

Moduli of sheaves

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Dedicated to Shigeru Mukai on the occasion of his 60th birthday

Abstract.

We survey classical and recent developments in the theory of moduli spaces of sheaves on projective varieties.

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§1. Introduction

The study of moduli is one of the fundamental aims of algebraic geometry. There are two basic kinds of moduli problems: absolute and relative ones. An *absolute* moduli problem speaks of parametrizing objects such as varieties themselves, standing alone and not in relation to any fixed choices. The moduli space of curves M_g is a basic example. On the other hand, a *relative* moduli problem starts with some initial choice and then speaks of parametrizing objects relative to that choice. The moduli of vector bundles is the prime example: we fix a variety X and look at the moduli space $M_X(\xi)$ of vector bundles over X with given topological invariants indicated by the *Mukai vector* ξ . Absolute moduli problems have the advantage of describing primary spaces which don't depend on anything else and play a fundamental role. However, there are not so many of them, and after M_g the program of understanding the moduli spaces of higher-dimensional varieties is arduous. Relative moduli problems, on the other hand, can give us a wide range of new spaces to study, on which a lot can be said with a multitude of available techniques. It is in this way that the study of the moduli of vector bundles, and later more generally sheaves, has been one of the main strands of research in algebraic geometry over the past half-century.

The basic constructions of moduli spaces of vector bundles were established beginning with the works of Weil, then Mumford, Narasimhan, Seshadri, Ramanan, and Tyurin for curves, and continuing with the work of Maruyama and Gieseker for higher-dimensional varieties. Starting from this period, there were on the one hand many investigations into the algebraic geometry of these moduli spaces, and on the other hand, the famous interaction with Yang-Mills theory through Donaldson's polynomial invariants of smooth manifolds. As a result, throughout the 1980's and 1990's, an enormous amount of progress has been made.

Our goal in the present paper is to give a survey, to discuss even if only very briefly some of the many themes of past and current research, and to formulate when possible open directions for future research.

It is a great pleasure to dedicate this paper to Shigeru Mukai. In surveying the multifold research on the moduli of vector bundles and sheaves on projective varieties, we touch upon a vast realm of progress in many different directions inspired by Mukai's fundamental contributions. His work on the Fourier-Mukai transform, on moduli of sheaves in particular over symplectic varieties, his point of view encompassing the globality of moduli spaces indexed by what is now known as the "Mukai vector", his studies relating geometry of objects over surfaces and threefolds with their position in the Kodaira classification, and many other works have inspired generations of young mathematicians. It was very impressive to do a Google Scholar search on *Mukai "stable bundles"*, which led to pages and pages of references, all highly relevant and really interesting papers. The widely different subjects we survey here all share Mukai's work as inspiration. This great diversity reflects Mukai's playful, but also forward-looking mathematical style.

We give a general overview of many areas which have been the subject of a fantastic amount of research, including rationality of moduli spaces, strange duality, jumping curves, Betti numbers, existence of universal families, as well as a discussion of the many moduli problems which are closely related to moduli of sheaves. For our discussion of wall-crossing, which is necessarily just as incomplete as the other parts, we provide a small illustration of a recent new subject stemming from work of Kontsevich and Soibelman, relating wall-crossing for Donaldson-Thomas invariants of moduli problems for sheaves, with wall-crossing in the Hitchin system. Our illustration is an explicit calculation showing what kinds of walls this theory is talking about on the base of an extremely simple Hitchin system. This is related to the second author's talk for the present conference.

The size of this subject means that many important directions will be left out. These include the relationship with gauge theory and Donaldson polynomials; Reidinger's new Hodge-type structures; all sorts of questions about moduli problems in positive characteristic; moduli of Higgs bundles, connections and the like; and only a brief mention will be made of things like moduli of principal bundles, parabolic structures, moduli of complexes and perverse coherent sheaves, twisted sheaves, . . .

Our survey is based in part on the first author's talk at the Kobe workshop on the geometry of moduli spaces in 2009. As well as giving an extended survey, we would also like to concentrate on some particular

aspects surrounding our subsequent work on moduli of vector bundles on quintic hypersurfaces, as was the subject of the workshop lectures by the second author in the first week of the present conference.

In the course of our work about hypersurfaces, we met at least two distinct ways—classically well-known—in which the theory of linear systems on curves leads to constructions of vector bundles on surfaces. These are two techniques which enter in an essential way: Serre’s construction using the Cayley-Bacharach property, and O’Grady’s method. We will look at them in some detail. They provide important links between the study of moduli of vector bundles on surfaces, and the study of Brill-Noether theory for curves in \mathbb{P}^3 which has been one of Mukai’s favorite topics.

To introduce the Cayley-Bacharach condition, we discuss Reider’s theorem giving precise bounds for linear series by applying the Bogomolov-Gieseker inequality. To illustrate O’Grady’s method, discussed at first in the middle of the paper, we give at the end a proof of Nijssse’s connectedness theorem, that the moduli spaces of bundles with $c_2 \geq 10$ on a very general quintic surface are connected. This is used in our recent proof of irreducibility, so it seemed useful to give an exposition of the proof. These techniques will be emphasized, because we feel that they should lead to fascinating topics for future research relating Mukai’s beautiful works on linear systems on curves [211], [210], . . . , and many directions in the study of moduli of sheaves on higher dimensional varieties.

The study of moduli spaces of vector bundles has evoked the interest of many mathematicians studying many aspects in great depth. While our knowledge has gone very far in some directions, there are other directions in which many questions remain open. These include the geography of moduli spaces for small values of \tilde{c}_2 , the existence of universal families, and the generalization of many results from rank $r = 2$ to arbitrary rank or other structure groups. And, even in the subjects which have been most extensively covered, there is much room for further progress. We hope that our indications will inspire the reader to dig more deeply into some of these subjects.

Unless otherwise stated, our discussion will take place over the field \mathbb{C} of complex numbers.

§2. Moduli of sheaves

The study of moduli of vector bundles on a curve got started with the work of André Weil, parametrizing vector bundles using adèles over the curve. This inspired the first wave of geometrical studies of the

question, with Narasimhan and Seshadri [221] relating stable bundles to unitary representations, and the constructions and study of the moduli spaces by Mumford [215], Tyurin [275], Seshadri [259], Ramanan [245] and many others since then.

Let X be a smooth projective variety of dimension n , and let $H \in H^2(X, \mathbb{Q})$ denote the first Chern class of an ample line bundle. If V is a vector bundle over X , its *degree* is by definition

$$\deg(V) := \int_X c_1(V) \cdot H^{n-1}$$

the integral sign meaning evaluation on the fundamental class $[X]$. The *slope* is the quotient of the degree by the rank

$$\mu(V) := \frac{\deg(V)}{\operatorname{rk}(V)}.$$

This definition extends to coherent sheaves of positive rank, and we say that V is *Mumford-Takemoto*, or “*slope*” *stable* if for any subsheaf $\mathcal{F} \subset V$ with $0 < \operatorname{rk}(\mathcal{F}) < \operatorname{rk}(V)$ we have

$$\mu(\mathcal{F}) < \mu(V).$$

We say that V is *Mumford-Takemoto*, or “*slope*” *semistable* if for any nontrivial subsheaf $\mathcal{F} \subset V$ we have

$$\mu(\mathcal{F}) \leq \mu(V).$$

These definitions extend immediately to the case where V is itself a torsion-free sheaf. The first main indication that stability is a useful notion comes from the strictness properties which follow.

Lemma 2.1. *The category of slope-semistable torsion-free sheaves of a given slope μ_0 , is abelian. A slope-stable torsion-free sheaf is simple, that is to say its endomorphism algebra is reduced to the scalars. A semistable sheaf of slope μ_0 admits a Jordan-Hölder filtration in the category, such that the subquotients are polystable i.e. direct sums of stable sheaves of slope μ_0 . This filtration may be chosen canonically by induction, using the socle, the maximal polystable subsheaf, in each successive quotient.*

Other well-known instances of strictness in algebraic geometry are closely related. For example, Penacchio applies Klyachko’s description [144] to say that we can interpret mixed Hodge structures as torus-equivariant semistable sheaves on \mathbb{P}^2 . Deligne’s strictness properties for

mixed Hodge structures become immediate consequences of the previous lemma [238].

The *Harder-Narasimhan filtration* of any torsion-free sheaf V is a canonical decreasing filtration by saturated subsheaves (i.e. subsheaves such that the quotient is also torsion-free) $V_{\geq a} \subset V$ such that

$$\mathrm{Gr}_a^{\mathrm{HN}}(V) := V_{\geq a}/V_{\geq a+\epsilon} \text{ is semistable of slope } a.$$

This measures the degree of instability: V is semistable of slope μ if and only if $V_{\geq \mu} = V$ and $V_{\geq \mu+\epsilon} = 0$. The slope of the highest nonzero piece in the associated-graded, is the maximum slope of any subsheaf of V .

Bridgeland has introduced a vast generalization of this circle of ideas. He observes that one might consider an analogous situation within any triangulated category. His definition will be recalled in more detail in Section 13 on wallcrossing.

The basic idea of the construction of the moduli space of vector bundles is that the stability condition is related to Geometric Invariant Theory stability of points in a natural parameter scheme under the group action which comes from the choices that needed to be made. This works perfectly well if X is a curve, yielding the moduli space $M_X(r, d)$ of vector bundles of rank r and degree d (hence slope $\mu = d/r$).

For higher-dimensional varieties, the construction most naturally incorporates a refined notion of stability. This is because the natural quantities which occur in the GIT stability of parameter points are the *Hilbert polynomials* of subsheaves. This is the motivation for the introduction of *Gieseker (semi)stability*. Let $P_H(\mathcal{F}, n) := \chi(\mathcal{F}(nH))$ denote the Hilbert polynomial of a sheaf with respect to the hyperplane class H . Then we say that a torsion-free sheaf V is *Gieseker semistable* if, for any nonzero subsheaf $\mathcal{F} \subset V$ we have

$$\frac{P_H(\mathcal{F}, n)}{\mathrm{rk}(\mathcal{F})} \leq \frac{P_H(V, n)}{\mathrm{rk}(V)} \text{ for } n \gg 0.$$

Say that V is *Gieseker stable* if a strict inequality holds for any nonzero subsheaf $\mathcal{F} \subset V$ different from V . Notice here that it is safe to include even subsheaves of the same rank as V ; in that case, as long as the subsheaf isn't V itself then its Hilbert polynomial will be smaller.

Recall that the rank gives the first term of P_H , and the degree gives the next term. It follows that the Gieseker notions are related to slope (semi)stability by the implications

$$\begin{aligned} \text{slope stable} &\Rightarrow \text{Gieseker stable} \\ \Rightarrow \text{Gieseker semistable} &\Rightarrow \text{slope semistable} . \end{aligned}$$

After the construction of moduli spaces of vector bundles over curves by Mumford [215], Tyurin [275] and Seshadri [259], the next main step was to construct the moduli spaces of vector bundles over surfaces and then higher-dimensional varieties. This was accomplished by Gieseker [84] and Maruyama [186, 188, 189]. Maruyama had first constructed the moduli space of bundles of rank $r = 2$ on a surface, but was missing a key Lemma [186, 4.1] for the case of $r > 2$. This was provided by Gieseker; with that addition Maruyama went on very quickly to construct the moduli space of stable sheaves of any rank over varieties of any dimension, and obtained many basic properties.

The next major development was the work of Donaldson, Uhlenbeck and Yau on Yang-Mills equations, gauge theory, and Donaldson's polynomial invariants. The scope of this theory extended well beyond holomorphic vector bundles, however the holomorphic case was a fundamental part of the theory. The gauge-theoretical motivations inspired people working on the algebraic geometrical side of things, and led to a lot of progress which we will touch upon briefly in the "geography" section below.

A very useful reference, combining a discussion of the moduli theory with a discussion of the gauge-theoretical aspects, is the book [160] about Atiyah-Bott theory for the moduli space of vector bundles on a curve.

Further developments included the beginnings of the moduli theory in characteristic p , with Maruyama's boundedness paper [190]. This problem has now been solved extensively by Langer [154, 156, 157], Gómez, Langer, Schmitt, Sols [89], and Zuo's group [153, 178].

Mukai considered the family of moduli spaces of vector bundles with different topological types, as a combinatorial structure enclosing interesting information in the way they vary as a function of the Chern invariants. We introduce what is now known as the *Mukai vector* $\xi = (r, c_1, c_2, \dots)$, and let $\mathcal{M}_X(\xi)$ denote the moduli stack of sheaves with Mukai vector ξ . The exact meaning of this notation depends on whether we think of the Chern classes as elements of a cohomology group or a Chow group. For example, in the case of surfaces, it is usual to let c_1 denote the determinant line bundle of V which shall be fixed. Mukai also introduced a twisting by the square root of the Todd class, making the Riemann-Roch formula more transparent, but we will not need that for our present level of discussion so we mostly keep to the notation (r, c_1, c_2, \dots) .

For Gieseker stable sheaves, the moduli problem becomes *separated*: there cannot be two distinct stable sheaves which are limits of the same family. For semistable sheaves the moduli problem is non-separated, but in a somewhat mild way. Given two different semistable sheaves which

are limits of the same family, then they must lie in the same *S-equivalence class*, meaning that the associated-graded sheaves of their Jordan-Hölder filtrations are isomorphic. From this non-separatedness, it follows that for any map from the moduli functor to a separated scheme, all sheaves in the same S-equivalence class must map to the same point.

The geometric invariant theory construction gives a moduli space $\overline{M}_{X,H}(\xi)$ whose closed points parametrize *S-equivalence classes* of Gieseker semistable (with respect to the hyperplane class H) torsion-free sheaves with Mukai vector ξ . It contains the open set

$$M_{X,H}(\xi) \subset \overline{M}_{X,H}(\xi)$$

corresponding to locally free sheaves i.e. vector bundles.

Partly because of the S-equivalence relation, but also because of more subtle questions about global obstructions to the existence of universal families, the moduli space $\overline{M}_{X,H}(\xi)$ is not *fine*, it doesn't represent a functor. It is, however, uniquely characterized by the property that it *universally co-represents* the moduli functor. Co-representing means that any map $\mathcal{M}_{X,H}(\xi) \rightarrow Z$ to a separated scheme of finite type factors through a unique map $\overline{M}_{X,H}(\xi) \rightarrow Z$. Universally co-representing is a technically useful strengthening of this condition by saying that it also applies to any pullback along $Y \rightarrow \overline{M}_{X,H}(\xi)$.

The classical approach to moduli problems of this kind concentrated on vector bundles or sheaves which are closely related, such as torsion-free or reflexive sheaves. In these cases, the sheaf is a global object over the whole variety X , in particular its rank is positive.

Mukai was the first to consider the moduli of sheaves supported on strict subvarieties, in his 1984 paper [209]. He was motivated by the idea of transferring the symplectic structure from the underlying variety such as a K3 surface, to the moduli space of sheaves. Mukai constructed the moduli space of simple sheaves using stack-theoretic ideas. This fit in very naturally with his consideration of the whole family of moduli spaces parametrized by their Mukai vectors, since there was not a natural numerical reason to avoid sheaves of rank 0.

The moduli space of *Higgs bundles*, first introduced by Hitchin with an analytic construction for rank 2 and odd degree [110], and constructed in general for the case of curves by Nitsure [227], may be viewed by the BNR correspondence [20] as a moduli space of sheaves supported on proper subvarieties in the cotangent bundle of X . This was the motivation for the introduction of a notion of Gieseker semistability and the GIT construction of the moduli spaces of sheaves of arbitrary pure dimension in [261] and [123].

The moduli of sheaves supported in positive codimension entered into Le Potier's theory of *coherent systems* and his consideration of jumping curves [162, 163]. Le Potier investigated extensively many of the general questions we shall formulate here, for the case of pure dimension 1 sheaves on \mathbb{P}^2 in [163]. He showed local factoriality, irreducibility, determined the Picard group, and showed under appropriate hypotheses rationality and existence of a universal family. In recent times, a closer look at the detailed geometric structure of these moduli spaces continues with Maican's work [180, 181], his work with Drézet [68], Iena [127], Yuan [285] and others. Freiermuth and Trautmann [78, 79] considered dimension 1 sheaves on \mathbb{P}^3 , prefiguring current interest in them over CY threefolds.

§3. The Bogomolov-Gieseker inequality

The *Bogomolov-Gieseker inequality* is one of the main facts determining the behavior of the moduli spaces of stable bundles on a surface. Basically, it bounds c_2 to be positive. It is convenient to introduce the quantity (defined in the rational Chow ring)

$$\Delta(E) := c_2(E \otimes c_1(E)^{\otimes -\frac{1}{r}}) = c_2(E) - \frac{r-1}{2r} c_1(E)^2$$

which is independent of tensoring by a line bundle.

Theorem 3.1 (Bogomolov, Gieseker, Donaldson, Uhlenbeck, Yau). *Suppose E is an H -stable vector bundle on a projective surface X , then $\Delta(E) \geq 0$. If equality holds, then E is a unitary projectively flat bundle.*

Donaldson and Uhlenbeck-Yau give proofs using Yang-Mills theory, which is how one gets the last part of the statement. This combines with the *restriction theorem* of Mehta-Ramanathan:

Theorem 3.2 (Mehta-Ramanathan). *Suppose E is an H -stable reflexive sheaf on a smooth projective variety X . Then for appropriately chosen complete intersections $Y \subset X$ of dimension ≥ 1 , obtained by intersecting divisors in multiples of the hyperplane class H , $E|_Y$ is again stable.*

We get the Bogomolov-Gieseker inequality on varieties of any dimension.

Corollary 3.3. *Suppose E is an H -semistable torsion-free sheaf on a smooth projective variety X of dimension n . Then*

$$\Delta(E).H^{n-2} \geq 0.$$

If equality holds then E is a projectively flat bundle coming from an extension of unitary projectively flat bundles.

Langer has given several far-reaching treatments of these statements in characteristic p starting with [154], and he also improves the bounds for the restriction theorem. See also the discussion in [157] and [123] of various other restrictions such as the Grauert-Mülich restriction theorem and its generalizations, and Flenner's theorem.

In his recent paper [158], Langer proves the Bogomolov-Gieseker inequality in characteristic p assuming that there exists a lifting to the Witt vectors W_2 of order two. This provides a sharp delimitation of counterexamples such as [213]. Along the way, he has also solved a longstanding problem posed by Narasimhan, how to give an algebraic proof of the Bogomolov-Gieseker inequality for Higgs bundles, and again it includes varieties in characteristic p with a lifting to W_2 .

The Bogomolov-Gieseker inequality has a fundamental consequence for the study of moduli spaces: the only range where $M_H(X; r, c_1, c_2)$ can possibly be nonempty is when $\Delta \geq 0$. And, for $\Delta = 0$ we get the moduli space of projectively flat unitary bundles on X which tends to be small—for example it contains only the trivial bundle if X is simply connected.

This situation suggests that the behavior of the moduli theory will strongly depend on Δ , which is to say on c_2 once c_1 is fixed, and that turns out to be the case at least for bundles on surfaces.

For $\Delta \gg 0$, the moduli space is irreducible, and is a local complete intersection of the expected dimension, with a good control on the high codimension of the singular locus.

On the other hand, for c_2 in the “intermediate range”, with $\Delta \geq 0$ but not very big with respect to the numerical invariants of X , very little is known. This will be the subject of further discussion later in the paper.

As we shall see starting in Section 7, the Serre construction allows us to present vector bundles on a smooth surface, particularly in rank 2, by means of zero-dimensional subschemes satisfying the “Cayley-Bacharach” condition—see there for the definition. The Bogomolov-Gieseker inequality gives an inequality on the length of the subscheme, but requires some hypotheses corresponding to stability and to the situation of our subscheme in a surface.

Question 3.4. *Can one obtain inequalities of Bogomolov-Gieseker type (cf Corollary 7.5 below), just from algebraic considerations of postulation for a zero-dimensional subscheme $Z \subset \mathbb{P}^3$ satisfying the Cayley-Bacharach condition?*

One can ask, in fact, for a small improvement of the inequality. This kind of question will come into play in our discussion surrounding Question 17.2 later on.

For bundles on varieties X of dimension ≥ 3 , the situation is more complicated. On the one hand, we would expect to have something similar to the Bogomolov-Gieseker inequality for the higher Chern classes c_3, \dots , but almost nothing is known here—see however Bayer, Macrì and Toda [19]. Furthermore, there is a nontrivial interaction with the *Hartshorne conjecture* saying that there should be very few smooth subvarieties of a higher dimensional variety, of small codimension. This constrains the existence of vector bundles of rank $\text{rk}(E) \leq \dim(X)/3$, indeed given such a bundle we could take a section of $E(n)$ for some $n \gg 0$, and that would be a smooth subvariety of small codimension. The problems seem to occur earlier, already for bundles of rank $\text{rk}(E) < \dim(X)$. In the remainder of the paper we shall mostly concentrate on bundles over surfaces.

§4. Symplectic structures

Classical deformation theory says that the tangent space to the moduli stack of bundles at a point E , is given by

$$T_E \mathcal{M} = \text{Ext}^1(E, E) = H^1(E^* \otimes E).$$

Mukai's paper on symplectic structures [209] starts from the fundamental observation that if X is a surface, then Serre duality says

$$H^1(E^* \otimes E)^* = H^1(E \otimes E^* \otimes K_X).$$

Thus, if $K_X \cong \mathcal{O}_X$ is trivial, which is the case when X is abelian or a K3, then the tangent space is autodual. Furthermore, it turns out that the resulting form

$$T\mathcal{M} \otimes T\mathcal{M} \rightarrow \mathcal{O}_X$$

is antisymmetric and closed, so it gives a symplectic structure.

Mukai furthermore observed that this may be extended quite naturally to the moduli stack of *sheaves*, indeed the *Ext*-version of Serre duality says

$$\text{Ext}^1(E, E)^* \cong \text{Ext}^1(E, E \otimes K_X)$$

and we still get a symplectic form. With this motivation, and also following [2], Mukai introduced the *moduli space of simple sheaves* [209]. A sheaf is *simple* if $\text{Hom}(E, E) = \mathbb{C}$, and the moduli stack of simple sheaves is actually an Artin algebraic space. This was the first time that

anyone had considered a moduli problem for sheaves supported on strict subvarieties.

An additional benefit of the symplectic situation is that the obstructions vanish. Indeed, the obstruction classes for deformations of E lie in the trace-free part $Ext_0^2(E, E)$, and by Serre duality this is the same as $Hom_0(E, E \otimes K_X)$. When $K_X = \mathcal{O}_X$, we just get $Hom_0(E, E)$. If E is simple, there are no trace-free endomorphisms, so the obstructions always vanish and the moduli space is smooth.

As mentioned before, one may draw a connection between Mukai's moduli spaces of sheaves on symplectic surfaces [209], and the moduli spaces of Higgs bundles on a curve introduced in [109, 110] [227]. The cotangent bundle of a curve is a symplectic surface, which however is noncompact. Sheaves of pure dimension 1 over the cotangent bundle, having compact support, are the same as Higgs bundles by the BNR correspondence [20] and comparison of stability conditions allows one to use this to construct the moduli space [261]. Mukai's calculation of the symplectic structure is purely local, so it holds equally well on an open surface, yielding the symplectic structure on the moduli space of Higgs bundles.

Because of the algebraic symplectic structure, the moduli spaces of sheaves on abelian and K3 surfaces have a very rich geometry, which has subsequently been studied by many authors.

One of the main features is the dependence of the moduli space on its Mukai vector. This phenomenon enters also into the notion of "strange duality" which we shall talk about later.

Several people including Yoshioka [287, 289, 291] and Gomez [88] prove irreducibility results for the moduli space of stable torsion-free sheaves on an abelian or K3 surface. For the most straightforward case, they assume that the Mukai vector is *irreducible*, that is to say it isn't a multiple. This includes, for example, the case of bundles whose rank and degree are coprime. Then all semistable sheaves are stable, in particular simple, and the moduli space is smooth and symplectic.

In cases where there can be non-simple sheaves, these correspond in general to singularities in the moduli space. Still, following Mukai's basic intuition, the singularities may be resolved in a natural way, as was shown by O'Grady in [233], using techniques of Kirwan [142]. Kaledin, Lehn and Sorger [133, 165] classify the singularities which can arise and show that O'Grady's resolution is symplectic. Yoshioka applied the Fourier-Mukai transform to prove irreducibility in [292]. See also Choy and Kiem [46, 47]. The study of singularities comes up again for Enriques surfaces in Yamada's work [283].

Tyurin [276, 277, 278] was also a pioneer in the study of the symplectic structures on moduli of vector bundles on surfaces. He provided a direct correspondence between a two form or pre-Poisson structure on the surface, and the same structure on the moduli space. Inspired by Beauville, Bottacin later took up this direction, showing that the Jacobi identity for a Poisson structure on the surface transfers to the corresponding identity on the moduli space [26]. At the same time he gets a direct proof for the closedness property of the two-form on the moduli space, in the symplectic case.

This subject has recently been generalized to shifted symplectic structures on higher derived stacks by Pantev, Toën, Vaquié and Vezzosi [236]. They provide an explanation coming from the world of derived stacks, for the phenomenon of transfer of symplectic (or in their more recent work Poisson) structures from the underlying variety to the moduli space.

4.1. Enriques surfaces

We have seen that the moduli spaces of vector bundles on a K3 surface have a rich structure as was foreseen in Mukai's seminal works. There is another class of surfaces very close by in Kodaira's classification: the Enriques surfaces. These are quotients of K3 surfaces by fixed-point free involutions. In particular, the moduli of vector bundles on the Enriques surface is approximately the same thing as the fixed point set of the involution on the moduli of bundles over the K3. A natural question is, what becomes of the symplectic structure? It turns out that the moduli of bundles over the Enriques surface is a *lagrangian subvariety* of the symplectic moduli space for the K3. This has been studied by a number of people; we would like to thank I. Dolgachev and G. Saccà for some useful discussions on this topic. The first author to treat this question was H. Kim, starting with his 1994 paper [137] then continuing with [138] and [139] where he set out the basic parameters of the theory stemming in large part from Mukai's work on moduli over K3's. Bundles over Enriques surfaces have been further studied by several people since then, including Hauzer [101], Yoshioka [292], Zowislok [296], Saccà [253, 254], and Yamada [283].

A lot is known about the moduli of the Enriques surfaces themselves, rationality for example [147]. The interaction between the moduli of Enriques surfaces and the moduli of vector bundles over each one of them, could provide a good source of problems. For example, the question of the existence of a Poincaré bundle or universal family over the total family of moduli spaces would be quite natural to consider. Drezet's techniques [66] should be useful.

§5. Geography

It would go way beyond our present scope to talk in depth about Donaldson's polynomial invariants. However, beyond the realm of calculations proper to that theory, the consideration of these invariants has led to some important abstract advances in our understanding of what the moduli spaces look like.

The Donaldson invariants were motivated originally by gauge theory, but nearly simultaneously Donaldson proved that stable bundles lead to solutions of the Hermitian Yang-Mills equations on a surface, and he and Uhlenbeck-Yau proved the same thing for higher dimensional varieties.

This almost makes the moduli space of bundles into a space on which the polynomial invariants can be calculated. However, there is a genericity assumption on the base metric, which in more modern terms means that we would like to use the virtual fundamental class of the moduli space. The possibility of using the moduli spaces of holomorphic bundles would allow one to do calculations using algebraic geometry. It therefore became a crucial question to understand when the fundamental class of the moduli space was the same as its virtual fundamental class.

The Bogomolov-Gieseker inequality now comes in: for small values of c_2 , the moduli space would be empty. For somewhat larger values, it parametrizes first flat bundles, then bundles with very small numbers of instantons, and the equations in this case can be highly overdetermined.

In terms of gauge theory, when the instanton number gets big, then the instantons themselves can tend to look more and more like things concentrated in small zones of the surface. This gives an intuitive picture in which it was expected that for high values of c_2 , the moduli space should become equivalent to its generic deformation. This would mean several things: first of all, that it should be generically smooth of dimension equal to the expected dimension which is calculated by using the Euler characteristic (appropriately taking into account the fact that the trace of an obstruction is automatically zero). Second, that the codimension of the singular locus becomes high; and third, that the moduli space becomes irreducible.

These things have now all been proven, by the work of several people including Friedman, Gieseker, Li, and specially O'Grady who developed a general method capable of giving the expected results in full generality. Friedman's book [76] gives an approach to the geography of these moduli spaces, informed by the gauge-theoretical and differential-geometric considerations originating from Donaldson's theory, but also representing the algebraic geometer's point of view. One can pose many questions

about the moduli spaces, and ask how they depend on the numerical invariants at hand.

One of the first aspects is the deformation theory and dimension of the moduli space. Suppose $Z = \overline{M}_{X,H}(\xi)_i$ is an irreducible component of the moduli space of H -Gieseker semistable torsion-free sheaves on a surface X , with Mukai vector ξ . We may distinguish two cases depending on how c_1 is designated within ξ : it is either

- (a) the rational Chern class $c_1 \in H^2(X, \mathbb{Q})$, or
- (b) the isomorphism class of $\det(E) \in \text{Pic}(X)$. These are different if X is irregular. For surfaces of irregularity zero (for example, simply connected ones), cases (a) and (b) are identical.

For $E \in Z$, we have a decomposition

$$\text{Ext}^i(E, E) = H^i(\mathcal{O}_X) \oplus \text{Ext}_0^i(E, E)$$

with the projection $\text{Ext}^i(E, E) \rightarrow H^i(\mathcal{O}_X)$ given by the trace. The *obstruction* to deforming E always takes values in the “trace-free” part which we denote by

$$\text{Obs}(E) := \text{Ext}_0^2(E, E) \cong \text{Ext}_0^0(E, E \otimes K_X)^*.$$

Assuming that E is simple (for example if it is stable), then we have $\text{Ext}_0^0(E, E) = 0$. The space of infinitesimal deformations of E is either $\text{Def}(E) := \text{Ext}^1(E, E)$ (in case (a)) or $\text{Def}(E) := \text{Ext}_0^1(E, E)$ (in case (b)). The Kuranishi deformation space of E is the analytic germ at the origin of an analytic germ of map

$$\Phi : \text{Def}(E) \rightarrow \text{Obs}(E).$$

Under the hypothesis that E is simple, the moduli space is locally analytically isomorphic to the Kuranishi space.

Corollary 5.1. *Define the expected dimension of the moduli space at E to be*

$$\text{e.d.}_E(\overline{M}_{X,H}(\xi)_i) := \dim(\text{Def}(E)) - \dim(\text{Obs}(E)).$$

If E is simple, then Kuranishi theory tells us that

$$\dim_E(\overline{M}_{X,H}(\xi)_i) \geq \text{e.d.}_E(\overline{M}_{X,H}(\xi)_i),$$

and if equality holds then the moduli space is a local complete intersection.

The expected dimension is a topological invariant. In case (b) it is the Euler characteristic

$$\text{e.d.}_E(\overline{M}_{X,H}(\xi)_i) = \chi_0(E, E) := \sum_{i=0}^2 (-1)^i \dim \text{Ext}_0^i(E, E)$$

which is a topological invariant depending only on ξ . In case (a), we just have to include the irregularity of X :

$$\text{e.d.}_E(\overline{M}_{X,H}(\xi)_i) = \chi_0(E, E) + h^1(\mathcal{O}_X).$$

Definition 5.2. *We say that an irreducible component of the moduli space $\overline{M}_{X,H}(\xi)$ is good, if it is generically reduced of dimension equal to the expected dimension.*

By Kuranishi theory, the moduli space is smooth of dimension equal to the expected dimension at E , if and only if the space of obstructions vanishes, that is to say $\text{Ext}_0^2(E, E) = 0$. Thus, our irreducible component $\overline{M}_{X,H}(\xi)_i$ is good if and only if the space of obstructions vanishes for a general point E . Notice that this condition is independent of whether we choose option (a) or (b) for fixing $c_1(E)$.

We can now state O'Grady's precise result, generalizing theorems due to Donaldson, Friedman, Zuo, Gieseker, Li and others [63, 86, 297].

Theorem 5.3 (O'Grady [230] Corollary B', Theorem D). *Suppose H is an ample divisor on a smooth surface X . There exist constants Δ_1 and Δ_2 depending only on the rank and the numerical invariants of X such that if $\Delta(\xi) \geq \Delta_1$ then every irreducible component of the moduli space is good, and if $\Delta(\xi) \geq \Delta_2$ then the moduli space is irreducible with a dense open set consisting of slope-stable bundles.*

For bundles of rank $r = 2$, for example, O'Grady can conclude, assuming $K_X^2 > 100$ and $h^2(\mathcal{O}_X) > 0$, the moduli space is good if

$$\tilde{c}_2 \geq 42K_X^2 + 15\chi(\mathcal{O}_X),$$

and with a similar bound it is also irreducible.

These bounds are the best we can say for general surfaces.

For surfaces of smaller Kodaira dimension, the situation becomes much better. As we have pointed out above, Mukai's observation gives vanishing of the obstructions, independently of the value of c_2 , when the surface is symplectic, i.e. for K3 or abelian surfaces. This led to many works for example by O'Grady, Gomez, Yoshioka, Qin and others giving precise details about the moduli spaces in these cases. Huybrechts shows

that the moduli spaces of sheaves on K3 surfaces are deformation equivalent to Hilbert schemes, which among other things yields a calculation of their Hodge numbers [120, 121].

For certain rational surfaces the effect goes in the same direction. For bundles on \mathbb{P}^2 , Maruyama showed that the moduli space is irreducible and smooth for any value of \tilde{c}_2 . O'Grady [231], Gómez [88], Yoshioka [290] extend this for rank 2 bundles on del Pezzo surfaces. Kapustka and Ranestad [134] look at vector bundles on genus 10 Fano varieties.

The question of the behavior of irreducible components, whether they are good and how many there are, seems to be open for general rational surfaces or, for example, for blow-ups of any kind of surfaces. Walter has a fairly general result on birationally ruled surfaces, together with some examples where irreducibility and goodness don't hold [281]. Several studies have been made of what can happen under blowing up, see [38], [219], [286], [218], [171], [182] and many others.

For surfaces of general type, in the intermediate range $0 < \Delta(\xi) \leq \text{const}(X, H)$ the behavior of M can be wild, and indeed little is known about what happens. For example, Gieseker originally constructed components whose dimension is bigger than the expected one [85].

O'Grady, the first author and others show that there can be several different irreducible components to the moduli space.

Consider for example a hypersurface $X \subset \mathbb{P}^3$ of degree d . In [196], starting by restriction of certain special vector bundles from \mathbb{P}^3 and then applying the stabilization construction $c_2 \mapsto c_2 + 1$, obtained by deforming torsion-free sheaves to locally free ones, gives a good irreducible component of $M_H(X; 2, 0, c_2)$ for

$$c_2 \geq d^3/4 - d^2/4.$$

On the other hand O'Grady constructs an irreducible component which is not good, whenever $c_2 < d^3/3 - 3d^2 + 26d/3 - 1$. Between these two bounds there are at least two irreducible components, and this happens whenever $d \geq 27$.

Question 5.4. *What is the smallest value of d for which, for a very general hypersurface $X \subset \mathbb{P}^3$ of degree d and for some c_2 , there are two or more components of the moduli space of vector bundles on X ?*

For $d \leq 4$ we fall into the situations envisioned previously, of K3 or del Pezzo surfaces.

In our work [198, 199, 200] on the quintic case $d = 5$, we show that the moduli space of bundles of odd degree is irreducible for all values of c_2 . So, the first case could occur either for bundles of even degree on a

quintic, or perhaps on a surface of degree $6, 7, \dots$. We feel that it should be possible to find the first cases of multiple irreducible components, and that these will undoubtedly come from interesting geometrical phenomena.

The middle part of the paper will be devoted to an overview of several kinds of natural questions which may be asked: rationality, strange duality, jumping curves, Betti numbers, obstructions to the existence of universal families, and wallcrossing particularly under change of polarization. Many mathematicians have contributed results in these directions, for vector bundles on varieties occupying various places in the classification. For example, in a whole series of articles, Zhenbo Qin studies many aspects of moduli of bundles on surfaces which will be a part of our discussion, including rationality questions [239], Picard groups [241], wallcrossing [240, 244], symplectic structures and other special properties of the moduli spaces of vector bundles on surfaces of Kodaira dimension zero [240, 243], and the role of the stability condition [242]. Yoshioka develops a wide range of results on K3, abelian surfaces, twisted sheaves, and on Fourier-Mukai transforms bringing in perverse coherent sheaves. Beyond the question of rationality, Li gives a general result on the Kodaira dimension of the moduli space [167].

A novel aspect which appears when the moduli spaces are considered from a global point of view is the relationship between spaces corresponding to different numerical invariants. This was pioneered by Mukai, with his introduction of the Fourier-Mukai transform and its noncommutative variants. In his paper on the noncommutative nature of the Brill-Noether problem [210], he uses the operation of taking a K3 surface and getting another one which is the moduli space of vector bundles with appropriate Mukai vector on the original one. This allows him to characterize in certain cases the K3 surfaces containing a given curve as nonabelian Brill-Noether loci, that is to say the analogue for vector bundles. The possibility of doing that kind of operation has led to current research in the direction of derived categories with Bridgeland's stability conditions, Mirror symmetry, twisted sheaves and perverse coherent sheaves, and wallcrossing.

In our discussion touching upon these topics, we shall try to mention a diverse collection of references. But, in writing this survey we have become aware of the vastness of the literature on this subject and it is impossible to include everything, so we would like to apologize in advance for the numerous references which are unfortunately left out. The

things we are able to mention should be considered as first indications—a brief look on the web will allow the reader to follow up by finding many more references in any direction.

After the theory of moduli of vector bundles on curves, the moduli of sheaves on surfaces has received by and large the most attention. Moduli of bundles and sheaves on varieties of dimension ≥ 3 varieties is considerably more difficult and the literature is correspondingly more sparse. Nonetheless, we are unable to adequately touch upon it in more than a scattered way, due to our own lack of competence in these matters (the reader will kindly consult the references of the few papers we cite below for many more on these topics).

One part of the theory concentrates on specific kinds of bundles. Aprodu, Farkas and Ortega [6] and others study *Ulrich bundles*, admitting fully linear resolutions, which are closely related to minimal resolution conjectures. See Arrondo [10], Ancona, Ottaviani, [4] Vallès [279] on the theory of Schwarzenberger and Steiner bundles. These classes of bundles are related to *EPW sextics* introduced by O’Grady, again in relation to minimal resolutions. Hirschowitz [107] considers rank two reflexive sheaves with good cohomology.

§6. Reider’s theorem

The Bogomolov-Gieseker inequality is one of the most powerful statements in the moduli theory of vector bundles. Its reach is very nicely illustrated by *Reider’s theorem*, giving an application to the theory of linear systems. Reider’s proof introduces many useful techniques, so it is worthwhile to review it here. See Lazarsfeld’s notes [159], Beltrametti *et al* [22], as well as Reider’s paper [247]. His hypothesis, as follows, will be in effect throughout this section.

Hypothesis 6.1. *Let X be a smooth projective surface with a line bundle \mathcal{L} which is nef, meaning that it is a limit of ample points in $NS(X)_{\mathbb{Q}}$. The divisor of \mathcal{L} will be denoted L . Assume that $L^2 \geq 5$, and that the linear system $|K_X + L|$ has a basepoint x .*

The idea is to consider vector bundles fitting into an exact sequence of the form

$$0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow J_{P/X} \otimes \mathcal{L} \rightarrow 0.$$

Here $P = \{x\}$ is a subscheme of X consisting of a single point x , and $J_{P/X}$ denotes its ideal sheaf.

The exact sequence corresponds to an extension class

$$e \in \text{Ext}^1(J_{P/X} \otimes \mathcal{L}, \mathcal{O}_X)$$

$$\cong \text{Ext}^1(\mathcal{O}_X, J_{P/X} \otimes \mathcal{L} \otimes K_X)^* = H^1(J_{P/X} \otimes \mathcal{L} \otimes K_X)^*.$$

The isomorphism is Serre duality. The Ext group on the right is an extension of a locally free sheaf \mathcal{O}_X so it is just the same as its cohomology group as we have written. In turn, this cohomology group fits into an exact sequence

$$\begin{aligned} H^0(J_{P/X} \otimes \mathcal{L} \otimes K_X) &\rightarrow H^0(\mathcal{L} \otimes K_X) \rightarrow H^0((\mathcal{L} \otimes K_X)_P) \rightarrow \\ &\rightarrow H^1(J_{P/X} \otimes \mathcal{L} \otimes K_X) \rightarrow H^1(\mathcal{L} \otimes K_X) = H^1(\mathcal{L}^*)^*. \end{aligned}$$

Since P is a single point, $H^0((\mathcal{L} \otimes K_X)_P) = \mathbb{C}$. We can rewrite our exact sequence as

$$H^0(J_{P/X} \otimes \mathcal{L} \otimes K_X) \rightarrow H^0(\mathcal{L} \otimes K_X) \rightarrow \mathbb{C} \rightarrow H^1(J_{P/X} \otimes \mathcal{L} \otimes K_X) \rightarrow \dots$$

The morphism in the middle is evaluation of sections in $H^0(\mathcal{L} \otimes K_X)$ at the point x .

We come to one of the main elements of Hypothesis 6.1, that x is a basepoint of the linear system $|\mathcal{L} \otimes K_X|$, which is equivalent to saying that this evaluation map is zero. It follows that \mathbb{C} injects into $H^1(J_{P/X} \otimes \mathcal{L} \otimes K_X)$, in particular there exists a nontrivial extension.

The next step is to note that this extension is locally free. This is a special case of a more general principle, applicable to any subscheme $P \subset X$ on which we get a similar exact sequence. See [159, Proposition 3.9]. Our extension class will be a linear function on $H^1(J_{P/X} \otimes \mathcal{L} \otimes K_X)$, and the condition that it correspond to a locally free extension is the same as saying that it restricts to a nonzero element on any submodule of length 1 in $\mathcal{O}_P \otimes \mathcal{L} \otimes K_X$. In general the existence of an extension satisfying that nonvanishing, is the famous *Cayley-Bacharach property* to be discussed in the next section. Back to our first case, under our hypothesis 6.1 that x is a basepoint, the space of extensions has dimension ≥ 1 and a general element restricts to a nonzero element on the only point of P , that is to say on the image of \mathbb{C} by the injective connecting map considered above. The basepoint hypothesis therefore insures exactly that there exists a nontrivial locally free extension, and we get a vector bundle E .

The Bogomolov-Gieseker inequality now gives some information.

Lemma 6.2. *In the situation of Hypothesis 6.1, for any ample line bundle H , the vector bundle E is not H -semistable.*

Proof. Suppose E were H -semistable. Then, by the Bogomolov-Gieseker inequality we would have $\Delta(E) \geq 0$. But from the exact sequence, we have

$$c_1(E) = L, \quad c_2(E) = 1, \quad \Rightarrow \quad \Delta(E) = 1 - \frac{L^2}{4}.$$

By hypothesis, $L^2 \geq 5$ so this gives $\Delta(E) < 0$, a contradiction. We conclude that E is not H -semistable. Q.E.D.

Corollary 6.3. *There is a nonzero effective divisor D such that $(L - 2D).H > 0$ for any ample divisor H . Hence $(L - 2D).L \geq 0$.*

Proof. Choose an ample H' and let $M \subset E$ be the H' -destabilizing subbundle, so $(M - L/2).H' > 0$. Suppose first that there is another ample H such that $(M - L/2).H \leq 0$. Then, for some ample H'' on the segment joining H to H' in the rational Neron-Severi group, we would have $(M - L/2).H'' = 0$. Let $N := (E/M)^{**}$, then we get $(N - L/2).H'' = 0$ too. It now follows that E is H'' -semistable, indeed $M.H = N.H = L.H/2$ and if $U \subset E$ is another line bundle, it has a nonzero map either to N or to M , either way we get $U.H \leq L.H/2$ showing semistability. But this contradicts Lemma 6.2. We conclude that $(M - L/2).H > 0$ for all ample divisors H .

Note in particular that $M.H > 0$, so M cannot be contained in $\mathcal{O}_X \subset E$. Thus, there is a nonzero map $M \rightarrow L$ and we can write $M = L - D$ for an effective divisor D . Notice that D is nonzero, since the exact sequence defining E cannot split. As before let $N := (E/M)^{**}$, but calculating we find that $N \cong \mathcal{O}_X(D)$. We get $(L - 2D).H > 0$ for any ample H , and for a nef divisor such as L which is a limit of ample ones, we get $(L - 2D).L \geq 0$. Q.E.D.

Lemma 6.4. *In the situation of the previous corollary,*

$$L.D - 1 \leq D^2 < L.D/2.$$

Proof. For N as in the proof above, we have an exact sequence

$$0 \rightarrow M \rightarrow E \rightarrow J_{Q/X} \otimes N \rightarrow 0$$

where $Q \subset X$ is a subscheme of finite length. We get $c_2(E) = M.N + |Q| \geq (L - D).D$. But from the original exact sequence, $c_2(E) = 1$, so we get

$$(L - D).D \leq 1.$$

This gives the first inequality.

For the second one, suppose to the contrary that $D^2 \geq L.D/2$, that is to say $(L - 2D).D \leq 0$. Notice that $L.D \geq 0$ since L is a limit of ample divisors and D effective, so $D^2 \geq 0$. Now both D and L are in the positive part of the cone of divisors whose square is positive, so D may be joined to L by a segment contained inside this cone. For an

ample H we have $D.H > 0$ and $L.H > 0$ so any element of the segment is nonzero. By the conditions

$$(L - 2D).L \geq 0, \quad (L - 2D).D \leq 0$$

it follows that somewhere on that segment we will get a nonzero divisor B such that $B^2 \geq 0$ and $(L - 2D).B = 0$. The Hodge index theorem then says $(L - 2D)^2 \leq 0$ (indeed, if $(L - 2D)^2 > 0$ then as $(L - 2D)$ is again a divisor on the positive side of this cone, it could be used in the Hodge index theorem and since B is orthogonal to it and nonzero, we would get $B^2 < 0$ a contradiction). Write out

$$L^2 - 4L.D + 4D^2 \leq 0 \quad \Rightarrow \quad 1 < \frac{L^2}{4} \leq (L - D).D$$

contradicting the first statement proven above. This contradiction shows the second claimed inequality. Q.E.D.

Theorem 6.5 (Reider [247]). *Suppose L is a nef divisor with $L^2 \geq 5$. If the linear system $|K_X + L|$ has a basepoint x , then there exists an effective divisor D through x such that either $L.D = 0$ and $D^2 = -1$, or $L.D = 1$ and $D^2 = 0$.*

Proof. From the above arguments, we have integers $a := L.D \geq 0$ and $b := D^2$ with $a - 1 \leq b < a/2$. The only possibilities are $a = 0, b = -1$ or $a = 1, b = 0$. Notice also that the inclusion of line bundles $M \rightarrow \mathcal{L}$ takes image in $J_{P/X} \otimes \mathcal{L}$, which says that our basepoint x is contained in D . These give the conclusion. Q.E.D.

There is a similar statement for separation of points, with hypothesis $L^2 \geq 10$. See [247, 159].

The reader is pointed towards Reider's recent work, going from the Serre construction in its very general setting, to a new Hodge-type structure for the moduli spaces of vector bundles which he calls the *nonabelian Jacobian* [248, 249, 250].

§7. Relation with linear systems on curves, I: the Cayley-Bacharach condition

In the previous section, we have seen a first example of the *Serre construction*, which builds up a rank 2 bundle as an extension of an ideal sheaf twisted by a line bundle, with another line bundle. One of the key points is to understand when such an extension will be locally free. For the ideal sheaf of a single point $\{x\}$, the condition was that x should be a basepoint of the linear series $|K_X + L|$. This is, in fact, the

first case of the Cayley-Bacharach condition. In this section we discuss further, with the goal of gaining an understanding of how this condition provides one relationship between vector bundles on hypersurfaces in \mathbb{P}^3 and Brill-Noether on curves.

Let X be a smooth projective surface. The Serre construction presents a rank 2 vector bundle E as an extension of the form

$$(7.1) \quad 0 \rightarrow U \rightarrow E \rightarrow J_{P/X} \otimes L \rightarrow 0$$

where U and L are line bundles, and $J_{P/X} \subset \mathcal{O}_X$ is the ideal sheaf of a 0-dimensional subscheme $P \subset X$.

We obtain such an extension for any rank one saturated subsheaf of E ; saturated means that the quotient is a torsion-free sheaf. Notice that any saturated subsheaf of a locally free sheaf on a surface is itself locally free so the subsheaf U will be a line bundle. We have $L = (E/U)^{**}$.

The subscheme P is the scheme of zeros of the section of $U^* \otimes E$. As such, it is locally defined by two equations, so it is locally a complete intersection subscheme in particular it is Gorenstein. One can say for example, at any point $x \in \text{Supp}(P)$ with maximal ideal \mathfrak{m}_x , that the numbers $\dim(\mathfrak{m}_x^k \mathcal{O}_P / \mathfrak{m}_x^{k+1})$ present a symmetry as a function of k , stepping up by 1 at successive values of k for a certain time, staying constant, and then stepping back down by 1 until vanishing. The last step is a 1-dimensional piece of the form $S_{P,x} := \mathfrak{m}_x^a \mathcal{O}_P \subset \mathcal{O}_P$ called the *socle*. It is the only ideal sheaf of length 1 in the local piece P_x of P supported at x , defining a unique colength 1 subscheme $P'_x \subset P$ with $J_{P'_x/P} = S_{P,x}$. So, because of the Gorenstein property we have a good control over the colength 1 subschemes $P' \subset P$ and this will be helpful for what follows.

The extension (7.1) is defined by an extension class $\eta \in \text{Ext}^1(J_{P/X} \otimes L, U)$. We would like to know for which extension classes the corresponding sheaf is locally free, in other words what is the property of η that corresponds to the fact that E was a vector bundle.

Use Serre duality with the canonical sheaf K_X , as well as the fact that an *Ext* group from a locally free sheaf to anything, is just the same as the appropriate cohomology group. These say

$$\begin{aligned} \text{Ext}^1(J_{P/X} \otimes L, U) &\cong \text{Ext}^1(U, J_{P/X} \otimes L \otimes K_X)^* \\ &\cong H^1(J_{P/X} \otimes L \otimes U^{-1} \otimes K_X)^*. \end{aligned}$$

Therefore our extension class η may be viewed as a linear function

$$\eta : H^1(J_{P/X} \otimes L \otimes U^{-1} \otimes K_X) \rightarrow \mathbb{C}.$$

Consider next the long exact sequence associated to the sequence

$$0 \rightarrow J_{P/X} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_P \rightarrow 0$$

tensored with the line bundle $L \otimes U^{-1} \otimes K_X$. The connecting morphism is

$$(L \otimes U^{-1} \otimes K_X)_P \xrightarrow{\delta} H^1(J_{P/X} \otimes L \otimes U^{-1} \otimes K_X).$$

In particular, our linear function η restricts to a linear function

$$\eta \circ \delta : (L \otimes U^{-1} \otimes K_X)_P \rightarrow \mathbb{C}.$$

Note that by choosing local trivializations of the line bundle, we can say (non-canonically) $(L \otimes U^{-1} \otimes K_X)_P \cong \mathcal{O}_P$ and the Gorenstein property tells us that linear functions on \mathcal{O}_P may be identified with elements of \mathcal{O}_P . Under this identification, it is easy to say what is the property of η that corresponds to E being locally free: it just means that when viewed as an element of \mathcal{O}_P , η should be invertible. The reduction of this section modulo a maximal ideal \mathfrak{m}_x just corresponds to the value of the original linear function on the socle (tensorized with the appropriate line bundle) $S_{P,x} \otimes L \otimes U^{-1} \otimes K_X$ at x . Thus, in more canonical terms the criterion is as follows:

Proposition 7.1. *The extension corresponding to a class η is locally free at a point $x \in \text{Supp}(P)$ if and only if its restriction to the socle $\eta \circ \delta|_{S_{P,x} \otimes \dots}$ is nonzero. Hence, E is locally free everywhere if and only if this condition holds at every point of the support of P . This may be rephrased as saying that for any colength 1 subscheme $P' \subset P$, the restriction of $\eta \circ \delta$ to $J_{P'/P} \otimes L \otimes U^{-1} \otimes K_X$ is nonzero.*

This is classical and we don't give the proof here. It is discussed for example in [159] and [69].

In order best to understand the statement, the reader should think mainly of the case where P consists of a disjoint union of reduced points. Then, a subscheme $P' \subset P$ of colength 1 just means a collection of all the points except for one.

The corollary of this proposition is a condition for the existence of an extension class defining a locally free sheaf, which is the celebrated *Cayley-Bacharach condition*. We treat the proof in detail for the reader's convenience.

Corollary 7.2. *Suppose X is a smooth surface and $P \subset X$ is a local complete intersection subscheme of dimension 0. Suppose U and L are line bundles. Then there exists an extension class $\eta \in \text{Ext}^1(J_{P/X} \otimes L, U)$ defining an extension (7.1) with E locally free, if and only if the following*

“Cayley-Bacharach condition” is satisfied:

—for any colength 1 subscheme $P' \subset P$, any section in $H^0(L \otimes U^{-1} \otimes K_X)$ which vanishes on P' , also vanishes on P .

We denote this condition by $CB(L \otimes U^{-1} \otimes K_X)$ or just by $CB(n)$ for a line bundle of the form $\mathcal{O}_X(n)$.

Proof. Suppose P satisfies $CB(L \otimes U^{-1} \otimes K_X)$. Let $\text{Im}(e_P)$ denote the image of the evaluation map

$$e_P : H^0(L \otimes U^{-1} \otimes K_X) \rightarrow (L \otimes U^{-1} \otimes K_X)_P.$$

The condition may be reinterpreted as saying that for any $P' \subset P$ of colength 1, the map

$$\text{Im}(e_P) \rightarrow (L \otimes U^{-1} \otimes K_X)_{P'}$$

is injective. If we let $R_{P'} \subset (L \otimes U^{-1} \otimes K_X)_P$ denote the kernel of the projection to $(L \otimes U^{-1} \otimes K_X)_{P'}$, that is to say $R_{P'} := J_{P'/P} \otimes_{\mathcal{O}_P} (L \otimes U^{-1} \otimes K_X)_P$, then $R_{P'}$ is a one-dimensional subspace and the Cayley-Bacharach condition says that $R_{P'} \cap \text{Im}(e_P) = \{0\}$ for all P' .

Consider the cokernel $Q : (L \otimes U^{-1} \otimes K_X)_P / \text{Im}(e_P)$, then by the CB condition the image of $R_{P'}$ in Q is again a line, in particular note that $Q \neq 0$.

Now, the connecting map δ provides an injection $Q \hookrightarrow H^1(J_{P/X} \otimes L \otimes U^{-1} \otimes K_X)$, so for any linear function on Q we may choose an extension class $\eta \in H^1(J_{P/X} \otimes L \otimes U^{-1} \otimes K_X)^*$ which restricts to that linear function. As the images of all the $R_{P'}$ are nontrivial lines in Q , a general linear function will restrict to a nonzero function on all of the $R_{P'}$. Hence, for a general choice of extension class η , the restriction of $\eta \circ \delta$ to $R_{P'}$ will be nonzero. This insures that the extension sheaf E is locally free, by Proposition 7.1. This completes the proof of one direction.

Conversely, if there exists an extension class η which restricts to something nonzero on all $R_{P'}$, it follows that each $R_{P'}$ injects into Q , that is to say it has trivial intersection with $\text{Im}(e_P)$, which exactly says that P satisfies $CB(L \otimes U^{-1} \otimes K_X)$. Q.E.D.

There is a useful property of transfer of the Cayley-Bacharach property to residual subschemes.

Lemma 7.3. *Suppose a 0-dimensional subscheme $Z \subset X$ satisfies $CB(L \otimes M)$ for line bundles L and M . Suppose $g \in H^0(L)$. Let Z' be the residual subscheme of Z with respect to the section g . Then Z' satisfies $CB(M)$.*

Proof. Locally, view g as a function. The residual subscheme, defined by the annihilator ideal of $g|_Z$, fits into an exact sequence

$$0 \rightarrow \mathcal{O}_{Z'} \xrightarrow{g} \mathcal{O}_Z \rightarrow \mathcal{O}_Z/(g|_Z) \rightarrow 0.$$

If $J \subset \mathcal{O}_{Z'}$ is an ideal of length 1 defining $Z'' \subset Z'$, then gJ is a length 1 ideal in \mathcal{O}_Z defining a colength 1 subscheme $P \subset Z$. Suppose $f \in H^0(M)$ vanishes on Z'' ; then $g \otimes f \in H^0(L \otimes M)$ vanishes on P . By $CB(L \otimes M)$, $g \otimes f$ vanishes on Z so f vanishes on Z' . This proves $CB(M)$ for the residual subscheme Z' . Q.E.D.

The Cayley-Bacharach condition is a strong numerical property, which has been studied a great deal by many mathematicians over a long period, partly because of its above relationship with the Serre construction for vector bundles, but also for a host of other reasons. The reader should consult the survey article by Eisenbud, Green and Harris [69].

The following two corollaries extend the technique of Reider's theorem to a general setting, and serve to make somewhat more precise Question 3.4.

Corollary 7.4. *Suppose X is a smooth surface with a fixed hyperplane class, suppose L is a line bundle of positive degree and suppose $P \subset X$ is a zero-dimensional local complete intersection subscheme satisfying $CB(L \otimes K_X)$. Suppose that $H^0(J_{P/X}(D)) = 0$ for any effective divisor D of degree $\deg(D) < \deg(L)/2$. Then the length of P is $\geq L^2/4$.*

Proof. By above, there is a vector bundle extension of $J_{P/X} \otimes L$ by $U := \mathcal{O}_X$. The hypothesis implies that it is semistable, so the Bogomolov-Gieseker inequality applies. Q.E.D.

Corollary 7.5. *Suppose $P \subset \mathbb{P}^3$ is a local complete intersection satisfying $CB(m)$, such that $H^0(J_{P/X}(n)) = 0$. If P is contained in a smooth surface of degree $d \leq m + 4$, and $2n \geq m + 2 - d$, then the length of P is at least $d(m + 4 - d)^2/4$.*

Proof. Let X be the smooth surface, so $K_X = \mathcal{O}_X(d - 4)$, and apply the previous corollary with $L := \mathcal{O}_X(m + 4 - d)$. The condition $2n \geq m + 2 - d$ ensures that the hypothesis of the previous corollary holds, so the length of P is at least $L^2/4 = d(m + 4 - d)^2/4$. Q.E.D.

Question 3.4 may be interpreted as asking, can we obtain a similar statement without the hypothesis that the surface containing P is smooth? We refer to Section 17.3 for a further discussion of these issues.

We would next like to describe how the Cayley-Bacharach property allows us to draw a relationship between the construction of bundles on surfaces, and Brill-Noether theory for curves. Adopt the setting of a smooth hypersurface $X \subset \mathbb{P}^3$ of degree d . Then, a subscheme $P \subset X$ may also be thought of as a subscheme of \mathbb{P}^3 . If we furthermore assume that the line bundles U , L and automatically $K_X = \mathcal{O}_X(d - 4)$ are restrictions of line bundles on \mathbb{P}^3 , then $L \otimes U^{-1} \otimes K_X = \mathcal{O}_X(n)$ for some n . We have $H^1(\mathbb{P}^3, \mathcal{O}_X(n-d)) = 0$, which says that sections of $\mathcal{O}_X(n)$ on X are restrictions of sections defined on \mathbb{P}^3 . Therefore, the conditions imposed by subschemes P or $P' \subset P$ on sections in $H^0(\mathcal{O}_{\mathbb{P}^3}(n))$ all factor through conditions imposed on $H^0(\mathcal{O}_X(n))$ (we don't need the restriction map to be an isomorphism for this to be true, its surjectivity is enough). Hence, the Cayley-Bacharach conditions are the same:

$$(7.2) \quad CB(\mathcal{O}_X(n)) \Leftrightarrow CB(\mathcal{O}_{\mathbb{P}^3}(n))$$

for subschemes $P \subset X$. We may forget about X , think of $P \subset \mathbb{P}^3$, and ask whether it satisfies Cayley-Bacharach there.

Now, very often our subscheme will sit on some curve $C \subset \mathbb{P}^3$, most likely not contained in the original surface. Typically, C is obtained as a complete intersection (or some component therein) of divisors in $|\mathcal{O}_X(m)|$ passing through P , whose existence is obtained using $CB(n)$ which implies $CB(m)$ for $m \leq n$.

We would like to remark, in passing, that the classification of curves in \mathbb{P}^3 becomes important for understanding which curves C to consider. For example, how can a complete intersection of bidegree (d, d') in \mathbb{P}^3 break up into irreducible curves of smaller genus?

In the most optimistic case, our curve C will be smooth, and the subscheme P constitutes a divisor on C . We may give a criterion:

Lemma 7.6. *Suppose $C \subset \mathbb{P}^3$ is a smooth curve and $P \subset C$ is a divisor. Put $A := \mathcal{O}_C(n) \otimes \mathcal{O}_C(-P)$. Suppose that for any point x in the support of P , we have*

$$h^0(A(x)) = h^0(A).$$

Then P satisfies the condition $CB(n)$ on \mathbb{P}^3 .

Proof. Suppose $P' \subset P$ is a subscheme of colength 1. Then $\mathcal{O}_C(-P') = \mathcal{O}_C(-P) \otimes \mathcal{O}_C(x)$. Thus, our hypothesis says that

$$H^0(C, \mathcal{O}_C(n)(-P)) \xrightarrow{\cong} H^0(C, \mathcal{O}_C(n)(-P')).$$

In other words, any section in $H^0(\mathcal{O}_C(n))$ vanishing on P' , also vanishes on P . This implies condition $CB(n)$ on \mathbb{P}^3 . Q.E.D.

Remark 7.7. *In fact, the lemma proves the condition $CB(\mathcal{O}_C(n))$ which implies $CB(\mathcal{O}_{\mathbb{P}^3}(n))$. These two conditions are equivalent if the restriction map $H^0(\mathcal{O}_{\mathbb{P}^3}(n)) \rightarrow H^0(\mathcal{O}_C(n))$ is surjective, which would be the case for example when C is a complete intersection.*

If the line bundle $A = \mathcal{O}_C(n)(-P)$ is not in the Brill-Noether locus, that is to say if $h^0(A) = \chi(A)$, then we have

$$h^0(A(x)) \geq \chi(A(x)) = h^0(A) + 1$$

so the condition of the lemma cannot hold. By the previous remark, in some cases such as when C is a complete intersection, this would actually mean that P is not $CB(n)$, but in any case it means that the method of Lemma 7.6 to get the $CB(n)$ condition doesn't apply.

So, in order to obtain Cayley-Bacharach, we need to have a line bundle $A = \mathcal{O}_C(n)(-P)$ in the Brill-Noether locus. Notice that P is a divisor in the linear system $|\mathcal{O}_C(n) \otimes A^{-1}|$. Some more can be said about when this will work. Suppose we fix the line bundle A , with $h^1(A) > 0$. For a point $x \in C$, the exact sequence

$$0 \rightarrow H^0(A) \rightarrow H^0(A(x)) \rightarrow \mathbb{C} \rightarrow H^1(A) \rightarrow H^1(A(x)) \rightarrow 0$$

is Serre dual to

$$\begin{aligned} 0 \rightarrow H^0(K_C \otimes A^{-1}(-x)) \rightarrow H^0(K_C \otimes A^{-1}) \rightarrow \mathbb{C} \rightarrow \\ \rightarrow H^0(K_C \otimes A^{-1}(-x)) \rightarrow H^0(K_C \otimes A^{-1}) \rightarrow 0. \end{aligned}$$

The hypothesis $H^0(K_C \otimes A^{-1}) \neq 0$ implies that for a point x which is general with respect to A , the evaluation map $H^0(K_C \otimes A^{-1}) \rightarrow \mathbb{C}$ will be surjective. Equivalently the connecting map $\mathbb{C} \rightarrow H^1(A)$ is injective, and we obtain the conclusion that $H^0(A) \rightarrow H^0(A(x))$ is an isomorphism as required for Lemma 7.6.

Corollary 7.8. *Suppose A is a line bundle in the Brill-Noether locus $h^1(A) > 0$, such that the linear system $|\mathcal{O}_C(n) \otimes A^{-1}|$ is effective and without basepoints. Then, for a general divisor $P \in |A \otimes \mathcal{O}_C(-n)|$, the hypothesis of Lemma 7.6 is satisfied and P satisfies $CB(n)$ on \mathbb{P}^3 .*

Proof. Since there are no basepoints, for a general divisor P in the linear system any point $x \in \text{Supp}(P)$ is general with respect to A . The previous discussion applies. Q.E.D.

Corollary 7.9. *In the situation of the previous corollary, suppose X is a smooth hypersurface of degree d containing P . Then there exists a vector bundle E on X fitting into an exact sequence*

$$0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow J_{P/X} \otimes \mathcal{O}_X(m) \rightarrow 0$$

where $m = n + 4 - d$.

Proof. Setting $U = \mathcal{O}_X$ and $L = \mathcal{O}_X(m)$, we have $U^{-1} \otimes L \otimes K_X = \mathcal{O}_X(m + d - 4) = \mathcal{O}_X(n)$. The Cayley-Bacharach condition $CB(n)$ on \mathbb{P}^3 deduced from the previous corollary implies the $CB(n)$ condition on X cf (7.2). Corollary 7.2 gives the vector bundle extension. Q.E.D.

One could refine the hypotheses through a more geometric discussion of how to apply Lemma 7.6. Notice that $h^0(A(x)) > h^0(A)$ if and only if, for any point $y \in C$, the divisor $A(x - y)$ is again in the Brill-Noether locus. Let $(x - C)$ denote the one-dimensional subset of $\text{Jac}^0(C)$ consisting of divisors of the form $x - y$ for $y \in C$. If

$$BN \subset \text{Jac}^r(C)$$

is some component of the Brill-Noether locus, such that for any x we have that $BN + (x - C)$ is not contained in BN , then the hypothesis of Lemma 7.6 will hold.

These considerations give a first kind of relationship between Brill-Noether theory and the construction of vector bundles. For the study of this locus as a moduli space itself, see for example King and Newstead [140]. Another kind of relation, based on elementary transformations, will come out of O'Grady's method which is our next subject.

We feel that the many beautiful things known about Brill-Noether theory by the work of Mukai and others, will lead to interesting new aspects of the geometry of vector bundles.

§8. The boundary of the moduli space

Let X be a smooth projective surface. The moduli space of stable vector bundles $M_{X,H}(r, c_1, c_2)$ is usually not compact, although some irreducible components can be compact. There are two main kinds of compactifications: the *Gieseker compactification*, which adds in torsion-free sheaves; and the minimal *Uhlenbeck compactification* which only keeps track of the locations and multiplicities of the singular points rather than their finer structure. The Uhlenbeck compactification was originally viewed in terms of solutions of the Yang-Mills equations, and indeed it makes sense on any 4-manifold, but it may also be seen as a quotient of the Gieseker compactification. For our discussion we will concentrate on the Gieseker compactification denoted $\overline{M}_{X,H}(r, c_1, c_2)$, which is a coarse moduli scheme for H -Gieseker-semistable torsion-free sheaves.

Being locally free is an open condition, so the moduli space of vector bundles is an open subset

$$M_{X,H}(r, c_1, c_2) \subset \overline{M}_{X,H}(r, c_1, c_2).$$

It is tempting to think of it as being a dense open subset, as will often be the case; but one should bear in mind that this is not always true—there exist examples of irreducible components of $\overline{M}_{X,H}(r, c_1, c_2)$ not containing any locally free point. When we speak in general terms about the “boundary” this usually means the complement of the locally free locus. However, to be more precise, if Z is a subset of $M_{X,H}(r, c_1, c_2)$ then we define $\partial Z := \overline{Z} - Z$. In what follows, the closure always means the closure inside the moduli space $\overline{M}_{X,H}(r, c_1, c_2)$ of torsion-free sheaves. Notice with this definition that $\partial M_{X,H}(r, c_1, c_2)$ can be smaller than $\overline{M}_{X,H}(r, c_1, c_2) - M_{X,H}(r, c_1, c_2)$ so the notation $\partial(\)$ might conflict with our general usage of the term “boundary”.

A first basic and very useful result is O’Grady’s lemma about the codimension of the boundary [230, Proposition 3.3]:

Lemma 8.1 (O’Grady). *Suppose $Z \subset M_X(r, c_1, c_2)$ is an irreducible closed subset. Then $\partial Z := \overline{Z} - Z$ has pure codimension 1, if it is nonempty.*

The main structure of the boundary is provided by what O’Grady calls the *double dual stratification*, closely related to the *Uhlenbeck stratification* from gauge theory. Suppose E is a torsion-free sheaf on X . Then E^{**} is locally free since X is regular of dimension 2, and we have an exact sequence

$$0 \rightarrow E \rightarrow E^{**} \rightarrow S \rightarrow 0$$

where S is a coherent sheaf of finite length, in particular of 0-dimensional support. The double dual stratification is defined by looking at the length of S , denoted $d := |S|$. Note that $c_2(E^{**}) = c_2(E) - d$.

For brevity suppose that H , r and c_1 are fixed and therefore dropped from the notation. This is permissible because, in the situation of the previous paragraph, the rank and c_1 of E and E^{**} are the same.

Let $M_X(c_2, c_2 - d)$ be the moduli space of torsion-free sheaves E with $c_2(E) = c_2$ and $c_2(E^{**}) = c_2 - d$, in other words $|S| = d$. These are locally closed subsets of $\overline{M}_X(c_2)$, giving a stratification

$$\overline{M}_X(c_2) = M_X(c_2) \sqcup \coprod_{d>0} M_X(c_2, c_2 - d).$$

The pieces are provided with fibrations

$$M_X(c_2, c_2 - d) \rightarrow M_X(c_2 - d)$$

sending E to E^{**} . One may need to be careful here about preservation of Gieseker semistability. Restrict for example to the case of bundles whose degree and rank are coprime, so that $M_X(c_2 - d)$ is locally a fine moduli space. The above map becomes a fibration in the analytic topology, with its fiber over a vector bundle F being the Grothendieck Quot scheme $\text{Quot}(F, d)$ of quotients $F \rightarrow S$ of length d .

Because of these fibrations on the boundary, several authors studied very early on the structure of the Quot scheme for rank 2. Li shows in [166, Proposition 6.4] that $\text{Quot}(E, d)$ is irreducible with a dense open subset U parametrizing quotients which are given by a collection of d quotients of length 1 supported at distinct points of X . Li's theorem was subsequently generalized for bundles of arbitrary rank by Ellingsrud and Lehn [72].

Theorem 8.2 (Li, Ellingsrud-Lehn). *Suppose E is a locally free sheaf of rank r on a smooth surface X . The quotient scheme parametrizing quotients of a locally free sheaf \mathcal{O}_X^r of rank r on a smooth surface X , located at a given point $x \in X$, and of length ℓ , is irreducible of dimension $r\ell - 1$. Thus for any $d > 0$, the global $\text{Quot}(E, d)$ is an irreducible scheme of dimension $(r + 1)d$. It contains a dense open subset parametrizing quotients which are direct sums of general rank 1 quotients over distinct general points of X .*

This theorem allows us to provide a fairly precise description of the possible boundary components. One should beware that a boundary piece $M_X(c_2, c_2 - d)$ is not always necessarily in the closure of $M_X(c_2)$, in which case the description would become more difficult. However, in many cases dimension considerations allow us to rule out such a thing.

Another useful consideration is as follows. If the moduli space is good, then it is a local complete intersection. This holds locally in the analytic topology by the Kuranishi deformation theory, which presents the moduli space as the zero set of a map from the Zariski tangent space to the space of obstructions and goodness says that the dimension of the moduli space is equal to the difference of the two dimensions.

In the situation of a local complete intersection, we have *Hartshorne's connectedness theorem* [255] [98]:

Theorem 8.3 (Hartshorne). *Suppose (Z, z) is a local analytic germ which is a complete intersection. Then, removing a subvariety of codimension ≥ 2 cannot disconnect Z .*

Corollary 8.4. *Suppose that the moduli space \overline{M} of torsion-free sheaves has dimension equal to the expected dimension at a point $[E]$. Then, if two different irreducible components meet, they must intersect*

in a codimension 1 subvariety. In particular, in cases where the singular locus is known to have codimension ≥ 2 , connectedness and irreducibility are the same thing.

This completes our discussion of what can be said about the boundary in very general terms.

§9. Relation with linear systems on curves, II: O'Grady's method

O'Grady introduced a method for studying the irreducibility and smoothness properties of moduli spaces of sheaves, based on geometric considerations of restriction to curves. One important piece of his construction comes from the notion of *elementary transformation*, introduced by Maruyama very early on [186, 187, 191]. It is the elementary transformation construction which provides a direct relation to the theory of linear systems on curves, and we start by recalling how it works, for simplicity in the case of surfaces.

Let X be a smooth surface, and suppose E is a vector bundle. Suppose $C \subset X$ is a smooth curve. Suppose we are given an exact sequence of bundles over C , written

$$0 \rightarrow U \rightarrow E|_C \rightarrow Q \rightarrow 0.$$

Let $i : C \rightarrow X$ denote the inclusion. Then we can consider the map of coherent sheaves on X

$$E \rightarrow i_*(Q).$$

The kernel of this map is a coherent subsheaf $T \subset E$ which, one may see, is in fact locally free itself. Thus, T is a new vector bundle called the *elementary transform of E along the quotient Q* . The salient fact is that we can restrict T to the curve. What remains of the original exact sequence becomes a map from $T|_C$ to the subbundle U , and fitting into the exact sequence

$$0 \rightarrow Q(-C) \rightarrow T|_C \rightarrow U \rightarrow 0.$$

The kernel is $Q \otimes \mathcal{O}_X(-C)|_C$ denoted $Q(-C)$ for short. We could now do the elementary transform of T along the quotient U , and we get back to the original bundle twisted by $\mathcal{O}_X(-C)$, a fact which may be recorded in the exact sequence

$$0 \rightarrow E(-C) \rightarrow T \rightarrow i_*(U) \rightarrow 0.$$

O'Grady adds a variation on this theme. The sheaf $i_*(Q)$ appearing above was of pure dimension 1. If we replace that by a more general

quotient sheaf $E \rightarrow \mathcal{Q}'$ such that \mathcal{Q}' contains some torsion elements, then the kernel subsheaf T' will be torsion-free but not locally free. Furthermore, we may consider the whole construction as parametrized by a point in the Grothendieck *Quot*-scheme of quotients of E with a fixed Hilbert polynomial. If we can deform from a pure dimension 1 quotient to a quotient that has torsion, then we get a deformation from the locally free elementary transform T to a torsion-free but not locally free one T' . This is O'Grady's method for obtaining the deformation towards the boundary of the moduli space. The boundary point T' comes from a moduli space of bundles (i.e. look at the double dual of T') with a smaller value of c_2 , setting up the possibility to do an inductive argument.

Before getting to a more detailed illustration, we can describe quickly how this construction relates to Brill-Noether theory on curves. Let us consider bundles of rank $r = 2$. The simplest example of a bundle to start with is just the trivial bundle $E = \mathcal{O}_X^2$. In that case, a quotient

$$E|_C \rightarrow Q$$

just consists of a line bundle Q on the curve C , plus a two-dimensional space of sections $\mathbb{C}^2 \rightarrow H^0(C, Q)$. Of course, if we take a map factoring through a one-dimensional subspace then our elementary transform will just be a direct sum of line bundles, so the construction is interesting only when we have a subspace $\mathbb{C}^2 \subset H^0(C, Q)$. In other words, this construction will apply any time we have a g_d^1 linear system on the curve C . We get a family of bundles parametrized by this piece of the Brill-Noether locus of the curve. To be more precise, the map is surjective when the two sections generate the line bundle Q . Deforming the elementary transformation to a boundary point (i.e. torsion-free but not locally free) corresponds to deforming the g_d^1 to one where the sections no longer generate the line bundle Q .

One seldom gets to start with a trivial bundle, and the study of line bundles admitting a morphism from a higher-rank vector bundle was the subject of Mukai's "nonabelian Brill-Noether theory" [211]. Copen [49] also studies irreducible components of the Brill-Noether loci in relation to restriction of stable bundles.

We now get to a more detailed illustration of O'Grady's method, with a view towards applying it in the case of vector bundles on a very general quintic hypersurface $X \subset \mathbb{P}^3$. For the general case, see [230].

The goal is to show that any irreducible component of $M(c_2)$ "meets the boundary", i.e. is a strict open subset of its closure in $\overline{M}(c_2)$. So, for this part, let us suppose that $Z \subset \overline{M}(c_2)$ is a closed irreducible subset which doesn't meet the boundary, that is to say $Z \subset M(c_2)$. The point

of departure therefore is to have a compact variety parametrizing stable vector bundles.

Suppose $Y \subset X$ is a smooth curve, intersection of X with a hyperplane section of class given by our ample divisor H . In the case we will later be interested in X is a quintic hypersurface in \mathbb{P}^3 and Y is a plane quintic curve of genus $g = 6$. The moduli space $M_Y(2, \mathcal{O}_Y(1))$ of rank two bundles on Y with determinant $\mathcal{O}_Y(1)$ is a smooth projective variety of dimension $3g(Y) - 3$, equal to 15 in the quintic case.

Consider the “theta-divisor” $\Theta_Y(2, \mathcal{O}_Y(1)) \subset M_Y(2, \mathcal{O}_Y(1))$. It is the subspace of bundles F on Y such that $h^0(Y, F \otimes L) > 1$ for an appropriate line bundle L (cf Section 11). It is known to be ample, indeed it is the divisor of a natural section of the determinant line bundle $\underline{\Theta}_Y(2, \mathcal{O}_Y(1))$ on $M_Y(2, \mathcal{O}_Y(1))$, which is ample [123]. We may write $c_1(\underline{\Theta}_Y(2, \mathcal{O}_Y(1))) = [\Theta_Y(2, \mathcal{O}_Y(1))]$.

O’Grady’s first main step is to get a curve on which a restricted bundle becomes unstable, Corollary 9.3 below. The proof is by contradiction. If the restrictions are always stable then ampleness of the determinant bundle gives a positive intersection number with a dimension bound, but the contrary bound on the dimension can be included as a hypothesis.

Proposition 9.1 (O’Grady [230] Proposition 1.18). *Suppose $Z \subset M_{X,H}(r, L, c_2)$ is a compact subvariety, and suppose $Y \subset X$ is a smooth curve in the linear system $|\mathcal{O}_X(H)|$. Suppose that for every $[E] \in Z$, the restriction $E|_Y$ is stable. Let $p : Z \rightarrow M_Y(r, L|_Y)$ be the resulting morphism to the moduli of bundles on Y . Then $p^*(\underline{\Theta}_Y(r, L|_Y))$ is ample on Z .*

Proof. We recall here the outline of the proof. The first step is to note that there is a smooth curve Y_k in the linear system $|\mathcal{O}_X(kH)|$ such that all points of Z restrict to stable bundles on Y_k too. See [230], Proposition 1.20. This type of uniform Mehta-Ramanathan restriction theorem [194] has more recently been vastly extended by Langer and others, see [157], and one may ask whether those techniques can improve O’Grady’s bounds.

In any case, let $p_k : Z \rightarrow M_{Y_k}(r, L|_{Y_k})$ be the corresponding morphism.

Next, O’Grady shows in [230, Lemma 1.21] that there is a constant $\lambda_k > 0$ such that

$$(9.1) \quad c_1(p_k^* \underline{\Theta}_{Y_k}(r, L|_{Y_k})) = \lambda_k c_1(p^* \underline{\Theta}_Y(r, L|_Y)).$$

This is a calculation using Grothendieck-Riemann-Roch.

The theorem is now proven by noting that for $k \gg 0$, the map $p_k : Z \rightarrow M_{Y_k}(r, L|_{Y_k})$ is an embedding. What we need may be seen easily as follows. Suppose E and E' are two bundles on X whose restriction to Y_k are isomorphic. We obtain a section of $H^0(Y_k, (E^* \otimes E')|_{Y_k})$. Look at the exact sequence

$$\begin{aligned} H^0(E^* \otimes E'(-kH)) &\rightarrow H^0(E^* \otimes E') \rightarrow H^0(Y_k, (E^* \otimes E')|_{Y_k}) \\ &\rightarrow H^1(E^* \otimes E'(-kH)) \end{aligned}$$

and note that for $k \gg 0$,

$$H^1(E^* \otimes E'(-kH)) \cong H^1((E')^* \otimes E(K_X + kH)) = 0.$$

Thus, our isomorphism over Y_k extends to a morphism over X , easily seen to be an isomorphism since both bundles are stable. This shows, at least, that the map p_k is quasifinite. That is enough to prove that the pullback by p_k of the ample line bundle $\underline{\mathcal{O}}_{Y_k}(r, L|_{Y_k})$ is ample on Z . By the formula (9.1), the proposition follows. Q.E.D.

Corollary 9.2. *Suppose $Z \subset M_{X,H}(r, L, c_2)$ is a compact subvariety, and suppose $Y \subset X$ is a smooth curve in the linear system $|\mathcal{O}_X(H)|$. Suppose that for every $[E] \in Z$, the restriction $E|_Y$ is stable. Let $p : Z \rightarrow M_Y(r, L|_Y)$ be the resulting morphism to the moduli of bundles on Y . Then*

$$p^* c_1(\underline{\mathcal{O}}_Y(r, L|_Y))^{\dim(Z)} > 0.$$

In particular,

$$\dim(Z) \leq \dim(M_Y(r, L|_Y)).$$

Proof. The top self intersection of an ample divisor is positive, showing the first statement. For the second, note that

$$p^* (c_1(\underline{\mathcal{O}}_Y(r, L|_Y)))^{\dim(Z)} = p^* (c_1(\underline{\mathcal{O}}_Y(r, L|_Y))^{\dim(Z)})$$

but if $\dim(Z) > \dim(M_Y(r, L|_Y))$ then we would have

$$c_1(\underline{\mathcal{O}}_Y(r, L|_Y))^{\dim(Z)} = 0$$

on $M_Y(r, L|_Y)$, a contradiction. This shows the second one. Q.E.D.

The contrapositive is how this conclusion will be used, see for example [225, Theorem 1.2, Step 1].

The present still somewhat general discussion will be continued in further detail, with the same notations, when we complete the proof of Nijssse's connectedness theorem [225] in Section 18 below.

For convenience in starting towards the discussion of Section 18, we adopt throughout the rest of the present section the assumption that $X \subset \mathbb{P}^3$ is a very general quintic hypersurface and we are looking at the moduli of rank 2 bundles of degree 1. The class H is $c_1(\mathcal{O}_X(1))$ and the curve Y is a plane section of X . It is therefore a smooth plane quintic curve, having genus 6 so $3g_Y - 3 = 15$.

Corollary 9.3. *On our quintic surface X , if $Z \subset M_X(2, 1, c_2)$ is a compact subvariety, suppose $\dim(Z) \geq \dim(M_Y(2, L|_Y)) + 1 = 16$. Then there exists a point $[E] \in Z$ such that $E|_Y$ is unstable.*

We may now choose a point $[E] \in Z$ such that $E|_Y$ is unstable, given by the previous lemma, and look at the destabilizing sequence:

$$(9.2) \quad 0 \rightarrow L_0 \rightarrow E|_Y \rightarrow Q_0 \rightarrow 0.$$

The kernel and cokernel are line bundles on Y with $Q_0 = L_0^{-1}(1)$ since $\det(E) = \mathcal{O}_X(1)$. The destabilizing condition says $d_L := \deg(L_0) > \deg(Q_0)$.

The kernel of the map of sheaves

$$0 \rightarrow T \rightarrow E \rightarrow i_*(Q_0) \rightarrow 0$$

where $i : Y \hookrightarrow X$ denotes the inclusion, is the *elementary transformation* of E along the exact sequence (9.2). The sheaf T is again locally free on X , and the elementary transformation yields an exact sequence

$$0 \rightarrow Q_0(-1) \rightarrow T|_Y \xrightarrow{f_0} L_0 \rightarrow 0.$$

Furthermore, from the quotient f_0 we can get back E from the exact sequence

$$0 \rightarrow E(-1) \rightarrow T \xrightarrow{i_*(f_0)} i_*(L_0) \rightarrow 0.$$

Let $\text{Quot}(T|_Y, d_L)$ denote the Grothendieck Quot-scheme parametrizing quotients

$$T|_Y \xrightarrow{f} L$$

such that L has Hilbert polynomial $p(n) = P_Y(L_0, n)$ with respect to $\mathcal{O}_Y(1)$. The quotient (L_0, f_0) is a particular point of $\text{Quot}(T|_Y, d_L)$.

For any point (L_1, f_1) of $\text{Quot}(T|_Y, d_L)$, we can form the kernel

$$0 \rightarrow E_1(-1) \rightarrow T \xrightarrow{i_*(f_1)} i_*(L_1) \rightarrow 0$$

and twisting back we get a sheaf E_1 . It is torsion-free, being a subsheaf of T . Furthermore, E_1 has the same Hilbert polynomial as E , and it

has the same determinant sheaf as well. We have the following basic property:

The new sheaf E_1 is locally free if and only if the quotient L_1 is locally free.

This may be applied to give a construction of deformation to the boundary, under the condition of retaining some control over stability. We can make the following general statement, under the hypotheses and notations in vigour above, and denoting $M(c_2) := M_X(2, 1, c_2)$.

Proposition 9.4. *Suppose we have a family of quotients (L_u, f_u) parametrized by $u \in U$ for an irreducible curve U , containing a point $0 \in U$ which corresponds to the original (L_0, f_0) . This leads to a family of sheaves E_u . Suppose $v \in U$ is a point such that E_v is stable. Then it is a point in $\overline{M}(c_2)$ which is in the closure of some irreducible component of $M(c_2)$ containing E . In particular, if L_v is not locally free, then the irreducible component of $M(c_2)$ containing E meets the boundary.*

Proof. Notice that E is stable, so there is a nonempty open subset $U' \subset U$ containing 0 such that E_u is stable for $u \in U'$. We obtain a morphism $U' \rightarrow \overline{M}(c_2)$. If E_v is stable, then $v \in U'$ and the point $[E_v]$ is joined to $[E]$ by an irreducible curve in $\overline{M}(c_2)$. Hence, $[E_v]$ is in the closure of some irreducible component of $M(c_2)$ containing $[E]$. If E_v is not locally free, we get a deformation to a boundary point. Q.E.D.

In order to complete the construction, we need to find a quotient such that L_v is not locally free, and such that the resulting bundle E_v is stable. Consider the question of stability first.

Lemma 9.5. *In the above situation, suppose $H^0(E) = 0$. Then all of the E_v are stable.*

Proof. Compose the morphisms

$$E_v(-1) \rightarrow T \rightarrow E.$$

Suppose E_v is unstable. It has determinant $\mathcal{O}_X(1)$, so by the hypothesis that X is very general, the destabilizing subsheaf would be a line bundle of the form $\mathcal{O}_X(k) \rightarrow E_v$ with $k \geq 1$. We get a nonzero map $\mathcal{O}_X(k-1) \rightarrow E_v(-1)$, hence an injective map

$$\mathcal{O}_X(k-1) \rightarrow E.$$

As $H^0(\mathcal{O}_X(k-1)) \neq 0$ we get a nonzero element of $H^0(E)$. Q.E.D.

In order to apply this to our situation, we need a bundle E such that $E|_Y$ is unstable, and $H^0(E) = 0$.

Corollary 9.6. *Suppose $c_2 \geq 16$. Then, in any irreducible component of $M(c_2)$ there exists a bundle E such that $E|_Y$ is unstable, and the deformed bundles E_v resulting from the above construction are stable.*

Proof. As Nijssse points out, by [230, Proposition 5.47], the locus of bundles with $E|_Y$ unstable has codimension at most $g_Y = 6$ if it is nonempty. Assuming that $4c_2 - 20 \geq 16$ we get that it is nonempty by above. In that case, it has dimension at least $4c_2 - 20$, whereas the locus $V(c_2)$ of bundles with $H^0(E) \neq 0$ has dimension $3c_2 - 11$. Thus, if $c_2 \geq 16$ hence $4c_2 - 26 > 3c_2 - 11$ so there must be a bundle whose restriction is unstable, and with $H^0(E) = 0$. Q.E.D.

We now consider the question of getting a quotient L_v which is not locally free. We simplify somewhat here Nijssse's discussion, as he considered an arbitrary subset of $M(c_2)$ rather than just an irreducible component. We start with a lemma which encloses the essential point of O'Grady's argument in our simplified situation.

Lemma 9.7. *Suppose Y is an irreducible smooth curve, $P \rightarrow Y$ is a \mathbb{P}^1 -bundle, and U is a smooth proper surface. Then any map $f : Y \times U \rightarrow P$, compatible with the projection back to Y , factors through a map from U to a curve.*

Proof. Fix $y \in Y$. It gives a map $f_y : U \rightarrow P_y = \mathbb{P}^1$, which has positive-dimensional fibers since U is a surface. Consider the Stein factorization

$$U \xrightarrow{g} C \rightarrow \mathbb{P}^1.$$

If V is an irreducible component of a fiber of g , then the restriction $f_y|_V$ is constant, in particular it has degree 0. Therefore, for any other point $z \in Y$, the restriction $f_z|_V$ which is a deformation of $f_y|_V$, also has degree zero so it must be constant too. It follows that the map f_z is constant on fibers of g since these fibers are connected. This shows that f factors as

$$Y \times U \rightarrow Y \times C \rightarrow P.$$

Q.E.D.

We apply this as follows in our situation:

Corollary 9.8 (O'Grady [230] Lemma 1.15). *Given a generically injective map from a smooth projective surface $U \rightarrow \text{Quot}(T|_Y, d_L)$, then U contains a point $v \in U$ such that L_v is not locally free.*

Proof. Consider the map of sheaves on $Y \times U$

$$p_1^*(T|_Y) \rightarrow \mathcal{L}_U \rightarrow 0$$

where \mathcal{L} is the universal quotient sheaf on $Y \times \text{Quot}(T|_Y, d_L)$ and \mathcal{L}_U denotes its restriction to $Y \times U$. If L_u is locally free for all $u \in U$ it means that \mathcal{L}_U is locally free. Let $P \rightarrow Y$ be the \mathbb{P}^1 -bundle of quotients of $T|_Y$. Then our quotient gives a map $Y \times U \rightarrow P$ compatible with the projection to Y . By the previous lemma, it would have to factor through a map to a curve $U \rightarrow C$, but this contradicts the assumption that $U \rightarrow \text{Quot}(T|_Y, d_L)$ is generically injective. Hence, at least one quotient has to contain a non-locally free point. Q.E.D.

Corollary 9.9. *If $\mathbf{U} \subset \text{Quot}(T|_Y, d_L)$ is an irreducible component of dimension at least 2, containing (L_0, f_0) , then E may be deformed to the boundary of the moduli space.*

This finishes our first discussion of the essence of O’Grady’s method. Some further work will be needed to treat the case where the *Quot* scheme has dimension 1. This part of the discussion, continuing in the same vein, will be deferred to Section 18 in order to complete the proof of Nijssse’s connectedness theorem. The above notations will come back into effect there.

§10. Rationality

The birational geometric type of the moduli spaces is an interesting and, in general, difficult question. There are many results about rationality of the moduli space, considering for example the case when X itself is rational.

Ballico [15] proved that if X is a rational surface, then for an appropriate choice of ample divisor H , the moduli space $M_H(r, c_1, c_2)$ is smooth, irreducible and unirational whenever it is nonempty.

For $X = \mathbb{P}^2$ itself, Ellingsrud and Strømme [73] and Maruyama [193] completed the proof, started by Hulek [118] and Barth [16], of (almost) rationality: for c_1 odd, or c_1 even and c_2 odd, the moduli space is rational when nonempty; when both are even there is a rational variety which is a \mathbb{P}^1 -bundle over the moduli space. See also Maeda [179].

Li and Qin consider rank 3 bundles [170], and by showing that a generic one may be presented as an extension of a rank 2 bundle by a line bundle, they are able to obtain rationality of the moduli of rank 3 bundles on \mathbb{P}^2 under certain numerical conditions. Katsylo proves rationality in a wide range of cases for higher rank [135]. His technique involves giving a birational equivalence with a quotient of a space of matrices. Schofield extends the result to whenever the gcd of the coefficients of the Mukai vector divides 420 [258], see also King-Schofield

[141]. They relate the moduli spaces to quiver representation spaces, a reduction closely related to the existence of a universal family.

Costa and Miro-Roig have a series of papers about rationality. For Hirzebruch surfaces they show rationality for large values of c_2 , for bundles of arbitrary rank [50]. They treat rank two bundles on Fano surfaces [51] and subsequently for arbitrary rational surfaces in [52], showing that the moduli space is rational for large values of c_2 . For such large values this removes, in particular, the need to go to a \mathbb{P}^1 -bundle in the previous results for $X = \mathbb{P}^2$.

Question 10.1. *An interesting question is to know whether there is a converse: if $M_H(c_1, c_2)$ is rational for $c_2 \gg 0$ then does that imply that X is rational?*

Costa and Miro-Roig have a program to attack this question [53], see also O’Grady’s review of their paper. We didn’t find a more recent reference on this question which is therefore apparently still open.

One may note that the condition $c_2 \gg 0$ is necessary: we will see some examples for hypersurfaces of high degree with a rational moduli space for very low values of c_2 .

Hoppe and Spindler considered the case when X is ruled over a curve C of higher genus [112]. They show, under some additional assumptions such as nonemptiness and existence of a universal family, that the moduli space is birationally equivalent to a projective space times an abelian variety (two copies of the Jacobian of C). Qin goes on, in this case, to give a computation of the Picard group of the moduli space [241].

Altogether we have a nice amount of information for the Fano case. By Yoshioka and others we also have a good amount of information for Calabi-Yau surfaces. Recent works are going towards an understanding for Enriques surfaces too. In the case of surfaces of general type, there can exist rational moduli spaces, for example $M_X(2, 1, 5)$ is an open subset of \mathbb{P}^3 when X is a very general quintic hypersurface [198, 200], but of course we expect that they are usually not rational. The delimitation between these situations doesn’t seem very clear.

These topics come together with the moduli of parabolic bundles in Mukai’s paper [212].

One crucial question is to understand what happens to the moduli space of bundles when we blow up a point. This has been discussed by Nakajima-Yoshioka [218], Nakashima [219], Yoshioka [286] and Brussee [38] among many others. Li and Qin relate this question to the S -duality conjecture [171]. Miró-Roig [202] shows irreducibility for rank 2 bundles on blow-ups of \mathbb{P}^2 , under a numerical condition that was used by Maruyama to obtain smoothness. Ballico [15] proves irreducibility and

smoothness for a rational surface, for a particular choice of polarization. It would be interesting to understand wallcrossing phenomena for the walls of Ballico's good chamber.

Going beyond these cases, the structure of a general rational surface can be very complicated, and it doesn't seem easy to fully understand the moduli spaces of sheaves over such surfaces. This should provide combinatorially rich environment for further study in relation with the other topics considered here.

§11. Strange duality

Strange duality relates moduli spaces with different Mukai vectors, so it provides some global structure to the family of all moduli spaces. On a moduli space associated to ξ , there is a determinant line bundle which depends on the choice of another "orthogonal" Mukai vector ζ . The strange duality pairing relates the space of sections of this determinant line bundle, with the space of sections over the ζ moduli space of the determinant line associated to ξ .

In the case of curves, it was also known as *rank-level duality*. This is because an SL_r moduli space depends only on the rank r , and the determinant line is uniquely determined up to a tensor power called the *level*. Strange duality relates sections of level k on the moduli space of bundles of rank r , with sections of level r on the moduli space of bundles of rank k .

Over curves, the strange duality phenomenon was extensively investigated and treated by many authors, and it is now proven by Belkale [21] and Marian and Oprea [183].

On a surface, we get a different moduli space for each Mukai vector $\xi = (r, c_1, c_2)$, and there are more parameters to play with. Strange duality is a conjectural statement relating sections of line bundles on $M_{X,H}(r, c_1, c_2)$ and $M_{X,H}(r', c'_1, c'_2)$, by analogy with the case of curves. The surface case is instructive because we see a little more clearly the role played by the choice of Mukai vectors.

The Mukai lattice has a product, corresponding to tensor product of bundles:

$$(r, c_1, c_2) \otimes (r', c'_1, c'_2) := \\ \left(rr', rc'_1 + r'c_1, rc'_2 + r'c_2 + (rr' - 1)c_1c'_1 + \frac{r(r-1)}{2}(c'_1)^2 + \frac{r'(r'-1)}{2}c_1^2 \right)$$

and applying the Euler characteristic operation, which depends only on the numerical data, we obtain a bilinear pairing

$$(\xi, \xi') \mapsto \chi(\xi \otimes \xi').$$

Strange duality concerns an orthogonal pair of Mukai vectors for this product, say ζ and ξ with $\chi(\zeta \otimes \xi) = 0$. On the moduli space $M_X(\xi)$, the orthogonal Mukai vector ζ allows us to define a *determinant line bundle*, or Θ -line bundle, denoted $\Theta_\zeta(\xi)$. This is a higher-rank and higher-dimensional generalization of the classical theta line bundle, whose sections are “theta-functions”, on an abelian variety.

It is defined by setting

$$\Theta_\zeta(\xi)(V) := \det(H^*(X, V \otimes G))$$

where G is any sheaf with Mukai vector ζ . The fact that $\chi(\zeta \otimes \xi) = 0$ means that we are taking the determinant of a complex with Euler characteristic zero, so multiplication by scalars on either of the factors acts trivially. This makes it so that the above definition descends to a well-defined line bundle on the moduli space. Some technical considerations are of course necessary, depending on the particular variety X which is considered, see the references.

Suppose now that we have an orthogonal pair ζ and ξ with $\chi(\zeta \otimes \xi) = 0$. Then, ζ determines a determinant line bundle on $M_X(\xi)$ and ξ determines a determinant line bundle on $M_X(\zeta)$. We can form their exterior tensor product:

$$\Theta_\zeta(\xi) \boxtimes \Theta_\xi(\zeta) \rightarrow M_X(\xi) \times M_X(\zeta).$$

Lemma 11.1. *Under some technical hypotheses, there is a canonical section $\sigma_{\xi, \zeta}$ of the exterior tensor product line bundle defined over $M_X(\xi) \times M_X(\zeta)$, nonzero if the cohomology $H^*(X, V \otimes G)$ is vanishing for generic $V \in M_X(\xi)$ and $G \in M_X(\zeta)$.*

Proof. Consider the line bundle over $M_X(\xi) \times M_X(\zeta)$ which to (V, G) associates $\det(H^*(X, V \otimes G))$. The technical hypotheses in question are needed to insure that this line bundle, which by definition restricts to the two determinant line bundles $\Theta_\zeta(\xi)$ and $\Theta_\xi(\zeta)$ on the copies $M_X(\xi) \times \{G\}$ and $\{V\} \times M_X(\zeta)$, is indeed the same as the exterior tensor product. Let us say that this is done.

Recall that the Euler characteristic is zero, so we can expect that the cohomology vanish for generic choices. Over the open set where it vanishes, the determinant line $\det(H^*(X, V \otimes G))$ is by definition trivialized. This gives a nonvanishing section of the bundle over the open set. One may see that it extends to a section, which vanishes on the locus where the cohomology is nonzero. If the cohomology is generically nonzero then the section is by definition zero. Q.E.D.

Suppose $\lambda \in H^0(M_X(\xi), \Theta_\zeta(\xi))^*$. For each $G \in M_X(\zeta)$ we have the section

$$\sigma_{\xi, \zeta}|_{M_X(\xi) \times \{G\}} \in \Theta_\xi(\zeta)(G) \otimes H^0(M_X(\xi), \Theta_\zeta(\xi)).$$

Applying λ we obtain an element of $\Theta_\xi(\zeta)(G)$. Considered as a function of G this gives a section of $H^0(M_X(\zeta), \Theta_\xi(\zeta))$. This constructs a map

$$H^0(M_X(\xi), \Theta_\zeta(\xi))^* \rightarrow H^0(M_X(\zeta), \Theta_\xi(\zeta))$$

which is called the *strange duality map*.

The *strange duality conjecture* says that this map should be an isomorphism, giving in particular equality between the dimensions of the spaces of sections of determinant line bundles over two different moduli spaces. This is not always expected to be true, indeed there are some cases where the section σ and hence the strange duality morphism itself could vanish. However, the statement or some variant is expected to be true in a surprisingly wide range of cases, many of which are now known thanks to the work of Le Potier, Beauville, Belkale, Marian, Oprea, Danila, O’Grady, and others.

See Marian and Oprea [184] for a complete overview. We just mention some cases here. A first case is Hilbert schemes of points. They give numerical calculations supporting the conjecture, i.e. identifying the dimensions of the spaces of sections, for abelian and K3 surfaces. They prove the strange duality isomorphism for abelian surfaces. A classical K3 case is the intersection of three quadrics in \mathbb{P}^5 , due to Mukai and generalized by O’Grady. Sawon applies Fourier-Mukai in cases where there doesn’t exist a universal family, getting a strange duality relationship with moduli spaces of twisted sheaves [256]. O’Grady treats extensively the case of elliptic K3 surfaces [234, Statement 5.15].

Le Potier formulated a strange duality conjecture for pure dimension 1 sheaves on \mathbb{P}^2 , that is for $M(\mathbb{P}^2, 0, c_1, c_2)$, some cases of which have been proven by G. Danila [56] and Abe [1]. The strange dual moduli space is of the form $M(\mathbb{P}^2, 2, 0, c'_2)$. A strong numerical condition is required, which is shown to hold for certain values of c_2 and c'_2 .

§12. Jumping curves

Consider a vector bundle E over \mathbb{P}^2 . For any line $L \subset \mathbb{P}^2$, the restriction $E|_L$ decomposes into a direct sum of line bundles by Grothendieck’s theorem. For general values of L , the integer degrees of these line bundles stay constant. However, for special values of L , the degrees in the Grothendieck decomposition will jump. We get a subvariety in the dual

projective space $D_E \subset \widehat{\mathbb{P}}^2$, consisting of those points L where the decomposition jumps. For $r = 2$ and $c_1 = 0$ it is a divisor.

The map $E \mapsto D_E$ is called the *Barth morphism* because it was first considered by Barth in [17].

The singularities of this curve of jumping lines for a vector bundle of rank 2 on \mathbb{P}^2 were studied by Maruyama in [192]. He uses the rationality of the base variety in order to obtain a rich supply of curves on which to look at elementary transformations, and gives a formula for the multiplicity of the jumping curve at a point, based on the numerical invariants of a sequence of transformations.

In Le Potier's point of view, the jumping curve construction was closely related to the strange duality conjecture for sheaves on \mathbb{P}^2 . In his famous paper, he introduces the notion of *coherent systems* [162]. A jumping curve gives rise to a coherent system which is a pure dimension 1 sheaf supported on the jumping curve, provided with a subspace of sections coming from sections of the original bundle. King and Newstead also introduced coherent systems on curves—see [30] for an overview. A coherent system may be viewed as a special case of a bundle with extra structure: in this case, the extra structure consists of a linear subspace of the space of sections (with or without a framing). These are related to the vast subject of Seiberg-Witten invariants, vortex equations and the like. There is a whole family of stability conditions depending on a real parameter, and this gave one of the first explicit and useable examples of wallcrossing, discussed in [30].

Several authors provide statements of various strengths on the injectivity or finiteness of the Barth morphism. Le Potier and Tikhomirov use coherent systems to prove that the Barth morphism is generically injective [164], and this result has been improved by others. Once we know that it is injective, it may be used to measure moduli. Tyurin, Le Potier, Tikhomirov, Ellingsrud and Strømme worked on the program of giving an algebraic calculation of the Donaldson invariants of surfaces, starting with \mathbb{P}^2 . In [71] an account is given, relating the theory to the Serre construction as we have considered in Section 7. G. Hein uses the Barth morphism to provide a construction [102] of the moduli space that doesn't use GIT, following Faltings' construction for moduli of bundles on a curve.

Looking at hypersurfaces $X \subset \mathbb{P}^3$ of degree d , after $X = \mathbb{P}^2$ which may be seen as the case $d = 1$, the next case to consider is $d = 2$ where X is a smooth quadric surface. Thus, $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ and again, X is covered by two families of projective lines. More interestingly, a general plane in \mathbb{P}^3 cuts out a plane conic $C \subset X$, which is itself a \mathbb{P}^1 and we can still study the variation of the Grothendieck decomposition of $E|_C$.

In this case, Huh [117] shows that the set of planes in \mathbb{P}^3 which cut out conics on which the decomposition jumps, is a hypersurface of degree $c_2(V) - 1$. Using a precise description of the singularities, he shows a sort of “Torelli” result that the jumping hypersurface determines the bundle, generically. With this method he can describe the moduli spaces $M_X(\mathcal{O}_X(1), c_2)$.

The general question of jumping in the stability properties of restrictions of bundles to subvarieties, is still undoubtedly most completely open. There have been a few forays in various related directions. We have seen an example of how these considerations enter in a general way in O’Grady’s method, where he looks for a curve such that the restriction is unstable. Langer and others prove strong restriction theorems which are sharp in some sense. Bruzzo formulates a conjectural statement about the relationship between stability of restrictions of Higgs bundles, and the case of equality in the Bogomolov-Gieseker inequality, which would generalize the theorem of Demailly, Peternell and Schneider for stable vector bundles [58]. Berlinger formulates a generic strong stability property for generic isomonodromic deformations of flat bundles [105]. Beyond these beginnings, there would seem to be a lot of room for studying the precise nature of the locus where the stability properties of restrictions jump.

§13. Wall-crossing

The moduli space $M_H(X; r, c_1, c_2)$ depends on the choice of H within the ample cone. It is independent of scaling by a positive multiple, so we can view $H \in NS(X) \otimes \mathbb{Q}$. The ample cone is divided into *chambers*; on the interior of each chamber the moduli space is constant as a function of H , but as H crosses a wall between two chambers it undergoes a birational transformation. Donaldson, Friedman, Morgan, Qin, Thaddeus, Hu, Li and many others since then have studied this chamber decomposition, and particularly the formula for the change of Donaldson’s invariants when moving between chambers. In general there are infinitely many chambers.

Thaddeus showed the way to use a precise analysis of the formulas giving the change in numerical invariants of the moduli space of Bradlow stable pairs, upon crossing walls of the real parameter governing stability, to obtain results on a moduli space of interest, in his proof of the Verlinde formula [270].

Hu, Dolgachev, Keel and others [61] [116] have developed a theory of variation of GIT quotients under change of the G -linearized polarization. In this case there are finitely many chambers, and it can be related

to the Mori chamber structure for different birational models of Fano varieties. Recent developments include the work of Halpern-Leistner [96] and Ballard *et al* [14].

The reason for the difference between the VGIT theory and the decomposition into chambers for $M_H(X; r, c_1, c_2)$ (which can have infinitely many pieces), is that the Hilbert scheme used to parametrize bundles in the GIT construction of the moduli space, itself depends on the choice of polarization H . It is an interesting question to try to relate these two theories.

One of the seminal works on wallcrossing was the paper of Hu and Li about the variation of Gieseker and Uhlenbeck compactifications when we vary the polarization [115] [114]. Subsequent work in this direction includes Sharpe's discussion of the location of the walls [260], Friedman and Qin [77], Yamada [282] and many others calculating the variation of Donaldson invariants, as well as a gauge-theoretic approach to that question by Hyun and Park [125], Göttsche on the change in Hodge numbers [90], Mochizuki on wall-crossing formulae for stacks [203], . . .

Wallcrossing phenomena have received particular attention in the case of moduli of bundles over rational surfaces, since one has usually a fairly large Picard group to play with. This includes works by Göttsche [90] and Qin [239, 240, 244]. A similar motivation applies for ruled surfaces, studied for example by Yoshioka [286].

The Nekrasov conjecture, prescribing a relationship between generating series for Donaldson invariants and periods of certain *Seiberg-Witten hyperelliptic curves*, also leads by consideration of higher correction terms and similarly to Witten's conjecture, to the prediction of an identity between counting invariants for vector bundles (instantons), and for certain coherent systems (Seiberg-Witten). After the original case by Nekrasov-Okounkov [222] and Nakajima-Yoshioka [218], it has been the subject of further work by Göttsche, Nakajima and Yoshioka [93, 94], Gasparim-Liu [83], Braverman, Etingof [31], and others. Göttsche, Nakajima and Yoshioka explain in [93] how the Nekrasov partition function enters into the wallcrossing formulae for Donaldson invariants, and they apply Mochizuki's wallcrossing formula [203, 204] to prove Witten's conjecture in [94].

Klyachko shows that reflexive sheaves on a toric variety, which are equivariant for the torus action, may be completely described in terms of systems of filtrations [144]. This has a number of applications. Penacchio's viewpoint on mixed Hodge structures as being semistable torus-equivariant sheaves on \mathbb{P}^2 [238] raises the interesting question of calculating the jumping subvarieties for the *second Chern class* of the bundle,

which he calls the \mathbb{R} -*splitting level*. Klyachko's result has been applied in heterotic string compactifications by Knutson and Sharpe [145, 146].

Diaconescu and Moore [60] relate wallcrossing with the theory of branes (boundary terms for gauge field theories) in physics. Chuang, Diaconescu, Donagi and Pantev [48] propose a program to derive the Hausel-Letellier conjectures on generating functions for the cohomology of character varieties, from BPS state counting and comparison with counting problems for parabolic Higgs sheaves.

13.1. Parabolic structures

A closely related direction is wall-crossing for moduli of parabolic bundles, started by Boden, Yokogawa, and Hu. In this case, changing the parabolic weights changes the notion of stability and again there is a chamber structure.

Recall that Seshadri introduced the notions of *quasi-parabolic structure* and *parabolic structure*, in order to parametrize equivariant vector bundles over ramified Galois coverings of a curve. The parabolic weights reflect the arguments of unitary monodromy eigenvalues in $U(1) \subset \mathbb{C}^*$. Parabolic bundles can therefore be put into the Narasimhan-Seshadri correspondence, extending it to quasiprojective curves. Thanks to work of Maruyama and Yokogawa, parabolic structures are extended to the higher-dimensional case, and to Higgs bundles and bundles with connection by numerous works culminating in Mochizuki's [205]. By Seshadri, Boalch and Bala \acute{c} i, the notion of parabolic structures may be extended in a not completely obvious way to other structure groups as *parahoric structures* [13], [24].

Parabolic structures enter into Kostov's *Deligne-Simpson problem* [151] through the work of Crawley-Boevey [55] using quivers, and this may be related to Gromov-Witten theory as pointed out for example by Teleman and Woodward [269] and others. Recently Soibelman [262] gives a different approach using ideas from geometric Langlands theory. Parabolic structures with rational weights are closely related to vector bundles over root stacks [25], [131], [267]. Szabo's work gives a Fourier-Nahm transform for parabolic Higgs sheaves [265].

This is a vast subject and to give a complete discussion would go beyond our present scope. Its close relation to the subject of wallcrossing comes about because the parabolic stability condition depends on the choice of parabolic weights. We get a chamber structure and birational modifications of the moduli spaces as we cross from one chamber to the next. These have been used by many authors to study the variation of counting invariants, cohomological invariants, and geometrical structures. The combinatorial structure of the system of walls has already

been the subject of many works, but it remains an important and largely open question for further study.

13.2. Bridgeland stability conditions

The notion of wallcrossing has recently taken on quite a new flavor, when combined with Bridgeland’s stability conditions. The foundational work of Kontsevich and Soibelman brings this into relation with BPS states in physics, asymptotics of differential equations, and mirror symmetry.

Bridgeland’s notion of stability condition on a triangulated category [34] gives rise to a *space of stability conditions* which has been studied a lot. One of the main motivations for the introduction of stability conditions on derived categories was the existence of Fourier-Mukai correspondences [208] between derived categories of different varieties. These correspondences do not, in general, preserve the original basic t -structures corresponding to the abelian categories of coherent sheaves. Therefore, it became quite natural to look for a notion of stability condition on a derived category, rather than an abelian category as had been considered by Rudakov [251].

Definition 13.1 (Bridgeland). *A stability condition on a triangulated category D consists of a central charge function $Z : K(D) \rightarrow \mathbb{C}$, and a collection of subcategories D_ϕ of objects whose central charge lies on the ray of angle $\pi\phi$, thought of as the “semistable objects of slope ϕ ”. These are required to satisfy some conditions, that $D_{\phi+1}$ is the shift of D_ϕ , that there are no extensions from objects of D_ϕ to objects of D_ψ for $\psi < \phi$, and there should exist “Harder-Narasimhan filtrations”.*

Beyond going towards the idea of working with the derived category, one of the new ideas here is the introduction of the “central charge” function, that is to say including the real number $|Z(E)|$ rather than just the phase or “slope” as data for an object E . Bridgeland’s motivation for this came from string theory and mirror symmetry, through the work of Douglas [64]. In this way the central charge function becomes a complex parameter, and the space of stability conditions gains (in many cases) a natural structure of complex manifold [34].

Here is a heuristic and somewhat vague way of thinking about the complexification of the central charge. Soibelman suggested that one should think that there is a relationship between the walls for parabolic stability conditions, and the walls in the Bridgeland space. We can interpret the central charge in this light. Recall that one of the original ways of thinking about vector bundles of nonzero degree on a curve,

was via the notion of a somewhat trivial parabolic structure at a single point, involving only one parabolic weight. This appeared already in Narasimhan-Seshadri [221]. The slope of the bundle is the parabolic weight which should be attached to a point in order to get a flat bundle. Locally, such a parabolic weight corresponds on the side of unitary flat bundles, to the angle of the scalar unitary monodromy transformation. Think of relaxing the condition that the monodromy be unitary; it then becomes a complex number, whose angle corresponds to the parabolic weight. In a heuristic way at least, we may think of the central charge as being a complex scalar monodromy transformation, whose angle corresponds to the slope of the bundle.

The study of the moduli space of Bridgeland stability conditions has recently become a major field of interest. In his thesis, Lowrey studies the action of autoequivalences on spaces of stability conditions [177]. Arcara, Bertram, Coskun and Huizenga study moduli spaces of Bridgeland stable objects over \mathbb{P}^2 [7], and Yoshioka looks at abelian surfaces [294]. Bayer and Macrì [18] use wallcrossing for Bridgeland stability conditions to study the geometric structures appearing in the minimal model program, particularly the nef, movable and effective cones. The link with BPS state counting and Donaldson-Thomas invariants leads to a whole subject with vast relationships to many of the other topics we have been considering.

13.3. Wallcrossing for Donaldson-Thomas invariants

Recent work by Kontsevich and Soibelman, as well as by Joyce and several other groups, gives a formula for wall-crossing for Donaldson-Thomas invariants. The Donaldson-Thomas invariants are a new version of Donaldson's classical invariants, involving counting virtual numbers for the moduli space of sheaves. Donaldson-Thomas [271] consider sheaves of rank 1, degree 0 and fixed classes c_2 and c_3 on a Calabi-Yau 3-fold. In this case, very analogously to Mukai's original observation for sheaves on $K3$ surfaces, the obstruction can be modified so that the virtual dimension of the moduli space is zero. On a CY 3-fold, $\text{Ext}^i(E, E)$ is dual to $\text{Ext}^{3-i}(E, E)$, so for a simple sheaf we have $\text{Ext}^3(E, E) \cong \mathbb{C}$ so the trace-free part vanishes. Then $\text{Ext}^1(E, E)$ is dual to $\text{Ext}^2(E, E)$ so the expected dimension is zero.

Therefore, we can define a 0-dimensional virtual fundamental class of the moduli space, an Euler class of the obstruction theory which is an integer, the *DT-invariant*. The analogy with 3-dimensional topology is why Thomas originally viewed this as a generalization of Casson's invariant.

Since their introduction, these invariants have been studied by many authors, and they provide a fertile ground for the study of wall-crossing phenomena. Pandharipande and Thomas generalize the DT-invariants in the direction of Le Potier’s “coherent systems” [235], see also Stoppa and Thomas [264], and Szendrői generalizes to a noncommutative setting [266]. In keeping with the analogy with 3-dimensional topology, Nagao and Nakajima [217] extend from sheaves to certain complexes, the *perverse coherent sheaves* of [9].

Kontsevich and Soibelman extend the DT-invariants to a *motivic DT-invariant*, then go on to propose a wall-crossing formula explaining how the invariant changes when we change the stability condition. Here, the notion of “stability condition” needs to be interpreted in Bridgeland’s sense, as a stability condition on a triangulated category. Kontsevich-Soibelman extend the notion of DT invariants to this categorical context, which among other things gives an extension of the theory to noncommutative CY 3-folds. See [132], [148, 149, 150], and Stoppa and Thomas [264] who give an introduction to the theory explaining how it works in a single wall-crossing relating DT and PT invariants.

13.4. Walls in the Hitchin base

The Kontsevich-Soibelman wallcrossing is conjecturally related, by mirror symmetry, to wallcrossing in the base of the Hitchin fibration on the moduli space of Higgs bundles [150].

This should fit into a theory of *stability Hodge structures* proposed by Kontsevich-Soibelman, Katzarkov and others, in which the moduli space of stability conditions gains a collection of structures very analogous to the moduli space of Higgs bundles, and which we could think of as a generalized type of “nonabelian Hodge structure”.

While it would go beyond our scope to delve into these matters here, we can do a very simple calculation which shows, in a very first case, the kinds of walls that we are talking about in the Hitchin base. Some people have asked the second author about that in recent conferences, and we hope that the following discussion can provide an indication to enter into this theory.

Consider the Hitchin fibration for the moduli space M_H of rank 2 Higgs bundles on $\mathbb{P}^1 - \{t_1, t_2, t_3, t_4\}$ with parabolic structure at the four points. The simplest possible case is when the parabolic weights are $0, 1/2$, and the corresponding monodromy representation has conjugacy classes

$$C_i \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

at each of the singularities. Denote by M_B the character variety of representations whose monodromies lie in these conjugacy classes.

This case, while formally similar to the general Painlevé VI situation, is actually a much easier special case. Indeed, let $p : Y \rightarrow \mathbb{P}^1$ be the elliptic curve branched over t_1, t_2, t_3, t_4 . If $E \in M_H$ then the pullback decomposes into a direct sum of two Higgs line bundles

$$p^*(E) \cong (L_1, \varphi_1) \oplus (L_2, \varphi_2)$$

on Y with trivial parabolic structure. They are interchanged by the elliptic involution of Y . Similarly, if $V \in M_B$ is a local system with monodromy in the conjugacy classes C_i , then its pullback decomposes into a direct sum of two rank 1 local systems

$$p^*(V) \cong U_1 \oplus U_2$$

which extend over the compact elliptic curve Y because the conjugacy classes C_i are of order two. Again U_1 and U_2 are interchanged by the elliptic involution of Y .

The determinant of any such local system V is always the rank 1 local system on $\mathbb{P}^1 - \{t_1, t_2, t_3, t_4\}$ with monodromies -1 at the singularities. It pulls back to the trivial local system on Y . Therefore, we have $U_2 \cong U_1^*$, and the space of possible pullbacks may be identified as the rank 1 character variety of Y . The choice of ordering of the two rank 1 local systems is not canonical, so M_B is the quotient by interchanging the factors U_i , which is the same as the action of the elliptic involution.

If we choose a basis $\{\gamma, \nu\}$ for $H_1(Y, \mathbb{Z})$ then we can write the rank 1 character variety of Y as $M_B(Y, 1) \cong \mathbb{G}_m^2$ with coordinates $a, b \in \mathbb{G}_m$ where a is the monodromy around γ and b the monodromy around ν . The elliptic involution acts on $M_B(Y, 1)$ by sending U to U^* , in other words by $(a, b) \mapsto (a^{-1}, b^{-1})$, and M_B is the quotient of $M_B(Y, 1)$ by this involution. Notice that it has four double points. Its classical *Fricke-Klein equation* is

$$(13.1) \quad xyz - x^2 - y^2 - z^2 + 4 = 0$$

where

$$\begin{aligned} x &:= a + a^{-1} \\ y &:= b + b^{-1} \\ z &:= (ab) + (ab)^{-1} \end{aligned}$$

The coordinates x, y, z are traces of monodromies around certain paths in $\mathbb{P}^1 - \{t_1, t_2, t_3, t_4\}$ chosen in relation to the choice of basis γ_1, γ_2 . It is easy check directly (13.1).

For more general choices of conjugacy classes C_i , one still has an equation of the form (13.1) but with a general linear term, not as easy to check (a shortcut method was discussed in the conference talk). Such an affine equation defines a cubic surface minus a triangle of projective lines, and Goldman and Toledo show that all such configurations arise as Painlevé VI character varieties as the 4-tuples of conjugacy classes range over all possibilities [87].

The triangle of lines is given by the homogeneous degree 3 equation $xyz = 0$ in the \mathbb{P}^2 at infinity of our affine 3-space. It basically says that x, y, z cannot all three become large at the same time. The lines are the places where one of the coordinates is small, and the intersection points are the places where two of the coordinates are small with respect to the third one.

Direct calculation allows one to identify these regions in terms of the Hitchin fibration on the moduli space M_H . The same discussion as above holds for M_H , in particular the two Higgs line bundles (L_j, φ_j) are dual to each other and interchanged by the involution. Thus $\varphi_2 = -\varphi_1$. The invariant determining the spectral curve of (E, ϕ) is the *quadratic differential* $\alpha := (\varphi_1)^2 = (\varphi_2)^2$. The Hitchin base is the space of such quadratic differentials, which one sees to have simple poles at the four points. Thus, the Hitchin base is \mathbb{C} . In terms of the elliptic curve, it is the quotient of $H^1(Y, \Omega_Y^1)$ by the involution (-1) .

In the rank 1 case, one can write explicit formulas for the correspondence between Higgs bundles and local systems. The line bundles L_j contribute unitary local systems, whose monodromies will be denoted generically by $e^{i\theta}$ with each occurrence of θ designating a different angle. Now, if U is the local system corresponding to (L_1, φ_1) , its monodromy transformations are

$$a = e^{i\theta} e^{\int_\gamma \operatorname{Re} \varphi_1}$$

and similarly for b with ν and (ab) with $\gamma + \nu$. The monodromy transformations for (L_2, φ_2) are, since $\varphi_2 = -\varphi_1$,

$$a = e^{i\theta} e^{-\int_\gamma \operatorname{Re} \varphi_1}$$

and similarly for b and (ab) . Thus, the monodromy for $p^*(V)$ is

$$x = a + a^{-1} = e^{i\theta} e^{\int_\gamma \operatorname{Re} \varphi_1} + e^{-i\theta} e^{-\int_\gamma \operatorname{Re} \varphi_1}$$

idem for y and z . Choose an initial quadratic differential α_0 , choose a square root φ_0 and let $A := \int_\gamma \varphi_0$ and $B := \int_\nu \varphi_0$ be the complex periods (one could normalize so that $A = 1$ and $B = \tau$). The period for $\gamma + \nu$ is just $A + B$. Now, another quadratic differential is $\alpha = t\alpha_0$, and

the above formula reads

$$x = e^{i\theta + \Re(A\sqrt{t})} + e^{-i\theta - \Re(A\sqrt{t})},$$

similarly for y with B and z with $(A + B)$. Asymptotically as $t \rightarrow \infty$ along a real ray, we have

$$|x| \sim e^{|\Re(A\sqrt{t})|}, \quad |y| \sim e^{|\Re(B\sqrt{t})|}, \quad |z| \sim e^{|\Re((A+B)\sqrt{t})|}.$$

We may now look at the zones where one of the coordinates is bigger than the other two: for example, $|x| \gg |y|, |z|$ whenever

$$|\Re(A\sqrt{t})| > |\Re(B\sqrt{t})|, |\Re((A + B)\sqrt{t})|.$$

and similarly for the others. These conditions may be seen to divide the complex t plane up into three triangular sectors near $t = \infty$; the angles of the lines separating the sectors depend on the periods A, B of the elliptic curve, hence on the cross-ratio of t_1, t_2, t_3, t_4 via the j -function. The complex t -plane is the Hitchin base, and the lines are the walls. A given sector between two walls, will be sent by the nonabelian Hodge correspondence to points very near to the corresponding one of the three double points of the three lines in the compactification of M_B . A very small neighborhood of the line separating two sectors will go to the points on the full \mathbb{P}^1 in between the two double points. Thus, we see a phenomenon whereby large sectors near the boundary of M_H go to small regions near the boundary of M_B , and small regions near the boundary in M_H go to large regions near the boundary in M_B .

This was a very simple explicit calculation, but it resumes the behavior which is described in much greater generality by Kontsevich and Soibelman in their wall-crossing theory ¹ of [150]. In recent work with Katzarkov, Noll and Pandit [136], we discuss some ideas relating the spectral networks of Gaiotto-Moore-Neitzke [80] with limiting harmonic maps to buildings of Parreau [237] and Kleiner-Leeb [143], which should, in principle, lead to an analytic description of the walls in the Hitchin base in general. The relationship between wallcrossing in the Hitchin base and the space of stability conditions on categories, following Kontsevich-Soibelman's program, is being studied by many people—we can cite for example Bridgeland-Smith [37] but to give a full treatment, even just of the references, would go beyond our scope.

¹We cannot, however, pretend to have given a full description of their theory as it pertains to the very simple Hitchin system discussed here—this would be a very nice and useful contribution if somebody could do it.

§14. Betti numbers

One very natural question about the moduli spaces of vector bundles is to ask for the calculation of their Betti numbers. For the moduli space of curves, the answer is given by a classical result of Harder and Narasimhan [97], using Tamagawa numbers to count points then apply Deligne's Weil conjectures; Desale (Bhosle) and Ramanan [59] who give a geometrization of this approach; and Atiyah and Bott [12] who give a gauge-theoretical argument.

For vector bundles on higher-dimensional varieties, the question remains quite open in the large majority of cases. Some results have nonetheless been obtained.

In 1992 in Toulouse, Maruyama presented Yoshioka's calculations of the Betti numbers of $M_H(\mathbb{P}^2, 2, -1, n)$, and from the table it is clear that for each i , $b_i(M)$ stabilizes as $n \gg 0$. In fact, this is the *Atiyah-Jones conjecture* which says that the map

$$M_H(X; r, c_1, c_2) \rightarrow \frac{\{\text{connections}\}}{\text{gauge equivalence}}$$

should induce an isomorphism on H_i for $i \leq k$ and $\tilde{c}_2 \gg_k 0$. On the right side the Betti numbers stabilize, and the inclusion of the Uhlenbeck boundary provides canonical maps

$$H_i(M(r, c_1, c_2), \mathbb{Z}) \rightarrow H_i(M(r, c_1, c_2 + 1), \mathbb{Z}).$$

The Atiyah-Jones conjecture says that these become isomorphisms for large \tilde{c}_2 .

The original Atiyah-Jones conjecture was for spaces of instantons on S^4 ; however, over a complex Kähler surface these are the same as the moduli spaces of stable bundles, by the Kobayashi-Hitchin correspondence.

Hurtubise and Milgram proved the Atiyah-Jones conjecture for ruled surfaces [119], and Gasparim [82] has shown that it is stable under blow-ups, so the AJC holds for all rational surfaces. These results are for bundles of rank 2.

The case of b_1 and b_2 for bundles of rank $r = 2$ on an arbitrary surface is treated by Li [169], see also O'Grady [232] for a discussion of the Hodge structure. More details in the special cases of \mathbb{P}^2 , K3 surfaces, elliptic surfaces, abelian surfaces, ... have been treated by many authors. Bridgeland relates these moduli spaces using Fourier-Mukai transform [33], leading to relations on the Hodge numbers as was pointed out for example in [36, 271]. Choi-Maică obtain the Betti and Hodge numbers for some moduli spaces of sheaves of dimension 1 on \mathbb{P}^2 [45].

A similar problem is to calculate the Picard group (we haven't included here the many references there). In some cases, bundles of rank $r > 2$ have also been treated. Kamiyama-Tezuka (2007) calculate the Chow ring of $M(\mathbb{P}^2, G, 0, 1)$ for classical groups G .

Göttsche and Huybrechts consider the Hodge numbers of the moduli spaces of stable bundles on a K3 surface [92]. They show that for a suitable polarization, and when the dimension of the moduli space is at least 9, then the Hodge numbers of the moduli space coincide with the Hodge numbers of an associated Hilbert scheme of points (this may also be seen as a consequence of the deformation equivalence of these spaces [120, 291]). Göttsche and Huybrechts use Le Potier's method with coherent systems, which is basically a wall-crossing formula. Göttsche expanded the method and applied it to rational surfaces in [91], and there are many related works in this direction on understanding the change of invariants under wallcrossing. One may note the proof in [92] of irreducibility of the moduli spaces when the dimension is at least 9, a statement they attribute originally to Mukai, for which we now have a general proof as has been mentioned above.

Calculation of the Betti numbers of moduli spaces of Higgs bundles, parabolic Higgs bundles, or equivalently character varieties for Riemann surfaces, constitutes an important open question. After Hitchin [110] originally introduced this question and provided the answer for rank 2, the case of rank 3 has been solved by García-Prada, Gothen and Muñoz [81]. Beyond these, we don't have a general theorem, but Heinloth and García-Prada have recently proposed a technique [104], and Chuang, Diaconescu, Donagi and Pantev propose a wall-crossing approach related to geometric Langlands [48]. Perhaps the most important recent progress is the conjectural answer formulated by Hausel, Rodríguez-Villegas and Letellier [99, 100]. They propose in fact a more precise answer for the mixed Hodge numbers, in a circle of ideas related to the $P = W$ conjecture [57] relating the weight filtration on the cohomology of the character varieties to the perverse Leray filtration for the Hitchin fibration.

§15. Poincaré bundles and universal families

Starting with an étale-locally fine moduli space, in what sense does the universal family exist globally?

Fix X , a hyperplane class H , and (r, c) . The *moduli functor* $\mathcal{F} = \mathcal{F}_H(X; r, c)$ assigns to any scheme S the set of isomorphism classes of torsion-free sheaves E of rank r on $X \times S$, flat over S , such that for any closed point $s \in S$ the fiber E_s is H -semistable with Chern classes $c_i(E) = c_i$. The moduli space $M = M_H(X; r, c)$ corepresents

the functor: there is a map of functors $\mathcal{F} \rightarrow M$ universal for maps from \mathcal{F} to schemes.

A *universal family* is a torsion-free sheaf E on $X \times M$, giving an element of $\mathcal{F}(M)$, such that the resulting map $M \rightarrow M$ is the identity. If $U \subset M$ is an open set, then a *universal family over U* is an element of $\mathcal{F}(U)$ whose projection to $M(U)$ is the inclusion $U \subset M$. Similarly for étale open sets.

The question of whether or not there exists a universal family, is important and interesting. However, not much is known about it.

An intermediate question is whether M is a *Zariski-fine moduli space*, meaning that M is covered by Zariski open sets on which there exist universal families. If M is Zariski-fine, and parametrizes simple sheaves then there exists a global universal family.

In a quite general relative situation X/S , Maruyama gave a sufficient criterion for existence of a universal family in his paper “Stable sheaves II”: let a_i denote the normalized integer coefficients of the Hilbert polynomial $P(n) = \sum a_i C_n^{n+i}$, then if $\gcd\{a_i\} = 1$ there exists a universal family on the moduli space of e -stable sheaves on X/S with the given Hilbert polynomial. He refers to the technique used for curves by Mumford and Newstead [216].

After results of Ramanan [245], Newstead [223] and others concerning curves, in a series of papers by Le Potier [161], Hirschowitz-Narasimhan [108], the first author [195], Drézet [66], Yoshioka [288], . . . , it was shown that an appropriate variant of Maruyama’s sufficient condition is also necessary, in the case of vector bundles of rank 2 over a surface, provided that \tilde{c}_2 is sufficiently large.

We recall here the geometric method of [195]. Suppose $c_1 = 2D$, in which case $\frac{1}{4}c_1^2$ and $\frac{1}{2}(c_1^2 + c_1K)$ have the same parity since $D^2 + KD$ is even. If $c_2 - \frac{1}{4}c_1^2$ is even, then by looking at elementary transformations one can show that there is no universal bundle.

Let $f : \tilde{X} \rightarrow \mathbb{P}^1$ be a Lefschetz pencil. If $\{x, y\} \subset \mathbb{P}^1$ is a pair of distinct points, let $C = C_{x,y} := f^{-1}(x) \cup f^{-1}(y)$, and consider vector bundles in an exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{X}}^2 \rightarrow E \rightarrow \mathcal{F}_C \rightarrow 0$$

where \mathcal{F}_C is a line bundle on C with the same negative degree on each component. Projecting back down to X we get a family of vector bundles which cannot admit a Poincaré family. If we fix C and \mathcal{F}_C , let C' and C'' be the two components and fix two-dimensional subspaces $V' \subset H^0(C', \mathcal{F}')$ and $V'' \subset H^0(C'', \mathcal{F}'')$. Then to give an elementary transformation vector bundle E as above essentially amounts to giving

an identification $V' \cong V''$ up to scalars, and this two-dimensional vector space would be $H^0(E)$. However, as we move around in the moduli space, the order of the two points x, y is in general changed; and the space of lines parametrized by such identifications on a pair of interchangeable projective lines, is a non-banal conic bundle, as can be seen by a geometric argument involving conic bundles degenerating to two lines of Hirschowitz-Narasimhan [108] and Newstead [224]. Thus, there can't be a Poincaré bundle. This geometric argument actually shows that the obstruction is topological, see Nitsure [226].

This method didn't allow to treat the case where c_1 is *numerically even* i.e. $\forall D, c_1 \cdot D \in 2\mathbb{Z}$ but not two times a divisor.

Yoshioka's method [288] was to let α denote the Mukai vector; and $Q(\alpha)^s$ the stable points of the Hilbert scheme whose quotient by $G = GL(N)$ is $M_H(\alpha)$. By GIT considerations, there is a universal family if and only if there is an element of the equivariant Neron-Severi group $NS_G(Q(\alpha)^s)$, whose restriction to the fiber over a point in $M_H(\alpha)^s$ has degree 1. Using results of Jun Li [168], Yoshioka calculates the G -equivariant Picard group of $Q(\alpha)^s$: in the case of irregularity 0 it is just the group of cycles modulo homological equivalence denoted $K(X)_{\text{alg}}$, and the map taking an element to its degree on a fiber, is $x \mapsto \chi(\alpha \otimes x)$. Thus the condition:

Theorem 15.1 (Yoshioka [288]). *For bundles of rank 2 and Mukai vector $\alpha \in K(X)_{\text{alg}}$, there is a universal family if and only if there exists $x \in K(X)_{\text{alg}}$ such that $\chi(\alpha \otimes x) = 1$.*

Problem 15.2. *Give a geometric demonstration of the necessary condition, in the case c_1 numerically even but not even. Is the obstruction topological in that case?*

The question of existence of a universal family is also closely related to the geography of moduli spaces: in the intermediate range $0 \leq \tilde{c}_2 \leq \text{const}(X, H)$, there may be several different irreducible components of the moduli space, and we expect in general that some of them might have universal families while others not.

In the cases of K3 and abelian surfaces where the moduli spaces are irreducible for all \tilde{c}_2 it would make sense to expect to have a simple numerical criterion for existence of a universal family; we don't know if this is known.

Drézet looks at "exotic fine moduli spaces", i.e. moduli spaces for which there exist universal families over Zariski open sets, but which don't correspond to stable sheaves [67]. For example, he constructs fine moduli spaces for *wide extensions*, extensions of line bundles with very different degrees.

As for many of the items discussed above, the generalization to rank $r \geq 3$ hasn't been done in the literature (although Drézet's partial result [66], concerns any r for rational surfaces). For existence of the universal family, since Maruyama's condition depends closely on the rank, it is an important open problem and gap in the literature.

We can also refine the question to become a question about the obstruction class in the Brauer group. Assume that there is a universal family étale-locally (eg for moduli of stable sheaves), then there is by definition a universal family on the *moduli stack* \mathcal{M} , and the map $\mathcal{M} \rightarrow M$ is a \mathbb{G}_m -gerb. This is classified by an element η of the *Brauer group* $H^2(M^{\text{ét}}, \mathbb{G}_m)$. The analytification morphism

$$H^2(M^{\text{ét}}, \mathbb{G}_m) \rightarrow H^2(M^{\text{an}}, \mathcal{O}^*)$$

induces an isomorphism between the Brauer group and the group of torsion elements of $H^2(M^{\text{an}}, \mathcal{O}^*)$. On the other hand the exponential exact sequence gives

$$H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathcal{O}) \rightarrow H^2(M^{\text{an}}, \mathcal{O}^*) \rightarrow H^3(M, \mathbb{Z}) \rightarrow H^3(M, \mathcal{O}),$$

so we get

$$0 \rightarrow \left(\frac{H^2(M, \mathcal{O})}{H^2(M, \mathbb{Z})} \right)_{\text{tors}} \rightarrow H^2(M^{\text{ét}}, \mathbb{G}_m) \rightarrow H^3(M, \mathbb{Z})_{\text{tors}} \rightarrow 0.$$

Question 15.3. *If we can calculate the group $H^2(M^{\text{ét}}, \mathbb{G}_m)$ explicitly using the above exact sequence, what is the class η in the case where there doesn't exist a universal family? For example what is its projection to the topological obstruction group $H^3(M, \mathbb{Z})_{\text{tors}}$?*

Problem 15.4. *Study the relationship between the obstruction class for existence of a universal family, and the stabilization maps on homology between $M_H(X; r, c_1, c_2)$ and $M_H(X; r, c_1, c_2 + 1)$. This might give an approach to studying the universal family question for smaller values of c_2 .*

The above questions can also be phrased when we have a family of varieties X/S . In this case we get a family of moduli spaces of vector bundles $M(X/S) \rightarrow S$.

Problem 15.5. *Determine when there exists a universal family on the full family of moduli spaces $M(X/S) \times_S X$.*

This is already an interesting question for families of curves ([197] ...), but is also reasonable to ask in the case of families of rational surfaces or K3 surfaces, the universal hypersurface, ...

§16. Related moduli problems

In our discussion up until now, we have mostly concentrated on moduli spaces of vector bundles or maybe coherent sheaves. There are a number of related moduli questions which, while looking like generalizations of the vector bundle case, actually introduce new phenomena of their own. The following light overview will be far from exhaustive.

16.1. Other structure groups

The theory of moduli of principal G -bundles brings with it several specific difficulties. For example, the classical Uhlenbeck and Gieseker compactifications of the moduli spaces of stable vector bundles are obtained by including all torsion-free sheaves into the moduli problem. This is very specific to the structure group $GL(r)$, and it is not at all obvious how to extend it to bundles with other, say reductive structure groups. This question has been a subject of recent research, by Langer, Gomez, Schmitt, Sols, Balaji, and others, see [257].

Somewhat similarly, the notion of “parabolic structure” needs a major modification, towards the notion of “parahoric structure”, in the case of most other structure groups [13] [24].

Another interesting question is how to define moduli problems combining the notion of sheaves supported on strict subvarieties, with the idea of principal bundles for structure groups other than GL_r . As a general matter, we don’t even know what that should mean.

Nonetheless, Sorger made important progress in this direction, investigating the moduli problem for sheaves with certain kinds of quadratic structures [263]. His idea was that a bilinear form on a vector bundle, used to define the notion of orthogonal vector bundle for example, should be replaced by a cohomological form defined using Ext sheaves. Thus, an orthogonal structure for a sheaf F could be something like a map

$$F \rightarrow \underline{Ext}^1(F, \omega_X).$$

An interesting problem for future research is to extend these ideas to many of the newer moduli problems discussed elsewhere in the present paper.

Even after the techniques which pose major new difficulties in going from vector bundles to principal bundles, most of the questions concerning moduli spaces that we have discussed above should also naturally be posed for moduli spaces of G -bundles. These have been considered in some cases (for example Tian considers the Atiyah-Jones conjecture for classical groups [272]), but not all that many, and most questions are pretty much open in this regard.

16.2. Bundles with extra structure

One of the first examples of moduli of bundles with extra structure were the *stable pairs* introduced by Hitchin [109] [110]. These consist of a vector bundle E together with a *Higgs field* $E \in H^0(\text{End}(E) \otimes \Omega_X^1)$. The terminology “Higgs bundle” was used by the second author.

It was originally envisioned in Higgs’ paper [106], that a field of extra structure in gauge theory could take values in any associated tensor field. In geometrical terms, it means that for a principal G -bundle, we could look at extra structure with values in any bundle obtained from the tensor algebra of representations of V together with the tangent bundle of the base manifold.

Perhaps the simplest case is that of a bundle with a section. This intervened in an important way in gauge theory with Seiberg-Witten invariants. In algebraic geometry, moduli problems for bundles with a section, or perhaps several sections providing a framing, have long played an important role, starting with Bradlow’s stability condition related to the vortex equations [27], Le Potier’s work on coherent systems [162, 163], and continuing work of Bradlow, Daskalopoulos, García-Prada, Gómez, Muñoz, Newstead and others on Bradlow pairs [28, 29, 30], with related considerations for objects such as *holomorphic chains* [3, 104]. In recent times, we can cite for example the work of Bruzzo, Sala and others on symplectic moduli spaces of framed sheaves [252] [39]. These appeared in the Nekrasov conjecture as discussed previously.

We should also mention moduli spaces of connections and logarithmic connections [228, 129, 130], which it would go beyond our scope to treat here. In positive characteristic, these have entered into the works of Ogus, Vologodsky, Langer, Zuo and others. Tortella provides a general framework in which to consider connections, or Higgs fields, along foliations and other Lie algebroids [274]. A vast subject way beyond our scope concerns the combinatorics and geometry surrounding isomonodromic deformations, see Boalch [23].

In the spirit of “noncommutativizability”, recent work of Hitchin and Schaposnik [111] points to an interesting new phenomenon. The classical fibers of the Hitchin map for complex groups are generically abelian varieties, being moduli spaces of rank 1 sheaves over spectral curves. Hitchin and Schaposnik show that for certain real groups closely related to the quaternions: $SL(m, \mathbb{H})$, $SO(2m, \mathbb{H})$ and $SP(m, m)$, the generic fiber of the Hitchin map is a moduli space of rank 2 bundles over a spectral curve. In these cases, the geometry of the Hitchin fibration will be related to noncommutative Brill-Noether.

For curves, one of Hitchin's original viewpoints was that the moduli space of Higgs bundles contained as a large open set, the cotangent bundle of the moduli space of stable vector bundles [109]. As we move to moduli spaces of vector bundles on surfaces, the analogous statement is to say that the space of pairs consisting of a bundle together with an endomorphism twisted by taking values in the dualizing line bundle, naturally appears as the space of *obstructed bundles* or more precisely, bundles together with a vector dual to the space of obstructions. This was first exploited by Donaldson [63] and then Zuo in his proof of generic smoothness [297], see also Langer [156] and [198].

16.3. Twisted sheaves

Inspired once again by the idea of looking at the relationships between moduli spaces for different numerical invariants, a very natural extension is to look at *twisted sheaves*. In the most concrete terms, twisted bundles may be viewed as defined by a modified or "twisted" version of the usual cocycle relation. A more abstract and usually technically more useful version is to view twisted bundles as being vector bundles provided with an action of a sheaf of *Azumaya algebras* representing an element of the Brauer group. An alternate point of view is to look at vector bundles over gerbs. The extension to a notion of twisted coherent sheaf becomes automatic.

The moduli theory of twisted sheaves, and its applications to the period-index problem, has been extensively investigated by Lieblich in a whole series of papers [172, 173]. See [176] for a survey. They have also been considered by several other authors, [293] It becomes natural to do a twisted version of the Fourier-Mukai transform, see Huybrechts-Stellari [124], Canonaco-Stellari [42], and Minamide-Yanagida-Yoshioka [201] for example. Căldăraru studied such derived equivalences in his thesis [40]. Very recently, Antieau classified completely the derived equivalences between affine schemes with twisting classes [5].

Following Lieblich's work, Reede looks at moduli spaces for modules over a sheaf of division rings in his thesis [246].

Connecting with various recent works on the Tate conjecture for K3 surfaces, Lieblich, Maulik and Snowden use twisted sheaves, and a close analysis of the action of twisted Fourier-Mukai transform on the Mukai lattice, to prove that the Tate conjecture for K3 surfaces is equivalent to the finiteness of the number of K3's over a given finite field [175].

16.4. Moduli of complexes

Several authors have started to investigate moduli spaces for complexes, rather than just sheaves. This is the most natural setting in which to consider Fourier-Mukai transforms [122].

Passing from sheaves to complexes of sheaves means we go from an abelian category to its derived triangulated category. Thus, the theory of moduli of complexes is closely related to the ideas of Bridgeland about stability conditions on triangulated categories, see Definition 13.1 above.

Bridgeland [35], and then Arinkin and Bezrukavnikov [9] introduced a notion of *perverse coherent sheaf* which is a perfect complex enjoying support properties analogous to those of usual perverse constructible sheaves. These behave sufficiently like coherent sheaves—for example one can decompose to simple objects which have trivial endomorphism algebras—that their moduli theory is very similar, in particular we can envision a moduli space.

Inaba [128] and Lieblich [174] consider moduli of perfect complexes. Arcara, Bertram and Lieblich apply this to Bridgeland stability [8].

Toën and Vaquié construct a higher derived moduli stack for all perfect complexes [273]. Here, the more complicated structure of the complexes is reflected in the more complicated kind of object that plays the role of moduli space: a higher derived stack that has both derived and stacky directions making it so the cotangent complex is nontrivial in both positive and negative degrees.

Hein and Ploog give a canonical stability-type condition for complexes on a curve, using Postnikov truncation and the question of existence of an orthogonal sheaf [103]. One main feature is that their notion of stability is preserved under Fourier-Mukai transform. They go on to use their stable complexes to compactify the moduli of rank two bundles on a ruled surface.

Yoshioka uses perverse coherent sheaves to extend the Fourier-Mukai type duality between K3 surfaces and their moduli spaces of sheaves, over certain singularities [295].

16.5. Non-projective surfaces

We have been discussing a lot about the geography of moduli spaces of vector bundles on surfaces, depending on the location of the surface in the Kodaira classification. Andrei Teleman and others have initiated a very interesting direction of research about vector bundles on non-algebraic surfaces, in particular on surfaces of Kodaira class VII. Teleman uses vector-bundle techniques to make progress in the still-open classification question for these surfaces. It would go beyond our scope to discuss this in detail here, the reader is referred to [268].

Voisin considers hypothetical vector bundles on non-projective Kähler varieties, showing that in some cases they cannot exist. Therefore, certain versions of the Hodge conjecture cannot be true in the Kähler case [280].

§17. Vector bundles on hypersurfaces

One way to concretize the question of geography is to look at vector bundles on a smooth hypersurface $X \subset \mathbb{P}^3$ of degree d . We pass from the rational case when $d = 3$, through the K3 case $d = 4$, into the range of surfaces of general type for $d \geq 5$. These particular cases have therefore attracted a certain amount of attention. The first general-type case, when $d = 5$, will be the subject of further discussion in the next two sections.

For hypersurfaces of higher degree, it is not easy to envision a full classification. Recall our Question 5.4, at which degree do we start to see several different irreducible components showing up? When the surface is rational, abelian or K3 then the moduli spaces are basically irreducible, and for quintic hypersurfaces we show that the phenomenon persists, as will be discussed in Section 19 below. By [196], non-irreducibility has to show up sometime before $d = 27$; we feel, based on the complicated nature of the discussion of [198, 199, 200] for $d = 5$, that the moduli space will probably start having several irreducible components around $d = 6$ or 7, but we have no firm evidence.

As the degree gets bigger, it will probably be difficult to understand the moduli space for the full range of intermediate values of c_2 . However, we should still expect to be able to say a lot about bundles whose c_2 are very small with respect to the degree of the hypersurface. Later in this section we formulate a question 17.2 concerning the case when $c_2 \leq d$. Before getting there, we mention a couple of other areas of investigation.

17.1. Arithmetically Cohen-Macaulay bundles

A vector bundle V over X is said to be *arithmetically Cohen-Macaulay (ACM)* if $H^i(V(m)) = 0$ for all m and any $0 < i < \dim(X)$. Faenzi completes the classification of these bundles over smooth cubic surfaces [74], after an initial construction by Arrondo and Costa [11].

Chiantini and Faenzi study ACM bundles on a general quintic surface [44]. They show that, up to a twist, there are only finitely many possibilities for the Mukai vector and they give the complete classification of the combinatorial possibilities. In our own work on the moduli spaces of stable bundles on a very general quintic hypersurface, looking at the component of moduli for $c_2 = 10$ corresponding to bundles with

seminatural cohomology plays an important role of junction between two ranges where differing techniques are needed [199].

17.2. Dimension 3 and more

We may also consider vector bundles on hypersurfaces of higher dimension, starting with dimension 3. This is a subtle and complicated subject, since it is related to Hartshorne's conjecture which should constrain low-codimensional cycles. Still, there have been a diverse collection of works. The paper of Arrondo and Costa [11] considered ACM bundles on Fano 3-folds. Chang studies bundles with small values of c_2 [43].

Going to even higher dimension, for general 4-dimensional hypersurfaces of degree at least 3 in \mathbb{P}^5 , Mohan Kumar, Rao and Ravindra prove that an ACM bundle must split [207].

In arbitrary dimension they introduce, generalizing Faltings, the *number of generators* of a bundle in the following sense: we say that V has r generators if it admits a surjection from a direct sum of r line bundles. They prove that if a bundle on an m -dimensional variety has r generators, then if $r < m$ or certain cases with $r = m$, the bundle had to be a direct sum of line bundles [206].

17.3. Bundles of very low c_2 on hypersurfaces

It is interesting to study stable vector bundles with very low values of c_2 . Drézet [65] considers the *extremal components* for the lowest values of c_2 such that the moduli space has positive dimension, for sheaves over \mathbb{P}^2 . Chang considers the question for sheaves on \mathbb{P}^3 , which has a somewhat different character [43].

Suppose X is a very general hypersurface of degree d in \mathbb{P}^3 . We have $K_X = \mathcal{O}_X(d - 4)$. Assuming $d \geq 4$, then $\text{Pic}(X) \cong \mathbb{Z}$ with generator $\mathcal{O}_X(1)$, in particular for any divisor $D \subset X$, we have $D \sim kH$ for some $k \in \mathbb{Z}$, where H denotes a hyperplane section of X .

Suppose E is a stable rank 2 bundle with $\det(E) \cong \mathcal{O}_X(1)$. Recall that $E^* = E(-1)$. We would like to consider the case when $c_2(E)$ is very low.

For X a very general hypersurface of degree $d = 5$ in \mathbb{P}^3 , the value $c_2 = 4$ is the lowest possible value and we obtained a simple and precise description of the bundles with $c_2 = 4, 5$ in [198, 200], as follows.

Consider a bundle $E \in M_X(2, 1, c)$ with $c = 4, 5$. An Euler characteristic argument gives $h^0(E) > 0$. Choosing an element $s \in H^0(E)$ gives an exact sequence

$$(17.1) \quad 0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow J_{P/X}(1) \rightarrow 0.$$

We showed in [198] that P is a subscheme of length $c = 4$ or $c = 5$ in the intersection $X \cap \ell$ with a line $\ell \subset \mathbb{P}^3$.

For a point $p \in \mathbb{P}^3$, let $G \cong \mathbb{C}^3$ be the space of linear generators of the ideal of p , that is to say $G := H^0(J_{p/\mathbb{P}^3}(1))$, and consider the natural exact sequence of sheaves on \mathbb{P}^3

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^3} \otimes G^* \rightarrow \mathcal{R}_p \rightarrow 0.$$

Here the cokernel sheaf \mathcal{R}_p is a reflexive sheaf of degree 1, and $c_2(\mathcal{R}_p)$ is the class of a line. The restriction $\mathcal{R}_p|_X$ therefore has $c_2 = 5$. If $p \in X$, it is torsion-free but not locally free, giving a point in the boundary stratum $M(5, 4)$. It turns out that these sheaves account for all of $M(4)$ and $M(5)$.

Proposition 17.1 ([200] Theorem 6.2). *For $E \in M_X(2, 1, 4)$, there is associated a point $p \in X$ such that E is generated by global sections outside of p , and $\mathcal{R}_p|_X$ is isomorphic to the subsheaf of E generated by global sections. This fits into an exact sequence*

$$0 \rightarrow \mathcal{R}_p|_X \rightarrow E \rightarrow S \rightarrow 0$$

where S has length 1, in particular $E \cong (\mathcal{R}_p|_X)^{**}$. The correspondence $E \leftrightarrow p$ establishes an isomorphism $M(4) \cong X$.

For $E' \in M_X(2, 1, 5)$ there exists a unique point $p \in \overline{\mathbb{P}^3 - X}$ such that $E' \cong \mathcal{R}_p|_X$. We get in this way an isomorphism $\overline{M(5)} \cong \overline{\mathbb{P}^3}$. The boundary component $M(5, 4) \cap \overline{M(5)}$ is exactly $X \subset \overline{\mathbb{P}^3}$, but $M(5, 4)$ is bigger and is a separate irreducible component of $\overline{M}_X(2, 1, 5)$.

For hypersurfaces of arbitrary degree, we formulate a question:

Question 17.2. *Suppose $X \subset \mathbb{P}^3$ is a very general smooth hypersurface of degree $d \geq 6$. Suppose E is a stable vector bundle of rank 2 with $\det(E) \cong \mathcal{O}_X(1)$. Is it true that:*

- (i) *we have $c_2(E) \geq d - 1$; and*
- (ii) *for $c_2(E) = d - 1$ or d , we have $h^0(E) > 0$?*

We note that the case $h^0(E) > 0$ may be understood as before.

Proposition 17.3. *In the situation of Question 17.2, suppose we have $c_2(E) \leq d$ and $H^0(E) \neq 0$. Then $c_2(E) = d - 1$ or d , and the subscheme of zeros Z of a section $s \in H^0(E)$ is contained in a line. Conversely, for any line $L \subset \mathbb{P}^3$, any subscheme $Z \subset L \cap X$ of length $d - 1$ or d satisfies $CB(d - 3)$ and we obtain a bundle E as an extension of $J_{Z/X}(1)$ by \mathcal{O}_X .*

Unfortunately, the argument of [198] when $d = 5$, doesn't transpose directly. The Euler characteristic argument used to get $h^0(E) > 0$, only gives $h^0(E(a)) > 0$ for $a \geq (d - 4)/2$ in general. We haven't found a proof or a counterexample. In Corollary 17.10 below, we will be able to partially answer the question for hypersurfaces of degree $d \leq 10$.

Suppose $a \geq 0$ is an integer such that there exists an element $s \in H^0(E(a))$ not vanishing on a divisor. Let $Z \subset X$ be the 0-dimensional scheme zeros of s , and consider the exact sequence

$$(17.2) \quad 0 \rightarrow \mathcal{O}_X(-a) \rightarrow E \rightarrow J_{Z/X}(a+1) \rightarrow 0.$$

Recall that Z is a local complete intersection satisfying $CB(K_X \otimes L^{-1} \otimes M)$ where $L = \mathcal{O}_X(-a)$ and $M = \mathcal{O}_X(a+1)$, that is to say Z satisfies $CB(d + 2a - 3)$.

Let $|Z|$ denote the length of the zero-dimensional subscheme Z . We have

$$c_2(E) = -a(a+1)H^2 + |Z|$$

so $|Z| = d(a^2 + a) + c_2(E)$.

The Cayley-Bacharach condition on a subscheme $Z \subset \mathbb{P}^3$ implies some important numerical properties of the postulation. This kind of thing is discussed in the wide survey of Eisenbud, Green and Harris [69]. We discuss in detail some aspects leading to Question 17.2 for $d \leq 10$, based on Lemma 7.3 transferring the CB property to residual subschemes.

Denote by $r_Z(n)$ the rank of the evaluation map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(n)) \rightarrow H^0(\mathcal{O}_Z(n)) \cong \mathbb{C}^{|Z|}.$$

We have the formulas

$$r_Z(n) = h^0(\mathcal{O}_{\mathbb{P}^3}(n)) - h^0(J_{Z/\mathbb{P}^3}(n))$$

and

$$r_Z(n) = |Z| - h^1(J_{Z/\mathbb{P}^3}(n)).$$

Proposition 17.4. *Suppose $Z \subset \mathbb{P}^3$ is a reduced zero-dimensional subscheme which satisfies $CB(n)$. Then, for any $m \leq n$, either there exists a section $s \in H^0(\mathcal{O}_{\mathbb{P}^3}(n-m))$ vanishing on Z , or else $r_Z(m) \leq |Z| - h^0(\mathcal{O}_{\mathbb{P}^3}(n-m))$.*

Proof. Suppose that no sections of $\mathcal{O}_{\mathbb{P}^3}(n-m)$ vanish on Z . That means that $r_Z(n-m) = h^0(\mathcal{O}_{\mathbb{P}^3}(n-m))$. Since Z is reduced, that is to say just a collection of distinct points, we can choose a subset of points $Z' \subset Z$ such that $|Z'| = h^0(\mathcal{O}_{\mathbb{P}^3}(n-m))$ and the points of Z' impose independent conditions on sections of $\mathcal{O}_{\mathbb{P}^3}(n-m)$. Let $Z'' \subset Z$ be the

complement. We have $|Z''| = |Z| - h^0(\mathcal{O}_{\mathbb{P}^3}(n-m))$. Thus, Z'' imposes no more than the number of conditions that we want. It therefore suffices to show that any section of $\mathcal{O}_{\mathbb{P}^3}(m)$ vanishing on Z'' , vanishes on Z . Suppose $f \in H^0(\mathcal{O}_{\mathbb{P}^3}(m))$ and f vanishes on Z'' but not on Z . Let V be the residual subscheme of Z along f , which in the case Z reduced, just means the subset of points of Z where f doesn't vanish. Then, V is in the complement of Z'' since by assumption f vanishes on Z'' , so $V \subset Z'$. But we now have a contradiction: on the one hand, V imposes independent conditions on sections of $\mathcal{O}_{\mathbb{P}^3}(n-m)$ because $V \subset Z'$ and Z' did so; on the other hand, by Lemma 7.3, V satisfies $CB(n-m)$. This is impossible, showing that our section f had to vanish on all of Z . This completes the proof. Q.E.D.

It would be very useful to have the same statement for arbitrarily subschemes, not just reduced ones. We can obtain a replacement result, with a weaker bound, as follows.

Lemma 17.5. *Suppose $X \subset \mathbb{P}^3$ is a smooth hypersurface of degree d and $Z \subset X$ satisfies $CB(n)$. Then for any $m \leq n$ we have*

$$r_Z(m) + r_Z(n-m) \leq |Z| + h^0(\mathcal{O}_X(d-4-m)).$$

Proof. For brevity set $k := d-4$ so $K_X = \mathcal{O}_X(k)$. By $CB(n)$ there exists an extension of the form

$$0 \rightarrow \mathcal{O}_X \rightarrow F \rightarrow J_{Z/X}(n-k) \rightarrow 0$$

with F a vector bundle. Twisting gives

$$0 \rightarrow \mathcal{O}_X(k+m-n) \rightarrow F(k+m-n) \rightarrow J_{Z/X}(m) \rightarrow 0.$$

Hence,

$$H^1(J_{Z/X}(m)) \rightarrow H^2(\mathcal{O}_X(k+m-n)) \rightarrow H^2(F(k+m-n))$$

so $h^1(J_{Z/X}(m)) \geq h^0(\mathcal{O}_X(n-m)) - h^2(F(k+m-n))$. Recall that

$$H^0(\mathcal{O}_X(m)) \rightarrow H^0(\mathcal{O}_Z(m)) \cong \mathbb{C}^{|Z|} \rightarrow H^1(J_{Z/X}(m)) \rightarrow 0.$$

By duality, $H^2(F(k+m-n)) \cong H^0(F^*(n-m))^*$ and we have an exact sequence

$$0 \rightarrow \mathcal{O}_X(k-n) \rightarrow F^* \rightarrow J_{Z/X} \rightarrow 0$$

from which,

$$0 \rightarrow \mathcal{O}_X(k-m) \rightarrow F^*(n-m) \rightarrow J_{Z/X}(n-m) \rightarrow 0.$$

We get

$$h^2(F(k+m-n)) = h^0(F^*(n-m)) \leq h^0(\mathcal{O}_X(k-m)) + h^0(J_{Z/X}(n-m)),$$

and putting this into the inequality of the previous sequence gives

$$\begin{aligned} h^1(J_{Z/X}(m)) &\geq h^0(\mathcal{O}_X(n-m)) - h^2(F(k+m-n)) \\ &\geq h^0(\mathcal{O}_X(n-m)) - h^0(\mathcal{O}_X(k-m)) - h^0(J_{Z/X}(n-m)). \end{aligned}$$

Recall that $r_Z(n-m) = h^0(\mathcal{O}_X(n-m)) - h^0(J_{Z/X}(n-m))$ and $r_Z(m) = |Z| - h^1(J_{Z/X}(m))$, so the above may be rewritten as

$$|Z| - r_Z(m) \geq r_Z(n-m) - h^0(\mathcal{O}_X(k-m)).$$

After rearranging this gives the desired statement.

Q.E.D.

Question 17.6. *How can we weaken the hypothesis that X is smooth in the previous lemma?*

The lemma allows us, in principle, to reduce a to about $3d/8$, and even a little better for low values. For $d \leq 10$ it gives $a \leq 1$.

Corollary 17.7. *Suppose X is a very general hypersurface of degree $d \leq 10$ in \mathbb{P}^3 , and suppose E is a stable bundle of rank 2 and degree 1, with $c_2(E) \leq d - 1$. Then $h^0(E(1)) > 0$.*

Proof. Let a be the smallest integer such that $h^0(E(a)) > 0$. By an Euler characteristic argument we get $a < d$, indeed even $a < (d - 4)/2$. We have an exact sequence (17.2), with a subscheme Z which is $CB(n)$ for $n = d + 2a - 3$ with $|Z| \leq d(a^2 + a + 1) - 1$. The assumption $h^0(E(a-1)) = 0$ says that $h^0(J_{Z/X}(2a)) = 0$, so $r_Z(2a) = h^0(\mathcal{O}_{\mathbb{P}^3}(2a))$. Put these into the conclusion of Lemma 17.5 with $n - m = 2a$, i.e. $m = d - 3$, to get

$$r_Z(d-3) \leq d(a^2 + a + 1) - 2 - \frac{(2a+3)(2a+2)(2a+1)}{6}.$$

Using just $r_Z(d-3) \geq r_Z(2a)$ we get

$$\frac{(2a+3)(2a+2)(2a+1)}{3} \leq d(a^2 + a + 1) - 1,$$

from which it follows when $d \leq 10$ that $a \leq 1$. For $a = 2$, say, the inequality would be $70 \leq 7d - 1$ which doesn't work if $d \leq 10$. Q.E.D.

We now discuss the situation $h^0(E(1)) > 0$ and $c_2(E) \leq d - 2$. Looking at the exact sequence (17.2) with $a = 1$, in order to reduce to the case $h^0(E) > 0$, the idea is to try to show that Z is contained in a quadric hypersurface. The condition $CB(d + 2a - 3) = CB(d - 1)$ on Z of length $3d - 2$ almost allows us to conclude even more, that Z is contained in a plane. However, there are some other cases which are contained in a quadric. The general statement is as follows.

Proposition 17.8. *Suppose $Z \subset \mathbb{P}^3$ satisfies $CB(n)$. We have $|Z| \geq n + 2$. If $|Z| \leq 2n + 1$ then Z is colinear (i.e. contained in a single line). If $|Z| \leq 3n + 1$ then Z is either coplanar, or contained in a union of two skew lines $\ell_1 \cup \ell_2$, or in a double line i.e. the subscheme defined by the square of the ideal of a line ℓ_1 . In the non coplanar cases, we have $|Z \cap \ell_i| \geq n + 1$ and $|Z| \geq 2n + 3$.*

Proof. The proof is by induction on n , it is easy to see for $n = 1$. Suppose $n \geq 2$ and it is known for $CB(n - 1)$ subschemes. If $|Z| \leq n + 1$ then since Z is $CB(n - 1)$ and $|Z| \leq 2(n - 1) + 1 = n + 1 + (n - 2)$, Z is colinear by the inductive hypothesis. It is easy to see that a subscheme of a line of length $\leq n + 1$ cannot be $CB(n)$. This proves the first statement.

Suppose $|Z| \leq 2n + 1$. Choose a plane meeting Z in a subscheme of length at least 2, and assume it doesn't contain Z . The residual subscheme Z' is $CB(n - 1)$ and has length $\leq 2(n - 1) + 1$. Thus, Z' is colinear, and $|Z'| \geq n + 1$. Choose a plane containing Z' . Then, the residual subscheme of this plane is $CB(n - 1)$ but it has length $\leq n$, so it must be empty. This shows that every plane containing Z' also contains Z . It follows that Z is in the same line as Z' . This proves the second part.

Suppose $2n + 2 \leq |Z| \leq 3n + 1$. If Z is contained in a plane then the proposition is proven, so suppose Z is not contained in a plane. Choose a plane H such that $k := |H \cap Z|$ is maximal, in particular $k \geq 3$. Let Z'' be the residual subscheme. It is $CB(n - 1)$ and has length $|Z| - k \leq 3(n - 1) + 1$. Therefore the inductive hypothesis applies: either Z'' is coplanar, or it is contained in two lines or a double line.

Inductively, if Z'' is not coplanar, then $|Z''| \geq 2(n - 1) + 3 = 2n + 1$, giving $k = |Z| - |Z''| \leq n$. But, at the same time, Z'' contains at least $(n - 1) + 1 = n$ points on a line. A plane through that line and contacting Z in another piece, would have order of contact at least $n + 1$ contradicting $k \leq n$. This rules out the possibility that the non-coplanar cases apply to Z'' .

We may now assume that Z'' is coplanar. Then by definition of k we have $|Z''| \leq k$ which implies $k \geq |Z|/2$ and $|Z''| \leq |Z|/2$. One calculates

(separating the cases $n = 2$ and $n \geq 3$) that $|Z''| \leq 2n - 1$. Recall that Z'' is $CB(n - 1)$ so the previous case of the proposition applies, that is to say Z'' is contained in a line and has length at least $n + 1$.

We now choose a line ℓ_1 whose order of contact $l := |\ell_1 \cap Z|$ is maximal. According to the previous arguments, $l \geq n + 1$.

Assume first that $l \geq n + 2$. If A is any plane containing ℓ_1 , then the residual V of Z with respect to A is $CB(n - 1)$ and has length at least $n + 1$ and at most $2n - 1$, so $V \subset \ell_2$. Note in passing that $|Z| = |Z \cap A| + |V| \geq (n + 2) + (n + 1) = 2n + 3$.

We now show, partly to be used later, that if ℓ_1 and ℓ_2 are two distinct lines each meeting Z at least $n + 1$ times, then the proposition holds.

—If ℓ_2 doesn't intersect ℓ_1 , then it is easy to see by the same style of argument that $Z \subset \ell_1 \cup \ell_2$, and the $CB(n)$ condition requires Z to have at least $n + 2$ points on each line. This gives the skew lines conclusion of the proposition.

—Suppose ℓ_1 and ℓ_2 are distinct but meet in a point. They define a plane B , and $Z \cap B$ has length at least $2n + 1$. But then the residual of Z with respect to B is $CB(n - 1)$ with at most n points, a contradiction, so Z is contained in B , contradicting the assumption that we weren't in the coplanar conclusion of the proposition.

This treats the case of two distinct lines each meeting Z at least $n + 1$ times.

Getting back to our hypothesis $l \geq n + 2$, we have treated the case $\ell_1 \neq \ell_2$ above. It follows that $\ell_1 = \ell_2$, but in particular $Z \subset 2A$ is contained in the double of the plane A . This holds for all choices of A passing through ℓ , which yields the conclusion that Z is contained in the double D defined by the square of the ideal of ℓ . Therefore, the proposition holds whenever we know $l \geq n + 2$.

We are reduced to the case where the maximal order of contact with a line is $l = n + 1$, and ℓ_1 is a line having this order of contact with Z .

Choose a plane A containing ℓ_1 and meeting Z in a subscheme of length at least $n + 2$. Recall that Z is assumed non coplanar, so it isn't contained in A .

Let V be the residual of Z with respect to A . It is $CB(n - 1)$ and has length at most $2n - 1$, so again by the previous case of the proposition, $V \subset \ell_2$ and $|V| \geq n + 1$. Notice again as before that $|Z| \geq 2n + 3$.

The proof done above shows that if ℓ_2 is distinct from ℓ_1 then the proposition holds, so we may assume that $\ell_1 = \ell_2 =: \ell$, and indeed there is no other line meeting Z in a subscheme of length $\geq n + 1$.

Let A' be a general plane through ℓ . Let V' be the residual of Z with respect to A' , which is $CB(n - 1)$ with $|V'| \leq 2n$. Applying

inductively the proposition, we conclude that V' must be coplanar (since $2n < 2(n - 1) + 3$), so V' is contained in a plane B' .

If V' is colinear, then from the discussion above, V' may be assumed to be contained in ℓ . Therefore $V' \subset A'$ so $Z \subset 2A'$ and as A' runs through all general choices, we obtain that Z is contained in the double line D defined by the square of the ideal of ℓ , completing the proof in this case. This applies notably whenever $|V'| \leq 2n - 1$ inductively from the second statement of the proposition for $n - 1$.

It remains to treat the case when $|V'| = 2n$ and V' is not colinear, so it is contained in a unique plane B' . We claim that $\ell \subset B'$. Suppose not. Then, choose a different general plane A'' through ℓ , leading to V'' and B'' as above again with B'' meeting ℓ at a single point. We have

$$Z \subset (A' \cup B') \cap (A'' \cup B'').$$

But this intersection defines the line ℓ outside of at most two points. It becomes impossible for Z to have a residual subscheme V with respect to the original plane A , of length $n + 1$ inside ℓ . Thus, $\ell \subset B'$.

Now, $Z \cap B'$ contains the subscheme $Z \cap \ell$ of length $n + 1$ in ℓ . Suppose that this subscheme were not contained in V' . Then $Z \cap B'$ is bigger than V' so $|Z \cap B'| \geq 2n + 1$. The residual of Z with respect to B' would have length $\leq n$, impossible since it satisfies $CB(n - 1)$, so we would get $Z \subset B'$ contradicting our original non coplanar supposition. This shows that V' contains the subscheme of length $n + 1$ inside ℓ . But also, V' is $CB(n - 1)$ as a subscheme of the plane B' . Let W be the residual of V' with respect to the line ℓ inside B' . Thus $|W| \leq 2n - (n + 1) = n - 1$. But W is $CB(n - 2)$, and we get a contradiction.

This completes, at last, the proof of the proposition. Q.E.D.

This proposition leads to a partial answer to Question 17.2 in the case where there is an exact sequence of the form (17.2) with $a = 1$, i.e. when $E(1)$ admits a section.

Corollary 17.9. *Suppose X is a very general surface of degree $d \geq 4$ in \mathbb{P}^3 , and suppose E is a stable bundle of rank 2 and determinant $\mathcal{O}_X(1)$. Suppose $h^0(E(1)) > 0$. Then $c_2(E) \geq d - 1$.*

Proof. Suppose $c_2(E) \leq d - 2$. Let $x \in H^0(E(1))$ leading to an exact sequence (17.2) with $a = 1$. The subscheme Z has length $|Z| = 2d + c_2(E) \leq 3d - 2$. It satisfies $CB(d + 2a - 3) = CB(n)$ for $n = d - 1$, and we have $|Z| \leq 3n + 1$. By the proposition, Z is either coplanar or contained in two skew lines or in a double line. In all of these cases, Z is contained in a hypersurface of degree 2 (not containing X), that is to say $h^0(J_{Z/X}(2)) > 0$. From the long exact sequence associated to

(17.2), we get $h^0(E) > 0$ and the discussion of Proposition 17.3 applies to give a contradiction to the hypothesis that $c_2(E) \leq d - 2$. Q.E.D.

Corollary 17.10. *Suppose X is a very general surface of degree d with $4 \leq d \leq 10$ in \mathbb{P}^3 , and suppose E is a stable bundle of rank 2 and determinant $\mathcal{O}_X(1)$. Then $c_2(E) \geq d - 1$.*

Proof. By Corollary 17.7, we have $h^0(E(1)) > 0$. Therefore, Corollary 17.9 applies, giving the conclusion. Q.E.D.

Exercise 17.11. *It is left to the reader to complete these informations to a more precise description of the moduli space along the lines of Propositions 17.1 and 17.3, in the cases $d \leq 10$ or under the hypothesis $h^0(E(1)) > 0$.*

We haven't been able to obtain techniques which would give the conclusion $c_2(E) \geq d - 1$ in general. It isn't clear whether it means that further more difficult or powerful techniques are needed, or whether some cases with $c_2(E) \leq d - 2$ could intervene for higher values of the degree d . Thus, Question 17.2 is left as an open problem.

§18. Vector bundles on quintic surfaces: Nijse's connectedness theorem

In the last two sections we specialize to the case when $X \subset \mathbb{P}^3$ is a very general quintic hypersurface defined by a general homogeneous polynomial of degree $d = 5$. The very general property means that its moduli point is not contained in some countable union of subvarieties, the subvarieties in question being determined by the needs of the argument. In particular, we may assume that $\text{Pic}(X) = \mathbb{Z}$ with generator $\mathcal{O}_X(1)$. The canonical class is $K_X = \mathcal{O}_X(1)$; it is this smallness property of the canonical class which makes possible many of the arguments specific to this case.

Consider rank 2 bundles of odd degree, which we normalize by $\bigwedge^2 E \cong \mathcal{O}_X(1)$. Let $M(c_2)$ denote the moduli space of stable rank 2 bundles with determinant $\mathcal{O}_X(1)$, with second Chern class c_2 . This moduli space is empty for $c_2 \leq 3$, and Proposition 17.1 gives an explicit description for $c_2 = 4, 5$. In [198] we have given a rough description for all $c_2 = 4, \dots, 9$, in particular seeing that it is irreducible. In [199] and [200], we complete the proof of irreducibility. Our argument makes use of the much anterior results of Nijse in 1995 [225].

Nijse's paper doesn't seem to have been published, so we will try to provide a fairly complete picture. This will constitute a very nice illustration of O'Grady's method which is fundamental to [225], continuing our discussion from Section 9.

Let $\overline{M}(c_2)$ denote the projective moduli space of rank 2 torsion-free sheaves of degree 1 with second Chern class c_2 . The moduli space of bundles is an open subset $M(c_2) \subset \overline{M}(c_2)$. For higher values of c_2 this tends to be a dense open subset, however it is possible that $\overline{M}(c_2)$ might contain irreducible components consisting entirely of non-locally free sheaves. This is the case, in fact, for all values $c_2 = 5, \dots, 10$ on our quintic surface, as may be seen from the dimensions in Table 2 in Section 19 below, but we don't yet have a full understanding of the phenomenon.

Recall the discussion leading up to Corollary 9.9, in Section 9 about O'Grady's method. We continue the discussion here, in particular we take up the same notations.

Recall that the first step was to obtain a bundle E such that $E|_Y$ is unstable where $i : Y \hookrightarrow X$ is the intersection with a plane. Then Q_0 is the quotient and L_0 the subbundle of the destabilizing sequence, and T is the kernel of the map $E \rightarrow i_*(Q_0)$. We are looking at quotients $T|_Y \rightarrow L$ which are deformations of the one $T|_Y \rightarrow L_0$ given by the elementary transformation defining T . Each time we have such a quotient we get a new sheaf E_1 ; the goal is to obtain E_1 non-locally free by getting a quotient L which is not locally free on the curve.

Deforming to a bundle with $H^0(E) \neq 0$ will be one of the conclusions of Proposition 18.5 below, so in our deformation arguments we shall assume $H^0(E) = 0$. This implies that the deformed sheaf E_1 is stable.

Corollary 9.9 said that the deformation procedure would be possible whenever the *Quot* scheme has dimension ≥ 2 . So, in order to complete the proof of deformation to the boundary, we need to treat the case where $\text{Quot}(T|_Y, d_L)$ has dimension 1. We were not able to understand a part of Nijse's argument, in particular whether the sections appearing in [225, Lemma 1.4] were supposed to be nonzero; but this might well be our own failing. Our discussion of this case will be a little more involved.

The idea is to globalize things over the moduli space of bundles deforming T . Notice first that T is stable, because of the hypothesis $H^0(E) = 0$.

Lemma 18.1. *In the situation of the destabilizing sequence (9.2), the dimension of $\text{Quot}(T|_Y, d_L)$ at the point $T|_Y \rightarrow L_0$ is greater than or equal to its expected dimension which is $\deg(L_0) - \deg(Q_0)$.*

Proof. The dimension of the Hilbert scheme at the point

$$0 \rightarrow Q_0(-1) \rightarrow T_Y \rightarrow L_0 \rightarrow 0$$

is at least

$$\begin{aligned} & h^0(Y, \text{Hom}(Q_0(-1), L_0)) - h^1(Y, \text{Hom}(Q_0(-1), L_0)) \\ &= \deg(Q_0(-1)^* \otimes L_0) + 1 - g_Y = \deg(L_0) - \deg(Q_0). \end{aligned}$$

Q.E.D.

Lemma 18.2. *Suppose that $\dim(\text{Quot}(T|_Y, d_L)) = 1$, in the situation of (9.2). Suppose also that $\text{Hom}(T|_Y, L)$ has dimension 1 (otherwise see Remark 18.3 below). Then the subset of $\text{Jac}^{\deg(L_0)}(Y)$ consisting of bundles L such that $\text{Hom}(T|_Y, L) \neq 0$, has dimension 1 and this is equal to its expected dimension. Therefore, this subset persists under deformations of the bundle T and the curve Y .*

Proof. The quantity in the previous lemma is ≥ 1 since L_0 is destabilizing. From the bound of the previous lemma and our present hypothesis, we therefore get $\deg(L_0) - \deg(Q_0) = 1$, that is $\deg(L_0) = 3$. Recall that $\deg(T|_Y) = 0$, so $\chi(T^* \otimes L) = 2\deg(L) - 10 = -4$. At a point where $h^0(T^* \otimes L) = 1$ we have $h^1(T^* \otimes L) = 5$ so the expected codimension of this part of the Brill-Noether locus is $1 \cdot 5 = 5$. As $\text{Jac}(Y)$ has dimension 6, the expected dimension of this locus is 1, and by the hypotheses, the locus has its expected dimension. It follows that it is preserved under deformations of T and Y . Q.E.D.

Remark 18.3. *If $\dim \text{Hom}(T|_Y, L) \geq 2$ then our map $T|_Y \rightarrow L$ can be deformed to a map which vanishes at a point, and we immediately get a deformation in $\text{Quot}(T|_Y, d_L)$ to a non-locally free quotient.*

If we are in the situation $\dim(\text{Quot}(T|_Y, d_L)) = 1$, we may think of the family of these *Quot* schemes as a fiber space

$$V \rightarrow M(\mathcal{O}_X, c_2(T))$$

over the moduli space of semistable bundles with determinant \mathcal{O}_X and second Chern class $c_2(T)$. We obtain a family of elementary transformations E_v parametrized by this fiber space.

Corollary 18.4. *In the above situation, suppose that T may be deformed, within $\overline{M}(\mathcal{O}_X, c_2(T))$, to a non-locally free sheaf. Then E may be deformed to a stable non-locally free sheaf, or to a stable sheaf with $H^0(E') \neq 0$.*

Proof. We are free to move the curve Y as well as the bundle T . Therefore, we may choose a deformation of T to a non-locally free sheaf and choose a deformation of Y to miss the singular locus of the limiting sheaf T' . By the above construction we obtain a family of elementary

transformations, whose limit will be an elementary transformation of T' , hence not locally free. We need to ask if the limiting E_v is stable. We have $E_v(-1) \subset T'$, so if E_v had a subsheaf of positive degree it would give $\mathcal{O}_X \subset (T')^{**}$. Thus, if $H^0((T')^{**}) = 0$ then the limiting E_v is stable and non-locally free.

Suppose $H^0((T')^{**}) \neq 0$. Let E' be the limiting stable sheaf for the family of sheaves E_u approaching E_v . There is a nonzero map $E'(-1) \rightarrow T'$. The subsheaf $\mathcal{O}_X \subset (T')^{**}$ corresponds to a saturated subsheaf $F \subset T'$ with $F^{**} = \mathcal{O}_X$, and an exact sequence

$$0 \rightarrow F \rightarrow T' \rightarrow G \rightarrow 0$$

with $G^{**} = \mathcal{O}_X$ also. Either the map $E'(-1) \rightarrow G \subset \mathcal{O}_X$ is nonzero, or else there is a map $E'(-1) \rightarrow F \subset \mathcal{O}_X$ and either way we get a nonzero map $E'(-1) \rightarrow \mathcal{O}_X$. If E' is non-locally free we are done, so we may assume it is locally free. Thus $E'(-1) = (E')^*$, and the transpose of $E'(-1) \rightarrow \mathcal{O}_X$ gives a nonzero element of $H^0(E')$, so we get the second case of the statement. Q.E.D.

Note that if T, L can be deformed to a bundle T' together with a quotient L' where $\dim(\text{Quot}(T'|_Y, d_L)) \geq 2$, then the same discussion can be applied starting from T' . The corresponding bundle E' given by the elementary transformation can then be deformed to a non-locally free sheaf. This process might in theory require us to move to a different irreducible component, but E will be connected to the new sheaf.

We have now transformed the problem of degenerating E to the boundary, to a problem of degenerating T to the boundary. Apply again the same strategy. Let $W \subset \overline{M}(\mathcal{O}_X, c_2(T))$ be an irreducible component. If it contains a point T' which is not locally free, then the previous corollary applies. If it contains a point T' with $H^0(T') \neq 0$ then as in the previous proof we would obtain a degeneration of E to a stable bundle E' with $H^0(E') \neq 0$. So, let us suppose that $H^0(T') = 0$ for all points of W (i.e. the semistable bundles are stable).

One calculates $c_2(T) = c_2(E) - 3$ and the expected dimension of W is $4c_2(T) - 15$. For $c_2(E) \geq 11$ this gives $c_2(T) \geq 8$ and the expected dimension is ≥ 17 . O'Grady's method applies, see Corollary 9.3 giving existence of a point $T' \in W$ such that $T'|_Y$ is not semistable. Look at the destabilizing sequence

$$0 \rightarrow L'_0 \rightarrow T'|_Y \rightarrow Q'_0 \rightarrow 0$$

which now has $\deg(L'_0) \geq 1$ and $\deg(Q'_0) \leq -1$. As before, letting F be the kernel of $T' \rightarrow i_*(Q'_0)$, we have a map $F|_Y \rightarrow L'_0$. We can again

consider the Hilbert scheme of quotients $F|_Y \rightarrow L'$. If there is a non-locally free quotient, this gives a deformation of T' to a boundary point (as $H^0(T') \neq 0$, the new boundary point would also be semistable). So we may assume that all of the quotients are locally free. As in Corollary 9.8, this implies that the dimension of the *Quot*-scheme is ≤ 1 . However, as in Lemma 18.1, the dimension of the *Quot*-scheme is $\geq \deg(L'_0) - \deg(Q'_0) \geq 2$. Here, we are helped by the fact that T has even degree. We get a contradiction, so it follows that T' may be deformed to a non-locally free sheaf.

For $c_2(E) = 10$, some further work is needed. In this case $c_2(T) = 7$ and the expected dimension of its moduli space is 13, less than the dimension of the moduli space of bundles on Y so Corollary 9.3 doesn't apply. This case may be treated more directly. From the Euler characteristic, $h^0(T(1)) \geq 3$ so we can choose a section and express T as an extension

$$0 \rightarrow \mathcal{O}_X(-1) \rightarrow T \rightarrow J_{P/X}(1) \rightarrow 0.$$

where P has length 12 and satisfies $CB(3)$. Arguing under the assumption that the situations covered by the previous discussions don't already apply (in particular by the previous paragraphs we may assume that the restrictions $T'|_Y$ are semistable), and using the techniques of [198, 199, 200], we find that for a general T in its component, the subscheme P is a *dodecad*: twelve points which form the complete intersection of two quadrics and a cubic. It is very similar to the case $c_2(E) = 8$ where E is a Cayley octad. In our situation, the dodecad P can be degenerated on X to a configuration of 11 points on a rational normal cubic, plus a twelfth point in general position. This is seen by showing that the configuration of $11 + 1$ points generizes in \mathbb{P}^3 , then applying the monodromy argument on the incidence variety that Hirschowitz suggested to us in [198]. The 11 points satisfy $CB(3)$ but the twelfth point imposes an independent condition. We get in this way a degeneration of T to a boundary point T' which is a torsion free sheaf with one singularity at the twelfth point, whose double dual with $c_2 = 6$ corresponds to the 11 points. By Corollary 18.4, this treats the case $c_2(E) = 10$.

We may sum up with the following proposition.

Proposition 18.5. *Suppose that $c_2 \geq 10$. Then any point in $M(c_2)$ is connected either to a boundary point (i.e. a stable non-locally free sheaf), or to a point E with $H^0(E) \neq 0$, by a sequence of families of deformations parametrized by irreducible curves.*

Proof. It has been described above.

Q.E.D.

Lemma 18.6. *For $c_2 \geq 10$, the space $V(c_2)$ consisting of stable bundles E with $H^0(E) \neq 0$, is irreducible of dimension $3c_2 - 11$.*

Proof. See Nijssse [225, Lemma 3.1], and also our discussion in [198].
Q.E.D.

We have now arrived at Nijssse's connectedness theorem. We give first the statement which can be seen easily from the preceding arguments.

Proposition 18.7. *For $c_2 \geq 11$, any irreducible component of $\overline{M}(c_2)$ meets the boundary (the moduli subset of non-locally free sheaves). For $c_2 = 10$, any irreducible component meets either the boundary or the subset $V(10)$ considered above.*

Proof. We use here the fact that $M(c_2)$ is good for $c_2 \geq 10$, proven in [198]. Furthermore, the singular locus consists of $V(c_2)$ plus things of strictly smaller dimension. Goodness implies that the moduli space is a local complete intersection, and this in turn tells us by Hartshorne's connectedness theorem, that if two different irreducible components meet, they must meet along a subset of codimension 1. Recall that $\dim(V(c_2)) = 3c_2 - 11$ for $c_2 \geq 10$. This has codimension 1 when $c_2 = 10$, and codimension ≥ 2 for $c_2 \geq 11$. Therefore, if two different irreducible components of $\overline{M}(c_2)$ meet, either they must meet along a boundary point (one or both could in fact be contained in the boundary), or else $c_2 = 10$ and they meet along points of $V(10)$.

Consider a point $[E] \in M(c_2)$. It is connected by a sequence of deformations, either to a boundary point or to a point of $V(c_2)$. However, we can only change irreducible components along points of these two kinds. Thus, we may say that any irreducible component of $M(c_2)$ contains either a boundary point, or a point of $V(c_2)$ in its closure. When $c_2 \geq 11$ it is also easy to see that $V(c_2)$ contains boundary points in its closure; thus, for $c_2 \geq 11$ any irreducible component meets the boundary.
Q.E.D.

Theorem 18.8 (Nijssse [225] Proposition 3.2). *The moduli space of stable torsion-free sheaves $\overline{M}(c_2)$ is connected for $c_2 \geq 10$.*

Proof. We follow fairly closely Nijssse's proof. By Lemma 18.6, $V(c_2)$ is connected for $c_2 \geq 10$. So, it defines a unique connected component of $\overline{M}(c_2)$. In order to prove the connectedness theorem, it suffices to show that there is not a different connected component. So, suppose $Z \subset \overline{M}(c_2)$ is an irreducible component which is in a connected component different from the one containing $V(c_2)$. By the previous

proposition, Z contains a boundary point $[F]$ with F not locally free. Let $E := F^{**}$. We have an exact sequence

$$0 \rightarrow F \rightarrow E \rightarrow S \rightarrow 0$$

where S is a skyscraper sheaf of length d , and $c_2(E) = c_2 - d$. By the theorem of Li-Ellingsrud-Lehn, Theorem 8.2, F may be deformed to a sheaf F' which is a kernel as above, but with quotient S' a direct sum of length 1 sheaves at distinct points of X .

If $c_2 \geq 11$ then, removing a single one of these points gives a sheaf $G \subset E$ with $c_2(G) = c_2 - 1$. We may work by induction on $c_2 \geq 10$, and assume that $\overline{M}(c_2 - 1)$ is connected. It leads to a connected piece of the boundary of $\overline{M}(c_2)$, and we have connected F to something in here. This completes the proof of connectedness for $c_2 \geq 11$ by induction, assuming it is known for $c_2 = 10$.

To treat the case $c_2 = 10$, we obtain F in our connected component, and E as above. As $c_2(E) \leq 9$, an Euler characteristic argument tells us that $H^0(E) \neq 0$. If we deform the quotient S to a quotient S' which respects this section, we obtain a deformation of F to a sheaf F' with $H^0(F') \neq 0$.

The section gives an exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow F' \rightarrow J_{P/X}(1) \rightarrow 0.$$

The subscheme $P \subset X$ of length 10 is the union of the corresponding one for E , denoted Q , and the collection of points W in the support of S' . The existence of a nonzero extension corresponds to having an element $e \in H^0(J_{P/X}(2))$. We may fix this quadric containing P , then deform P to a general collection of 10 points on that quadric (which is to say, on the curve of intersection of our quintic X with the corresponding quadric in \mathbb{P}^3). This general collection now satisfies Cayley-Bacharach, so the extension is locally free. This gives a generalization from F' to a point of $V(10)$, showing that F' is in the closure of $V(10)$. But F' was connected to our original point, contradicting the hypothesis that we had a different component than that containing $V(10)$. This completes the proof of the connectedness theorem. Q.E.D.

§19. Classification of vector bundles on a very general quintic surface

We finish with a brief description of our proof that the moduli space is irreducible [200]. In [198] we have closely investigated the moduli spaces $M(4), \dots, M(9)$, in particular they are all irreducible, which fills

in the rows up to $c_2 = 9$ in the following table. We thank I. Dolgachev for pointing out the terminology *Cayley octad* for a set of 8 points which is the complete intersection of three quadrics.

Table 1. Stable bundles on a quintic

c_2	dim	e.d.	properties	CB subscheme
≤ 3			empty	none
4	2	-4	irred., $\cong X$	$Z \subset \text{line} \cap X$
5	3	0	irred., $\cong \mathbb{P}^3 - X$	$Z = \text{line} \cap X$
6	7	4	irred., gen. smooth	$Z \subset \text{planar conic} \cap X$
7	9	8	irred., nonreduced	general $Z \subset \text{plane} \cap X$
8	13	12	irred., gen. smooth	Cayley octad
9	16	16	irred., nonreduced	$Z \subset \text{ell. curve} \cap X$
10	20	20	irred., good, gen. semin. coh.	CB(4)
≥ 11	$4c_2 - 20$	$4c_2 - 20$	irred., good	...

In [199], we showed that for $c_2 = 10$ the subset of the moduli space consisting of bundles with seminatural cohomology (meaning that for each n , at most one of the $h^i(E(n))$ is nonzero), is nonempty and irreducible.

Boundary points in $\overline{M}(c_2)$ come from stable bundles in $M(c_2 - d)$ for $d \geq 1$. It follows by Li-Ellingsrud-Lehn, see Theorem 8.2, that an irreducible component of $M(c_2 - d)$ leads to an irreducible subvariety of the boundary of $\overline{M}(c_2)$, denoted $M(c_2, c_2 - d)$, whose dimension is what we would expect:

$$\dim M(c_2, c_2 - d) = \dim M(c_2 - d) + 3d.$$

This allows us to fill in the following table with the dimensions of all the strata, see [200, Table 2]. The column labeled $c_2 - d$ gives the dimension of the stratum $M(c_2, c_2 - d)$.

Table 2. Dimensions of boundary strata

c_2	$\dim(M)$	c_2-1	c_2-2	c_2-3	c_2-4	c_2-5	c_2-6	c_2-7
4	2	—	—	—	—	—	—	—
5	3	5	—	—	—	—	—	—
6	7	6	8	—	—	—	—	—
7	9	10	9	11	—	—	—	—
8	13	12	13	12	14	—	—	—
9	16	16	15	16	15	17	—	—
10	20	19	19	18	19	18	20	—
11	24	23	22	22	21	22	21	23
≥ 12	$4c_2-20$	$4c_2-21$	$\leq 4c_2-22$					

From O’Grady’s lemma 8.1 the boundary has pure codimension 1. In the lines $c_2 = 10, 11$ we emphasize in boldface the strata which can contribute a codimension 1 piece of the boundary. For $c_2 = 11$, there are two possibilities, the usual $M(11, 10)$ and $M(11, 4)$. For $c_2 = 10$ the possibilities are $M(10, 9)$, $M(10, 8)$, $M(10, 6)$ which can give pieces of codimension 1, and furthermore $M(10, 4)$ which must be a separate irreducible component since it has the same dimension 20 as the full moduli space.

Using Hartshorne’s connectedness theorem (Corollary 8.4), and the bounds we know for the singular locus inside the moduli space of bundles (Lemma 18.6 for the main piece V which itself meets the boundary, and the other pieces are even smaller [198]), we may conclude that if two different irreducible components meet, they must both meet the boundary. There exists at least one irreducible component meeting the boundary, for example one containing V . Therefore, Nijssse’s connectedness theorem (18.8) implies that any irreducible component has to intersect the boundary starting from $c_2 = 10$ onwards.

From the explicit description of [198], we are able to conclude that the boundary pieces $M(10, 9)$, $M(10, 8)$, and $M(10, 6)$ have to attach to moduli spaces of bundles with seminatural cohomology; thus, by [199] these must all go to the unique seminatural component for $c_2 = 10$. Some further work is needed to treat the codimension 1 intersection between $\overline{M(10)}$ and $M(10, 4) = \overline{M(10, 4)}$ because the intersection will be a codimension 1 subvariety in the latter, hence not including a general point, in $M(10, 4)$. But, this discussion again leads to the seminatural condition, allowing us to conclude that the only irreducible component for $c_2 = 10$ is the seminatural one considered in [199]. This completes the proof for the row $c_2 = 10$. For $c_2 = 11$, irreducibility is obtained by noting that the two boundary components $M(11, 10)$ and $M(11, 4)$ have

a nontrivial intersection. For $c_2 \geq 12$ there is only a single possible codimension 1 boundary component so irreducibility follows by induction.

A lot is left to be done. On the one hand to study more carefully the irreducible components of the moduli space—all of the questions we have discussed above: rationality, Betti numbers, existence of Poincaré families and so on, are open. And, on the other hand to try to extend the investigation to other surfaces. One direction would be to go to higher degrees: what is the first place where there are two irreducible components, for a very general hypersurface of degree d (Question 5.4)? What happens for low values of c_2 (Question 17.2)? The other direction is to relax the assumption that X is very general. In that case, the Picard group has higher rank, and there is a choice of polarization to define stability, so wallcrossing phenomena can be expected.

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