight of y_0 can be calculated from that of $(x_k/x_i)x_k^{\xi-1}$ times other variables whose M coefficient is nonnegative; but this has M coefficient > 0, whereas y_0 has M coefficient $= -1/2$, a contradiction.

Now compare x_iy_0 and $m = y_1^{\eta}L^{\lambda}M^{\mu}A^{\alpha}B^{\beta}$: their impartial coordinates are

$$\begin{array}{rcl} x_iy_0 & = & \begin{pmatrix} \frac{i-1}{2} & \frac{i-1}{2} & \frac{2k-i-1}{2} & \frac{2k-1+1}{2} \end{pmatrix} \\ y_1^{\eta}L^{\lambda}M^{\mu}A^{\alpha}B^{\beta} & = & \begin{pmatrix} \frac{\eta}{2}+\lambda & \frac{\eta}{2}+\mu & \frac{\eta}{2}+\alpha & \frac{\eta}{2}+\beta \end{pmatrix}. \end{array}$$

Since $\alpha \geq 0$, it follows from the coefficient of A that $\eta/2 \leq (2k-i-1)/2$, so now from the coefficient of B we have $\beta \geq 1$. In other words, B divides the monomial m, and $m \in I_{D_1}$ as required. Q.E.D.

3.2. Case $[4,1]$ with even $l = 2k$

The odd numbered x_i are redundant generators, and omitting them gives Figure 3.2. The diptych variety has variables $x_{0\dots k}$, $y_{0\dots 4}$, A, B, L, M with the two bottom equations

$$x_1y_0 = A^{k-1}B^{2k-1}y_1 + x_0^3L \quad \text{and} \quad x_0y_1 = A^kB^{2k+1} + y_0M.$$

We adjoin y_2, then $x_2, \dots, x_k$ by a game of pentagrams centred on a long

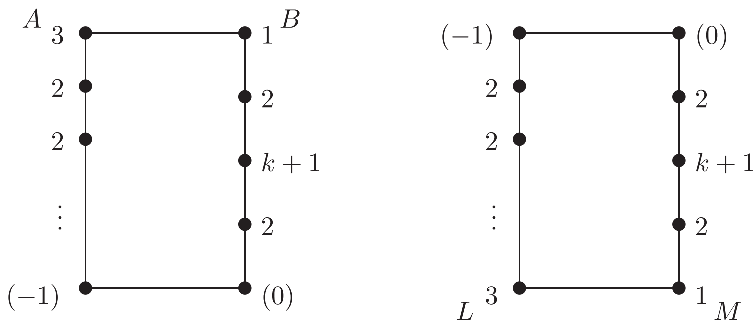


Figure 3.2. Case $[4, 1]$ with even $l = 2k$

rally of flat pentagrams, with y_2 against $x_{i-1}, x_i, x_{i+1}, x_{i+2}$ and Pfaffian equations

$$y_2 x_i = AB^2 x_{i+1} + LM^2 x_{i-1},$$

$$x_{i-1}x_{i+1} = x_i^2 + (AB^2)^{k-i-1}(LM^2)^{i-1}BM$$

$$\text{and} \quad x_{i-1}x_{i+2} = x_i x_{i+1} + (AB^2)^{k-i-2}(LM^2)^{i-1}BMy_2$$

These are the equations of V_k after the substitution

$$(a, b, c, z) \mapsto (LM^2, -y_2, AB^2, BM). \tag{3.4}$$

Lemma 3.4. *In the impartial basis L, M, A, B, the monomials x_0, $\dots$, y_4 have $\mathbb{T}$-weights as listed in Table 1.*

Proof. Once more, either observe that these vectors satisfy all the tag relations of the pair of long rectangles, or plug in the formulas from [BR1], Proposition 4.1, then delete every alternate x variable (the ones tagged with a 1) and relabel to get these $x_{0\dots k}$. Q.E.D.

The proof below that we can make the remaining unprojections is similar to that of Theorem 3.3, so we restrict ourselves to setting out the steps and indicating how to modify them for this case.

Theorem 3.5. *The diptych variety on the pair of long rectangles of Figure 3.2 exists.*

$$
\begin{array}{rcccccc}
 & & L & M & A & B & \\
x_0 & = (& -\frac{1}{4} & 0 & \frac{2k-1}{4} & k &) \\
x_1 & = (& \frac{1}{4} & 1 & \frac{2k-3}{4} & k-1 &) \\
x_2 & = (& \frac{3}{4} & 2 & \frac{2k-5}{4} & k-2 &) \\
 & \vdots & & & & & \\
x_i & = (& \frac{2i-1}{4} & i & \frac{2k-2i-1}{4} & k-i &) \\
 & \vdots & & & & & \\
x_{k-1} & = (& \frac{2k-3}{4} & k-1 & \frac{1}{4} & 1 &) \\
x_k & = (& \frac{2k-1}{4} & k & -\frac{1}{4} & 0 &) \\
 & & & & & & \\
y_0 & = (& 0 & -1 & k & 2k+1 &) \\
y_1 & = (& \frac{1}{4} & 0 & \frac{2k+1}{4} & k+1 &) \\
y_2 & = (& \frac{1}{2} & 1 & \frac{1}{2} & 1 &) \\
y_3 & = (& \frac{2k+1}{4} & k+1 & \frac{1}{4} & 0 &) \\
y_4 & = (& k & 2k+1 & 0 & -1 &)
\end{array}
$$

Table 1. $x_0, \dots, y_4$ in the impartial basis L, M, A, B.

Proof. First construct the 6-fold $W_0 \subset \mathbb{A}^{k+6}_{\langle x_{0\dots k}, y_2, A, B, L, M\rangle}$ as the pullback of V_k by the morphism (3.4). From the equations (1.1) of V_k, one sees that $D_0 \subset W_0$, where $I_{D_0} = (x_{1\dots k}, M)$, and we can unproject this to construct W_1 with new ambient variable y_1.

We define $D_1 \subset \mathbb{A}^{k+7}_{\langle x_{0\dots k}, y_1, y_2, A, B, L, M\rangle}$. To show that $D_1 \subset W_1$ we check that any monomial m with the same $\mathbb{T}$-weight as a generator of I_{W_1} that has not already been considered is already in I_{D_1}. For example, if $m \overset{\mathbb{T}}{\sim} x_i y_1$, for any $i = 1, \dots, k$, then we can suppose without loss of generality that $m = x_0^\xi L^\lambda M^\mu A^\alpha B^\beta$. By Lemma 3.4, in impartial L, M, A, B coordinates we see that

$$x_i y_1 = (\tfrac{i}{2}, i, k - \tfrac{i}{2}, 2k - i + 1).$$

His M-coordinate is $i \geq 1$, and since $x_0 = (-1/4, 0, (2k-1)/4, k)$, the only contribution to the M-coordinate on the right comes from M^μ, so $\mu \geq 1$. In other words, M divides m, so $m \in I_{D_1}$ as required.

The only other equation to check has leading term $x_0 y_2 \overset{\mathbb{T}}{\sim} m = x_0^\xi y_1^\eta L^\lambda M^\mu A^\alpha B^\beta$. Since both x_0 and y_1 have zero M coefficient, the same argument works again. Thus $D_1 \subset W_1$, and we can unproject with new variable y_0 to obtain $W_2 \subset \mathbb{A}^{k+8}_{\langle x_{0\dots k}, y_{0\dots 2}, A, B, L, M\rangle}$. The pentagrams confirm the tag equations at the bottom corners.

We continue to unproject y_3 and then y_4 to conclude. For the first of these, define $D_2 \subset \mathbb{A}^{k+8}$ by the ideal $I_{D_2} = (x_{0\dots k-1}, y_{0\dots 1}, B)$ and check that $D_2 \subset W_2$. We check the critical equations (those that are not automatically in I_{D_2} as a corollary of previous checks). First suppose that $x_k y_0 \overset{\mathbb{T}}{\sim} m = y_2^\eta L^\lambda M^\mu A^\alpha B^\beta$. Since

$$x_k y_0 = (\tfrac{2k-1}{4}, k-1, k-\tfrac{1}{4}, 2k+1) \quad \text{and} \quad y_2 = (\tfrac{1}{2}, 1, \tfrac{1}{2}, 1)$$

consideration of the A-coordinate shows that $\eta < 2k$, so the B-coordinate shows that $\beta \geq 2$; in particular, $m \in I_{D_2}$ as required.

Now consider $y_0 y_2 \overset{\mathbb{T}}{\sim} m = x_k^\xi L^\lambda M^\mu A^\alpha B^\beta$. We have

$$y_0 y_2 = (1/2, 0, k+1/2, 2(k+1)) \quad \text{and} \quad x_k = ((2k-1)/4, k, -1/4, 0),$$

so $\beta \geq 2(k+1)$, whence B divides m and $m \in I_{D_2}$.

Thus we obtain $W_3 \subset \mathbb{A}^{k+9}_{\langle x_{0\dots k}, y_{0\dots 3}, A, B, L, M\rangle}$ by unprojecting $D_2 \subset W_2$. Finally we observe that $D_3 \subset W_3$, where $I_{D_3} = (x_{0\dots k-1}, y_{0\dots 2}, B)$ for similar reasons. For example, if $y_0 y_3 \overset{\mathbb{T}}{\sim} m = x_k^\xi L^\lambda M^\mu A^\alpha B^\beta$, then $y_0 y_3 = (\frac{2k+1}{4}, k, k+1/4, 2k+1)$ and $x_k = (\frac{2k-1}{4}, k, -1/4, 0)$ shows that $\beta \geq k+1$, so again B divides m and so $m \in I_{D_3}$. Unprojecting $D_3 \subset W_3$ gives the diptych variety we seek. Q.E.D.

3.3. Case $[1,4]$ with even $l = 2k$

Omit the even numbered x_i, giving Figure 3.3. The diptych variety

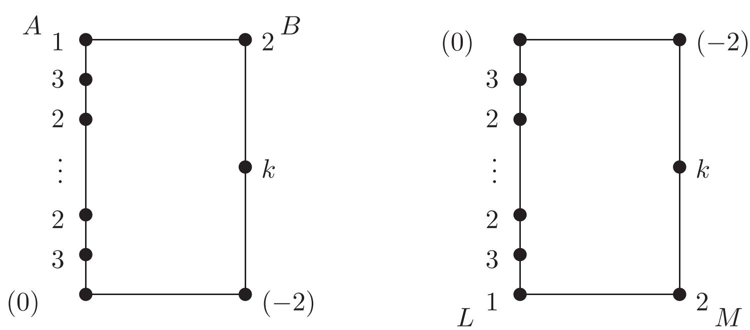


Figure 3.3. Case $[4,1]$ with even $l = 2k$

has variables $x_{0\dots k}$, $y_{0\dots 2}$, A, B, L, M with the two bottom equations

$$x_1 y_0 = A^{2k-1} B^k + x_0 L \quad \text{and} \quad x_0 y_1 = x_1^2 A^2 B + y_0^2 M.$$

As before, adjoining $x_2, \dots, x_k$ features a long rally of flat pentagrams, with y_1 against $x_{i-1}, x_i, x_{i+1}, x_{i+2}$ and Pfaffian equations

$$y_1 x_i = A^2 B x_{i+1} + L^2 M x_{i-1},$$

$$x_{i-1} x_{i+1} = x_i^2 + (A^2B)^{k-i-1} (L^2M)^{i-1} AL$$

$$\text{and} \quad x_{i-1} x_{i+2} = x_i x_{i+1} + (A^2B)^{k-i-2} (L^2M)^{i-1} BMy_2.$$

These are the equations of V_k after the substitution

$$(a, b, c, z) \mapsto (L^2M, -y_1, A^2B, BM).$$

We omit the formal statement and proof of the analogue of Theorem 3.5: the diptych variety on the pair of long rectangles of Figure 3.3 exists, and after the substitution the proof unprojects y_0 and y_2 by similar arguments in impartial coordinates.

3.4. Case $[1, 4]$ with odd $l = 2k + 1$

This is $[1, 4]$ read from the top, but $[4, 1]$ read from the bottom, so is a mix of the two preceding cases. Omit the odd numbered x_i, giving Figure 3.4. The diptych variety has variables $x_{0\dots k}$, $y_{0\dots 3}$, A, B, L, M

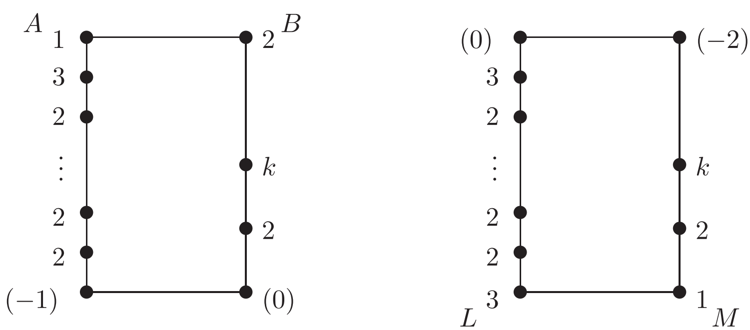


Figure 3.4. Case $[1, 4]$ with odd $l = 2k + 1$

with the two bottom equations

$$x_1 y_0 = y_1 A^{2k-3} B^{k-1} + x_0^3 L \quad \text{and} \quad x_0 y_1 = A^{2k-1} B^k + y_0 M.$$

Adjoin y_2 then x_2 by

$$\begin{pmatrix} y_1 & A^2B & M & y_2 \\ & y_0 & A^{2k-3}B^{k-1} & x_0^2 L \\ & & x_0 & y_1 \\ & & & x_1 \end{pmatrix} \quad \text{then} \quad \begin{pmatrix} y_2 & x_1 & M & x_2 \\ & y_1 & A^2B & x_0 LM \\ & & x_0 & y_2 A^{2k-5} B^{k-2} \\ & & & x_1 \end{pmatrix}$$

After this, adjoining $x_3, \dots, x_{k-1}$ is the usual long rally of flat pentagrams, with y_2 against $x_{i-1}, x_i, x_{i+1}, x_{i+2}$ and

$$\begin{pmatrix} y_2 & x_{i+1} & LM^2 & x_{i+2} \\ & x_{i-1} & A^2B & x_i \\ & & x_i & (A^2B)^{k-i-3}(LM^2)^{i-1}ABMy_2 \\ & & & x_{i+1} \end{pmatrix}$$

and the Pfaffian equations

$$y_2 x_i = A^2 B x_{i+1} + LM^2 x_{i-1},$$

$$x_{i-1}x_{i+1} = x_i^2 + (A^2B)^{k-i-2}(LM^2)^{i-1}ABMy_2$$

$$\text{and} \quad x_{i-1}x_{i+2} = x_i x_{i+1} + (A^2B)^{k-i-3}(LM^2)^{i-1}ABMy_2^2.$$

These are the equations of $V(k-1)$ after the substitution

$$(a, b, c, z) \mapsto (LM^2, -y_2, A^2B, BM).$$

We again omit the formal statement and proof: the diptych variety on the pair of long rectangles of Figure 3.4 exists, and after the substitution the proof unprojects y_3, y_1 and y_0 by arguments in impartial coordinates.

§4. The apolar varieties W_d and diptychs with $k \leq 3$

By [BR1], Classification Theorem 3.3, (3.7), when $de < 3$, the cases to treat are

$$\text{(4.1)} \qquad \begin{array}{ll} (d,e) = (1,1), & k \leq 2 \\ (d,e) = (1,2), & k \leq 3 \\ (d,e) = (1,3), & k \leq 5 \end{array} \qquad \begin{array}{ll} (d,e) = (2,1), & k \leq 3 \\ (d,e) = (3,1), & k \leq 5 \end{array}$$

The case $k = 1$ is already in [BR1], (3.9): for any values of d, e we get the codimension 2 complete intersection

$$\big(x_1 y_0 = B + L x_0^e,\ x_0 y_1 = A x_1^d + M\big) \subset \mathbb{A}^8_{\langle x_0, x_1, y_0, y_1 A, B, L, M\rangle}.$$

In §4.1 we discuss the case $k = 2$ for arbitrary d, e: again there is an almost homogeneous variety W_d that serves as a model for the equations.

The cases with $k \geq 3$ have some x_i variables with tags $= 1$, which, by the tag relations (2.2), are therefore redundant generators. Eliminating them leaves a variety in low codimension that we can specify by equations. For $k \geq 3$, the reduced models are as follows (for odd k, top-to-bottom symmetry swaps d, e; we only list the cases with $d = 1$):

k	V_{AB} tags	codim as given	reduced codim
3	$[1, 2, 1, (0)]$	4	2
3	$[1, 3, 1, (0)]$	5	4
4	$[1, 3, 1, 3, (0)]$	5	4
4	$[3, 1, 3, 1, (0)]$	6	3
5	$[1, 3, 1, 3, 1, (0)]$	6	2

Eliminating the redundant generators is convenient to establish that the varieties exist, but leaving them in has its own advantages. It allows us to write their equations more naturally (in fact, usually as Tom unprojections, in the language of [TJ], 2.2–2.3), sometimes in closed Pfaffian formats. In addition, we can put an extra deformation parameter as coefficient in front of each variable tagged with 1, thus exhibiting the variety as a section of a bigger key variety.

4.1. Case $k = 2$, any d, e; the apolar variety W_d

For any $d, e \geq 1$, the variables and tags on V_{AB} are as follows: going up the lefthand side we have x_0, x_1, x_2 tagged with $(0), e, d$, against $y_{0\dots d}$ tagged with $(-e+1), 2, \dots, 2, 1$. In V_{AB} the projection sequence first eliminates the variables $y_d, y_{d-1}, \dots, y_2$, and then the top left corner x_2; in V_{LM} the sequence of projections is $y_0, y_1, \dots, y_{d-2}$, then the bottom left corner x_0. Following the model equations (2.3) (or [BR1], 1.2), one calculates the two equations at the bottom of the long rectangle as

$$x_1 y_0 = AB^d + Lx_0^d \quad \text{and} \quad x_0 y_1 = -x_1^{e-1} AB^{d-1} + y_0 M.$$

One can then restore variables in the reverse order to the projection sequence using magic pentagrams, as in (2.4). The 5×5 matrixes can be combined into a single $(d+4) \times (d+4)$ skew matrix

$$\text{(4.2)} \qquad \begin{pmatrix} C & -x_0 & B & y_0 & y_1 & \dots & y_{d-1} \\ & -M & x_2 & y_1 & y_2 & \dots & y_d \\ & & x_1 & AB^{d-1} & AB^{d-2}x_2 & \dots & Ax_2^{d-1} \\ & & & Lx_0^{d-1} & LMx_0^{d-2} & \dots & LM^{d-1} \\ & & & & & \text{see (4.3)} & \end{pmatrix}$$

in which we have replaced x_1^{e-1} by the token C in m_{12}; the bottom right entries are

$$\text{(4.3)} \qquad m_{i+5,j+5} = ALC(x_0 B)^{d-j-1}(x_2 M)^i \cdot \frac{(x_0 x_2)^{j-i} - (BM)^{j-i}}{x_0 x_2 - BM}$$

for $0 \le i < j \le d-1$. The 4×4 Pfaffians of this $(d+4)\times(d+4)$ skew matrix provide the remaining equations.

If we treat C as an independent variable, then the Pfaffians of (4.2) generate the ideal of a 7-fold

$$W_d \subset \mathbb{A}^{d+9}_{\langle x_{0\dots 2}, y_{0\dots d}, A,B,L,M,C\rangle}.$$

It can be realised by serial unprojection following [BR1], 1.2: the equations appearing in pentagrams are

$$\begin{aligned}
x_0x_2 &= -x_1C + BM\\
y_{i-1}y_{i+1} &= y_i^2 + ALC^2(x_0B)^{d-i-1}(x_2M)^{i-1}\\
x_0y_i &= -x_2^{i-1}AB^{d-i}C + y_{i-1}M\\
x_1y_i &= Ax_2^iB^{d-i} + Lx_0^{d-i}M^i\\
x_2y_i &= y_{i+1}B - x_0^{d-i-1}CLM^i
\end{aligned}$$

The equation for x_0x_2 and for all x_iy_j are contained among the Pfaffians of the first 4 rows of (4.2). Beyond the 4th row, each entry $m_{i+5,j+5}$ of (4.3) appears in just one generating relation, namely

$$\text{(4.4)}\qquad \mathrm{Pf}_{2,3,i+5,j+5} = Cm_{i+5,j+5} - y_iy_{j+1} + y_{i+1}y_j.$$

These varieties are interesting in several ways. Replacing x_1^{e-1} by the token C in m_{12} displays V_{ABLM} as the section $C = x_1^{e-1}$ of the 7-fold W_d, that is a almost homogeneous variety under $\mathrm{GL}(2)\times\mathbb{G}_m^3$. Setting $C=0$ or $C=1$ gives invariant 6-fold sections that are also almost homogeneous. The case $d=1$ is just the affine cone $W(1) = \mathrm{aGr}(2,5)$ on $\mathrm{Gr}(2,5)$.

Exercise 4.1. Write U for the given representation of $\mathrm{GL}(2)$. Use $y_{0\dots d}$ as coefficients of a binary form $f = \sum\binom{d}{i}y_iu^{d-i}v^i \in \mathrm{Sym}^d U$ and (B,x_2), (x_0,M) as those of two linear forms $g = Bu + x_2v$, $h = x_0u + Mv \in U$. Then the 4×4 Pfaffians of (4.2) take the form
(4.5)

$$\begin{aligned}
x_1f &= Ag^d + Lh^d, & Cx_1 &= \det\begin{vmatrix} B & x_0\\ x_2 & M\end{vmatrix} = \frac{g\wedge h}{u\wedge v},\\
Mf_u - x_0f_v &= dACg^{d-1}, & &\\
-x_2f_u + Bf_v &= dLCh^{d-1}, & f_u\wedge f_v &= d^2ALC^2\times\frac{g^{d-1}\wedge h^{d-1}}{g\wedge h},
\end{aligned}$$

where of course $f_u = \frac{\partial f}{\partial u}$ and $f_v = \frac{\partial f}{\partial v}$. As we saw in (4.3), $g^{d-1}\wedge h^{d-1}$ written out as 2×2 minors is identically divisible by $BM - x_0x_2$, so the final set of equations give (4.4). This form of the equations is manifestly $\mathrm{GL}(2) = \mathrm{GL}(U)$ invariant. A typical solution of (4.5) is $x_0 = x_2 = 0$

and $x_1 = A = C = L = B = M = 1$, giving $g = u$, $h = v$, $f = u^d + v^d$, and one sees that W_d is the orbit closure of this typical solution under $\mathrm{GL}(2) \times \mathbb{G}_m^3$.

At the level of the matrix (4.2), the GL(2) action replaces rows 1 and 2 by their general linear combinations, and the d rows-and-columns $5, 6, \dots, d+4$ by the linear combinations corresponding to the $(d-1)$st symmetric power. For example, adding λ times row 2 to row 1 (and the same for the columns to preserve skew symmetry),

$$\lambda^{j-i} \times \text{binomial coefficient} \times \text{column } (5+j)$$

to column $5+i$ for $j = i+1, \dots, d$ does $x_0 \mapsto x_0 + \lambda M$, $B \mapsto B + \lambda x_2$ and $y_i \mapsto \sum \lambda^{i+j} y_j + (d-i)\lambda y_{i+1} + \text{etc.}$, meaning $f(u,v) \mapsto f(u+\lambda v, v)$.

4.2. Case $k = 3$; floating factors and crazy Pfaffians

We only need to do $e = 1$; this covers $d = 1$ after top-to-bottom reflection. The case $e = 1$ differs from $e \geq 2$ in the order of elimination in V_{AB}, as we discuss systematically in [BR3]: projecting V_{AB} from the top, we eliminate x_2 and all the y_i for $i = d-1, d-2, \dots, 2$ before it becomes possible to eliminate x_3. This qualitative change prevents us from treating cases with $e = 1$ as a limit of $e \geq 2$.

Consider the general case $k = 3$, $d \geq 2$. In V_{AB} we have $x_{0\dots3}$ tagged with $(0), d, 1, d$ against $y_{0,\dots,d-1}$ tagged with $(-d+2), 2, \dots, 2, 1$. The equations of V_{ABLM} not involving x_0 are those of a single vertebra, and we can see them as the 4×4 Pfaffians of the $(d+3) \times (d+3)$ matrix

$$\begin{pmatrix} -C & x_1 & B & y_0 & y_1 & \dots & y_{d-2} \\ & LM & x_3 & y_1 & y_2 & \dots & y_{d-1} \\ & & x_2 & x_3AB^{d-2} & x_3^2AB^{d-3} & \dots & x_3^{d-1}A \\ & & & x_1^{d-2}L & x_1^{d-3}L^2M & \dots & L^{d-1}M^{d-2} \\ & & & & \text{see } (4.7) & & \end{pmatrix} \tag{4.6}$$

with

$$m_{i+5,j+5} = x_3ALC(x_1B)^{d-2-j}(x_3LM)^i \frac{(x_1x_3)^{j-i} - (BLM)^{j-i}}{x_1x_3 - BLM}. \tag{4.7}$$

For general d, this is the regular pullback of the apolar 7-fold $W(d-1)$ constructed in 4.1 under the substitution

$$\begin{aligned} &(x_{0\dots2}, y_{0\dots d-1}, A, B, L, M, C) \\ &\qquad\qquad \mapsto (-x_1, x_2, x_3, y_{0\dots d-1}, x_3A, B, L, LM, -C). \end{aligned}$$

The diptych variety V_{ABLM} comes from this pullback on adjoining x_0 by unprojection of the divisor

$$\begin{aligned} D_0 &= \mathbb{A}^6_{\langle x_1,y_0,A,B,M,C\rangle} \\ &= (x_2 = x_3 = y_{1\dots d-1} = L = 0) \subset \mathbb{A}^{d+8}_{\langle x_{1\dots 3},y_{0\dots d-1},A,B,L,M,C\rangle}. \end{aligned}$$

The Pfaffians of (4.6) clearly vanish on D_0, so D_0 is contained in the pullback and we can unproject it to get V_{ABLM}.

For our application, this proves that V_{ABLM} exists (for any $d \ge 2$), and we could stop there. However, this case still has a general point to teach us: namely, how the Pfaffians of (4.6) fit together with the unprojection equations of x_0.

Starting from the bottom, as in (2.3), we have

$$x_1y_0 = AB^{d-1}C^2 + Lx_0 \quad \text{and} \quad x_0y_1 = x_1^{d-2}AB^{d-2}C + My_0^2.$$

(We add a variable C as annotation on x_2, making its tag equation $Cx_2 = x_1x_3$ in V_{AB} and V_{LM}.) It contains the unprojection divisor $D : (x_0 = y_0 = AB^{d-2}C = 0)$, leading to the pentagram x_1, y_0, y_0, y_1, ξ and the 4×4 Pfaffians of

$$\begin{pmatrix} x_1 & BC & -L & -\xi \\ & x_0 & AB^{d-2}C & -My_0 \\ & & y_0 & x_1^{d-2} \\ & & & y_1 \end{pmatrix}. \tag{4.8}$$

The unprojection variable ξ here must be x_3 (rather than x_2 with the tag $e = 1$), as one sees for example from the Pfaffian $\mathrm{Pf}_{12.35} = x_1^{d-1} - x_0\xi + BMCy_0$.

We link the equations together by adding a final $(d+4)$th column to (4.6):

$$\begin{pmatrix} -C & x_1 & B & y_0 & y_1 & \dots & y_{d-2} & x_0 \\ & LM & x_3 & y_1 & y_2 & \dots & y_{d-1} & y_0M \\ & & x_2 & x_3AB^{d-2} & x_3^2AB^{d-3} & \dots & x_3^{d-1}A & AB^{d-1}M \\ & & & x_1^{d-2}L & x_1^{d-3}L^2M & \dots & L^{d-1}M^{d-2} & x_1^{d-1} \\ & & & & & \dots & & \end{pmatrix} \tag{4.9}$$

with the same lower right entries $m_{i+5,j+5}$ as (4.7), and the last column ending in

$$m_{4+i,4+d} = -AC(Bx_1)^{d-1-i} \times \frac{(x_1x_3)^i - (BLM)^i}{x_1x_3 - BLM} \text{ for } i = 1,\dots,d-1.$$

The 4×4 Pfaffians of (4.9) provide all but one of the equations of V_{ABLM}. Comparing (4.8) with (4.9), we see that the equation

$$x_1y_0 = -AB^{d-1}C^2 + x_0L$$

is missing, although M times it is the Pfaffian $\mathrm{Pf}_{12.3(d+4)}$ (in fact its multiples by x_1^{d-2}, x_2, x_3, $y_1,\dots,y_{d-1}$ are also in the ideal of Pfaffians of (4.9)).

The little problem we face is how to cancel the common factor M in the entries $m_{2,3}$, $m_{2,d+4}$ and $m_{3,d+4}$ of (4.9), or in the 3×3 submatrix $\left(\begin{smallmatrix} LM & y_0M \\ & AB^{d-1}M \end{smallmatrix}\right)$ formed by rows and colums $2,3,d+4$, without spoiling the other Pfaffians. We do this by *floating* M from the entries with indices $2,3,d{+}4$ to the complementary entries with $1,4,\dots,d{+}3$, adding the 4×4 Pfaffians of the floated matrix, including the equation for x_1y_0, to those of (4.9).

The full set of equations is a mild form of *crazy Pfaffian*, analogous to Riemenschneider's quasi-determinantal [R]: rather than floating M as a factor in two matrixes, we can view it as a multiplier between entries with indices $2,3,d+4$ and those with $1,4,\dots,d+3$; when evaluating a crazy Pfaffian, we include M as a factor whenever a product crosses between these two regions. Thus the factors M in the triangle $m_{2,3}$, $m_{2,d+4}$ and $m_{3,d+4}$ of (4.9) appear as before in most Pfaffians, but not in $\mathrm{Pf}_{12.3(d+4)}$ or $\mathrm{Pf}_{23.i(d+4)}$ for $i=4,\dots,d+3$.

We discussed a case of floating in [TJ], 9.1, especially around (9.4), but the present instance displays the phenomenon in a particularly clear form. This type of crazy Pfaffians or floating factors occur frequently in our experience of working with Gorenstein rings of codimension ≥ 4, and seem to be a basic device in understanding how one vertebra links to the next. We expect to return to this in future publications.

§5. The cases $de=3$ and parallel unprojection

In 5.1, we construct all remaining cases $de=3$ with $k=4$ or 5 of (4.1) to complete the construction of all diptych varieties with $de\leq 4$. Finally, in 5.2, we observe that each of these can be realised as a regular pullback from a single *key variety*, a 10-fold $W\subset\mathbb{A}^{16}$.

5.1. Small diptychs by pentagrams

When $k=4$, the cases $(d,e)=(1,3)$ or $(3,1)$ are distinct. In each case, we pass to the reduced model, which is isomorphic to the diptych variety we seek but easier to treat because it has lower codimension, and then adjoin the redundant generators using pentagrams.

Case $[3,1,3,1]$. Write x_0,x_1,x_2,x_3,x_4 with V_{AB} tags $[(0),1,3,1,3]$ opposite y_0,y_1,y_2. We work up from the reduced model, that has only x_0,x_4 against y_0,y_1,y_2; we eliminate y_2 from this getting the codimension 2 complete intersection

$$x_0y_1 = AB + My_0 \quad \text{and} \quad x_4y_0 = By_1^2 + Lx_0,$$

and adjoin y_2 by the pentagram x_4,x_0,y_0,y_1,y_2 and its Pfaffian matrix

$$M_1 = \begin{pmatrix} x_4 & y_1^2 & -L & -y_2 \\ & x_0 & B & -M \\ & & y_0 & A \\ & & & y_1 \end{pmatrix} \qquad \begin{aligned} x_0y_2 &= x_4A + My_1^2, \\ y_0y_2 &= y_1^3 + AL, \\ x_4y_1 &= y_2B + LM. \end{aligned}$$

These five Pfaffian equations define the reduced model in codimension 3.

We recover the full set of equations by adjoining the redundant x_2, then x_1 and x_3 in either order. Adjoin x_2 by the pentagram x_0,x_0,y_0,y_1,x_2:

$$M_2 = \begin{pmatrix} x_0 & AB & -M & -x_2 \\ & y_0 & 1 & -y_1B \\ & & y_1 & Lx_0 \\ & & & x_4 \end{pmatrix} \qquad \begin{aligned} x_2 &= x_0x_4 - y_1BM \\ & \text{and} \\ x_2y_0 &= y_1AB^2 + Lx_0^2, \\ x_2y_1 &= x_4AB + LMx_0. \end{aligned}$$

Adjoin x_1 by the pentagram x_0,y_1,x_4,x_2,x_1:

$$M_3 = \begin{pmatrix} x_0 & x_2 & -BM & -x_1 \\ & y_1 & 1 & -AB \\ & & x_4 & LMx_0 \\ & & & x_2 \end{pmatrix} \qquad \begin{aligned} x_1 &= x_0x_2 - AB^2M \\ & \text{and} \\ x_1x_4 &= x_2^2 + x_0BLM^2, \\ x_1y_1 &= x_2AB + LMx_0^2. \end{aligned}$$

Finally adjoin x_3 by the pentagram x_2,x_0,y_1,x_4,x_3:

$$M_4 = \begin{pmatrix} x_2 & x_4AB & -LM & -x_3 \\ & x_0 & 1 & -BM \\ & & y_1 & x_2 \\ & & & x_4 \end{pmatrix} \qquad \begin{aligned} x_3 &= x_2x_4 - BLM^2 \\ & \text{and} \\ x_0x_3 &= x_2^2 + x_4AB^2M, \\ x_3y_1 &= x_4^2AB + LMx_2. \end{aligned}$$

The five Pfaffians of M_1 together with the three equations for x_1,x_2,x_3 define $V_{ABLM} \subset \mathbb{A}^{11}_{\langle x_{0\dots4},y_{0\dots1},A,B,L,M\rangle}$.

Case $[1,3,1,3]$**.** Write x_0,x_1,x_2,x_3,x_4 with V_{AB} tags $[(0),3,1,3,1]$ against y_0,y_1. The reduced model is in codimension 4 on variables x_0,x_1,x_3,x_4,y_0,y_1; eliminating x_4 then x_3 from this leaves two equations

$$x_0y_1 = Ax_1 + y_0^2M \quad \text{and} \quad x_1y_0 = A^2B + Lx_0$$

To recover the reduced model, we adjoin x_3 and then x_4. Adjoin x_3 by the pentagram x_1,x_0,y_0,y_1,x_3:

$$M_1 = \begin{pmatrix} x_1 & AB & -L & -x_3 \\ & x_0 & A & -My_0 \\ & & y_0 & x_1 \\ & & & y_1 \end{pmatrix} \qquad \begin{array}{rcl} x_0x_3 &=& x_1^2 + y_0ABM, \\ x_3y_0 &=& y_1AB + x_1L, \\ x_1y_1 &=& x_3A + LMy_0. \end{array}$$

The unprojection divisor of x_4 is $(x_0 = x_1 = y_0 = A)$, so that the reduced model exists. We adjoin x_4 by the pentagram x_3,x_1,y_0,y_1,x_4:

$$M_2 = \begin{pmatrix} x_3 & y_1B & -L & -x_4 \\ & x_1 & A & -LM \\ & & y_0 & x_3 \\ & & & y_1 \end{pmatrix} \qquad \begin{array}{rcl} x_1x_4 &=& x_3^2 + y_1BLM, \\ x_3y_1 &=& x_4A + L^2M, \\ x_4y_0 &=& y_1^2B + x_3L. \end{array}$$

These 8 equations define the reduced model in codimension 4 together with a residual $\mathbb{A}^4_{\langle x_0,x_4,B,M\rangle}$. Calculating with syzygies or saturating against y_0 (say) recovers the *long equation*

$$x_0x_4 = x_1x_3 + y_0y_1BM + ABLM.$$

In terms of the Tom and Jerry unprojections of [TJ], the calculation to this point is a standard double Jerry; see [TJ] Section 9.2 which gives a closed form statement of the result, apart from the long equation.

Finally, we adjoin the redundant generator x_2 by the pentagram x_1,y_0,y_1,x_3,x_2:

$$M_3 = \begin{pmatrix} x_1 & x_3A & -LM & -x_2 \\ & y_0 & 1 & -AB \\ & & y_1 & x_1L \\ & & & x_3 \end{pmatrix} \qquad \begin{array}{rcl} x_1x_3 &=& x_2 + ABLM, \\ x_2y_0 &=& x_3A^2B + x_1^2L, \\ x_2y_1 &=& x_3^2A + x_1L^2M. \end{array}$$

Thus the diptych in this case is the graph of $x_2 = x_1x_3 - ABLM$ over its reduced model, in codimension 5 with 10×25 resolution.

Remark 5.1. Since x_2 has tag 1, it makes sense to give him annotation C; in the pentagram equations above, this can be done simply by replacing the 1 in M_3 by C. Computer algebra experiments (after saturating these pentagram equations against y_0LM) show that this gives a 7-fold V_{ABCLM} in codimension 5 with 14×35 resolution and serial unprojection form. (The webpage [Dip] has files to download and run in Magma [Ma] to run this calculation and other experiments.)

Case $[1,3,1,3,1]$**.** When $k = 5$, we consider tags $[1,3,1,3,1]$ on V_{AB}; this also covers the case $[3,1,3,1,3]$ by top-to-bottom reflection. Write $x_0, x_1, x_2, x_3, x_4, x_5, y_0, y_1$ with V_{AB} tags $[0,1,3,1,3,1]$. The reduced model has only x_0, x_5 against y_0, y_1, with two equations

$$x_0y_1 = A + y_0M \quad \text{and} \quad x_5y_0 = y_1^3B + L.$$

The diptych variety is isomorphic to $\mathbb{A}^6_{\langle x_0,x_5,y_0,y_1,B,M\rangle}$, and is the graph over it of A, L, x_4, x_2, x_1, x_3 expressed as functions by

$$\begin{array}{rclcrcl} & & & \qquad & x_1 & = & x_0x_2 - A^2BM, \\ A & = & x_0y_1 - y_0M, & & x_2 & = & x_0x_4 - y_1ABM, \\ L & = & x_5y_0 - y_1^3B, & & x_3 & = & x_2x_4 - ABLM^2, \\ & & & & x_4 & = & x_0x_5 - y_1^2BM. \end{array}$$

It is a fun exercise to compute all of this with magic pentagrams as in previous cases.

5.2. A key variety by parallel unprojection

There is a uniform treatment of the cases $k = 4$ and 5 and $de = 3$ as regular pullbacks of a key 10-fold W that is given by a parallel unprojection construction similar to that of Papadakis and Neves [PN]. We start from the codimension 2 complete intersection $W_0 \subset \mathbb{A}^{12}_{\langle u_{1\dots4},s_{1\dots4},a_{1\dots4}\rangle}$ given by

$$u_1u_3 = a_2s_1s_2u_2 + a_4s_3s_4u_4,$$
$$u_2u_4 = a_1s_1s_4u_1 + a_3s_2s_3u_3,$$

which is a normal 10-fold containing as divisors the four codimension 3 complete intersections

$$(s_1,u_3,u_4), \quad (s_2,u_4,u_1), \quad (s_3,u_1,u_2), \quad (s_4,u_2,u_3).$$

Parallel unprojection of these four divisors gives a codimension 6 Gorenstein subvariety $W \subset \mathbb{A}^{16}_{\langle u_{1\dots4},v_{1\dots4},s_{1\dots4},a_{1\dots4}\rangle}$ with a 20×66 resolution, by standard application of the Kustin–Miller unprojection theorem. The full set of equations is obtained as follows. Each individual unprojection variable v_i is adjoined by a pentagram, giving three linear unprojection equations such as

$$(5.1)\quad \begin{pmatrix} u_2 & a_1s_4u_1 & -a_3s_2s_3 & -v_1 \\ & u_3 & s_1 & -a_4s_3s_4 \\ & & u_4 & a_2s_2u_2 \\ & & & u_1 \end{pmatrix} \qquad \begin{aligned} s_1v_1 &= u_1u_2 - a_3a_4s_2s_3^2s_4, \\ u_4v_1 &= a_1s_4u_1^2 + a_2a_3s_2^2s_3u_2, \\ u_3v_1 &= a_1a_4s_3s_4^2u_1 + a_2s_2u_2^2. \end{aligned}$$

In addition, there are 6 bilinear equations for v_iv_j, making $2+4\times3+6 = 20$ equations. Four of these also come from pentagrams, the first of which gives

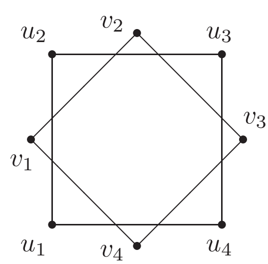
$$(5.2)\qquad v_1v_2 = a_2u_2^3 + a_1a_3a_4^2s_3^3s_4^3,$$

whereas the remaining two are "long equations"

$$\begin{aligned} v_1v_3 &= a_1a_4s_4^3v_4 + a_2a_3s_2^3v_2 + 3a_1a_2a_3a_4s_1s_2^2s_3s_4^2, \\ v_2v_4 &= a_1a_2s_1^3v_1 + a_3a_4s_3^3v_3 + 3a_1a_2a_3a_4s_1^2s_2s_3^2s_4 \end{aligned}$$

that can be computed using syzygies.

The construction has 4-fold cyclic symmetry (1234), apparent in the picture



We view the v_i as tagged by 1 and annotated by s_i (by the first equation of (5.1)), and the u_i as tagged by 3 and annotated by a_i (by (5.2)). We get Gorenstein projections on eliminating any subset of the v_i, but we can only eliminate u_i after projecting out the neighbouring v_{i-1} and v_i.

We use this variety as a model for diptych varieties. The diptychs with $de = 3$ and $k = 4, 5$ of 5 arise by pullback from W on making the following substitutions:

$$\text{Case } [3,1,3,1]: \quad \begin{array}{llll} v_1 = x_1 & u_1 = x_0 & a_1 = L & s_1 = 1 \\ v_2 = x_3 & u_2 = x_2 & a_2 = 1 & s_2 = 1 \\ v_3 = y_2 & u_3 = x_4 & a_3 = A & s_3 = B \\ v_4 = y_0 & u_4 = y_1 & a_4 = 1 & s_4 = M \end{array}$$

$$\text{Case } [1,3,1,3]: \quad \begin{array}{llll} v_1 = x_2 & u_1 = x_1 & a_1 = 1 & s_1 = 1 \\ v_2 = x_4 & u_2 = x_3 & a_2 = 1 & s_2 = A \\ v_3 = z & u_3 = y_1 & a_3 = B & s_3 = 1 \\ v_4 = x_0 & u_4 = y_0 & a_4 = M & s_4 = L \end{array}$$

$$\text{Case } [3,1,3,1,3]: \quad \begin{array}{llll} v_1 = x_1 & u_1 = x_0 & a_1 = L & s_1 = 1 \\ v_2 = x_3 & u_2 = x_2 & a_2 = 1 & s_2 = 1 \\ v_3 = x_5 & u_3 = x_4 & a_3 = 1 & s_3 = A \\ v_4 = y_0 & u_4 = y_1 & a_4 = B & s_4 = M \end{array}$$

where, in the second case, $z = y_0y_1 - AL$ is a redundant generator.

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