Advanced Studies in Pure Mathematics 68, 2016
School on Real and Complex Singularities in São Carlos, 2012
pp. 267–373

Centers and limit cycles in polynomial systems of ordinary differential equations

Valery G. Romanovski and Douglas S. Shafer

Abstract.

A polynomial system of differential equations on the plane with a singularity at which the eigenvalues of the linear part are complex can be placed, by means of an affine transformation and a rescaling of time, in the form $\dot{x} = \lambda x - y + P(x, y)$, $\dot{y} = x + \lambda y + Q(x, y)$. The problem of determining, when $\lambda = 0$, whether the origin is a spiral focus or a center dates back to Poincaré. This is the center problem. We discuss an approach to this problem that uses methods of computational commutative algebra. We treat generalizations of the center problem to the complex setting and to higher dimensions. The theory developed also has bearing on the cyclicity problem at the origin, the problem of determining bounds on the number of isolated periodic orbits that can bifurcate from the origin under small perturbation of the coefficients of the original system. We also treat this application of the theory. Some attention is also devoted to periodic solutions on center manifolds and their bifurcations.

Received November 1, 2013.
Revised April 17, 2014.
2010 *Mathematics Subject Classification.* 34C05, 34C07, 34C20, 34C23, 34C45, 37G05, 37G10.
Key words and phrases. center, cyclicity, focus quantity.
Acknowledgement. The first author acknowledges the support of this work by the Slovenian Research Agency and FP7-PEOPLE-2012-IRSES-316338.

Introduction

A polynomial system of differential equations on the plane with a singularity at which the eigenvalues of the linear part are complex can be placed, by means of an affine transformation and a rescaling of time, in the form

$$\dot{x} = \lambda x - y + P(x, y), \quad \dot{y} = x + \lambda y + Q(x, y). \tag{0}$$

When $\lambda = 0$ the origin is either a focus (in the phase portrait either every trajectory near the origin spirals towards the origin or every nearby trajectory spirals away from it) or a center (every nearby trajectory is an oval). The center problem, which dates back to Poincaré, is the problem of characterizing the center case. When P and Q are polynomials that are parametrized by their coefficients the systems with a center at the origin are picked out by the simultaneous vanishing of infinitely many polynomials in the coefficients. The set of solutions of polynomial equations is determined by the ideal in the ring of all polynomials that the polynomials in question generate and forms the affine variety of that ideal. Thus the systems in family (0) that have a center at the origin correspond to points of a variety in the space of coefficients, the *center variety*. The problem of finding the relevant polynomials, understanding the structure of the ideal they generate, and ultimately the corresponding center variety in the space of coefficients of (0) is one to which the methods of computational algebra are well suited. Thus we begin in Section 1 with a description of the algebraic machinery used in a computational algebra approach to the center problem. Section 2 then treats the center problem and its natural generalization to the complex setting in order to take advantage of working over the algebraically closed field $\mathbb{C}$ in place of $\mathbb{R}$.

In Section 3 we describe a generalization of the center problem to higher dimensional settings. It is here that we present, in the middle two subsections, important concepts that also apply in the original two-dimensional setting. Section 5 also presents a generalization to higher dimensions, but in a different context.

The question of how many limit cycles (isolated closed orbits) can emerge from a center under arbitrarily small perturbation of the coefficients in P and Q is called the cyclicity problem. Its intimate connection to the center problem through the ideal of polynomials that yield the center variety is described in Section 4.

In order to illustrate how the ideas and techniques that we describe are actually applied in practice we have concluded Sections 2, 3, and 5 with discussions of specific families of differential equations. These

discussions illustrate how questions of the feasibility of computations arise and provide an opportunity to introduce methods for extending our computational reach.

Finally, in Section 6 we bring the various disparate, independently developed ideas from earlier sections together in a complete treatment of systems of the form (0) in which P and Q are homogeneous quadratic polynomials. The definitive solutions of the center and cyclicity problems for these systems is one of the crowning achievements of the qualitative theory of differential equations and an excellent vehicle for illustrating the concepts that we present.

Much but not all of the material in Sections 1, 2, 4, and 6 is a streamlined but completely independent presentation of ideas and results developed in more detail in our book [70], to which we sometimes refer the reader for more elaboration or details of proofs. Conversely, some of the ideas in [70] are treated more fully here, such as the question of convergence of normalizing transformations that is addressed in Subsection 2.3. The material on higher dimensional systems in Sections 3 and 5 is not treated in [70].

For the convenience of the reader in navigating this article we present here an outline organized by sections and subsections.

§1. Algebraic Background

In this section we lay out the algebraic and geometric framework in which we will be working. Proofs and elaborations on what is presented here can be found, for example, in [1], [8], [26], and [70], and for the purely algebraic results, in [58] and [83]. The reader can also consult the notes of Hans Schönemann in the present volume.

1.1. Ideals and Their Varieties

Let $k[x_1, \ldots, x_n]$ denote the ring of polynomials in n indeterminates with coefficients in the field k, which will typically be the set $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers. We will be interested in solutions of systems of polynomial equations

$$f_1 = 0, \quad f_2 = 0, \quad f_3 = 0, \quad \ldots \qquad f_j \in F \subset k[x_1, \ldots, x_n] \tag{1.1}$$

for specific collections F of polynomials, where the polynomials are here regarded as mappings from k^n into k. Associate to F the set $\langle F \rangle$ of all finite linear combinations of elements of F, that is, all polynomials $h_1 f_1 + \cdots + h_s f_s$ for all choices of $s \in \mathbb{N}$, of $h_j \in k[x_1, \ldots, x_n]$, and of $f_j \in F$. The set $\langle F \rangle$ is an *ideal* of $k[x_1, \ldots, x_n]$ (a subset of $k[x_1, \ldots, x_n]$ that is closed under addition and multiplication by any element of $k[x_1, \ldots, x_n]$), the ideal generated by F, whose elements are called its *generators*. It is immediate that an element $\mathbf{a}$ of k^n is a solution of system (1.1) if and only if $f(\mathbf{a}) = 0$ for every $f \in \langle F \rangle$. Thus the

solution set of (1.1) is determined by the ideal generated by F, not the specific set of generators F.

Important concepts connected with ideals that we need to describe the set of solutions of (1.1) are the following. The *radical* of an ideal I, denoted $\sqrt{I}$, is the ideal

$$\sqrt{I} = \{f \in k[x_1, \ldots, x_n] : \text{ there exists } p \in \mathbb{N} \text{ such that } f^p \in I\}.$$

An ideal $I \subset k[x_1, \ldots, x_n]$ is called a *radical ideal* if $I = \sqrt{I}$. A proper ideal I is *prime* if $fg \in I$ only if f or g is in I. Every prime ideal is radical, but not conversely.

The following result is of fundamental importance.

Theorem 1.1.1 (Hilbert Basis Theorem)**.** *If k is a field, then every ideal I in $k[x_1, \ldots, x_n]$ is finitely generated: there exist finitely many $f_j \in I$ such that $I = \langle f_1, \ldots, f_s\rangle = \{h_1 f_1 + \cdots + h_s f_s : h_j \in k[x_1, \ldots, x_n]\}$.*

Thus finitely many of the elements of F suffice to determine the set of solutions of (1.1).

Definition 1.1.2. *An* affine variety *is a subset of k^n that is the solution set of a system of equations of the form* (1.1)*. It will be denoted by $\mathbf{V}(I)$, where I is the ideal $\langle F\rangle$, or, when a finite set of generators of I is specified, by $\mathbf{V}(f_1, \ldots, f_s)$. A variety is* irreducible *if it is not the union of finitely many proper subsets, each of which is itself a variety.*

Some authors (e.g., Hartshorne [38]) refer to the set of solutions of (1.1) as an "algebraic set" and reserve the term "variety" for the case that the field k is algebraically closed and the ideal I is prime, in which case the set of solutions is irreducible. (In general an algebraic set V is irreducible if and only if the ideal $\mathbf{I}(V)$ of Definition 1.1.3 below is a prime ideal.) However, we are often interested in just the real solutions of (1.1) and prefer to refer to the irreducible components of a variety as themselves varieties.

Definition 1.1.3. *Given any subset S of k^n, the ideal $\mathbf{I}(S)$ of S is defined as the set of elements of $k[x_1, \ldots, x_n]$ that vanish on S.*

Denote by $\mathbb{V}$ the set of all affine varieties of k^n and by $\mathbb{I}$ the set of all polynomial ideals in $k[x_1, \ldots, x_n]$. Then Definitions 1.1.2 and 1.1.3 define maps $\mathbf{V} : \mathbb{I} \to \mathbb{V}$ and $\mathbf{I} : \mathbb{V} \to \mathbb{I}$. The maps $\mathbf{I}$ and $\mathbf{V}$ are inclusion-reversing, $\mathbf{I}$ is one-to-one (injective) and $\mathbf{V}$ is onto (surjective). Furthermore, for any variety $V \subset k^n$, $\mathbf{V}(\mathbf{I}(V)) = V$, and for any two varieties V and W, $V = W$ if and only if $\mathbf{I}(W) = \mathbf{I}(V)$.

Our goal is to understand the set of solutions of system (1.1). To do so means to obtain a description of the variety $\mathbf{V}(F)$ that is as complete

and as explicit as possible. Every affine variety Vas we have described it can be decomposed into finitely many irreducible components. This is because any chain of varieties

$$V \supsetneqq V_1 \supsetneqq V_2 \supsetneqq V_3 \supsetneqq \cdots$$

generates by an application of $\mathbf{I}$ a strictly increasing chain of ideals

$$\mathbf{I}(V) \subsetneqq \mathbf{I}(V_1) \subsetneqq \mathbf{I}(V_2) \subsetneqq \mathbf{I}(V_3) \subsetneqq \cdots$$

in $k[x_1, \ldots, x_n]$ (the inclusions are proper because $\mathbf{I}$ is injective) and it is a consequence of Theorem 1.1.1 that any ascending chain of ideals in $k[x_1, \ldots, x_n]$ must terminate. Thus V is expressible as

$$V = V_1 \cup \cdots \cup V_m, \tag{1.2}$$

where each V_j is irreducible and $V_j \not\subset V_k$ if $j \neq k$, and in fact this decomposition is unique up to the order of the V_j. Thus to solve (1.1) we wish to find the decomposition (1.2) for $V = \mathbf{V}(F)$. Knowing that this decomposition exists, apply $\mathbf{I}$ to (1.2). Since for any two varieties V and W, $\mathbf{I}(V \cup W) = \mathbf{I}(V) \cap \mathbf{I}(W)$ (which is always itself an ideal) we obtain

$$\mathbf{I}(\mathbf{V}(F)) = \mathbf{I}(V_1) \cap \cdots \cap \mathbf{I}(V_m).$$

The ideals on the right hand side are all prime. The ideal on the left hand side is not typically $\langle F \rangle$, but is identified by the following theorem, if we are working over an algebraically closed field.

Theorem 1.1.4 (Strong Hilbert Nullstellensatz)**.** *Let I be an ideal in $\mathbb{C}[x_1, \ldots, x_n]$. Then*

$$\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}. \tag{1.3}$$

If the ideal $\langle F \rangle$ is radical then the following theorem applies.

Theorem 1.1.5. *Every radical ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$ can be represented in a unique way as an intersection of prime ideals, $I = \cap_{j=1}^{m} P_j$, where $P_r \not\subset P_s$ if $r \neq s$.*

Thus starting with a finite set of generators of $\langle F \rangle$ we can use a computer algebra system such as Macaulay2 [56], Maple [57], Mathematica [59], REDUCE [66], or Singular [29] to decompose $\langle F \rangle$ as

$$\langle F \rangle = P_1 \cap \cdots \cap P_m,$$

where each P_j is a prime ideal, and know that

$$V = \mathbf{V}(P_1) \cup \cdots \cup \mathbf{V}(P_m).$$

If the ideal $\langle F\rangle$ is not radical then in order to decompose it into an intersection of ideals we need a weaker condition on the components of the decomposition. Namely, an ideal $I \subset k[x_1,\ldots,x_n]$ is called a *primary ideal* if for any pair $f, g \in k[x_1,\ldots,x_n]$, $fg \in I$ only if either $f \in I$ or $g^p \in I$ for some $p \in \mathbb{N}$. An ideal I is primary if and only if $\sqrt{I}$ is prime; $\sqrt{I}$ is called the *associated prime ideal of* I. A *primary decomposition* of an ideal $I \subset k[x_1,\ldots,x_n]$ is a representation of I as a finite intersection of primary ideals Q_j, $I = \cap_{j=1}^m Q_j$. The decomposition is called a *minimal* primary decomposition if the associated prime ideals $\sqrt{Q_j}$ are all distinct and $\cap_{i\neq j} Q_i \not\subset Q_j$ for any j. A minimal primary decomposition of a polynomial ideal always exists, but it is not necessarily unique.

Theorem 1.1.6 (Lasker–Noether Decomposition Theorem)**.** *Every ideal I in $k[x_1,\ldots,x_n]$ has a minimal primary decomposition. All such decompositions have the same number m of primary ideals and the same collection of associated prime ideals.*

Then as in the case that $\langle F\rangle$ is radical we use a computer algebra system to perform a primary decomposition of $\langle F\rangle$, which typically computes the prime decomposition of $\sqrt{\langle F\rangle}$ as well. Since over any field, for any ideal I of $k[x_1,\ldots,x_n]$, $\mathbf{V}(I) = \mathbf{V}(\sqrt{I})$, we obtain the prime ideals corresponding to the irreducible components in the variety corresponding to (1.1). Two efficient algorithms for primary decomposition, both implemented in the SINGULAR library `primdec.lib` ([28]), are those developed by P. Gianni, B. Trager, and G. Zacharias ([35]) and by T. Shimoyama and K. Yokoyama ([76]).

1.2. Monomial Orders and Gröbner Bases

Two important problems that we will need to be able to solve are:

1. The ideal membership problem: given an ideal I of $k[x_1,\ldots,x_n]$ and a polynomial f, determining whether or not f is an element of I.
2. The ideal equality problem: given two ideals I and J, each represented by a finite set of generators, determining whether or not $I = J$.

Consider the first one. In the special case that $n = 1$ every ideal I is generated by a single element f_1. Moreover the Division Algorithm states that for every polynomial f there exist unique polynomials q and r such that $f = qf_1 + r$ and either $r = 0$ or $\deg(r) < \deg(f)$; f is in I if and only if the "remainder" r of this division of f by f_1 is the zero polynomial. To generalize these ideas to the case $n > 1$ we must decide how to order the monomials in a polynomial and, for a collection $\{f_1,\ldots,f_s\}$ in $k[x_1,\ldots,x_n]$, be able to algorithmically obtain $g_1,\ldots,g_s$ and r such that $f = g_1f_1 + \cdots g_sf_s + r$.

When $n = 1$ the monomials in $k[x]$ are naturally ordered by degree. For $n > 1$ there are numerous natural orders of the monomials. Let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Exploiting the one-to-one correspondence between elements $\alpha = (\alpha_1, \ldots, \alpha_n)$ of $\mathbb{N}_0^n$ and monomials $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in $k[x_1, \ldots, x_n]$ (once an order $x_1 > \cdots > x_n$ of the indeterminates has been specified) we define a *monomial order* to be a total order $>$ on $\mathbb{N}_0^n$ (i.e., a partial order in which any two elements of $\mathbb{N}_0^n$ can be compared) such that (a) for all α, β, and γ in $\mathbb{N}_0^n$, if $\alpha > \beta$ then $\alpha + \gamma > \beta + \gamma$, and (b) every non-empty subset of $\mathbb{N}_0^n$ has a smallest element.

Three commonly used monomial orders are the following (where addition and rescaling in $\mathbb{Z}^n$ is performed componentwise: for α, $\beta \in \mathbb{Z}^n$ and $p \in \mathbb{Z}$ the jth entry of $\alpha + p\beta$ is the jth entry of α plus p times the jth entry of β):

(a) Lexicographic Order. Define $\alpha >_{\text{lex}} \beta$ if and only if, reading left to right, the first nonzero entry in the n-tuple $\alpha - \beta \in \mathbb{Z}^n$ is positive.

(b) Degree Lexicographic Order. Define $\alpha >_{\text{deglex}} \beta$ if and only if

$$|\alpha| = \sum_{j=1}^{n} \alpha_j > |\beta| = \sum_{j=1}^{n} \beta_j \qquad \text{or} \qquad |\alpha| = |\beta| \text{ and } \alpha >_{\text{lex}} \beta\,.$$

(c) Degree Reverse Lexicographic Order. Define $\alpha >_{\text{degrev}} \beta$ if and only if either $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and, reading right to left, the first nonzero entry in the n-tuple $\alpha - \beta \in \mathbb{Z}^n$ is negative.

For example, for $\alpha = (2, 2, 3, 1)$ and $\beta = (2, 0, 4, 2)$, α is greater than β with respect to all three orders.

Once a monomial order has been fixed then the *standard form* for any nonzero element f of $k[x_1, \ldots, x_n]$ is

$$f = a_1\mathbf{x}^{\alpha_1} + a_2\mathbf{x}^{\alpha_2} + \cdots + a_s\mathbf{x}^{\alpha_s},$$

where $a_j \neq 0$ for $j = 1, \ldots, s$, $\alpha_i \neq \alpha_j$ for $i \neq j$ and $1 \leq i, j \leq s$, and where, with respect to the specified term order, $\alpha_1 > \alpha_2 > \cdots > \alpha_s$. When $f \neq 0$ is written in standard form then

(a) the *leading term* $LT(f)$ of f is the term $a_1\mathbf{x}^{\alpha_1}$,

(b) the *leading monomial* $LM(f)$ of f is the monomial $\mathbf{x}^{\alpha_1}$, and

(c) the *leading coefficient* $LC(f)$ of f is the coefficient a_1.

For $n > 1$ the Division Algorithm generalizes to a Multivariate Division Algorithm by which for any polynomial f and any *ordered* set $F = \{f_1, \ldots, f_s\}$ of non-zero polynomials (the generators of the ideal $I = \langle F \rangle$) there exist polynomials $u_1, \ldots, u_s$ and r such that $f = u_1 f_1 + \cdots + u_s f_s + r$ and either $r = 0$ or r is *reduced* with respect to the set F in the sense that none of its monomials is divisible by any element of the set $\{LM(f_1), \ldots, LM(f_s)\}$ (where we say that a monomial

$\mathbf{x}^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ *divides* a monomial $\mathbf{x}^\beta = x_1^{\beta_1} \dots x_n^{\beta_n}$, written $\mathbf{x}^\alpha \mid \mathbf{x}^\beta$, if $\beta_j \geq \alpha_j$ for all j, $1 \leq j \leq n$). The inequality $\deg(r) \leq \deg(f)$ always holds, but if no further restrictions are placed on the elements of F then the remainder could change when the order of the elements of F is changed and could be nonzero for every order even when f is in I. These problems are eliminated when we restrict to a Gröbner basis of I, introduced by B. Buchberger in [12].

Definition 1.2.1. *Fix a monomial order on $k[x_1, \dots, x_n]$.*

(a) *A* Gröbner basis *(also called a* standard basis*) of a nonempty ideal I in $k[x_1, \dots, x_n]$ is a finite nonempty subset $G = \{g_1, \dots, g_m\}$ of $I \setminus \{0\}$ with the property that for every nonzero $f \in I$, there exists $g_j \in G$ such that $LT(g_j) \mid LT(f)$.*

(b) *A Gröbner basis $G = \{g_1, \dots, g_m\}$ is called* minimal *if for all $i, j \in \{1, \dots, m\}$, $LC(g_i) = 1$ and for $j \neq i$, $LM(g_i)$ does not divide $LM(g_j)$.*

(c) *A Gröbner basis $G = \{g_1, \dots, g_m\}$ is called* reduced *if for all i, $1 \leq i \leq m$, $LC(g_i) = 1$ and no term of g_i is divisible by any $LT(g_j)$ for $j \neq i$ (i.e., g_i is reduced with respect to $G \setminus \{g_i\}$).*

Every nonzero ideal in $k[x_1, \dots, x_n]$ has a unique reduced Gröbner basis (which of course may depend on the monomial order selected), which can be computed algorithmically. The procedure is available on all popular computer algebra systems. Thus the issue of membership of a polynomial f in an ideal I can be settled by computing its remainder upon division by a Gröbner basis with respect to any monomial order: f is in I if and only if the remainder is zero (in SINGULAR the remainder is computed with the command `reduce`). The issue of the equality of two ideals can be settled by comparing their unique reduced Gröbner bases with respect to any monomial order: two ideals are equal if and only if they have the same reduced Gröbner bases.

1.3. Ideals and the Geometry of Varieties

We have already noted that over any ground field k, $\mathbf{V}(I) = \mathbf{V}(\sqrt{I})$. It is also true in general that two affine varieties V and W are the same if and only if $\mathbf{I}(V) = \mathbf{I}(W)$. For ideals I and J in $\mathbb{C}[x_1, \dots, x_n]$, $\mathbf{V}(I) = \mathbf{V}(J)$ if and only if $\sqrt{I} = \sqrt{J}$.

If V and W are affine varieties in k^n then $V \cap W$ and $V \cup W$ are also varieties. The set-theoretic difference $V \setminus W$ need not be a variety, but has important applications, so for any set S in k^n we define the *Zariski closure* of S, denoted $\overline{S}$, to be the smallest variety with respect to inclusion that contains S; it is the intersection of all varieties that contain S. If $V = \mathbf{V}(I)$ and $W = \mathbf{V}(J)$ then we would like to explore

the connection between the ideals I and J and the new varieties $V \cap W$, $V \cup W$, and $\overline{V \setminus W}$. This involves the sum, intersection, and quotient of the ideals I and J.

Given ideals I and J in $k[x_1, \ldots, x_n]$ the sum of I and J is the ideal defined by

$$I + J := \{f + g : f \in I \text{ and } g \in J\}.$$

This corresponds to the intersection of the corresponding affine varieties:

$$\mathbf{V}(I + J) = \mathbf{V}(I) \cap \mathbf{V}(J). \tag{1.4}$$

To get a basis of $I + J$ we can simply take the union of any basis of I and any basis of J.

The set-theoretic intersection of any two ideals in $k[x_1, \ldots, x_n]$ is also an ideal in $k[x_1, \ldots, x_n]$ and this corresponds to the union of the corresponding affine varieties:

$$\mathbf{V}(I \cap J) = \mathbf{V}(I) \cup \mathbf{V}(J). \tag{1.5}$$

A Gröbner basis G of $I \cap J$ can be computed using the following procedure. If $I = \langle f_1, \ldots, f_u \rangle$ and $J = \langle g_1, \ldots, g_v \rangle$ in $k[x_1, \ldots, x_n]$:

1. Compute a Gröbner basis G' of

$$\langle\, tf_1(\mathbf{x}), \ldots, tf_u(\mathbf{x}),\, (1-t)g_1(\mathbf{x}), \ldots, (1-t)g_v(\mathbf{x}) \,\rangle$$

   in $k[t, x_1, \ldots, x_n]$ with respect to lexicographic order with $t > x_1 > \cdots > x_n$.
2. $G = G' \cap k[x_1, \ldots, x_n]$

In SINGULAR one can simply use the command `intersect`.

The situation with the set-theoretic difference of two varieties is somewhat more complicated. Given two ideals I and J their *ideal quotient* $I : J$ is the ideal

$$I : J = \{f \in k[x_1, \ldots, x_n] : fg \in I \text{ for all } g \in J\}.$$

In SINGULAR the quotient is computed with the command `quotient`. In general if I and J are ideals in $k[x_1, \ldots, x_n]$ then

$$\overline{\mathbf{V}(I) \setminus \mathbf{V}(J)} \subset \mathbf{V}(I : J) \tag{1.6}$$

(recall that the overline indicates Zariski closure). If $k = \mathbb{C}$ and I is a radical ideal then this improves to

$$\overline{\mathbf{V}(I) \setminus \mathbf{V}(J)} = \mathbf{V}(I : J). \tag{1.7}$$

1.4. Elimination, Implicitization, and the Radical Membership Test

We state here three theorems that are most helpful in investigating solutions of (1.1).

Let I be an ideal in $k[x_1, \ldots, x_n]$ (with the implicit ordering of the variables $x_1 > \cdots > x_n$) and fix $\ell \in \{0, 1, \ldots, n-1\}$. The *$\ell$-th elimination ideal* of I is the ideal $I_\ell = I \cap k[x_{\ell+1}, \ldots, x_n]$.

Theorem 1.4.1 (Elimination Theorem)**.** *Let G be a Gröbner basis for an ideal I of $k[x_1, \ldots, x_n]$ with respect to lexicographic order with $x_1 > x_2 > \cdots > x_n$. Then for every ℓ, $0 \le \ell \le n-1$, the set*

$$G_\ell := G \cap k[x_{\ell+1}, \ldots, x_n]$$

is a Gröbner basis for the ℓ-th elimination ideal I_ℓ.

To eliminate a group of variables it is more efficient to use an appropriate elimination order instead of the lexicographic order (see the notes of Hans Schönemann in this volume).

Consider now the following problem: given a rational or a polynomial parametrization of a subset S of k^n, try to eliminate the parameters so as to express the set in terms of polynomials in $x_1, \ldots, x_n$. Hence suppose we are given the system of equations

$$x_1 = \frac{f_1(t_1, \ldots, t_m)}{g_1(t_1, \ldots, t_m)}, \ \ldots, \ x_n = \frac{f_n(t_1, \ldots, t_m)}{g_n(t_1, \ldots, t_m)}, \tag{1.8}$$

where $f_j, g_j \in k[t_1, \ldots, t_m]$ for $j = 1, \ldots, n$. Let $W = \mathbf{V}(g_1 \cdots g_n)$. Equations (1.8) define a function

$$F : k^m \setminus W \to k^n$$

by the formula

$$F(t_1, \ldots, t_m) = \left(\frac{f_1(t_1, \ldots, t_m)}{g_1(t_1, \ldots, t_m)}, \ \ldots, \ \frac{f_n(t_1, \ldots, t_m)}{g_n(t_1, \ldots, t_m)} \right). \tag{1.9}$$

The image of $k^m \setminus W$ under F, which we denote by $F(k^m \setminus W)$, is not necessarily an affine variety. Consequently, we look for the smallest affine variety that contains $F(k^m \setminus W)$, that is, its Zariski closure $\overline{F(k^m \setminus W)}$. The problem of finding $\overline{F(k^m \setminus W)}$ is known as the problem of *rational implicitization*. If the right-hand sides of (1.8) are polynomials, then it is the problem of *polynomial implicitization*. The terminology comes from the fact that the collection of polynomials $f_1, \ldots, f_s$ that determine the variety $V(f_1, \ldots, f_s)$ defines it only implicitly. The following theorem gives an algorithm for rational implicitization.

Theorem 1.4.2 (Rational Implicitization Theorem)**.** *Let k be an infinite field, let $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ be elements of $k[t_1, \ldots, t_m]$, let $W = \mathbf{V}(g_1 \cdots g_n)$, and let $F : k^m \setminus W \to k^n$ be the function defined by equations* (1.9)*. Set $g = g_1 \cdots g_n$. Consider the ideal*

$$J = \langle f_1 - g_1 x_1, \ldots, f_n - g_n x_n, 1 - gy \rangle \subset k[y, t_1, \ldots, t_m, x_1, \ldots, x_n],$$

and let

$$J_{m+1} = J \cap k[x_1, \ldots, x_n] \tag{1.10}$$

be the $(m+1)$st elimination ideal. Then $\mathbf{V}(J_{m+1})$ is the smallest variety in k^n containing $F(k^m \setminus W)$.

The following statement is called the Radical Membership Test.

Theorem 1.4.3. *Let $I = \langle f_1, \ldots, f_s \rangle$ be an ideal in $k[x_1, \ldots, x_n]$. Then $f \in \sqrt{I}$ if and only if $1 \in \langle f_1, \ldots, f_s, 1 - wf \rangle \subset k[x_1, \ldots, x_n, w]$.*

Thus to determine if a polynomial f is an element of the ideal $\sqrt{\langle f_1, \ldots, f_s \rangle} \subset k[x_1, \ldots, x_n]$ we merely compute the (unique) reduced Gröbner basis G of the ideal $\langle f_1, \ldots, f_s, 1 - wf \rangle \subset k[x_1, \ldots, x_n, w]$ with respect to any monomial order and have an affirmative answer if and only if $G = \{1\}$. In the case that $k = \mathbb{C}$ we can use this test to check whether a polynomial f vanishes on the variety of the ideal I. By Theorems 1.1.4 and 1.4.3 $f \in \mathbf{I}(\mathbf{V}(I)) = \sqrt{I}$ if and only if the reduced Gröbner basis G (with respect to any fixed monomial order) of the ideal $\langle f_1, \ldots, f_s, 1 - wf \rangle \subset k[x_1, \ldots, x_n, w]$ is $G = \{1\}$.

1.5. Irreducible Decomposition of Varieties Using Modular Arithmetic

At present there are few algorithms for the primary decomposition of polynomial ideals (see for example the notes of Hans Schönemann in this volume). All of them involve rather laborious calculations. If we are interested in decomposition of varieties then it is sufficient to find the minimal associated primes of the ideal of the variety (see Definition 1.1.3 and the paragraphs that follow it); in SINGULAR this be done with the routines `minAssGTZ` and `minAssChar` ([28]).

When the variety consists of a finite number of points in $\mathbb{C}^n$ computation of a Gröbner basis with respect to a lexicographic order solves the problem. In such a case a Gröbner basis with respect to lexicographic order is always in "triangular" form, like in the following example from

[3]. Consider the system $f_1 = f_2 = f_3 = f_4 = 0$ where

$$\begin{aligned} f_1 &= 8x^2y^2 + 5xy^3 + 3x^3z + x^2yz \\ f_2 &= x^5 + 2y^3z^2 + 13y^2z^3 + 5yz^4 \\ f_3 &= 8x^3 + 12y^3 + xz^2 + 3 \\ f_4 &= 7x^2y^4 + 18xy^3z^2 + y^3z^3. \end{aligned} \tag{1.11}$$

The reduced Gröbner basis for the ideal generated by f_1, f_2, f_3, f_4 in $\mathbb{Q}[x, y, z]$ with respect to degree reverse lexicographic order with the variables ordered $x > y > z$ is

$$g_1 = x, \quad g_2 = y^3 + \tfrac{1}{4}, \quad g_3 = z^2. \tag{1.12}$$

Thus the system $f_1 = f_2 = f_3 = f_4 = 0$ is equivalent to the system $g_1 = g_2 = g_3 = 0$ and finding the solutions to the latter system is straightforward. Of course for a general system (1.1) a Gröbner basis will usually be much more complicated than in this example. However, if system (1.1) has only a finite number of solutions then any reduced Gröbner basis with respect to lexicographic order *must* contain a polynomial in one variable, $g_1(x_1)$, say. Then there exists a collection of polynomials in the Gröbner basis depending on x_1 and one additional variable, say $g_2(x_1, x_2), \ldots, g_t(x_1, x_2)$, and so on through the full list of variables. Thus we first solve (perhaps only numerically) the equation $g_1(x_1) = 0$. Then for every solution x_1^* of $g_1(x_1) = 0$ we find the solutions of $g_2(x_1^*, x_2) = \cdots = g_t(x_1^*, x_2) = 0$, a system of polynomials in the single variable x_2. Continuing the process we obtain all solutions of the system (1.1). Thus in the case of a finite number of solutions, at least theoretically a Gröbner basis computation provides the complete solution to the problem. In practice, however, calculation of Gröbner bases, especially with respect to lexicographic orders, often generates enormous computational difficulties. In particular the size of the coefficients of the so-called S-polynomials involved in the computation of a Gröbner basis grow exponentially. For instance, as pointed out in [3], for the simple polynomials of (1.11) the following polynomial appears in intermediate computations of the Gröbner basis (1.12):

$$y^3 - 1735906504290451290764747182 \cdots . \tag{1.13}$$

The integer in the second term of this polynomial contains roughly 80,000 digits. It is the numerator of a rational number with roughly an equal number of digits in the denominator.

This notorious computational difficulty in Gröbner basis calculations over the field of rational numbers is an essential obstacle for using the

Gröbner basis theory for real world applications, in particular since the algorithms for computing decompositions of ideals and varieties rely on multiple computations of Gröbner bases of ideals and modules. We now describe an approach based on the use of modular arithmetic which can drastically simplify the problem of finding the set of solutions of a polynomial system (1.1).

To perform modular computations we choose a prime number p and carry out all calculations modulo p, that is, in the finite field of characteristic p (the field $\mathbb{Z}_p = \mathbb{Z}/p$). It turns out that modular calculations still keep essential information about the original system and it is often possible to extract this information from the result of the calculations done in $\mathbb{Z}_p$ and thereby obtain the exact solution of system (1.1) over the field of rational numbers.

To carry out the rational reconstruction, that is, to reconstruct the element r/s of $\mathbb{Q}$ given its image $t \in \mathbb{Z}_p$, we use the following algorithm of [86] (where the symbol $\lfloor\cdot\rfloor$ stands for the floor function).

Step 1. Define $u = (u_1, u_2, u_3) := (1, 0, p)$, $v = (v_1, v_2, v_3) := (0, 1, c)$.
Step 2. While $\sqrt{p/2} \le v_3$
do $\{q := \lfloor u_3/v_3 \rfloor,\ r := u - qv, u := v,\ v := r\}$.
Step 3. If $|v_2| \ge \sqrt{p/2}$ then error().
Step 4. Return v_3, v_2.

Given an integer c and a prime number p the algorithm produces integers v_2 and v_3 such that $v_3/v_2 \equiv c \pmod p$. However, v_3/v_2 need not exist and if this is the case the algorithm returns "error()." A MATHEMATICA code to perform the rational reconstruction by means of this algorithm is given in [36].

For example, computing the Gröbner basis of (1.11) over the field of characteristic 32003 we obtain $G = \{x,\ y^3 + 8001,\ z^2\}$. Using the algorithm of rational reconstruction given above we immediately find $8001 \equiv 1/4 \pmod{32003}$. The reconstruction gives the polynomials (1.12). Now, of course, no large numbers (as in (1.13)) appear in the intermediate polynomials; all numbers have at most five digits. Thus the speed of calculation increases and memory consumption falls drastically.

An approach suggested in [69] for solving systems (1.1) that involve large polynomials, that is, for finding the decomposition of the variety of the ideal $I = \langle f_1, \ldots, f_m \rangle$, consists of the following five steps.

Step 1. Choose a prime number p and compute the minimal associated primes $\tilde{Q}_1, \ldots, \tilde{Q}_s$ of I in $\mathbb{Z}_p[x_1, \ldots, x_n]$.
Step 2. Using the rational reconstruction algorithm lift the ideals $\widetilde{Q}_j$ $(1 \le j \le s)$ to ideals Q_j in $\mathbb{Q}[x_1, \ldots, x_n]$ by replacing all

the coefficients of $\widetilde{Q}_j$ by the corresponding rational numbers computed with the reconstruction algorithm.

Step 3. For each $j = 1, \ldots, s$ use the radical membership test to check whether the polynomials $f_1, \ldots, f_s$ are in the radicals of the ideals Q_j, that is, whether the reduced Gröbner basis of the ideal $\langle 1 - wf, Q_j \rangle$ is equal to $\{1\}$. If "yes" then go to Step 4, otherwise pick another prime number p and go back to Step 1.

Step 4. Compute $Q = \cap_{j=1}^{s} Q_j \subset \mathbb{Q}[x_1, \ldots, x_n]$.

Step 5. Check that $\sqrt{Q} = \sqrt{I}$, that is, that for any $g \in Q$ the reduced Gröbner basis of the ideal $\langle 1 - wg, I \rangle$ is equal to $\{1\}$ and for any $f \in I$ the reduced Gröbner basis of the ideal $\langle 1 - wf, Q \rangle$ is equal to $\{1\}$. If this is the case then $\mathbf{V}(I) = \cup_{j=1}^{s} \mathbf{V}(Q_j)$. If not then pick another prime number p and go back to Step 1.

We will not discuss whether the algorithm must terminate for general polynomial systems. However we will mention that this procedure has been successfully applied to the decomposition of polynomial varieties defined by large polynomials arising in the study of the center problem for many systems of ordinary differential equations (see, for example, [69] and the references given there).

We note that the first two steps of the algorithm are well-known and have been widely used in solving polynomial systems. The difference in the approach presented above is that it proposes a procedure (Steps 3–5) for checking that the set of solutions obtained is complete, that is, that no solution has been lost using modular calculations.

We use the approach presented here to study the center problem for a two-dimensional system in the next section. As mentioned there, if the computation with one prime number does not give the correct result then instead of recomputing with another prime (going back to Step 1) sometimes it is preferable to try to make a "lucky guess" about the correct polynomials in the decomposition and use them to obtain true components of the decomposition of the variety of interest (see, for example, [68]) .

§2. The Center Problem for Two-dimensional Real Systems and Its Generalizations

If a real analytic system of differential equations on an open subset of the plane has a singular point then by a translation the singularity can be placed at the origin so that the system takes the form

$$\dot{u} = au + bv + \widetilde{U}(u, v), \quad \dot{v} = cu + dv + \widetilde{V}(u, v)\,, \tag{2.1}$$

where $\widetilde{U}$ and $\widetilde{V}$ are convergent series whose expansions begin with terms of degree at least two.

In the generic case that $ad-bc<0$ or $ad-bc>0$ but $a+d\neq 0$ the eigenvalues of the linear part of (2.1) at $\mathbf{0}$ have nonzero real part and by the Hartman-Grobman Theorem the phase portrait in a neighborhood of $\mathbf{0}$ is the same, up to homeomorphism, as that of the linear approximation

$$\dot{u}=au+bv,\quad \dot{v}=cu+dv, \tag{2.2}$$

hence is a topological saddle, sink, or source. In fact in the latter two cases it is either a *node* (every trajectory tends towards or away from $\mathbf{0}$ with a limiting tangent direction) or a *focus* (every trajectory spirals towards or away from $\mathbf{0}$). When $ad-bc>0$ and $a+d=0$ a new possibility emerges, depending on the nonlinearities: $\mathbf{0}$ could be a *center*, a singularity for which there exists a neighborhood Ω such that the trajectory of any point in $\Omega\backslash\{\mathbf{0}\}$ is a simple closed curve γ that contains $\mathbf{0}$ in its interior (the bounded component of the complement $\mathbb{R}^2\setminus\{\gamma\}$ of γ). (In this case the *period annulus* of the center is the set $\Omega_M\setminus\{\mathbf{0}\}$, where Ω_M is the largest such neighborhood Ω, with respect to set inclusion.) Most importantly, the phase portrait near $\mathbf{0}$ of the full system (2.1) could differ topologically from that of the linear approximation. For example, when written in polar coordinates the system

$$\begin{aligned}\dot{u}&=-v-u(u^2+v^2)\\ \dot{v}&=\ \ u-v(u^2+v^2)\,.\end{aligned} \tag{2.3}$$

takes the form $\dot{r}=-r^3$, $\dot{\varphi}=1$, making it clear that the origin is a stable focus, whereas for the linear approximation the origin is a center.

In this article by the *center problem* for system (2.1) we mean the problem of determining, for a particular family of the form (2.1) for which $ad-bc>0$ and $a+d=0$, computable conditions on the nonlinearities that distinguish between a focus and a center at the origin. (The condition $ad-bc\neq 0$ classifies the singularity as *elementary*. There is also the problem of determining when a non-elementary singularity of an analytic system is a center.)

The center problem dates back a little over a hundred years with the early work of Dulac and Kapteyn ([31, 44, 45]), followed by that of Frommer ([33]), and later by that of many others. The literature devoted to the subject is vast, and we will discuss only a few aspects of the problem. Among other references the reader may consult [2, 22, 23, 51, 70, 74, 75, 78, 91] and the references that they contain.

Intimately connected to the center problem is the existence of first integrals, according to the following definition.

Definition 2.0.1.

a. *A* first integral *for a system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *of analytic ordinary differential equations on an open set* U *in* $\mathbb{R}^n$ *or* $\mathbb{C}^n$ *is a nonconstant function* $\Psi : U \to \mathbb{C}$ *that is constant on trajectories of the system in* U.

b. *A* formal first integral *for such a system is a nontrivial formal power series* $\Psi(\mathbf{x}) = \sum v_\alpha \mathbf{x}^\alpha$ *such that, regarding* $\mathbf{f}$ *as a vector field on* U, *the scalar product* $(\mathbf{f}, grad\, \Psi)$ *is zero (which can be viewed as meaning that* $\frac{d}{dt}[\Psi(\mathbf{x}(t)] \equiv 0$ *under term by term differentiation).*

c. *First integrals* $\Psi_1, \ldots, \Psi_s$ *are* functionally independent *on* U *if their gradients as vectors in* $\mathbb{R}^n$ *or* $\mathbb{C}^n$ *are linearly independent on a dense open subset of* U *(see, for example,* [42, 88]*).*

2.1. The First Return Map and the Lyapunov Quantities

Probably the most natural way to try to distinguish between a center and a focus is using polar coordinates. We discuss this approach first.

By a nonsingular linear coordinate change any system of the form (2.1) for which $ad - bc > 0$ can be written in the form

$$\begin{aligned} \dot{u} &= \alpha u - \beta v + \widetilde{P}(u, v) \\ \dot{v} &= \beta u + \alpha v + \widetilde{Q}(u, v)\,, \end{aligned} \tag{2.4}$$

where $\widetilde{P}$ and $\widetilde{Q}$ are convergent series starting with quadratic terms. Changing to polar coordinates $u = r\cos\varphi$, $v = r\sin\varphi$, near the origin we may take φ as the independent variable and write

$$\frac{dr}{d\varphi} = \frac{r^2 F\,(r, \cos\varphi, \sin\varphi)}{1 + rG\,(r, \cos\varphi, \sin\varphi)} = R\,(r, \varphi)\,.$$

The function $R(r, \varphi)$ is 2π-periodic in φ and analytic for sufficiently small $|r|$ and all φ. Thus we can expand $R(r, \varphi)$ in a convergent power series in r to obtain

$$\frac{dr}{d\varphi} = r^2 R_2(\varphi) + r^3 R_3(\varphi) + \cdots\,. \tag{2.5}$$

Choose a sufficiently short line segment $\Sigma = \{(u, v) : v = 0, 0 \le u \le r^*\}$ and consider the solution of (2.5) with the initial condition $r(0) = r_0$. Expanding it in a power series in r_0 we obtain

$$r(\varphi, r_0) = w_1(\varphi) r_0 + w_2(\varphi) r_0^2 + w_3(\varphi) r_0^3 + \cdots\,. \tag{2.6}$$

The $r(\varphi, r_0)$ from (2.6) is a solution of (2.5); inserting it into (2.5) yields a system of recurrence differential equations for the functions $w_j(\varphi)$ from

which the functions w_j are determined successively. Setting $\varphi = 2\pi$ in the solution $r(\varphi, r_0)$ we obtain the value $r(2\pi, r_0)$ corresponding to the point of Σ at which trajectory $r(\varphi, r_0)$ first intersects Σ again. The reader can consult, for example, [70, §3.1] for a fuller discussion of the derivation procedure. It is illustrated in some detail in the proof of parts (a) through (d) of Theorem 5.2.2 below.

Definition 2.1.1. *Fix a system of the form* (2.4).

(a) The function

$$\mathscr{R}(r_0) = f(2\pi, 0, r_0) = \tilde{\eta}_1 r_0 + \eta_2 r_0^2 + \eta_3 r_0^3 + \cdots \tag{2.7}$$

(defined for $|r_0| < r^*$), where $\tilde{\eta}_1 = w_1(2\pi)$ and $\eta_j = w_j(2\pi)$ for $j \geq 2$, is called the Poincaré first return map *or just the* return map.

(b) The function

$$\mathscr{D}(r_0) = \mathscr{R}(r_0) - r_0 = \eta_1 r_0 + \eta_2 r_0^2 + \eta_3 r_0^3 + \cdots \tag{2.8}$$

is called the difference function.

(c) The coefficient η_j, $j \in \mathbb{N}$, is called the jth Lyapunov quantity.

The first Lyapunov quantity has the value $\eta_1 = \tilde{\eta}_1 - 1 = e^{2\pi\alpha/\beta} - 1$. Thus if $\alpha < 0$ then the origin is a stable focus and if $\alpha > 0$ then it is an unstable focus. It is also easy to see that the first nonzero coefficient of the expansion (2.8) is the coefficient of an odd power of r_0.

Zeros of (2.8) correspond to *cycles* (closed orbits, that is, orbits that are ovals) of system (2.4); *isolated* zeros correspond to *limit cycles* (isolated closed orbits). Note that analyticity of system (2.1) insures that $\mathscr{D}$ has but finitely many isolated zeros in a neighborhood of $\mathbf{0}$, so that the singularity at $\mathbf{0}$ must be either a focus or a center. Other possibilities exist for C^∞ systems.

It is apparent that system (2.4) has a center at the origin if all the Lyapunov quantities are zero and that if for some $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$

$$\eta_1 = \eta_2 = \cdots = \eta_{2k} = 0, \ \eta_{2k+1} \neq 0, \tag{2.9}$$

then the origin is a focus, which is stable if $\eta_{2k+1} < 0$ and is unstable if $\eta_{2k+1} > 0$. When (2.9) holds for $k > 0$ it is termed a *weak (or fine) focus of order k*. As is the case with a center, in this case it is possible that small isolated closed orbits of the system could emerge from the singularity when the right hand sides of (2.1) are perturbed slightly. This issue will be addressed in Section 4.

This approach to distinguishing foci and centers, which is illustrated in the proof of Theorem 5.2.2, reduces the problem to that of computing

the Lyapunov quantities. Although conceptually straightforward, it is rather inefficient from the computational point of view since the procedure requires computations of integrals of trigonometric polynomials, a relatively difficult problem that becomes infeasible when the degree of the polynomial increases.

2.2. Lyapunov Functions and the Center Problem

A *Lyapunov function* for a smooth system

$$\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u}) \tag{2.10}$$

on a neighborhood of $\mathbf{0}$ in $\mathbb{R}^n$ for which $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ is a function W from an open neighborhood Ω of $\mathbf{0}$ into $\mathbb{R}$ that is continuous on Ω, continuously differentiable and strictly positive on $\Omega \setminus \{\mathbf{0}\}$, and is nonincreasing on trajectories of (2.10) in $\Omega \setminus \{\mathbf{0}\}$. This last condition is equivalent to the condition that the scalar product $\dot{W}(\mathbf{u}) := (\operatorname{grad} W(\mathbf{u}), \mathbf{f}(\mathbf{u}))$ be nonpositive on $\Omega \setminus \{\mathbf{0}\}$. W is a *strict Lyapunov function* if $\dot{W}$ is negative. The Lyapunov Stability Theorem (see, for example, [9, 17, 70]) states that if there exists a Lyapunov function for (2.10) then $\mathbf{0}$ is stable in the sense that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $|\mathbf{u}_0| < \delta$ then the trajectory $\mathbf{u}_0(t)$ through $\mathbf{u}_0$ satisfies $|\mathbf{u}_0(t)| < \epsilon$ for all $t \geq 0$, and that if there exists a strict Lyapunov function for (2.10) then $\mathbf{0}$ is asymptotically stable, that is, it is stable and additionally there exists a $\Delta > 0$ such that if $|\mathbf{u}_0| < \Delta$ then the trajectory $\mathbf{u}_0(t)$ through $\mathbf{u}_0$ satisfies $|\mathbf{u}_0(t)| \to 0$ as $t \to \infty$. Moreover $\mathbf{0}$ is unstable, i.e., not stable, if there exists what would otherwise be a Lyapunov function W except that $\dot{W} > 0$ on $\Omega \setminus \{\mathbf{0}\}$.

Another approach to the center problem, which is computationally much better than that presented in the previous subsection, is based on making use of Lyapunov functions, which for the system

$$\begin{aligned} \dot{u} &= -v + \widetilde{U}(u,v) = U(u,v) \\ \dot{v} &= \quad u + \widetilde{V}(u,v) = V(u,v)\,, \end{aligned} \tag{2.11}$$

where U and V are convergent series starting with quadratic terms, have the form

$$\Phi(u,v) = u^2 + v^2 + \sum_{j+k\geq 3} \phi_{jk} u^j v^k. \tag{2.12}$$

Letting $\mathcal{X} = U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v}$, the vector field associated to the right hand side of (2.11), if we can find a function (2.12) such that for some $k \in \mathbb{N}$

$$\mathcal{X}\Phi = \frac{\partial \Phi}{\partial u} U + \frac{\partial \Phi}{\partial v} V = g \cdot (u^2 + v^2)^k + \cdots \tag{2.13}$$

then by the Lyapunov Stability Theorem the origin is a stable focus if $g < 0$ and is an unstable focus if $g > 0$. If

$$\mathcal{X}\Phi = \frac{\partial \Phi}{\partial u} U + \frac{\partial \Phi}{\partial v} V \equiv 0 \tag{2.14}$$

then $\Phi(u, v)$ is a first integral of system (2.11). In this case the origin is stable (in the sense of Lyapunov), but not asymptotically stable. That is, it is a center.

Along the lines of the discussion above a characterization of elementary centers at the origin of (2.11) is given by the following theorem. Among many other places a proof may be found in [70, §3.1]. A proof can be also derived from Theorem 2.4.4 proved below.

Theorem 2.2.1 (Poincaré–Lyapunov Theorem)**.** *System* (2.11) *on* $\mathbb{R}^2$ *has a center at the origin if and only if there exists an analytic or formal first integral that has the form* $\Psi(u, v) = u^2 + v^2 + \cdots$. *Moreover in the latter case there exists an analytic first integral of the same form.*

As already mentioned, usually we are interested in the behavior of trajectories, not of a particular system, but of some family of systems, for example, all systems (2.11) in which $\widetilde{U}$ and $\widetilde{V}$ are arbitrary homogeneous polynomials of degree two. Thus the coefficients of (2.11) are not fixed numbers but parameters. In such a case we can hardly expect that either (2.13) or (2.14) will hold for all values of the parameters. Rather, it is usual that for some values (2.13) holds while for the others there is a first integral, that is, (2.14) holds. In the parametric case the coefficients ϕ_{jk} of (2.12) prove to be polynomials in the parameters of (2.11). If in trying to satisfy the identity (2.14) we encounter obstacles to its fulfillment then these obstacles will constitute necessary conditions for the existence of a first integral of the form (2.12) for system (2.11).

A computational procedure for finding the first m necessary conditions for integrability of (2.11) in the case that U and V are polynomials is as follows.

1. Write down the initial string of (2.12) up to order $2m + 1$,

$$\Phi_{2m+1}(u, v, w) = u^2 + v^2 + \sum_{j+k=3}^{2m+1} \phi_{jk} u^j v^k.$$

2. For each $i \in \{3, \ldots, 2m + 1\}$ equate the coefficients of the terms of order i in the expression

$$\frac{\partial \Phi_{2m+1}}{\partial u} \widetilde{U} + \frac{\partial \Phi_{2m+1}}{\partial v} \widetilde{V} \tag{2.15}$$

to zero, obtaining $2m-2$ systems of linear equations in the unknowns ϕ_{jk}.

3. Successively solve the $2m-2$ linear systems, beginning with that arising from index $i=3$, using the values obtained for the ϕ_{jk} by means of the system corresponding to index i in the systems of higher index. The systems that correspond to odd $i=2i_0-1$ always have unique solutions. For systems that correspond to even $i=2i_0$ there is one more equation than there are unknowns, but by dropping a suitable equation one obtains a system that has a unique solution. After solving the system assign the value 0 to the as yet undetermined $\phi_{j_0k_0}$ (with $j_0+k_0=2i_0$).

4. Evaluate (2.15) with the currently known ϕ_{jk} $(i+j\le 2i_0)$ and find the coefficient of $u^{j_0}v^{k_0}$, which we denote by g_{i_0-1}.

Computing in this way one obtains a list of polynomials $g_1, g_2, g_3, \ldots$ in the parameters of system (2.11). Each polynomial g_i gives a necessary condition for integrability so that system (2.11) admits an integral of the form (2.12) if and only if

$$g_1 = g_2 = g_3 = \cdots = 0. \tag{2.16}$$

Thus the set in the parameter space corresponding to systems with a center at the origin (equivalently, systems having a first integral of the form (2.12)) is the set of common zeros of system (2.16), that is, is the variety of the ideal $I=\langle g_i : i\in\mathbb{N}\rangle$, which we call the *center variety.*. By the Hilbert Basis Theorem (Theorem 1.1.1) there is an integer m that system (2.16) is equivalent to the system $g_1=\cdots=g_m=0$, but it is a difficult problem to find such m.

A practical approach to finding the center variety is as follows. We compute the polynomials g_i until the chain of varieties (considered as complex varieties) $\mathbf{V}(I_1)\supset\mathbf{V}(I_2)\supset\mathbf{V}(I_2)\supset\ldots$ stabilizes, that is, until we find k_0 such that $\mathbf{V}(I_{k_0})=\mathbf{V}(I_{k_0+1})$, where $I_k=\langle g_1,\ldots,g_k\rangle$. To check that two varieties are equal one can use the Radical Membership Test (Theorem 1.4.3) and the fact that over $\mathbb{C}$, $\mathbf{V}(I)=\mathbf{V}(J)$ if and only if $\sqrt{I}=\sqrt{J}$. Once we have found such a k_0 we expect that $\mathbf{V}(I)=\mathbf{V}(I_{k_0})$, but all we know so far is the inclusion $\mathbf{V}(I)\subset\mathbf{V}(I_{k_0})$. The first step in establishing the reverse inclusion is to find the irreducible decomposition (1.2) of $V:=\mathbf{V}(I_{k_0})$. Having found the decomposition we must then prove that for each point of V_i, $1\le i\le m$, the corresponding system admits a first integral of the form (2.12). If this can be done for every i then we conclude that the variety $\mathbf{V}(I)$ coincides with $\mathbf{V}(I_{k_0})$ and the center problem for the family is solved.

We will discuss this computational procedure in more detail in Subsection 2.4, where a more general system is treated. A concrete example

is the discussion in Subsection 6.1 leading up to the statement of Theorem 6.1.1.

2.3. Normal Forms

Deep insight in the center problem is provided by the theory of normal forms so before proceeding further we recall some of the main facts of the theory of Poincaré-Dulac normal forms for systems of differential equations on $\mathbb{C}^n$. Consider the system

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{X}(\mathbf{x}), \tag{2.17}$$

where $\mathbf{x} \in \mathbb{C}^n$, A is a possibly complex $n \times n$ matrix, and each component $X_k(\mathbf{x})$ of $\mathbf{X}$, $1 \leq k \leq n$, is a convergent power series, possibly with complex coefficients, that contains no constant or linear terms. In brief the idea is to make an analytic or formal change of variables to reduce (2.17) to a form more amenable for study, typically by eliminating as many unnecessary terms as possible. Since there is always an invertible linear transformation that places the linear part A of (2.17) in Jordan normal form, we will assume without loss of generality that this has already been done. For simplicity from now on we assume additionally that the Jordan normal form is diagonal, that is, that the *matrix A is a diagonal matrix with eigenvalues $\kappa_1, \ldots, \kappa_n$.* We denote the ordered n-tuple of the eigenvalues by $\kappa = (\kappa_1, \ldots, \kappa_n)$.

We say that system (2.17) is *formally equivalent* to a system

$$\dot{\mathbf{y}} = A\mathbf{y} + \mathbf{Y}(\mathbf{y}) \tag{2.18}$$

if there is a change of variables

$$\mathbf{x} = \mathbf{H}(\mathbf{y}) = \mathbf{y} + \mathbf{h}(\mathbf{y}) \tag{2.19}$$

that transforms (2.17) into (2.18), where the coordinate functions of $\mathbf{Y}$ and $\mathbf{h}$, Y_j and h_j, $j = 1, \ldots, n$, are formal power series that begin with terms of at least degree two. If all Y_j and h_j are convergent power series (and all X_j are as well) then by the Inverse Function Theorem the transformation (2.19) has an analytic inverse on a neighborhood of $\mathbf{0}$ and we say that (2.17) and (2.18) are *analytically equivalent.*

Henceforth we will use the following notation. For any multi-index α, the coefficient of the monomial $\mathbf{x}^\alpha$ in the mth component X_m of $\mathbf{X}$ will be denoted $X_m^{(\alpha)}$. We will use the same notational convention for $\mathbf{Y}$ and $\mathbf{h}$.

Let $\mathcal{H}^s$ denote the vector space of functions from $\mathbb{C}^n$ to $\mathbb{C}^n$ each of whose components is a homogeneous polynomial function of degree

s; elements of $\mathcal{H}^s$ will be termed vector homogeneous polynomials. If $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{C}^n$,

$$\mathbf{e}_j = (0, \ldots, 0, \overset{j}{1}, 0, \ldots, 0)^T,$$

then a basis for $\mathcal{H}^s$ is the collection of vector homogeneous functions

$$\mathbf{v}_{j,\alpha} = \mathbf{x}^\alpha \mathbf{e}_j \tag{2.20}$$

for all j such that $1 \leq j \leq n$ and all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ such that $|\alpha| = s$ (where $|\alpha| = \alpha_1 + \cdots + \alpha_n$). Note that (2.20) is the product of a monomial and a vector. Supposing that by a series of transformations all terms in the right hand side of (2.17) that are in $\mathcal{H}^s$ for $2 \leq s \leq s_0 - 1$ that can be eliminated by a transformation of the form (2.19) have already been so removed, terms in $\mathcal{H}^{s_0}$ that can be removed are those that lie in the image of the linear operator on $\mathcal{H}^{s_0}$ defined by

$$\mathcal{L}\mathbf{p}(\mathbf{y}) = \mathrm{d}\mathbf{p}(\mathbf{y}) A\mathbf{y} - A\mathbf{p}(\mathbf{y}), \tag{2.21}$$

where $\mathbf{p}(\mathbf{y})$ denotes a vector homogeneous polynomial and $\mathrm{d}\mathbf{p}(\mathbf{y})$ stands for the Jacobi matrix of $\mathbf{p}(\mathbf{y})$. $\mathcal{L}$ is called the *homological operator*. Its eigenvalues are given by the following lemma. See for example [9, 70] for a proof. For α, $\beta \in \mathbb{C}^n$, (α, β) will denote the scalar product

$$(\alpha, \beta) = \sum_{j=1}^{n} \alpha_j \beta_j .$$

Lemma 2.3.1. *Let A be an $n \times n$ matrix with eigenvalues $\kappa_1, \ldots, \kappa_n$ and let $\mathcal{L}$ be the corresponding homological operator on $\mathcal{H}^s$ given by* (2.21)*. Let $\kappa = (\kappa_1, \ldots, \kappa_n)$. Then the eigenvalues λ_j of $\mathcal{L}$ are*

$$\lambda_j = (\alpha, \kappa) - \kappa_m,$$

where m ranges over the initial segment $\{1, \ldots, n\} \subset \mathbb{N}$ and α ranges over the set $\{\beta \in \mathbb{N}_0^n : |\beta| = s\}$.

Suppose $m \in \{1, \ldots, n\}$ and $\alpha \in \mathbb{N}_0^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n \geq 2$, are such that

$$(\alpha, \kappa) - \kappa_m = 0 .$$

Then m and α are called a *resonant pair*, the corresponding coefficient $X_m^{(\alpha)}$ of the monomial $\mathbf{x}^\alpha$ in the mth component of $\mathbf{X}$ is called a *resonant coefficient*, and the corresponding term is called a *resonant term* of the *vector field* $\mathcal{X}$ associated to (2.17). Index and multi-index pairs, terms, and coefficients that are not resonant are called *nonresonant*.

If $F = \sum_\alpha f_\alpha \mathbf{x}^\alpha$ is a convergent or formal series we say that $\mathbf{x}^\alpha$ is a *resonant monomial* of the *series* F if $(\kappa, \alpha) = 0$.

Definition 2.3.2. *A* normal form *for system* (2.17) *is a system* (2.17) *in which every nonresonant coefficient is equal to zero. A* normalizing transformation *for system* (2.17) *is any (possibly merely formal) change of variables* (2.19) *that transforms* (2.17) *into a normal form; it is called* distinguished *if for each resonant pair* m *and* α *the corresponding coefficient* $h_m^{(\alpha)}$ *is zero, in which case the resulting normal form is likewise termed* distinguished.

Note however that although a normal form is the simplest form that we are certain to be able to obtain in general, for a particular system it might not be the absolute simplest. A normalizing transformation that eliminates all the nonresonant terms could very well produce resonant terms, which the original system did not have. Thus when normalizing a polynomial system, which of course contains but a finite number of terms, we can obtain a normal form that contains infinitely many terms.

Every system is at least formally equivalent to a normal form which, unless we restrict to distinguished normalizing transformations, need not be unique.

Theorem 2.3.3. *Every system* (2.17) *can be transformed to its distinguished normal form and the distinguished normalizing transformation that produces it is unique.*

Proof. Assume that a substitution (2.19) transforms system (2.17) to (2.18). We write $\mathbf{Y}(\mathbf{y}) = \sum_{s=2}^{\infty} \mathbf{Y}_s(\mathbf{y})$ and $\mathbf{h}(\mathbf{y}) = \sum_{s=2}^{\infty} \mathbf{h}_s(\mathbf{y})$, where $\mathbf{Y}_s$ and $\mathbf{h}_s$ are homogeneous vector-valued polynomials of degree s.Then $\mathbf{Y}_s$ and $\mathbf{h}_s$ satisfy

$$\mathrm{d}\mathbf{h}_s(y)A\mathbf{y} - A\mathbf{h}_s = [\mathbf{X}]_s - \sum_{j=2}^{s-1} \mathrm{d}\mathbf{h}_j(\mathbf{y})\mathbf{Y}_{s+1-j}(\mathbf{y}) - \mathbf{Y}_s(\mathbf{y}) \tag{2.22}$$

where by $[\mathbf{X}]_s$ we denote the degree s homogeneous vector-valued polynomial obtained when $\mathbf{X}(\mathbf{y} + \mathbf{h}(\mathbf{y}))$ is expanded in a power series in $\mathbf{y}$. Decompose $\mathcal{H}^s$ as the direct sum $\mathcal{H}^s = \mathcal{H}_r^s \oplus \mathcal{H}_n^s$, in which $\mathcal{H}_r^s$ consists of resonant homogeneous polynomials and $\mathcal{H}_n^s$ consists of nonresonant homogeneous polynomials (that is, $\mathcal{H}_r^s$ is the kernel of the operator $\mathcal{L}$ of (2.21) and $\mathcal{L}$ is invertible on $\mathcal{H}_n^s$).

To find the distinguished normal form we separate the components on the right-hand side of (2.22) into two parts according to the decomposition $\mathcal{H}^s = \mathcal{H}_r^s \oplus \mathcal{H}_n^s$. For the part belonging to $\mathcal{H}_n^s$, since $\mathcal{L}$ is invertible

on this subspace, we choose $\mathbf{Y}_s = 0$ and then $\mathbf{h}_s$ is the corresponding solution of (2.22). For the part belonging to $\mathcal{H}_r^s$ we choose $\mathbf{h}_s \equiv 0$ and

$$\mathbf{Y}_s(\mathbf{y}) = [\mathbf{X}]_s - \sum_{j=2}^{s-1} \mathrm{d}\mathbf{h}_j(\mathbf{y})\mathbf{Y}_{s+1-j}(\mathbf{y}).$$

Performing this procedure for all s we obtain the distinguished normal form of (2.17). Q.E.D.

Lemma 2.3.1 yields the following theorem.

Theorem 2.3.4. *Let $\kappa_1, \dots, \kappa_n$ be the eigenvalues of the $n \times n$ matrix A in* (2.17) *and* (2.18)*, set $\kappa = (\kappa_1, \dots, \kappa_n)$, and suppose that*

$$(\alpha, \kappa) - \kappa_m \neq 0 \tag{2.23}$$

for all $m \in \{1, \dots, n\}$ and for all $\alpha \in \mathbb{N}_0^n$ for which $|\alpha| \geq 2$. Then systems (2.17) *and* (2.18) *are formally equivalent for all $\mathbf{X}$ and $\mathbf{Y}$, and the equivalence transformation* (2.19) *is uniquely determined by $\mathbf{X}$ and $\mathbf{Y}$.*

We now discuss the convergence of the normalizing transformation. A classical result is the Poincaré-Dulac Theorem. Define the *Poincaré domain* in $\mathbb{C}^n$ to be all points $(z_1, \dots, z_n)$ such that the convex hull of the set $\{z_1, \dots, z_n\} \subset \mathbb{C}$ does not contain the origin. Then the theorem asserts that if the vector $(\kappa_1, \dots, \kappa_n)$ of eigenvalues of A in (2.17) lies in the Poincaré domain then there exists a convergent normalizing transformation (see Chapter 5 of [4] for a complete exposition).

The following theorem was proved by C. L. Siegel ([79]).

Theorem 2.3.5. *Suppose there exist positive constants $C > 0$ and $\nu > 0$ such that for all $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| > 1$ and for all $k \in \{1, \dots, n\}$ the inequality*

$$\left| \sum_{i=1}^{n} \alpha_i \kappa_i - \kappa_k \right| \geq C|\alpha|^{-\nu} \tag{2.24}$$

holds. Then there exists a convergent transformation of (2.17) *to normal form.*

Siegel's theorem is strong in the sense that condition (2.24) is satisfied for almost all $\kappa \in \mathbb{C}^n$ with respect to Lebesgue measure. However the condition of the theorem in fact implies that the normal form should be linear, so the normal form itself is not interesting. An essential further step in the investigation of the convergence of normalizing transformation is due to V. A. Pliss [65].

Theorem 2.3.6. *Suppose that for system* (2.17)

(i) the nonzero elements among the $\sum_{j=1}^{n} \alpha_j \kappa_j - \kappa_k$ *satisfy condition* (2.24) *and*

(ii) some formal normal form of (2.17) *is linear.*

Then there exists a convergent transformation to normal form.

Although this theorem still deals with linear normal forms the result is very useful and it finds a generalization in the work of A. D. Bryuno, who has achieved fundamental insights into convergence and divergence problems in his work ([10, 11]). In particular Bryuno introduced two conditions that together are sufficient for existence of a convergent normalizing transformation:

Condition ω: for $w_\ell = \min(\alpha, \kappa)$ over all $\alpha \in \mathbb{N}_0^n$ for which $(\alpha, \kappa) \neq 0$ and $|\alpha| \leq 2^\ell$, $\sum 2^{-\ell} \ln w_\ell < \infty$;

Condition A (simplified version): some normal form has the form

$$\dot{\mathbf{y}} = (1 + g(\mathbf{y}))A\mathbf{y}, \tag{2.25}$$

that is, $\dot{y}_j = \kappa_j y_j (1 + g(\mathbf{y}))$ for some scalar function $g(\mathbf{y})$.

Clearly Condition ω is a sharper version of (2.24) and Condition A extends the condition of Pliss. Thus we will say that system (2.17) satisfies the Pliss-Bryuno condition if it can be transformed to (2.25) by a substitution of the form (2.19).

To reiterate, if Condition ω and Condition A are satisfied then a convergent normalizing transformation exists ([10]).

As indicated in [85], Bryuno's criteria have been the standard against which convergence results are measured. Nevertheless see [81] for significant progress, including criteria for divergence of the formal normal form. For more on the convergence properties of normal forms the reader can consult [84, 85, 87, 88].

We conclude this subsection with three lemmas that will be needed later on. The proofs are from [87].

Lemma 2.3.7. *Suppose* $\Phi(\mathbf{x})$ *is an analytic or formal first integral of* (2.17) *and that* (2.18) *is the distinguished normal form of* (2.17) *that arises by means of the distinguished normalization* (2.19). *Then* $\widetilde{\Phi}(\mathbf{y}) = \Phi(\mathbf{y} + \mathbf{h}(\mathbf{y}))$ *is a first integral of* (2.18) *and it contains only resonant terms.*

Proof. The first assertion is just the chain rule. Write $\widetilde{\Phi}(\mathbf{y})$ as a sum of homogeneous polynomials, $\widetilde{\Phi}(\mathbf{y}) = \sum_{k=\ell}^{\infty} \widetilde{\Phi}_k(\mathbf{y})$, where $\ell > 0$. We prove the second assertion by induction on the degree k.

Basis step. Because $\widetilde{\Phi}(\mathbf{y})$ is a first integral of (2.18)

$$\sum_{j=1}^{n} \kappa_j y_j \frac{\partial \widetilde{\Phi}_\ell}{\partial y_j} = 0$$

(that is, $(\mathrm{d}\widetilde{\Phi}_\ell, A\mathbf{y}) = 0$), which means that $\widetilde{\Phi}_\ell$ contains only resonant terms.
Inductive step. Suppose that for $\ell \le j \le m-1$, $\widetilde{\Phi}_j$ contains only resonant terms. A computation yields

$$(\mathrm{d}\widetilde{\Phi}_m, A\mathbf{y}) + \sum_{j=2}^{m} (\mathrm{d}\widetilde{\Phi}_{m+1-j}(\mathbf{y}), Y_j) = 0.$$

Since Y_j and $\widetilde{\Phi}_{m+1-j}$ are resonant homogeneous polynomials in the vector field and in the function, respectively, the second summand contains, as a function, only resonant terms, hence the same is true of the first. Thus $\widetilde{\Phi}$ contains only resonant terms. Q.E.D.

Lemma 2.3.8. *If system* (2.17) *has* $n-1$ *functionally independent analytic or formal first integrals then its distinguished normal form satisfies the Pliss-Bryuno condition, that is, is of the form* (2.25).

Proof. Let $\widetilde{\mathcal{X}} := (\kappa_1 y_1 + Y_1(\mathbf{y}), \dots, \kappa_n y_n + Y_n(\mathbf{y}))$ be the vector field corresponding to the distinguished normal form (2.23) and suppose that $H_1(\mathbf{x}), \dots, H_{n-1}(\mathbf{x})$ are the $n-1$ functionally independent first integrals of (2.17). By Lemma 2.3.7 the vector field $\widetilde{\mathcal{X}}$ has $n-1$ first integrals $\widetilde{H}_1(\mathbf{y}), \dots, \widetilde{H}_{n-1}(\mathbf{y})$, which are functionally independent and all resonant.

Let Ω denote the $(n-1)$-dimensional linear space spanned by $\mathrm{d}\widetilde{H}_1$, $\mathrm{d}\widetilde{H}_2, \dots, \mathrm{d}\widetilde{H}_{n-1}$. Since the $\widetilde{H}_j(\mathbf{x})$ are first integrals of the vector field $\widetilde{\mathcal{X}}$, $\widetilde{\mathcal{X}}$ is orthogonal to Ω and $\widetilde{H}_i \to \widetilde{H}_j$. Since all the first integrals $\widetilde{H}_i(\mathbf{y})$ contain only resonant terms this implies that $(\mathrm{d}\widetilde{H}_j(\mathbf{y}), A\mathbf{y}) = 0$ for each j, hence $(\kappa_1 y_1, \dots, \kappa_n y_n)$ is also orthogonal to Ω. Since we are in n-dimensional space, the two vector fields $\widetilde{\mathcal{X}}$ and $(\kappa_1 y_1, \dots, \kappa_n y_n)$ must be parallel at each point $\mathbf{y}$ in a neighborhood of the origin. Hence there exists a function of the form $1 + g(\mathbf{y})$ such that

$$\widetilde{\mathcal{X}} = (\kappa_1 y_1 (1 + g(\mathbf{y})), \dots, \kappa_n y_n (1 + g(\mathbf{y}))).$$

Q.E.D.

Lemma 2.3.9. *If system* (2.17) *has* $n-1$ *functionally independent analytic or formal first integrals on a neighborhood of* $\mathbf{0}$ *in* $\mathbb{C}^n$ *then*

the distinguished normalization (2.19) *that transforms* (2.17) *into the distinguished normal form* (2.25) *is convergent.*

Proof. We give the proof for the case that not all the eigenvalues of the matrix A are zero.

By Lemma 2.3.8 the distinguished normal form has the form (2.25). By the proof of Theorem 2.3.3 the coefficients $h_j^{(\alpha)}$ satisfy

(2.26)
$$((\alpha,\kappa)-\kappa_j)h_j^{(\alpha)} = [X_j(\mathbf{y}+\mathbf{h}(\mathbf{y}))]^{(\alpha)} - \kappa_j g^{(\alpha-\mathbf{e}_j)} - \sum_{i=1}^{n}\sum_{\substack{\ell\in\mathbb{N}_0^n\\ \ell\prec\alpha}} \ell_i\kappa_i g^{(\alpha-\ell)}h_j^{(\ell)}$$

where $[X_j(\mathbf{y}+\mathbf{h}(\mathbf{y}))]^{(\alpha)}$ (which henceforth will be abbreviated to $[X_j]^{(\alpha)}$) denotes the coefficient of $\mathbf{y}^\alpha$ when $X_j(\mathbf{y}+\mathbf{h}(\mathbf{y}))$ is expanded in powers of $\mathbf{y}$ and $\ell \prec \alpha$ means $\alpha-\ell\in\mathbb{N}_0^n$.

If $(\alpha,\kappa)-\kappa_j = 0$ then by Theorem 2.3.3 (2.26) yields

$$h_j^{(\alpha)} = 0 \quad\text{and}\quad g^{(\alpha-\mathbf{e}_j)} = \kappa_j^{-1}\left([X_j]^{(\alpha)} - \sum_{\substack{\ell\in\mathbb{N}_0^n\\ \ell\prec\alpha}} (\ell,\kappa)h_j^{(\ell)}g^{(\alpha-\ell)}\right). \tag{2.27}$$

But since $g^{(\alpha-\ell)}$ is the coefficient of a resonant term, $(\alpha-\ell,\kappa)=0$, so $(\ell,\kappa)=(\alpha,\kappa)=\kappa_j$, hence $h_j^{(\ell)}$ is the coefficient of a resonant term, hence is zero (since the normalization is distinguished), and in fact (2.27) is

$$h_j^{(\alpha)} = 0 \quad\text{and}\quad g^{(\alpha-\mathbf{e}_j)} = \kappa_j^{-1}[X_j]^{(\alpha)}. \tag{2.28}$$

If $(\alpha,\kappa)-\kappa_j \neq 0$ then making the choice $g^{(\alpha-\ell)}=0$, (2.26) yields

$$h_j^{(\alpha)} = ((\kappa,\alpha)-\kappa_j)^{-1}\left([X_j]^{(\alpha)} - \sum_{\substack{\ell\in\mathbb{N}_0^n\\ \ell\prec\alpha}} (\ell,\kappa)h_j^{(\ell)}g^{(\alpha-\ell)}\right). \tag{2.29}$$

Thus in the distinguished normalizing transformation $\mathbf{x}=\mathbf{y}+\mathbf{h}(\mathbf{y})$ the coefficients of $\mathbf{h}$ are given by (2.29) and the distinguished normal form is

$$\dot{y}_j = \kappa_j y_j\left(1 + \sum_{\substack{\alpha-\mathbf{e}_j\in\mathbb{N}_0^n\\ \alpha-\mathbf{e}_j\neq\mathbf{0}}} g^{(\alpha-\mathbf{e}_j)}y^{(\alpha-\mathbf{e}_j)}\right), \qquad 1\le j\le n$$

in which $(\alpha-\mathbf{e}_j,\kappa)=0$ and the $g^{(\alpha-\mathbf{e}_j)}$ satisfy (2.28).

By the results of [15] and [53], since system (2.17) has $n-1$ functionally independent analytic first integrals on a neighborhood of $\mathbf{0}$ in $\mathbb{C}^n$ the eigenvalues $\kappa_1, \ldots, \kappa_n$ of the matrix A satisfy

$$\begin{aligned} s_{1,1}\kappa_1 + \cdots + s_{1,n}\kappa_n &= 0, \\ &\vdots \\ s_{n-1,1}\kappa_1 + \cdots + s_{n-1,n}\kappa_n &= 0, \end{aligned} \tag{2.30}$$

where the $n-1$ vectors $(s_{1,1}, \ldots, s_{1,n}), \ldots, (s_{n-1,1}, \ldots, s_{n-1,n}) \in \mathbb{N}_0^n$ are linearly independent. Without loss of generality we can assume that the solution of (2.30) is

$$\kappa_1 = \frac{\nu_1}{\mu_1}\kappa_n, \ldots, \kappa_{n-1} = \frac{\nu_{n-1}}{\mu_{n-1}}\kappa_n$$

where $\mu_j \in \mathbb{Z}\setminus\{0\}$, $\nu_j \in \mathbb{N}_0$, and each pair μ_j and ν_j are relatively prime for $j = 1, \ldots, n-1$. Thus for any $\alpha \in \mathbb{N}_0^n$ and j for which $|\alpha| \geq 2$ and $|(\alpha,\kappa) - \kappa_j| \neq 0$ we have that for some $M_j \in \mathbb{N}$,

$$|(\alpha,\kappa) - \kappa_j| = \frac{M_j}{\mu_1 \cdots \mu_{n-1}}|\kappa_n| \geq \frac{|\kappa_n|}{\mu_1 \cdots \mu_{n-1}},$$

where the last inequality holds because of our assumption that at least one eigenvalue of A is nonzero. Setting $\delta = \mu_1 \cdots \mu_{n-1}/|\kappa_n|$, for any $\alpha \in \mathbb{N}_0^n$ for which $|\alpha| \geq 2$ and $|(\alpha,\kappa) - \kappa_j| \neq 0$,

$$|(\alpha,\kappa) - \kappa_j|^{-1} \leq \delta. \tag{2.31}$$

Then using the fact that $(\ell,\kappa) = (\alpha,\kappa) = \kappa_j$,

$$\begin{aligned} |(\alpha,\kappa) - \kappa_j|^{-1} &\left| \sum_{\substack{\ell \in \mathbb{N}_0^n \\ \ell \prec \alpha}} (\ell,\kappa)\mathbf{h}^{(\ell)} g^{(\alpha-\ell)} \right| \\ &\leq \sum_{\substack{\ell \in \mathbb{N}_0^n \\ \ell \prec \alpha}} (1 + |(\alpha,\kappa) - \kappa_j|^{-1}\,|\kappa_j|)|\mathbf{h}^{(\alpha)} g^{(\alpha-\ell)}| \\ &\leq \sum_{\substack{\ell \in \mathbb{N}_0^n \\ \ell \prec \alpha}} (1 + \delta|\kappa_j|)|\mathbf{h}^{(\alpha)} g^{(\alpha-\ell)}|. \end{aligned}$$

Setting $\rho = \max\{1 + \delta|\kappa_j| : 1 \leq j \leq n\}$ we obtain from (2.29)

$$|h_j^{(\alpha)}| \leq \delta|[X_j]^{(\alpha)}| + \rho \sum_{\substack{\ell \in \mathbb{N}_0^n \\ \ell \prec \alpha}} |h_j^{(\alpha)} g^{(\alpha-\ell)}|.$$

By the Cauchy estimates there exists a polydisk

$$D = \{|y_j| < r : 1 \le j \le n\}$$

on which $|[X_j]^{(\alpha)}| \le Mr^{-|\alpha|}$, where $M = \max_j \sup_{\partial D} |X_j|$.

Recall that a power series $\xi = \sum \xi^{(\alpha)}\mathbf{z}^\alpha$ is said to majorize a power series $\zeta = \sum \zeta^{(\alpha)}\mathbf{z}^\alpha$, denoted $\zeta \preccurlyeq \xi$, if $|\zeta^{(\alpha)}| \le \xi^{(\alpha)}$ for all $\alpha \in \mathbb{N}_0^n$. Let $\widehat{X}(\mathbf{x}) = M\sum_{|\alpha|=2}^{\infty} r^{-|\alpha|}\mathbf{x}^\alpha$. Then $\widehat{X}$ is analytic on D and majorizes X_j for each j. Otherwise for any series w we let $\widehat{w}$ denote the majorant of w obtained by replacing each coefficient in w by its modulus. Then for $\nu = \max\{|\kappa_j|^{-1} : 1 \le j \le n\}$,

$$\sum_{j=1}^{n} h_j + g \preccurlyeq \sum_{j=1}^{n} \widehat{h}_j + \widehat{g} \preccurlyeq (n\delta + \nu)\widehat{X}(\mathbf{y} + \widehat{\mathbf{h}}) + (\rho + 1)\sum_{j=1}^{n} \widehat{h}_j\widehat{g}. \tag{2.32}$$

Define

$$W(u) = \left[\sum_{j=1}^{n} \widehat{h}_j + \widehat{g}\right]\Bigg|_{\mathbf{y}=(u,u,\ldots,u)}.$$

Then $W(u)$ has the form $W(u) = uV(u)$ and by (2.32)

$$uV(u) \preccurlyeq (n\delta + \nu)u^2\widehat{X}_*(1 + V(u)) + (\rho + 1)V(u)^2u^2 \tag{2.33}$$

where $\widehat{X}_*(1 + V(u)) = \widehat{X}(u + \widehat{h}_1(u, \ldots, u), \ldots, u + \widehat{h}_n(u, \ldots, u))/u^2$. Defining

$$\Gamma(u, s) := s - (n\delta + \nu)u\widehat{X}_*(1 + s) - (\rho + 1)s^2u, \tag{2.34}$$

it is clear that Γ is analytic on a neighborhood of the origin and satisfies $\Gamma(0, 0) = 0$ and $\frac{\partial\Gamma}{\partial s}(0, 0) = 1$. By the Implicit Function Theorem there is a unique analytic solution $s(u)$ of $\Gamma(u, s) = 0$ satisfying $s(0) = 0$ in a neighborhood of the origin. A comparison of (2.33) and (2.34) shows that $s(u)$ majorizes $V(u)$, hence V is analytic in a neighborhood of the origin, hence W is, which in turn implies convergence of $\sum_{j=1}^{n} \widehat{h}_j + \widehat{g}$, hence ultimately of h_j and g. Q.E.D.

In the course of proving Lemma 2.3.9 we showed that the existence of $n - 1$ functionally independent first integrals implies the estimate (2.31). This means that in this case Condition ω is fulfilled. Thus the convergence of the normalizing transformation follows from the result of Bryuno. However we have presented a detailed proof to demonstrate the usage of the method of majorants, which is the main tool for proving convergence of normalizing transformations in the theory of normal forms.

2.4. Complexification and the $p:-q$ Resonant Center Problem

Computation of the obstacles for integrability of systems (2.4) will be further simplified if we consider a generalization of the center problem suggested by Dulac ([31]). We introduce the complex variable

$$x_1 = u + iv, \tag{2.35}$$

thus regarding the real plane (u, v) as a complex line. Differentiating (2.35) we obtain the complex differential equation

$$\dot{x}_1 = (\alpha + i\beta)x_1 + X_1(x_1, \bar{x}_1), \tag{2.36}$$

where $X_1 = P + iQ$ and P and Q (from (2.4)) are evaluated at $(x_1 + \bar{x}_1)/2, (x_1 - \bar{x}_1)/(2i)$. Adjoining to this equation its complex conjugate we obtain the system

$$\begin{aligned} \dot{x}_1 &= (\alpha + i\beta)x_1 + X_1(x_1, \bar{x}_1) \\ \dot{\bar{x}}_1 &= (\alpha - i\beta)\bar{x}_1 + \overline{X_1(x_1, \bar{x}_1)}\,. \end{aligned} \tag{2.37}$$

We observe that if a system (2.11), which is a particular case of (2.4), has a first integral of the form (2.12) then the corresponding system (2.37) admits a first integral of the form

$$\Psi = x_1\bar{x}_1 + \cdots .$$

It is convenient to consider $\bar{x}_1$ as a distinct variable x_2 that is independent of x_1, the function $\overline{X_1(x_1, \bar{x}_1)}$ as a new independent function $X_2(x_1, x_2)$, and in place of (2.37) work with the more general system

$$\begin{aligned} \dot{x}_1 &= (\alpha + i\beta)x_1 + X_1(x_1, x_2) \\ \dot{x}_2 &= (\alpha - i\beta)x_2 + X_2(x_1, x_2)\,. \end{aligned} \tag{2.38}$$

If in going from (2.37) to (2.38) we maintain the condition that $X_2(x_1, \bar{x}_1) = \overline{X_1(x_1, \bar{x}_1)}$ then system (2.38) on $\mathbb{C}^2$ is the *complexification* of the real system (2.4) on $\mathbb{R}^2$. In this case the complex line $\Pi := \{(x_1, x_2) : x_2 = \bar{x}_1\}$ is invariant for system (2.38); viewing Π as a two-dimensional hyperplane in $\mathbb{R}^4$, the flow on Π is precisely the original flow of system (2.4) on $\mathbb{R}^2$. In this sense the phase portrait of the real system has been embedded in an invariant set in the phase portrait of a complex one.

For the center problem the case of interest is $\alpha = 0$. In this situation a time rescaling by $i\beta$ yields from (2.38) the system

$$\begin{aligned} \dot{x}_1 &= x_1 + X_1(x_1, x_2) \\ \dot{x}_2 &= -x_2 + X_2(x_1, x_2)\,. \end{aligned} \tag{2.39}$$

This latter family is a particular subfamily of the family

$$\begin{aligned} \dot{x}_1 &= \quad px_1 + X_1(x_1, x_2) \\ \dot{x}_2 &= -qx_2 + X_2(x_1, x_2) \end{aligned} \tag{2.40}$$

where p and q are relatively prime positive integers.

It is easy to see from condition (2.43) below that if system (2.40) has a first integral represented by the formal power series

$$\Psi(x_1, x_2) = \sum_{\substack{i+j>0 \\ i,j\in\mathbb{N}_0}} u_{ij} x_1^i x_2^j$$

that begins with terms of order at most $p+q$ then up to rescaling by a non-zero constant Ψ must be of the form

$$\Psi(x_1, x_2) = x_1^q x_2^p + \sum_{\substack{i+j>p+q \\ i,j\in\mathbb{N}_0}} v_{i-q,j-p} x_1^i x_2^j, \tag{2.41}$$

where the indexing has been chosen so as to simplify formulas that we will obtain below.

The observations presented above suggest the following generalization of the concept of a center to systems of the form (2.40) (see [91]).

Definition 2.4.1. *System* (2.40) *has a* $p : -q$ resonant center *at the origin if it admits an analytic first integral of the form* (2.41)*.*

We consider polynomial systems (2.40) on $\mathbb{C}^2$, which we write in the form

$$\begin{aligned} \dot{x} &= \quad px - \sum_{(i,j)\in S} a_{ij} x^{i+1} y^j = P(x, y), \\ \dot{y} &= -qy + \sum_{(i,j)\in S} b_{ji} x^j y^{i+1} = Q(x, y), \end{aligned} \tag{2.42}$$

where $p, q \in \mathbb{N}$, $GCD(p, q) = 1$, and where, for some $\ell \in \mathbb{N}$, S is an ordered ℓ-element subset

$$S = \{(u_1, v_1), (u_2, v_2), \ldots, (u_1, v_\ell)\}$$

of $\mathbb{N}_{-1} \times \mathbb{N}_0$, every element of which satisfies $u_k + v_k \geq 1$, where $\mathbb{N}$ denotes the set of natural numbers and for a non-negative integer n, $\mathbb{N}_{-n} = \{-n, \ldots, -1, 0\} \cup \mathbb{N}$. The notation (2.42) simply emphasizes that we take into account only non-zero coefficients of the polynomials of interest and will simplify formulas that occur later. We denote by

$(a,b) = (a_{u_1,v_1}, a_{u_2,v_2}, \dots, a_{u_\ell,v_\ell}, b_{v_\ell,u_\ell}, \dots, b_{v_1,u_1})$ the ordered vector of the coefficients of system (2.42), by $E(a,b) = \mathbb{C}^{2\ell}$ the parameter space of (2.42), and by $\mathbb{C}[a,b]$ the polynomial ring in the variables a_{ij}, b_{ji}.

Writing $\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ for the vector field corresponding to (2.42), the condition that a function $\Psi(x,y)$ be a first integral of (2.42) is the identity

$$\mathcal{X}\Psi \stackrel{\text{def}}{=} \frac{\partial \Psi}{\partial x}P(x,y) + \frac{\partial \Psi}{\partial y}Q(x,y) \equiv 0\,, \tag{2.43}$$

which for functions of the form (2.41) is

$$\begin{aligned}&\Big(qx^{q-1}y^p + \sum_{i+j>p+q} i v_{i-q,j-p}x^{i-1}y^j\Big)\Big(px - \sum_{(m,n)\in S} a_{mn}x^{m+1}y^n\Big)\\ &+\\ &\Big(px^q y^{p-1} + \sum_{i+j>p+q} j v_{i-q,j-p}x^i y^{j-1}\Big)\Big(-qy + \sum_{(m,n)\in S} b_{nm}x^n y^{m+1}\Big)\\ &\equiv 0\,.\end{aligned} \tag{2.44}$$

We augment the set of coefficients in (2.41) with the collection

$$J = \{v_{-q+s,q-s} : s = 0, \dots, p+q\}, \tag{2.45}$$

where in agreement with formula (2.41) we set $v_{00} = 1$ and $v_{mn} = 0$ for all other elements of J, so that elements of J are the coefficients of the terms of degree $p+q$ in $\Psi(x,y)$. We also set $a_{mn} = b_{nm} = 0$ for $(m,n) \notin S$. With these conventions, for $(k_1,k_2) \in \mathbb{N}_{-q} \times \mathbb{N}_{-p}$, the coefficient g_{k_1,k_2} of $x^{k_1+q}y^{k_2+p}$ in (2.44) is zero for $k_1+k_2 \le 0$ and for $k_1+k_2 \ge 1$ is

$$\begin{aligned}&g_{k_1,k_2} =\\ &(pk_1 - qk_2)v_{k_1,k_2}\\ &- \sum_{\substack{s_1+s_2=0\\ s_1\ge -q,\ s_2\ge -p}}^{k_1+k_2-1} \left[(s_1+q)a_{k_1-s_1,k_2-s_2} - (s_2+p)b_{k_1-s_1,k_2-s_2}\right] v_{s_1,s_2}.\end{aligned} \tag{2.46}$$

This formula can be used recursively in an attempt to construct a formal first integral Ψ for system (2.42), at the first stage finding all v_{k_1,k_2} for which $k_1+k_2 = 1$, at the second all v_{k_1,k_2} for which $k_1+k_2 = 2$, and so on. For any pair k_1 and k_2, if

$$qk_1 \ne pk_2\,, \tag{2.47}$$

and if all coefficients v_{ℓ_1,ℓ_2} are already known for $\ell_1+\ell_2 < k_1+k_2$, then v_{k_1,k_2} is uniquely determined by (2.46) and the condition that g_{k_1,k_2} be

zero. But at each of the stages $k_1 + k_2 = k(p+q)$, $k \in \mathbb{N}$ (but only at these stages, since $GCD(p,q) = 1$) there occurs the one resonant pair $(k_1, k_2) = (kq, kp)$ for which (2.47) does not hold, hence for which (2.46) becomes

(2.48)
$$g_{kq,kp} = -\sum_{\substack{s_1+s_2=0 \\ s_1 \geq -q, s_2 \geq -p}}^{kq+kp-1} \left[(s_1+q)a_{k_1-s_1,k_2-s_2} - (s_2+p)b_{k_1-s_1,k_2-s_2}\right] v_{s_1,s_2} \,,$$

so that the process of constructing a first integral Ψ succeeds at this step only if the expression on the right hand side of (2.48) is zero. In this case the value of $v_{k_1,k_2} = v_{kq,kp}$ is not determined by equation (2.46) and may be assigned arbitrarily.

We remark that even though it is not generally true that an integral of the form (2.41) exists, the construction process described above always yields a formal series of the form (2.41) for which $\mathcal{X}\Psi = \Psi_x P + \Psi_y Q$ reduces to

$$\mathcal{X}\Psi = g_{q,p}(x^q y^p)^2 + g_{2q,2p}(x^q y^p)^3 + g_{3q,3p}(x^q y^p)^4 + \cdots \,, \tag{2.49}$$

where the polynomials $g_{kq,kp}$ are computed by (2.48).

It is evident from (2.46) that for all indices $(k_1, k_2) \in \mathbb{N}_{-q} \times \mathbb{N}_{-p}$, v_{k_1,k_2} is a polynomial function of the coefficients of (2.42), that is, is an element of the ring that we have denoted $\mathbb{C}[a,b]$, hence by (2.48) so are the expressions $g_{kq,kp}$ for all k. As above we would like to regard the polynomial $g_{kq,kp}$ as the kth "obstruction" to the existence of the integral (2.41). It is certainly true that if at a point (a^*, b^*) of our parameter space $E(a,b)$, $g_{kq,kp}(a^*, b^*) \neq 0$, then the construction process fails at that step. However, although $g_{q,p}$ is uniquely determined, for $k > 1$ $g_{kq,kp}$ is not, since for $\ell < k$ $v_{\ell q,\ell p}$ was arbitrary. Thus although it is true that the vanishing of $g_{kq,kp}(a^*, b^*)$ for all $k \in \mathbb{N}$ is sufficient for the existence of a formal first integral of the form (2.41), it is not clear *a priori* that it is necessary. In Theorem 2.4.4 below we prove that the condition is indeed necessary, independently of the choices of the $v_{\ell q,\ell p}$.

Definition 2.4.2. *Fix a set S. The* kth focus quantity *of the family* (2.42) *is the polynomial $g_{kq,kp}$. A* Bautin ideal *of the family* (2.42) *is the ideal $\mathcal{B} = \langle g_{q,p}, g_{2q,2p}, \ldots, g_{jq,jp}, \ldots \rangle$ in $\mathbb{C}[a,b]$. The* center variety *of the family* (2.42) *is the variety of the Bautin ideal,*

$$\begin{aligned} V_{\mathcal{C}} &= \mathbf{V}(\langle g_{q,p}, g_{2q,2p}, \ldots, g_{jq,jp}, \ldots \rangle) \\ &= \{(a,b) : g_{jq,jp}(a,b) = 0 \text{ for all } j \in \mathbb{N}\}. \end{aligned}$$

We now prove that the variety $V_{\mathcal{C}}$ is the same for all choices of the polynomials $v_{jq,jp}$, $j \in \mathbb{N}$ that determine $g_{kq,kp}$ and thus that the center variety $V_{\mathcal{C}}$ is well-defined. For that purpose we will need the following lemma.

Lemma 2.4.3. *Suppose the system $\dot{x} = Ax$ admits $n-1$ independent polynomial first integrals. If system (2.17) admits $n-1$ functionally independent analytic or formal first integrals (i.e., possibly some of each) then it also admits $n-1$ functionally independent analytic first integrals.*

Proof. If system (2.17) admits $n-1$ analytic or formal first integrals then by Lemmas 2.3.8 and 2.3.9 the system is transformed to the distinguished normal form satisfying the Pliss-Bryuno condition by a convergent transformation. By hypothesis the system in normal form has $n-1$ polynomial first integrals. Applying to these integrals the inverse of the distinguished normalizing transformation we obtain $n-1$ functionally independent analytic first integrals of (2.17). Q.E.D.

Theorem 2.4.4. *Consider a family of systems of the form (2.42), with parameter space $E(a,b) = \mathbb{C}^{2\ell}$, where $p, q \in \mathbb{N}$, $GCD(p,q) = 1$.*

1. *Let Ψ be an analytic or formal series of the form (2.41) and let $g_{q,p}$, $g_{2q,2p}$, $g_{3q,3p}$, ... be polynomials in $\mathbb{C}[a,b]$ that satisfy (2.49) with respect to the system (2.42). Then the system in family (2.42) corresponding to the choice of coefficients $(a^*, b^*) \in E(a,b)$ has a center at the origin if and only if $g_{kq,kp}(a^*, b^*) = 0$ for all $k \in \mathbb{N}$.*
2. *Let Ψ and $g_{qk,pk}$ be as above and suppose there exists another function Ψ' of the form (2.41) and polynomials $g'_{q,p}$, $g'_{2q,2p}$, $g'_{3q,3p}$, ... in $\mathbb{C}[a.b]$ that satisfy (2.49) with respect to the family (2.42). Then $V_{\mathcal{C}} = V'_{\mathcal{C}}$, where*

$$V_{\mathcal{C}} = \mathbf{V}(\langle g_{q,p}(a,b), g_{2q,2p}(a,b), \ldots \rangle)$$

and

$$V'_{\mathcal{C}} = \mathbf{V}(\langle g'_{q,p}(a,b), g'_{2q,2p}(a,b), \ldots \rangle).$$

Proof. 1) Suppose that family (2.42) is as in the statement of the theorem. Let Ψ be an analytic or formal series of the form (2.41) and let $\{g_{kq,kp}(a,b) : k \in \mathbb{N}\}$ be polynomials in (a,b) that satisfy (2.49).

If, for $(a^*, b^*) \in E(a,b)$, $g_{kq,kp}(a^*, b^*) = 0$ for all $k \in \mathbb{N}$ then Ψ is an analytic or formal first integral for the corresponding family in (2.42). By Lemma 2.4.3 there also exists an analytic first integral, so by Definition 2.4.1 the system has a center at the origin of $\mathbb{C}^2$.

For the converse, suppose that there exists a $k \in \mathbb{N}$ and a choice (a^*, b^*) of the parameters such that $g_{jq,jp}(a^*, b^*) = 0$ for $1 \leq j \leq k-1$

but $g_{kq,kp}(a^*,b^*)\neq 0$. Let $\mathbf{H}(x_1,y_1)$ be the distinguished normalizing transformation (2.19) that produces the distinguished normal form

$$\begin{aligned}\dot{x}_1 = &\quad px_1 + x_1\sum_{j=1}^{\infty} X^{(jq+1,jp)}(x_1^q y_1^p)^j = px_1 + x_1 X(x_1^q y_1^p)\\ \dot{y}_1 = &-qy_1 + y_1\sum_{j=1}^{\infty} Y^{(jq,jp+1)}(x_1^q y_1^p)^j = -qy_1 + y_1 Y(x_1^q y_1^p).\end{aligned} \tag{2.50}$$

and consider the function $F=\Psi\circ\mathbf{H}$. By the proof of Lemma 2.3.7 we see that

$$\begin{aligned}F(x_1,y_1) &= x_1^q y_1^p + f_2\,(x_1^q y_1^p)^2 + \cdots + f_k\,(x_1^q y_1^p)^k + U(x_1,y_1)\\ &= f(x_1^q y_1^p) + U(x_1,y_1)\end{aligned}$$

where $U(x_1,y_1)$ begins with terms of order no less than $k(p+q)+1$. Thus

$$x_1\frac{\partial F}{\partial x_1} = qx_1^q y_1^p f'(x_1^q y_1^p) + \alpha(x_1,y_1)$$

and

$$y_1\frac{\partial F}{\partial y_1} = px_1^q y_1^p f'(x_1^q y_1^p) + \beta(x_1,y_1)$$

where $\alpha(x_1,y_1)$ and $\beta(x_1,y_1)$ begin with terms of order no less than $k(p+q)+1$, and so the left hand side of (2.43) is

$$\begin{aligned}&p\,\alpha(x_1,y_1) - q\,\beta(x_1,y_1)\\ &\quad + (qX(x_1^q y_1^p) + pY(x_1^q y_1^p))\,x_1^q y_1^p\,f'(x_1^q y_1^p)\\ &\quad\quad + X(x_1^q y_1^p)\,\alpha(x_1,y_1) + Y(x_1^q y_1^p)\,\beta(x_1,y_1)\,.\end{aligned}$$

Hence if we subtract

$$p\,\alpha(x_1,y_1) - q\,\beta(x_1,y_1) + X(x_1^q y_1^p)\,\alpha(x_1,y_1) + Y(x_1^q y_1^p)\,\beta(x_1,y_1)\,,$$

which begins with terms of order at least $k(p+q)+1$, from each side of (2.43) we obtain

$$G(x_1^q y_1^p)\,x_1^q y_1^p\,f'(x_1^q y_1^p) = g_{kq,kp}(a^*,b^*)(x_1^q y_1^p)^k + \cdots\,, \tag{2.51}$$

where $G(x_1^q y_1^p) = qX(x_1^q y_1^p) + pY(x_1^q y_1^p)$.

Now suppose, contrary to what we wish to show, that system (2.42) for the choice $(a,b)=(a^*,b^*)$ has a center at the origin of $\mathbb{C}^2$, so that it admits a first integral $\Phi(x,y) = x^q y^p + \cdots$. Then by Lemma 2.3.8 the distinguished normal form (2.50) satisfies the Pliss-Bryuno condition.

But this means that the function G vanishes identically, hence the left hand side of (2.51) is identically zero, whereas the right hand side is not, a contradiction.

2) If $V_{\mathcal{C}} \neq V'_{\mathcal{C}}$ then there exists (a^*, b^*) that belongs to one of the varieties $V_{\mathcal{C}}$ and $V'_{\mathcal{C}}$ but not to the other, say $(a^*, b^*) \in V_{\mathcal{C}}$ but $(a^*, b^*) \notin V'_{\mathcal{C}}$. The inclusion $(a^*, b^*) \in V_{\mathcal{C}}$ means that the system corresponding to (a^*, b^*) has a center at the origin. Therefore by part (1) $g'_{kq,kp}(a^*, b^*) = 0$ for all $k \in \mathbb{N}$. This contradicts our assumption that $(a^*, b^*) \notin V'_{\mathcal{C}}$. Q.E.D.

The center variety therefore corresponds exactly to those systems of the form (2.42) for which there is a center at the origin of $\mathbb{C}^2$, in the sense of Definition 2.4.1. In the case that (2.42) is a system on $\mathbb{R}^2$ rather than $\mathbb{C}^2$ the polynomials $g_{jq,jp}$ are still defined but in that context are called the *saddle quantities* of the system.

2.5. Integrability of a Cubic Family

As an illustration and application of the ideas developed so far, in this subsection we derive a set of sufficient conditions for existence of a $2:-3$ resonant center in a plane cubic system. We begin, however, with a general observation.

When investigating integrability one sometimes obtains integrals or integrating factors that are defined for some but not all values of the parameters. Instead of looking for explicit forms of first integrals in the remaining "degenerate" cases one can often conclude that integrals exist using the geometric argument given in the following lemma.

Lemma 2.5.1. *Suppose the parameters in the family of systems* (2.42) *are the coefficients of the polynomials, so that the parameter space is* $E(a, b) = \mathbb{C}^{2\ell}$, *and let* V *and* W *be varieties in* $E(a, b)$ *such that system* (2.42) *admits a local analytic first integral of the form* (2.41) *for all values of the parameters in* $V \setminus W$. *If*

$$\overline{V \setminus W} = V \tag{2.52}$$

(where the overline denotes the Zariski closure) then the system admits a local analytic first integral of the form (2.41) *for all values of the parameters in* V.

Proof. By Theorem 2.4.4 the set of all systems that admit a local analytic first integral of the form (2.41) is the variety of a Bautin ideal $\mathbf{V}(\mathcal{B})$. By hypothesis $V \setminus W \subset \mathbf{V}(\mathcal{B})$. Taking the Zariski closure we obtain $\overline{V \setminus W} \subset \mathbf{V}(\mathcal{B})$. Thus by (2.52) $V \subset \mathbf{V}(\mathcal{B})$ and all systems from V admit a local analytic first integral of the form (2.41). Q.E.D.

For some varieties it is straightforward to check whether (2.52) holds. In more complicated situations, if $V = \mathbf{V}(I)$ and $W = \mathbf{V}(J)$, if I is a radical ideal then since $k = \mathbb{C}$ by (1.7) $\overline{V \setminus W} = \mathbf{V}(I : J)$, so (2.52) holds if $I : J = I$. If I is not radical then it may be replaced by $\sqrt{I}$, since $\mathbf{V}(I) = \mathbf{V}(\sqrt{I})$.

The system that we treat in this subsection is

$$\begin{aligned} \dot{x} &= x(2 - a_{20}x^2 - a_{11}xy - a_{02}y^2), \\ \dot{y} &= y(-3 + b_{20}x^2 + b_{11}xy + b_{02}y^2), \end{aligned} \tag{2.53}$$

which was studied in [36]. We first recall the following result of [16] which we will need for our analysis. It was first observed in [36] that the result is valid for non-integer values of q when the result is stated as we do here (compare with Definition 2.4.1).

Theorem 2.5.2. *The system*

$$\begin{aligned} \dot{x} &= x(1 - a_{20}x^2 - a_{11}xy - a_{02}y^2), \\ \dot{y} &= y(-q + b_{20}x^2 + b_{11}xy + b_{02}y^2), \end{aligned} \tag{2.54}$$

with $q \in \mathbb{R}$ and $q > 1$ has a first integral of the form

$$\Psi(x, y) = x^q y + \sum_{\substack{i+j>1+q \\ i,j\in\mathbb{N}_0}} v_{ij} x^i y^j$$

if at least one of the following conditions holds:

(1) $a_{11} = b_{20} = b_{11} = 0$*;*

(2) $a_{11} = (q-2)a_{20} - b_{20} = 0$*;*

(3) $qa_{20}a_{11} + a_{11}b_{20} + (q-2)a_{20}b_{11} - b_{20}b_{11} = 0$,
$qa_{20}a_{02} + (q-1)a_{20}b_{02} - b_{20}b_{02} = 0$, *and*
$qa_{11}a_{02} - qa_{02}b_{11} + (2q-1)a_{11}b_{02} - b_{11}b_{02} = 0$.

As a preliminary result we first derive a collection of sufficient conditions for linearizability of system (2.53). Computational constraints that will arise later force the restriction to the three cases based on the values of a_{11} and b_{11}. The condition $a_{20} = b_{20}$ arises from the method of proof.

Theorem 2.5.3. *System* (2.53) *has a linearizable resonant center at the origin if at least one of conditions* (α), (β), *and* (γ) *holds:*

(α) $a_{11} = b_{11} = 1$ *and*
(1) $b_{20} = a_{20} = 0$ *or*
(2) $a_{20} - b_{20} = 0$ *and*
$27a_{02}^2 b_{20}^2 - 9a_{02}b_{20}^2 b_{02} + 144a_{02}b_{20} - 28b_{20}b_{02} + 48 = 0$*;*

(β) $a_{11} = 1$, $b_{11} = 0$, *and*
  (1) $b_{20} = a_{20} = 0$ *or*
  (2) $a_{20} - b_{20} = 0$ *and*
  $27a_{02}^2 b_{20}^2 - 9a_{02}b_{20}^2 b_{02} + 396a_{02}b_{20} - 52b_{20}b_{02} + 360 = 0$;

(γ) $a_{11} = 0$, $b_{11} = 1$, *and*
  (1) $a_{02} = a_{20} - b_{20} = 0$ *or*
  (2) $a_{20} - b_{20} = 3a_{02}b_{20} - b_{20}b_{02} + 6 = 0$.

Proof. Following the method developed in [23] we make the substitutions $v = xy$ and $w = y^2$. When $a_{20} = b_{20}$ we obtain a system of ordinary differential equations on the whole (v, w)-plane:

$$\begin{aligned} \dot{v} &= -v + (b_{11} - a_{11})v^2 + (b_{02} - a_{02})vw \\ \dot{w} &= -6w + 2b_{20}v^2 + 2b_{11}vw + 2b_{02}w^2. \end{aligned} \tag{2.55}$$

Since the convex hull in $\mathbb{C}$ of the set eigenvalues $\{-1, -6\}$ of the linear part does not contain the origin of $\mathbb{C}$, by the Poincaré-Dulac Theorem (Subsection 2.3) there exists an analytic change of coordinates $(v_1, w_1) = \xi(v, w)$ bringing (2.55) to its normal form

$$\dot{v}_1 = -v_1, \quad \dot{w}_1 = -nw_1 + av_1^6. \tag{2.56}$$

Thus system (2.55) is linearizable if and only if the coefficient a in the resonant monomial av_1^6 in the normal form is zero. In terms of the coefficients of (2.53) the coefficient a is

(2.57)
$$\begin{aligned} \tfrac{1}{36}b_{20}\big(&360a_{11}^4 - 516a_{11}^3 b_{11} + 240a_{11}^2 b_{11}^2 - 36a_{11}b_{11}^3 \\ &+ 396a_{02}a_{11}^2 b_{20} - 52a_{11}^2 b_{02}b_{20} - 306a_{02}a_{11}b_{11}b_{20} \\ &+ 24a_{11}b_{02}b_{11}b_{20} + 54a_{02}b_{11}^2 b_{20} + 27a_{02}^2 b_{20}^2 - 9a_{02}b_{02}b_{20}^2\big). \end{aligned}$$

Setting $a_{11} = b_{11} = 1$ in (2.57) gives that $a = 0$ provided

$$b_{20}(48 + 144a_{02}b_{20} - 28b_{02}b_{20} + 27a_{02}^2 b_{20}^2 - 9a_{02}b_{02}b_{20}^2) = 0,$$

which yields (α).

Setting $a_{11} = 1$ and $b_{11} = 0$ in (2.57) gives that $a = 0$ provided

$$b_{20}(360 + 396a_{02}b_{20} - 52b_{02}b_{20} + 27a_{02}^2 b_{20}^2 - 9a_{02}b_{02}b_{20}^2) = 0,$$

which yields (β).

Finally, setting $a_{11} = 0$ and $b_{11} = 1$ in (2.57)) gives that $a = 0$ provided

$$a_{02}b_{20}^2(6 + 3a_{02}b_{20} - b_{02}b_{20}) = 0,$$

which yields subcases (γ).

The linear system has the first integral $\Psi(v_1, w_1) = v_1^6/w_1$ off the v_1-axis. By invariance of the w-axis $v = 0$ in (2.55) the normalizing transformation has the form $(v_1, w_1) = \xi(v, w) = (v(1 + \overset{(1)}{\cdots}), w + \overset{(2)}{\cdots})$, where the symbol (k) over the dots indicates the lowest order of the omitted terms, so that we obtain an analytic first integral (where defined)

$$\begin{aligned}\Phi(x, y) &= \Psi\Big(xy(1 + \overset{(2)}{\cdots}), y^2 + \overset{(4)}{\cdots}\Big) \\ &= \frac{x^6 y^6 \Big(1 + \overset{(2)}{\cdots}\Big)^6}{y^2 \Big(1 + \overset{(2)}{\cdots}\Big)} = x^6 y^2 (1 + \overset{(2)}{\cdots}).\end{aligned}$$

But this last object is a formal first integral of (2.53), which by taking square roots gives a formal first integral of the form $x^3 y + \cdots$, which implies existence of a local analytic first integral of the same form. Q.E.D.

The following theorem gives a collection of sufficient conditions for integrability of system (2.53). Along with the results of [30] these conditions constitute the list of necessary and sufficient conditions for integrability.

Theorem 2.5.4. *System* (2.53) *has a resonant center at the origin if at least one of conditions* (α), (β), (γ), *and* (δ) *holds:*

(α) $a_{11} = b_{11} = 1$ *and at least one of the following holds:*
- (1) $a_{20} - b_{20} = 0$ *and* $27a_{02}^2 b_{20}^2 - 9a_{02} b_{20}^2 b_{02} + 144 a_{02} b_{20} - 28 b_{20} b_{02} + 48 = 0$;
- (2) $b_{20} = a_{20} = 0$;
- (3) $b_{02} = a_{20} = 0$;

(β) $a_{11} = 1$, $b_{11} = 0$, *and at least one of the following holds:*
- (1) $a_{20} - b_{20} = 0$ *and* $27a_{02}^2 b_{20}^2 - 9a_{02} b_{20}^2 b_{02} + 396 a_{02} b_{20} - 52 b_{20} b_{02} + 360 = 0$;
- (2) $b_{20} = a_{20} = 0$;
- (3) $a_{20} b_{02} + 6 = b_{20} = a_{02} - b_{02} = 0$;
- (4) $a_{20} b_{02} - 6 = b_{20} = 3a_{02} + 4b_{02} = 0$;
- (5) $b_{20} b_{02} + 18 = 3a_{02} + 4b_{02} = a_{20} + 3b_{20} = 0$;
- (6) $3a_{02} + 4b_{02} = 3a_{20} + 2b_{20} = 0$;

(γ) $a_{11} = 0$, $b_{11} = 1$, *and at least one of the following holds:*
- (1) $b_{02} = a_{02} = 0$;
- (2) $3a_{02} b_{20} - b_{20} b_{02} + 6 = a_{20} - b_{20} = 0$;
- (3) $a_{02} = a_{20} - b_{20} = 0$;

(4) $a_{20} + 2b_{20} = 0$;

(δ) $a_{11} = b_{11} = 0$ *and at least one of the following holds:*

(1) $3a_{20}a_{02} + a_{20}b_{02} - 2b_{20}b_{02} = 0$;

(2) $a_{02} = 0$;

(3) $b_{20} = 0$;

(4) $a_{20} + 2b_{20} = 0$.

Proof. Computing the conditions. Using formulas (2.46) and (2.48) we compute the first twelve focus quantities $g_{3,2}, \ldots, g_{36,24}$. We find that $g_{q(2k+1),p(2k+1)} = 0$ for $k = 0, \ldots, 5$ and that (after rescaling by 504)

$$\begin{aligned} g_{6,4} = {} & 1512a_{11}^4a_{20} + 216a_{02}a_{11}^2a_{20}^2 + 36a_{11}^2a_{20}^2b_{02} - 1764a_{11}^3a_{20}b_{11} \\ & - 288a_{02}a_{11}a_{20}^2b_{11} - 72a_{11}a_{20}^2b_{02}b_{11} + 672a_{11}^2a_{20}b_{11}^2 + 72a_{02}a_{20}^2b_{11}^2 \\ & + 20a_{20}^2b_{02}b_{11}^2 - 84a_{11}a_{20}b_{11}^3 + 1008a_{11}^4b_{20} + 2196a_{02}a_{11}^2a_{20}b_{20} \\ & + 63a_{02}^2a_{20}^2b_{20} + 576a_{11}^2a_{20}b_{02}b_{20} + 21a_{02}a_{20}^2b_{02}b_{20} - 1848a_{11}^3b_{11}b_{20} \\ & - 1386a_{02}a_{11}a_{20}b_{11}b_{20} - 272a_{11}a_{20}b_{02}b_{11}b_{20} + 1008a_{11}^2b_{11}^2b_{20} \\ & + 198a_{02}a_{20}b_{11}^2b_{20} + 20a_{20}b_{02}b_{11}^2b_{20} - 168a_{11}b_{11}^3b_{20} + 360a_{02}a_{11}^2b_{20}^2 \\ & + 126a_{02}^2a_{20}b_{20}^2 - 976a_{11}^2b_{02}b_{20}^2 - 468a_{02}a_{11}b_{11}b_{20}^2 + 512a_{11}b_{02}b_{11}b_{20}^2 \\ & + 108a_{02}b_{11}^2b_{20}^2 - 40b_{02}b_{11}^2b_{20}^2 - 84a_{02}b_{02}b_{20}^3. \end{aligned}$$

The rest are too long to be presented here, but the interested reader should have no trouble computing them using (2.46) and (2.48) with any popular computer algebra system.

The next step is to find the irreducible decomposition of the variety of the ideal $\mathcal{B}_{12} = \langle g_{6,4}, g_{12,8}, \ldots, g_{36,24} \rangle$. Even using the routine `minAssGTZ` of the special purpose program SINGULAR and working in modular arithmetic this proved to be computationally infeasible. Indeed, even after applying a rescaling $x \to \alpha x$, $y \to \beta y$ in order to reduce the number of parameters by assigning specific values to pairs of nonzero coefficients the computations are still very difficult. Therefore we limit our considerations to the following four cases: (α) $a_{11} = b_{11} = 1$, (β) $a_{11} = 1$, $b_{11} = 0$, (γ) $a_{11} = 0$, $b_{11} = 1$, and (δ) $a_{11} = b_{11} = 0$.

In case (δ), computing over $\mathbb{Q}$ we obtain conditions (δ) of the theorem.

In case (β) it was not possible to complete the computations with `minAssGTZ` working over $\mathbb{Q}$ so we used the modular approach presented in Subsection 1.5. Executing `minAssGTZ` with coefficients in the field of

characteristic 32003 yields the following list of associated prime ideals:

$$
\begin{aligned}
P_1 &= \langle a_{20} - b_{20}, \\
&\qquad a_{02}^2 b_{20}^2 - 10668 a_{02} b_{20}^2 b_{02} - 10653 a_{02} b_{20} - 8299 b_{20} b_{02} + 10681 \rangle \\
P_2 &= \langle b_{20}, a_{20} \rangle \\
P_3 &= \langle a_{02} + 10669 b_{02}, a_{20} - 10667 b_{20} \rangle \\
P_4 &= \langle a_{02} + 10669 b_{02}, a_{20} + 3 b_{20}, b_{20} b_{02} + 18 \rangle \\
P_5 &= \langle b_{20}, a_{02} + 10669 b_{02}, a_{20} b_{02} - 6 \rangle \\
P_6 &= \langle b_{20}, a_{02} - b_{02}, a_{20} b_{02} + 6 \rangle
\end{aligned}
$$

Using the rational reconstruction algorithm given in Subsection 1.5 we obtain the conditions (β) of the theorem.

To check the correctness of the decomposition we follow the procedure described in Subsection 1.5. A simple check shows that each of conditions (β) yields the vanishing of all the generators of the ideal $\mathcal{B}_{12}$. However, since modular computations were used some components of the irreducible decomposition of the variety $\mathbf{V}(\mathcal{B}_{12})$ might have been lost. To check that the decomposition is correct we use the function `intersect` of SINGULAR to compute $P = \cap_{k=1}^{6} P_k$. Then using the Radical Membership Test (see Subsection 1.4) we verify that each polynomial of P vanishes on $\mathbf{V}(\mathcal{B}_{12})$. This means that conditions (1) through (6) of (β) give the correct decomposition of the variety of the ideal $\mathcal{B}_{12}$ when $a_{11} = 1$ and $b_{11} = 0$.

In the same manner we obtain the conditions (α) and (γ) of the theorem.

Proof of sufficiency. It must now be shown that in each of the seventeen cases in Theorem 2.5.4 the corresponding system is indeed integrable. One of the most useful techniques for doing so is the method of Darboux, which is based on the construction of first integrals or integrating factors using algebraic invariant curves and which will be described in detail in Subsection 3.2.

In twelve of the seventeen cases in Theorem 2.5.4 integrability follows directly from Theorem 2.5.2 or Theorem 2.5.3 and in three of the remaining cases a Darboux integrating factor can be constructed, as indicated in Table 1. Table 2 lists the invariant curves and Darboux integrating factors for the latter three cases, as well as for a fourth case in which a Darboux integrating factor can be constructed. As Table 2 shows, in the case (β)(2) the integrating factor does not exist, hence integrability is not guaranteed, when $b_{02} = 0$, but this situation, as well

Case in Theorem 2.5.4	Case in Theorem 2.5.2 or Theorem 2.5.3	Other method
2.5.4(α)(1)	2.5.3(α)(2)	
2.5.4(α)(2)	2.5.3(α)(1)	also series
2.5.4(α)(3)	2.5.2(3)	
2.5.4(β)(1)	2.5.3(β)(2)	
2.5.4(β)(2)	2.5.3(β)(1)	also Darboux IF
2.5.4(β)(3)		Darboux IF
2.5.4(β)(4)		Darboux IF
2.5.4(β)(5)		
2.5.4(β)(6)	2.5.2(3) [when $q = \frac{3}{2}$]	
2.5.4(γ)(1)		
2.5.4(γ)(2)	2.5.3(γ)(2)	
2.5.4(γ)(3)	2.5.3(α)(1)	
2.5.4(γ)(4)	2.5.2(2)	
2.5.4(δ)(1)	2.5.2(3)	
2.5.4(δ)(2)		Darboux IF
2.5.4(δ)(3)	2.5.2(1)	
2.5.4(δ)(4)	2.5.2(2)	

Table 1. Cases in Theorem 2.5.4.

as several others like it, is covered by Lemma 2.5.1. We also note that in the first three cases in Table 2 appeal must be made to Theorem 3.2.7. In the last case in Table 2, using the integrating factor one can construct the explicit first integral, in terms of the hypergeometric function,

$$\Psi = \frac{3x^3y^2}{3\left(1-\frac{a_{20}x^2}{2}\right)^{\frac{3}{2}-\frac{b_{20}}{a_{20}}} - b_{02}y^2\,{}_2F_1\left(-\frac{3}{2}, \frac{b_{20}}{a_{20}} - \frac{1}{2}; -\frac{1}{2}; \frac{a_{20}x^2}{2}\right)}.$$

As indicated in Table 1, case (α)(2) can also be handled by a different method, which we now describe in detail in order to illustrate the technique. Writing just a for a_{02} and just b for b_{02} the corresponding system is

$$\dot{x} = 2x - x^2y - axy^2, \quad \dot{y} = -3y + xy^2 + by^3. \tag{2.58}$$

Case	Invariant curves	Integrating factor
$(\beta)(2)$	$f_{3,4} = 1 \pm \sqrt{\frac{b_{02}}{3}}y$	$f_1^{-2} f_2^{-\frac{5}{3}} (f_3 f_4)^{-\frac{2}{3} - \frac{a_{02}}{2b_{02}}}$
$(\beta)(3)$	$f_{3,4} = 1 \pm \sqrt{\frac{b_{02}}{3}}y$ $f_5 = 1 + \frac{3}{b_{02}}x^2 - 2xy$	$f_1^{-\frac{5}{2}} f_2^{-2} f_3^{\frac{3}{4}} f_4^{\frac{3}{4}} f_5^{-\frac{1}{4}}$
$(\beta)(4)$	$f_{3,4} = 1 \pm \sqrt{\frac{b_{02}}{3}}y$ $f_{5,6} = 1 \pm \left(\sqrt{\frac{3}{b_{02}}}x + \sqrt{\frac{b_{02}}{3}}y\right)$	$f_1^{-\frac{5}{2}} f_2^{-2} (f_3 f_4)^{-\frac{5}{4}} (f_5 f_6)^{-\frac{1}{4}}$
$(\delta)(2)$	$f_{3,4} = 1 \pm \sqrt{\frac{a_{20}}{2}}x$	$f_1^{-4} f_2^{-3} (f_3 f_4)^{-\frac{1}{2} - \frac{b_{20}}{a_{20}}}$

Table 2. Integrating factors and invariant curves in addition to $f_1 = x$ and $f_2 = y$.

We look for a formal first integral of the form

$$\Psi(x, y) = \sum_{k=2}^{\infty} h_k(x) y^k.$$

Inserting this expression for Ψ into (2.43) and equating the coefficient of y^k to zero gives a sequence of first order linear ordinary differential equations that the unknown functions h_k must satisfy:

$$2xh_k' - 3kh_k = x^2 h_{k-1}' - (k-1)xh_{k-1} + axh_{k-2}' - (k-2)bh_{k-2} \tag{2.59}$$

(where we initialize by setting $h_0 = h_1 = 0$). We will prove by induction on k that there exists a sequence of polynomial functions h_k with $\deg h_k \le k+1$ that solves the sequence of differential equations (2.59).
Basis step. It is immediate that $h_2(x) = x^3$ solves (2.59) for $k = 2$ (and that $h_3(x) = -x^4$ solves (2.59) for $k = 3$).
Inductive step. Suppose that for $k = 2, \ldots, m-1$ there exist polynomials h_k satisfying (2.59) for which $\deg h_k \le k+1$. The solution of (2.59) for $k = m$ is

$$h_m(x) = \tfrac{1}{2} x^{3m/2} \int x^{-1-3m/2} g_m(x)\, dx \tag{2.60}$$

where $g_m(x) = x^2 h_{m-1}' - (mk-1)xh_{m-1} + axh_{m-2}' - (m-2)bh_{m-2}$. By the inductive hypothesis g_m is a sum of constants times powers of x

for which the powers all lie in the set $\{0, 1, 2, \ldots, m, m+1\}$, hence the integrand is is a sum of constants times powers of x for which the powers all lie in the set $\{-\frac{3}{2}m - 1 + s : 0 \le s \le m+1\}$, which cannot contain -1 since $m \ge 4$ implies $s < \frac{3}{2}m$. Thus no logarithm is produced, so that if we choose the constant of integration to be zero the right hand side of (2.60) produces a sum of constants times powers of x for which the powers all lie in the set $\{(-\frac{3}{2}m - 1 + s) + 1 + \frac{3}{2}m : 0 \le s \le m+1\}$, whose largest element is $m+1$.

Thus (2.58) admits a formal first integral which by our choice of h_2 is of the form $x^3y^2 + \cdots$. Consequently by (1) of Theorem 2.4.4 it has an analytic first integral of such a form around the origin.

There remain cases $(\beta)(5)$ and $(\gamma)(1)$. The latter is just like case $(\beta)(2)$ with a_{11}, a_{20}, and b_{20} swapped with b_{11}, a_{02}, and b_{02}, respectively. In particular there exist the two additional invariant curves $1 \pm \sqrt{a_{20}/2}x$. As for case $(\beta)(5)$, when $b_{02} \neq 0$ the system is

$$\dot{x} = 2x - \tfrac{54}{b_{02}}x^3 - x^2y + \tfrac{4}{3}b_{02}xy^2, \quad \dot{y} = -3y - \tfrac{18}{b_{02}}x^2y + b_{02}y^3.$$

When $b_{02} > 0$ the change of variables

$$x' = \frac{\sqrt{3}}{\sqrt{b_{02}}}\,x, \qquad y' = \frac{\sqrt{3}}{\sqrt{b_{02}}}\,y$$

has the effect of making $b_{02} = 3$. Retaining the notation x and y in the new system we find that the substitution

$$X = \frac{4x^2}{(-1 + 3xy + y^2)^2}, \quad Y = \frac{8x^3y(3x + y - 1)(1 + 3x + y)}{(-1 + 3xy + y^2)^4}$$

transforms it to

$$\dot{X} = 4X - 2Y - 9X^2, \quad \dot{Y} = 3Y(1 - 8X),$$

which has a node at the origin. The transformed system admits a first integral of the form $\widetilde{\Phi} = Y^4/(X + kY + O(2))^3$, which pulls back to a first integral of the form

$$\Phi = x^6y^4(k + O(1)), \quad (k > 0 \text{ is a constant}).$$

We can then take square roots to obtain a suitable first integral of the original system.

If b_{02} is negative we can effectively make $b_{02} = -3$ by means of a similar transformation along with x replaced by ix and y by y/i, which is still real once we multiply out the terms corresponding to the product $(3x + y - 1)(1 + 3x + y)$ in the transformation above.

The situation $b_{02} = 0$ is covered by Lemma 2.5.1. Q.E.D.

§3. Higher Dimensional Systems

3.1. The Center Problem For Higher Dimensional Systems

Normal form theory provides a natural generalization of the center problem to the case of higher dimensional systems. We have seen above that the existence of a center in system (2.40) is equivalent to local analytic integrability of the system. It is also equivalent to the fact that the distinguished Poincaré-Dulac normal form of the system satisfies the Pliss-Bryuno condition. An n-dimensional system

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{X}(\mathbf{x}), \tag{3.1}$$

is said to be *completely integrable* if it admits $n-1$ functionally independent first integrals. Thus a natural extension of the concept of a center for system (2.40) to the case of system (3.1) is the condition that the latter system have $n-1$ functionally independent analytic or formal first integrals. This property has been studied in [53, 85, 87, 88], with the result stated in Theorem 3.1.1. As above, for simplicity we assume that the matrix A in (3.1) is diagonal. We will also assume that it has at least one nonzero eigenvalue and will let $\kappa = (\kappa_1, \ldots, \kappa_n)$ be the n-tuple of its eigenvalues. Let

$$\mathfrak{R} = \{\alpha \in \mathbb{N}_0^n : |\alpha| > 0 \text{ and } (\kappa, \alpha) = 0\} \tag{3.2}$$

and let r denote the rank of the $\mathbb{Z}$-module spanned by elements of the set $\mathfrak{R}$.

Theorem 3.1.1. *System* (3.1) *has* $n-1$ *functionally independent analytic or formal first integrals on a neighborhood of* $\mathbf{0}$ *in* $\mathbb{C}^n$ *if and only if* $r = n-1$ *and the distinguished normal form of* (3.1) *satisfies the Pliss-Bryuno condition.*

Proof. Suppose $r = n-1$ and the distinguished normal form of (3.1) satisfies the Pliss-Bryuno condition. Then there exist $n-1$ independent vectors $\mu_i \in \mathbb{N}_0^n$ such that $(\mu_i, \kappa) = 0$ for each i. It is easy to see that $\mathbf{y}^{\mu_i}$, $i = 1, \ldots, n-1$, are $n-1$ functionally independent first integrals of the normal form (2.25). Since the distinguished normalizing transformation (2.19) is convergent, its inverse

$$\mathbf{y} = \mathbf{x} + \hat{\mathbf{h}}(\mathbf{x}) \tag{3.3}$$

is convergent as well. Applying this transformation to $\mathbf{y}^{\mu_i}$ we obtain $n-1$ functionally independent analytic or formal first integrals of (2.17), which is (3.1).

Conversely, if system (3.1), which is (2.17), has $n-1$ functionally independent analytic or formal first integrals, then by Lemma 2.3.8 its distinguished normal form is of the form (2.25) and by Lemma 2.3.9 the distinguished normalizing transformation is convergent. Q.E.D.

The normal form of system (3.1) is not uniquely defined since it depends on a particular choice of resonant coefficients in the normalizing transformation. Similarly, if we look for a convergent or formal power series $\Phi(\mathbf{x}) = \mathbf{x}^\alpha + \cdots$ that satisfies the condition $\mathcal{X}\Phi \equiv 0$, where $\mathcal{X}$ is the vector field associated to (3.1), we see that resonant coefficients in Φ can be chosen arbitrarily. Thus as in the two-dimensional case discussed above if for some power series Φ we have that $\mathcal{X}\Phi \not\equiv 0$, this does not yet mean that the system does not have a power series first integral of the form $\Phi(\mathbf{x}) = \mathbf{x}^\alpha + \cdots$. The problem of such uncertainty is addressed in the following generalization of Theorem 2.4.4 ([71]).

Theorem 3.1.2. *Let $\mathcal{X}$ be the vector field associated to system* (3.1) *and let $\mathfrak{R}$ be the set defined by* (3.2).

(a) There exists a series $\psi(\mathbf{x})$ with arbitrary resonant monomials such that

$$\mathcal{X}\psi(\mathbf{x}) = \sum_{\alpha\in\mathfrak{R}} p_\alpha \mathbf{x}^\alpha, \tag{3.4}$$

where p_α are polynomials in the coefficients of $\mathcal{X}$.

(b) If the vector field $\mathcal{X}$ has $n-1$ functionally independent analytic or formal first integrals then for any ψ satisfying (3.4),

$$p_\alpha = 0 \text{ for all } \alpha \in \mathfrak{R}. \tag{3.5}$$

The assertion in part (a) of the theorem is that any particular p_α is a polynomial in finitely many of the coefficients of X, with coefficients in the fixed underlying field. Henceforth we will assume that coefficients in (3.1) depend in a polynomial way on a finite number of parameters. Thus the p_α for which (3.4) holds are polynomials in these parameters, and although they are not uniquely defined, when the system admits $n-1$ analytic first integrals the variety that they determine is uniquely specified.

We denote by $\mathcal{B}$ the ideal generated by the polynomials p_α corresponding to $n-1$ functionally independent functions $\psi^{(1)}, \ldots, \psi^{(n-1)}$ satisfying (3.4), that is,

$$\mathcal{B} = \langle p_\alpha^{(i)} : \alpha \in \mathfrak{R}, \quad i = 1, \ldots, n-1 \rangle. \tag{3.6}$$

In analogy with the two-dimensional case we call $\mathcal{B}$ a *Bautin ideal* of system (3.1).

By Theorem 3.1.2 the variety $\mathbf{V}(\mathcal{B})$ of $\mathcal{B}$ is the set of *all points* in the space of parameters of system (3.1) such that the corresponding systems have $n-1$ functionally independent first integrals. In a kind of analogy with the two-dimensional case we call the variety of the Bautin ideal $\mathcal{B}$ the *integrability variety* of system (3.1). As mentioned above it follows from Theorem 3.1.2 that the integrability variety is well-defined, that is, is independent on a particular choice of the resonant coefficients in the series $\psi^{(i)}$ and is the same as the variety of any particular ideal $\langle p_\alpha^{(1)}, \dots, p_\alpha^{(n-1)} : \alpha \in \mathfrak{R}\rangle$, where $p_\alpha^{(1)}, \dots, p_\alpha^{(n-1)}$ are polynomials corresponding to any $n-1$ functionally independent series $\psi^{(1)}, \dots, \psi^{(n-1)}$ satisfying (3.4). It follows from the proof of Theorem 3.1.2 (see also [53]) that any series $\psi^{(s)}$ satisfying (3.4) is of the form

$$\psi^{(s)}(\mathbf{x}) = \mathbf{x}^{\alpha_s} + \cdots, \tag{3.7}$$

where $\alpha_s \in \mathfrak{R}$ and the dots stand for terms of order greater than $|\alpha_s|$. Moreover, if functions (3.7) for $s = 1, \dots, n-1$ are independent first integrals of our system then by Theorem 3.1.1 $r = n-1$ and the system satisfies the Pliss-Bryuno condition. Hence there are also integrals of the form (3.7) with lowest order terms corresponding to elements of $\mathfrak{R}$. Thus to find the variety of the ideal (3.6) we can choose $n-1$ linearly independent vectors from $\mathfrak{R}$, say $\alpha_1, \dots, \alpha_{n-1} \in \mathfrak{R}$. Then $x^{\alpha_1}, \dots, x^{\alpha_{n-1}}$ are functionally independent and we look for $n-1$ functions $\psi^{(s)}(\mathbf{x}) = \mathbf{x}^{\alpha_s} + \cdots$ satisfying

$$\mathcal{X}\psi^{(s)}(\mathbf{x}) = \sum_{\alpha \in \mathfrak{R}} p_\alpha^{(s)} \mathbf{x}^\alpha. \tag{3.8}$$

In actual calculations we can find only a finite number of polynomials $p_\alpha^{(s)}$, so writing (for each $k \in \mathbb{N}$)

$$\mathcal{B}_k = \langle p_\alpha^{(1)}, \dots, p_\alpha^{(n-1)} : \alpha \in \mathfrak{R}, |\alpha| \le k\rangle,$$

we compute successive $\mathcal{B}_k$ until the chain of radical ideals

$$\sqrt{\mathcal{B}_1} \subset \sqrt{\mathcal{B}_2} \subset \sqrt{\mathcal{B}_3} \subset \cdots$$

stabilizes (that is, until we find an m such that $\sqrt{\mathcal{B}_m} = \sqrt{\mathcal{B}_{m+1}}$). Then using various methods we try to show that $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_m)$, that is, that all systems corresponding to points in $\mathbf{V}(\mathcal{B}_m)$ have $n-1$ functionally independent analytic or formal first integrals. An example in which we study the center problem for a three-dimensional quadratic family will be given in Subsection 3.4.

3.2. Darboux Integrals and Integrating Factors

In 1878 G. Darboux ([27]) showed that for some systems of planar polynomial ordinary differential equations it is possible to construct first integrals using the invariant algebraic curves they possess. In particular, he proved that if a planar polynomial ordinary differential system of degree m has at least $m(m+1)/2+1$ invariant algebraic curves then it has a first integral, which has an easy expression as a function of the invariant algebraic curves.

The Darboux method for finding first integrals has been used and further extended by many authors (see, for instance, [20, 23, 52] and the references given there). We give here some of the main results of the Darboux theory of integrability in the n-dimensional setting, following mainly the survey [55]. An application of the method was already given in the proof of Theorem 2.5.4.

We consider the system of differential equations

$$\dot{x}_1 = P_1(\mathbf{x}), \dots, \dot{x}_n = P_n(\mathbf{x}), \tag{3.9}$$

where $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{C}^n$, $P_i = P_i(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$, and P_i and P_j have no common factor if $i \neq j$. As usual we let $\mathcal{X}$ denote the vector field on $\mathbb{C}^n$ associated to (3.9). The *degree* of $\mathcal{X}$ is the number $d = \max\{\deg P_1, \dots, \deg P_n\}$. For any polynomial $f \in \mathbb{C}[\mathbf{x}]$ the corresponding codimension-one algebraic surface $\mathbf{V}(f)$ is invariant for $\mathcal{X}$ if and only if $\mathbf{V}(f) \subset \mathbf{V}(\mathcal{X}f)$.

Definition 3.2.1. *A nonconstant polynomial $f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$ is called an* algebraic partial integral *of system* (3.9) *if there exists a polynomial $K(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$ such that*

$$\mathcal{X}f = Kf. \tag{3.10}$$

The polynomial K is termed a cofactor *of f.*

It is easy to see that K has degree at most $d-1$. The following facts are also apparent:

1. if f is an algebraic partial integral for (3.9) with cofactor K, then any constant multiple of f is also an algebraic partial integral for (3.9) with cofactor K;
2. if f_1 and f_2 are algebraic partial integrals for (3.9) with cofactors K_1 and K_2, then $f_1 f_2$ is an algebraic partial integral for (3.9) with cofactor $K_1 + K_2$;
3. if f is an algebraic partial integral for (3.9) then $\mathbf{V}(f)$ is an algebraic invariant surface of system (3.9).

The converse of the third point holds as well.

Proposition 3.2.2. *Fix $f \in \mathbb{C}[\mathbf{x}]$. $\mathbf{V}(f)$ is an algebraic invariant surface of system* (3.9) *if and only if f is an algebraic partial integral of system* (3.9)*.*

Proof of the "only if" part. The polynomial f factors as a product $f = f_1^{\alpha_1} \cdots f_s^{\alpha_s}$ of irreducible factors. The inclusion $\mathbf{V}(f) \subset \mathbf{V}(\mathcal{X}f)$ clearly implies that for all j, $\mathbf{V}(f_j) \subset \mathbf{V}(\mathcal{X}f_j)$, hence applying the mapping $\mathbf{I}$ of Section 1.1, $\mathcal{X}f_j \in \mathbf{I}(\mathbf{V}(\mathcal{X}f_j)) \subset \mathbf{I}(\mathbf{V}(f_j))$. Since f_j is irreducible, $\langle f_j \rangle$ is prime, hence radical, so by Theorem 1.1.4 $\mathbf{I}(\mathbf{V}(f_j)) = \langle f_j \rangle$ and we conclude that $\mathcal{X}f_j \in \langle f_j \rangle$ for all j. Therefore $\mathcal{X}f_j = K_j f_j$, for some $K_j \in \mathbb{C}[\mathbf{x}]$, so that every polynomial f_j is an algebraic partial integral of (3.9), hence a product of powers of them is. Q.E.D.

Remark. A function that meets the condition of Definition 3.2.1 is frequently termed an *algebraic invariant surface*, in keeping with the characterization given in the proposition.

For polynomials f and g that do not have a common factor the function $\exp(g/f)$ is an *exponential factor* for the system (3.9) with vector field $\mathcal{X}$ if there exists a polynomial $L \in \mathbb{C}[\mathbf{x}]$ of degree at most $d-1$ such that $\mathcal{X}(\exp(g/f) = L\exp(g/f)$. The polynomial L is the *cofactor* of the exponential factor. If $\exp(g/f)$ is an exponential factor then $\mathbf{V}(f)$ is an invariant algebraic surface ([20]).

Definition 3.2.3. *A* Darboux first integral *of system* (3.9) *is a first integral of the form*

$$f_1^{\alpha_1} \cdots f_s^{\alpha_s} \exp(g/h), \tag{3.11}$$

where f_i, g and h are polynomials and α_i are complex numbers.

If sufficiently many algebraic invariant surfaces can be found then they can be used to construct a Darboux first integral, as the following theorem, which goes back to Darboux, shows.

Theorem 3.2.4 (Darboux)**.** *Set $N = \binom{n+d-1}{n}$. If the polynomial vector field of degree d associated to* (3.9) *has at least $N+1$ irreducible, pairwise coprime algebraic invariant surfaces $f_1, \ldots, f_p$, then system* (3.9) *admits a Darboux first integral.*

If $p > N + n$ then the system admits a rational first integral ([43]). However, even the condition $p \geq N+1$ stated in the theorem occurs very seldom. Fortunately it is often possible to find a Darboux first integral using a smaller number of invariant algebraic surfaces.

The following result, apart from the reference to exponential factors, also goes back to Darboux.

Theorem 3.2.5. *Suppose the polynomial vector field $\mathcal{X}$ of degree d in $\mathbb{C}^n$ admits p irreducible, pairwise coprime invariant algebraic surfaces f_i with cofactors K_i and q exponential factors $\exp(g_j/h_j)$ with cofactors L_j. Then there exist $\alpha_i, \beta_j \in \mathbb{C}$, not all zero, such that*

$$\sum_{i=1}^{p} \alpha_i K_i + \sum_{j=1}^{q} \beta_j L_j = 0,$$

if and only if the (multi-valued) function

$$H = f_1^{\alpha_1} \cdots f_s^{\alpha_p} (\exp(g_1/h_1))^{\beta_1} \ldots (\exp(g_q/h_q))^{\beta_q} \tag{3.12}$$

is a first integral of $\mathcal{X}$.

If a first integral of system (3.9) cannot be found then we turn our attention to the possible existence of an integrating factor. Classically, an *integrating factor* of the differential equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{3.13}$$

for differentiable functions M and N on an open set Ω is a differentiable function $\mu(x,y)$ on Ω such that $\mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y)\,dy = 0$ is an exact differential, which is the case if and only if

$$\frac{\partial(\mu M)}{\partial y} - \frac{\partial(\mu N)}{\partial x} \equiv 0\,.$$

Let $\operatorname{div}\mathcal{X}$ denote the divergence of the vector field $\mathcal{X}$ corresponding to (3.9), $\operatorname{div}\mathcal{X} = \sum \partial P_j/\partial x_j$.

Definition 3.2.6. *An* integrating factor *on an open set $\Omega \in \mathbb{C}^n$ for system* (3.9) *is a differentiable function $\mu(\mathbf{x})$ on Ω such that*

$$\mathcal{X}\mu = -\mu \operatorname{div}\mathcal{X} \tag{3.14}$$

holds throughout on Ω. An integrating factor on Ω of the form (3.11) *is termed a* Darboux integrating factor *on Ω.*

The reciprocal of μ, where defined, is also of great importance for investigation of the behavior of trajectories of system (3.9). It is called the inverse Jacobi multiplier (see for example the survey paper [34]).

Under the conditions of Theorem 3.2.5 a function μ of the form (3.12) is a Darboux integrating factor if and only if

$$\sum_{j=1}^{p} \alpha_j K_j + \sum_{j=1}^{q} \beta_j L_j + \operatorname{div}\mathcal{X} \equiv 0\,.$$

Darboux's method is one of the most efficient tools for studying the center problem for polynomial systems (3.9). In particular, if we are able to construct a Darboux first integral (3.11) with algebraic surfaces $f_j = 0$ that do not pass through the origin, then we are sure to have a first integral that is analytic in a neighborhood of the origin. However existence of a Darboux integrating factor in a two-dimensional system (2.40) does not necessarily yield existence of an analytic first integral of the system. We mention the following result, which is a particular case of a theorem in [23] and which allows one to conclude existence of an analytic first integral of (2.40) from the form of the Darboux integrating factor.

Theorem 3.2.7. *Suppose system* (2.40) *has a local integrating factor of the form* $\mu = \prod_{i=1}^{m} F_i^{\alpha_i}$ *in which* $\alpha_i \neq 0$ *and* F_i *is analytic in* x_1 *and* x_2. *Then the system admits a first integral of the form* (2.41) *if one of the following conditions holds:*

(i) $F_i(0,0) \neq 0$ *for all* i;

(ii) $F_i(0,0)$ *vanishes for at most one value of* i *and the corresponding Darboux factor has either the form* $F_i(x_1,x_2) = x_1 + o(x_1,x_2)$ *or the form* $F_i(x_1,x_2) = x_2 + o(x_1,x_2)$; *or*

(iii) exactly two factors, denote them F_1 *and* F_2 *vanish at the origin, they have the form* $F_1(x_1,x_2) = x_1 + o(x_1,x_2)$ *and* $F_2(x_1,x_2) = x_2 + o(x_1,x_2)$, *and at most one of the coefficients* α_1 *and* α_2 *is an integer less than* -1.

3.3. Time-reversibility and Integrability

An important class of systems in the study of the center problem are those that are time-reversible according to the following definition.

Definition 3.3.1. *System*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad (\mathbf{x} \in \mathbb{R}^n \text{ or } \mathbf{x} \in \mathbb{C}^n), \tag{3.15}$$

is called time-reversible if there exists an invertible $n \times n$ *matrix* T *such that the system is invariant under the transformation* $\mathbf{y} = T\mathbf{x}$ *and the time inversion* $t \to -\tau$.

To illustrate the ideas we first consider the two-dimensional case, for which time-reversibility has a simple geometric meaning when the variables are real. Thus we let $\mathbf{x} = (x_1, x_2) \in \mathbb{C}^2$ (or $\mathbb{R}^2$) and further limit ourselves to the transformations of the form

$$T : x_1 \mapsto \gamma x_2, \ x_2 \mapsto \gamma^{-1} x_1 \tag{3.16}$$

with $\gamma \in \mathbb{C}$ (or $\mathbb{R}$) (note that T defined by (3.16) is an involution).

A straight line L is an *axis of symmetry* of a real autonomous two-dimensional system of ordinary differential equations if as point-sets (ignoring the sense of the parametrization by time t) the orbits of the system are symmetric with respect to the line L. There are two types of symmetry of a real system with respect to a line L: *mirror symmetry*, meaning that when the phase portrait is reflected in the line L it is unchanged; and *time-reversible symmetry*, meaning that when the phase portrait is reflected in the line L and then the sense of every trajectory is reversed (corresponding to a reversal of time) the original phase portrait is obtained. The symmetry (3.16) can be considered as a generalization of the second type of reflection to the case of complex two-dimensional systems, as we now explain.

Consider a parametric family of real systems

$$\dot{u} = U(u, v), \quad \dot{v} = V(u, v). \tag{3.17}$$

Introducing a complex structure on the plane (u, v) by setting $x = u + iv$ we obtain from (3.17) the equation

$$\dot{x} = P(x, \bar{x}) \qquad (P = U + iV). \tag{3.18}$$

As described above we can treat $\bar{x}$ as an independent variable y to obtain the complexification of (3.17) in the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y). \tag{3.19}$$

We will assume that system (3.19) has the form (2.42). Let a denote the vector of coefficients of the polynomial $P(x, y)$ in (3.18) that arises from the real system (3.17) by setting $x = u + iv$. It is easy to see that if $a = \pm\bar{a}$ (meaning that either all the coefficients are real or all are pure imaginary) then the u-axis is an axis of symmetry of the real system (3.17) and of the corresponding complex differential equation (3.18). Thus the u-axis is an axis of symmetry for (3.18) if

$$P(\bar{x}, x) = -\overline{P(x, \bar{x})} \tag{3.20}$$

(the case $a = -\bar{a}$) or if

$$P(\bar{x}, x) = \overline{P(x, \bar{x})} \tag{3.21}$$

(the case $a = \bar{a}$). We now observe that if condition (3.20) is satisfied then under the change $x \to \bar{x}$, $\bar{x} \to x$ equation (3.18) is transformed into its negative,

$$\dot{x} = -P(x, \bar{x}) \tag{3.22}$$

while if condition (3.21) holds then (3.18) is unchanged. Thus condition (3.21) means that the system is reversible with respect to reflection across the u-axis (i.e., the transformation does not change the system) while condition (3.20) corresponds to time-reversibility with respect to the same transformation.

If the line of reflection is not the u-axis but a distinct line L then we can apply the rotation $x_1 = e^{-i\varphi}x$ through an appropriate angle φ to make L the u-axis. In the new coordinates we have

$$\dot{x}_1 = e^{-i\varphi}P(e^{i\varphi}x_1, e^{-i\varphi}\bar{x}_1)\,.$$

By the discussion in the paragraph following (3.22) this system is time-reversible with respect to the line $\operatorname{Im} x_1 = 0$ if (3.20) holds, meaning that

$$e^{i\varphi}\overline{P(e^{i\varphi}x_1, e^{-i\varphi}\bar{x}_1)} = -e^{-i\varphi}P(e^{i\varphi}\bar{x}_1, e^{-i\varphi}x_1).$$

Hence, reverting to the variable x, (3.18) is time-reversible when there exists a φ such that

$$e^{2i\varphi}\overline{P(x,\bar{x})} = -P(e^{2i\varphi}\bar{x}, e^{-2i\varphi}x). \tag{3.23}$$

In fact we have shown that if system (3.19) is the complexification of a real system (3.17) and it admits a symmetry (3.16) with $\gamma_0 = e^{2i\varphi_0}$ then the line $v = u\tan\varphi_0$ is a line of symmetry of the trajectories (as point-sets) of the real system.

Direct calculation shows that the system (3.19) is time-reversible with respect to a transformation (3.16) if and only if for some γ

$$\gamma Q(\gamma y, x/\gamma) = -P(x,y), \quad \gamma Q(x,y) = -P(\gamma y, x/\gamma)\,. \tag{3.24}$$

We will limit our study to the case of polynomial systems of the form (2.42) and use the notation introduced in the paragraph following (2.42).

The condition (3.24) immediately yields that system (2.42) is time-reversible if and only if

$$b_{qp} = \gamma^{p-q}a_{pq}, \qquad a_{pq} = b_{qp}\gamma^{q-p}. \tag{3.25}$$

Eliminating γ from (3.25) using Theorem 1.4.2 we obtain equations defining the set of time-reversible systems in the space of parameters of (2.42). From formula (4.15) below it is not difficult to see that if system (2.42) with $p = q = 1$ is time-reversible (that is, condition (3.25) is fulfilled) then it has a first integral of the form (2.41).

A similar approach works for some higher-dimensional systems where the study is based on a result obtained in [53]. We consider a three-dimensional system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{X}(\mathbf{x})$ for which the eigenvalues of the

matrix A are 0, -1, and 1, which we write in the form

$$\begin{aligned}
\dot{x}_1 &= \sum_{\substack{i+j+k=2\\ i,j,k\geq 0}}^{m} a_{i,j,k}x_1^i x_2^j x_3^k = P(x)\\
\dot{x}_2 &= -x_2 + \sum_{\substack{i+j+k=2\\ i,j,k\geq 0}}^{m} b_{i,j-1,k}x_1^i x_2^j x_3^k = Q(x)\\
\dot{x}_3 &= x_3 + \sum_{\substack{i+j+k=2\\ i,j,k\geq 0}}^{m} c_{i,j,k-1}x_1^i x_2^j x_3^k = R(x).
\end{aligned} \tag{3.26}$$

Let u, v, and w be the number of parameters in the first, second, and third equations, respectively. By (a, b, c) we denote a $(u + v + w)$-tuple of parameters of system (3.26).

By Proposition 11 of [53], if system (3.26) is time-reversible with respect to a linear transformation that permutes x_2 and x_3 then it admits two functionally independent analytic first integrals. Time-reversibility means that there exists an invertible matrix T such that

$$T^{-1} \circ \mathbf{f} \circ T = -\mathbf{f}, \tag{3.27}$$

where $\mathbf{f}$ is the vector function of the right-hand side of (3.26). We look for a transformation T of the form

$$T = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & \gamma\\ 0 & 1/\gamma & 0 \end{pmatrix}. \tag{3.28}$$

Note that $T = T_1T_2$, where

$$T_1 = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0 \end{pmatrix}$$

and $T_2 = \operatorname{diag}(1, 1/\gamma, \gamma)$, that is, T_2 is the matrix of an orthogonal transformation. Thus T is a composition of the orthogonal transformation T_2 and the permutation T_1. Easy computations show that (3.27) is satisfied for T defined by (3.28) if and only if

$$a_{jkl} = -\gamma^{l-k}a_{jlk}, \quad b_{mnp} = -\gamma^{p-n}c_{mpn}. \tag{3.29}$$

Define the ideal

$$H = \langle 1 - y\gamma, a_{jkl} + \gamma^{l-k}a_{jlk}, b_{mnp} + \gamma^{p-n}c_{mpn}\rangle \subset k[y, \gamma, a, b, c], \tag{3.30}$$

where y is a new variable and k is $\mathbb{R}$ or $\mathbb{C}$. The following theorem identifies systems in the family (3.26) that are time-reversible with respect to (3.28). We call the ideal I_S defined in the theorem the *Sibirsky ideal*.

Theorem 3.3.2. *The Zariski closure of all systems in the family* (3.26) *(with coefficients in either $k = \mathbb{R}$ or $k = \mathbb{C}$) that are time-reversible with respect to* (3.28) *is the variety* $\mathbf{V}(I_S)$ *of the ideal*

$$I_S = k[a, b, c] \cap H. \tag{3.31}$$

A generating set for I_S is obtained by computing a Gröbner basis for H with respect to any elimination order with $\{y, \gamma\} > \{a, b, c\}$ and choosing from the output list the polynomials that do not depend on y and γ.

Proof. System (3.26) is time-reversible for particular parameters a, b, and c if there exists $\gamma \neq 0$ such that (3.29) holds. To find all such parameter strings we have to add to equations (3.29) the equation $1 - y\gamma = 0$ (which imposes the condition $\gamma \neq 0$) and eliminate from the system obtained the variables γ and y. Geometrically, the elimination means the projection of the variety $\mathbf{V}(H)$ of the ideal H onto the affine $(u + v + w)$-dimensional space of parameters (a, b, c). It is easily seen that the image of such a projection is not necessarily an algebraic set. However the Zariski closure of the projection can be computed using Theorem 1.4.2. By this theorem it is equal to the variety of the second elimination ideal of H, that is, to the variety of the ideal I_S. By the Elimination Theorem (Theorem 1.4.1), in order to compute a Gröbner basis for I_S one can compute a Gröbner basis of H with respect to lexicographic order (or any elimination order) with $\{y, \gamma\} > \{a, b, c\}$ and choose from the output list the polynomials that depend only on a, b, and c and not on y or γ. Q.E.D.

Corollary 3.3.3. *Let I_S be the ideal* (3.31) *of system* (3.26)*. Then all systems that correspond to points in* $\mathbf{V}(I_S)$ *are integrable.*

Proof. Denote by $\mathcal{S}$ the set in the parameter space of all systems of the form (3.26) for which (3.29) holds with $\gamma \neq 0$. All systems from $\mathcal{S}$ are time-reversible, therefore, by Proposition 11 of [53] they also are integrable. Thus $\mathcal{S} \subset \mathbf{V}(\mathcal{B})$, where $\mathcal{B}$ is a Bautin ideal of (3.26) defined in Subsection 3.1. Hence $\bar{\mathcal{S}} = \mathbf{V}(I_S) \subset \mathbf{V}(\mathcal{B})$, where $\bar{\mathcal{S}}$ is the Zariski closure of $\mathcal{S}$. As shown in Subsection 3.1, $\mathbf{V}(\mathcal{B})$ is the set of all integrable systems of family (3.26). Therefore all systems from $\mathbf{V}(I_S)$ are integrable. Q.E.D.

We note in passing that there is also a strong connection between time-reversibility and invariants of certain groups (see [70, 77, 78]).

3.4. Integrability of a Three-dimensional Quadratic System with an Invariant Plane

Recently the condition of complete analytic integrability of quadratic $(1:-1:1)$, $(2:-1:1)$, and $(1:-2:1)$ resonant Lotka-Volterra systems were obtained in [6]. In this subsection we consider $(0:-1:1)$ resonant systems of the form

(3.32)
$$\begin{aligned}\dot{x} &= x(a_{200}x + a_{110}y + a_{101}z)\\ \dot{y} &= -y + b_{200}x^2 + b_{110}xy + b_{101}xz + b_{020}y^2 + b_{011}yz + b_{002}z^2\\ \dot{z} &= z + c_{200}x^2 + c_{110}xy + c_{101}xz + c_{020}y^2 + c_{011}yz + c_{002}z^2.\end{aligned}$$

This family was studied in [39], where necessary and sufficient conditions for the existence of two functionally independent first integrals were obtained. We will show how the theory developed in earlier sections applies to this family, describe the computational difficulties encountered and ways around them, and give a full description of the result in a special case.

The following statement is a more general version of Lemma 2.5.1 and it is proved similarly.

Lemma 3.4.1. *Suppose* (2.17) *is a family of polynomial systems whose coefficients are the parameters and let* V *and* W *be varieties in the affine space of the parameters such that system* (2.17) *is integrable for all values of the parameters in* $V \setminus W$*. If*

$$\overline{V \setminus W} = V \tag{3.33}$$

(where the overline denotes the Zariski closure) then the system is integrable for all values of the parameters in V*.*

With the help of the results of the previous section finding the following sufficient conditions for integrability is straightforward.

Theorem 3.4.2. *The Zariski closure of the set of all time-reversible systems in family* (3.32) *is the variety of the ideal* I_S *generated by the polynomials listed in Table 3. Moreover, all systems from* $\mathbf{V}(I_S)$ *are integrable.*

Proof. The ideal H of (3.30) for family (3.32) is

$$\begin{aligned}H = \langle 1 - \gamma w, 2a_{200}, b_{110} + c_{101}, &a_{101} + a_{110}\gamma,\\ c_{002} + b_{020}\gamma, b_{011} + c_{011}\gamma, &b_{200} + c_{200}\gamma,\\ b_{101} + c_{110}\gamma^2, &b_{002} + c_{020}\gamma^3\rangle.\end{aligned}$$

a_{200}	$b_{110}+c_{101}$
$b_{011}c_{200}-b_{200}c_{011}$	$b_{020}b_{011}-c_{011}c_{002}$
$a_{110}b_{011}-a_{101}c_{011}$	$b_{200}b_{020}-c_{200}c_{002}$
$a_{101}b_{020}-a_{110}c_{002}$	$a_{110}b_{200}-a_{101}c_{200}$
$b_{020}b_{101}c_{011}+b_{011}c_{110}c_{002}$	$b_{011}b_{101}c_{020}+b_{002}c_{110}c_{011}$
$b_{002}c_{200}c_{110}+b_{200}b_{101}c_{020}$	$b_{020}b_{002}c_{110}+b_{101}c_{020}c_{002}$
$a_{110}b_{002}c_{110}+a_{101}b_{101}c_{020}$	$b_{011}^2c_{110}+b_{101}c_{011}^2$
$b_{200}b_{011}c_{110}+b_{101}c_{200}c_{011}$	$a_{101}b_{011}c_{110}+a_{110}b_{101}c_{011}$
$b_{101}c_{200}^2+b_{200}^2c_{110}$	$b_{020}b_{101}c_{200}+b_{200}c_{110}c_{002}$
$a_{110}b_{101}c_{200}+a_{101}b_{200}c_{110}$	$b_{020}^2b_{101}+c_{110}c_{002}^2$
$a_{110}b_{020}b_{101}+a_{101}c_{110}c_{002}$	$a_{110}^2b_{101}+a_{101}^2c_{110}$
$b_{020}b_{002}c_{011}^2-b_{011}^2c_{020}c_{002}$	$b_{020}b_{002}c_{200}c_{011}-b_{200}b_{011}c_{020}c_{002}$
$b_{020}^2b_{002}c_{011}-b_{011}c_{020}c_{002}^2$	$a_{110}b_{020}b_{002}c_{011}-a_{101}b_{011}c_{020}c_{002}$
$b_{020}b_{101}^2c_{020}-b_{002}c_{110}^2c_{002}$	$b_{011}^3c_{020}-b_{002}c_{011}^3$
$b_{200}b_{011}^2c_{020}-b_{002}c_{200}c_{011}^2$	$a_{101}b_{011}^2c_{020}-a_{110}b_{002}c_{011}^2$
$b_{200}^2b_{011}c_{020}-b_{002}c_{200}^2c_{011}$	$a_{101}b_{200}b_{011}c_{020}-a_{110}b_{002}c_{200}c_{011}$
$a_{101}^2b_{011}c_{020}-a_{110}^2b_{002}c_{011}$	$b_{011}b_{002}c_{110}^2-b_{101}^2c_{020}c_{011}$
$b_{200}b_{002}c_{110}^2-b_{101}^2c_{200}c_{020}$	$a_{101}b_{002}c_{110}^2-a_{110}b_{101}^2c_{020}$
$b_{002}c_{200}^3-b_{200}^3c_{020}$	$b_{020}b_{002}c_{200}^2-b_{200}^2c_{020}c_{002}$
$a_{110}b_{002}c_{200}^2-a_{101}b_{200}^2c_{020}$	$b_{020}^2b_{002}c_{200}-b_{200}c_{020}c_{002}^2$
$a_{110}b_{020}b_{002}c_{200}-a_{101}b_{200}c_{020}c_{002}$	$a_{110}^2b_{002}c_{200}-a_{101}^2b_{200}c_{020}$
$b_{020}^3b_{002}-c_{020}c_{002}^3$	$a_{110}b_{020}^2b_{002}-a_{101}c_{020}c_{002}^2$
$a_{110}^2b_{020}b_{002}-a_{101}^2c_{020}c_{002}$	$a_{110}^3b_{002}-a_{101}^3c_{020}$
$b_{002}^2c_{110}^3+b_{101}^3c_{020}^2$	

Table 3. Generators of the ideal I_S for Family (3.32).

Computing the second elimination ideal of H (Subsection §1.4) we obtain the ideal I_S given in the statement of the theorem. By Corollary 3.3.3 any system from $\mathbf{V}(I_S)$ is integrable. Q.E.D.

Turning to necessary conditions for integrability of members of family (3.32), as mentioned in the discussion after Theorem 3.1.2 to find the integrability variety of system (2.18) one can look for any series (3.7) satisfying (3.8). For system (3.32) $\mathfrak{R}=\{\alpha\in\mathbb{N}_0^3:\alpha_2=\alpha_3\}$ and two lowest order elements of $\mathfrak{R}$ are $(1,0,0)$ and $(0,1,1)$. So in our case we look for series ϕ and ψ in the forms

$$\phi=x+\sum_{i+j+k>1}\phi_{ijk}x^iy^jz^k \tag{3.34}$$

and

$$\psi = yz + \sum_{i+j+k>2} \psi_{ij} x^i y^j z^k, \tag{3.35}$$

such that

$$\frac{\partial \psi_1}{\partial x_1} P + \frac{\partial \psi_1}{\partial x_2} Q + \frac{\partial \psi_1}{\partial x_3} R = \sum_{\alpha \in \mathfrak{R}} g_\alpha(a,b,c) x^\alpha \tag{3.36}$$

and

$$\frac{\partial \psi_2}{\partial x_1} P + \frac{\partial \psi_2}{\partial x_2} Q + \frac{\partial \psi_2}{\partial x_3} R = \sum_{\alpha \in \mathfrak{R}} h_\alpha(a,b,c) x^\alpha, \tag{3.37}$$

where P, Q, R are the right hand sides of (3.32) and g_α, h_α ($\alpha \in \mathfrak{R}$) are polynomials in (a,b,c).

One computes, for small $|\alpha|$, polynomials g_α and h_α defined according to (3.36) and (3.37). In [39] computations were done for $|\alpha| \leq 8$ using the computer algebra system MATHEMATICA. This yielded an ideal $B_8 = \langle g_\alpha, h_\alpha : \alpha \in \mathfrak{R}, |\alpha| \leq 8 \rangle$. Since the expressions of the polynomials are large we do not present them here, but the interested reader should be able to compute them using any available computer algebra system.

The next step is to find the irreducible decomposition of the variety $\mathbf{V}(B_8)$. This is computationally difficult; even using the routine `minAssGTZ` ([28]) of the computer algebra system SINGULAR ([29]), which performs calculations according to the algorithm of [35], on most generally available computing systems it will probably prove infeasible. However, the change of coordinates

$$y \mapsto by, \quad z \mapsto cz,$$

for $bc \neq 0$, transforms (3.32) into a quadratic system with the same linear part but with b_{011} changed to b_{011}/c and c_{011} changed to c_{011}/b. Thus if $b_{011} c_{011} \neq 0$ one may assume that $b_{011} = c_{011} = 1$ by choosing suitable b and c. Hence to obtain necessary conditions for integrability of system (3.32) it is enough to consider separately the four cases: (i) $b_{011} = c_{011} = 0$, (ii) $b_{011} = 0$, $c_{011} = 1$, (iii) $b_{011} = 1$, $c_{011} = 0$, and (iv) $b_{011} = c_{011} = 1$. For each of these cases the computations with SINGULAR become feasible and produce a collection of necessary conditions, each corresponding to an irreducible component of $\mathbf{V}(B_8)$. We present here the case $b_{011} = c_{011} = 0$, and refer the reader to [39] for the complete treatment.

I.

$$a_{101}c_{110} + b_{101}c_{020} - c_{110}c_{002} - a_{110}c_{101} \quad b_{110} + c_{101} \quad b_{200}$$
$$a_{110}b_{101} - b_{020}b_{101} + b_{002}c_{110} + a_{101}c_{101} \quad c_{200}$$

II.

$$b_{002}^2c_{110}^3 + b_{101}^3c_{020}^2 \qquad b_{020}b_{101}^2c_{020} - b_{002}c_{110}^2c_{002}$$
$$b_{020}b_{002}c_{110} + b_{101}c_{020}c_{002} \qquad b_{020}^2b_{101} + c_{110}c_{002}^2$$
$$b_{020}^3b_{002} - c_{020}c_{002}^3 \qquad -b_{020}^2b_{002}c_{200} + b_{200}c_{020}c_{002}^2$$
$$b_{020}b_{101}c_{200} + b_{200}c_{110}c_{002} \qquad b_{002}c_{200}c_{110} + b_{200}b_{101}c_{020}$$
$$b_{200}b_{002}c_{110}^2 - b_{101}^2c_{200}c_{020} \qquad b_{200}b_{020} - c_{200}c_{002}$$
$$-b_{020}b_{002}c_{200}^2 + b_{200}^2c_{020}c_{002} \qquad b_{101}c_{200}^2 + b_{200}^2c_{110}$$
$$-b_{002}c_{200}^3 + b_{200}^3c_{020} \qquad -a_{101}b_{020} + a_{110}c_{002}$$
$$a_{110}b_{101}c_{200} + a_{101}b_{200}c_{110} \qquad -a_{101}b_{002}c_{110}^2 + a_{110}b_{101}^2c_{020}$$
$$a_{110}b_{002}c_{110} + a_{101}b_{101}c_{020} \qquad a_{110}b_{002}c_{200}^2 - a_{101}b_{200}^2c_{020}$$
$$a_{110}b_{020}b_{101} + a_{101}c_{110}c_{002} \qquad a_{110}b_{020}b_{002}c_{200} - a_{101}b_{200}c_{020}c_{002}$$
$$a_{110}b_{020}^2b_{002} - a_{101}c_{020}c_{002}^2 \qquad a_{110}b_{200} - a_{101}c_{200}$$
$$a_{110}^2b_{101} + a_{101}^2c_{110} \qquad a_{110}^2b_{002}c_{200} - a_{101}^2b_{200}c_{020}$$
$$a_{110}^2b_{020}b_{002} - a_{101}^2c_{020}c_{002} \qquad a_{110}^3b_{002} - a_{101}^3c_{020}$$
$$b_{110} + c_{101}$$

III.

$$c_{002},\ b_{020},\ b_{110} + c_{101},\ a_{101},\ a_{110}$$

Table 4. Integrability conditions for system (3.32) with $b_{011} = c_{011} = 0$.

Theorem 3.4.3. *System* (3.32) *with* $b_{011} = c_{011} = 0$ *is integrable if and only if* $a_{200} = 0$ *and all the polynomials in at least one of the lists I, II, and III in Table 4 vanish.*

Proof. System (3.32) is integrable only if all the conditions in at least one of the three lists are met. We must show conversely that if all the conditions in any one of the lists are met then system (3.32) is integrable.

If each polynomial in list III is zero then in (3.32) $\dot{x} = 0$ and the system restricted to any invariant plane $x = c$ is Hamiltonian. If each polynomial in list II is zero then each generator of I_S listed in Table 3 vanishes so by Theorem 3.4.2 system (3.32) is integrable. If each polynomial in list I is zero then by a linear transformation we may make $b_{002} = b_{101} = a_{110} = 1$ and thus reduce the parameter space to $\mathbb{C}^4$:

$(a_{101}, b_{110}, b_{020}, c_{002})$. As always $f_0(x, y, z) = x = 0$ is an invariant plane with cofactor $K_0 = y + a_{101}z$. We look for other invariant planes in the form $f(x, y, z) = 1 + u_1x + u_2y + u_3z$ with cofactors of the form $k = c_1x + c_2y + c_3z$. Calculations give

$$c_1 = 0, \quad c_2 = c_{002}u - u^2, \quad c_3 = u,$$

where u is a root of the cubic equation

$$(3.38) \quad (-a_{101} + a_{101}b_{020} + b_{110} + a_{101}^2 b_{110} + c_{002} - a_{101}b_{110}c_{002}) - (b_{020} + c_{002}^2)u + 2c_{002}u^2 - u^3 = 0.$$

Let D be the discriminant of (3.38). Then off the surfaces $D = 0$ and $F := -a_{101} + a_{101}b_{020} + b_{110} + a_{101}^2 b_{110} + c_{002} - a_{101}b_{110}c_{002} = 0$ in the space of the parameters equation (3.38) has three distinct and non-zero roots. Let f_1 and f_2 be invariant surfaces corresponding to any two of them, call them r_1 and r_2, respectively. Referring to Theorem 3.2.5 we look for a Darboux first integral in the form

$$(3.39) \quad \phi(x, y, z) = x\, f_1^{\alpha_1} f_2^{\alpha_2},$$

where α_1 and α_2 must satisfy

$$K_0 + \alpha_1 K_1 + \alpha_2 K_2 = 0.$$

Inserting $K_0 = y + a_{101}z$ and $K_j = (c_{002} - r_j)r_j y + r_j z$ for $j = 1, 2$ and equating the coefficients of y and z to zero gives a system of two linear equations in α_1 and α_2 whose determinant is $r_1r_2(r_1 - r_2)$, which is different from zero since the roots of (3.38) are non-zero and distinct. Thus we obtain a non-trivial solution and conclude by Theorem 3.2.5 that system (3.32) admits a first integral of the form (3.39). Since the divergence of the vector field corresponding to (3.32) is equal to $y + 2b_{020}y + a_{101}z + 2c_{002}z$ and also does not depend on x similar considerations show that the system also has a Jacobi multiplier of the form $M = x\, f_1^{\gamma_1} f_2^{\gamma_2}$. Thus by Theorem 1.3 of [6] for all values of the parameters for which $D \neq 0$ and $F \neq 0$ it also admits a first integral of the form (3.35). By Lemma 3.4.1 system (3.32) is integrable for all values of the parameters for which each polynomial in list III is zero. Q.E.D.

§4. The Cyclicity Problem

4.1. Counting Positive Zeros of Real Analytic Functions

A problem of both practical and theoretical importance in the geometric theory of differential equations on the plane is the creation of

limit cycles (isolated closed orbits) in the phase portrait when relevant parameters are perturbed slightly. In general limit cycles can bifurcate from critical points, other cycles, saddle loops, and other more elaborate sets. In this section we investigate the creation of limit cycles from an isolated singularity of a *polynomial* system $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u})$, which without loss of generality we may assume to be located at the origin.

If the eigenvalues of the linear part at the origin have non-zero real parts, so that the singularity is hyperbolic, then by the Hartman-Grobman Theorem and the structural stability of hyperbolic linear systems no small limit cycles can be created under small perturbation. The simplest non-hyperbolic case is that in which the determinant of $\mathrm{d}\mathbf{f}(\mathbf{0})$ is positive but its trace is zero, so that the singularity is either a focus or a center and every nearby system has an isolated singularity near the origin, which without loss of generality may be assumed to also be located at the origin. Thus after a possible time-rescaling the original system and any system near it can be expressed in the respective forms

$$\begin{aligned} \dot{u} &= -v + \widetilde{P}(u,v) \\ \dot{v} &= \quad u + \widetilde{Q}(u,v) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \dot{u} &= \lambda u - v + \widetilde{P}(u,v) \\ \dot{v} &= u + \lambda v + \widetilde{Q}(u,v). \end{aligned} \tag{4.2}$$

Loosely speaking the *cyclicity* of the origin in system (4.1) with respect to perturbation within family (4.2) is the maximum number of limit cycles that can appear in an arbitrarily small neighborhood of the origin under an arbitrarily small perturbation of λ (from 0) and the coefficients of the polynomials $\widetilde{P}$ and $\widetilde{Q}$, but in such a way that the perturbation remains in some predetermined family of systems. Note that when the origin is originally a center limit cycles can simultaneously bifurcate both from the center itself and from cycles within the period annulus that surrounds it. In this discussion we ignore the "large" cycles that might appear from cycles in the period annulus. By the *cyclicity problem* we mean the problem of determining the cyclicity of the origin.

The precise definition of cyclicity is as follows. Let Γ denote the parameter space that specifies the family of the form (4.1) of interest and γ an element of Γ. For example, if we are interested in all quadratic systems of the form (4.2), then

$$\widetilde{P}(u,v) = A_{20}u^2 + A_{11}uv + A_{02}v^2 \text{ and } \widetilde{Q}(u,v) = B_{20}u^2 + B_{11}uv + B_{02}v^2$$

so that a parameter string is $\gamma = (A_{20}, A_{11}, A_{02}, B_{20}, B_{11}, B_{02})$, which we will typically abbreviate to just $\gamma = (A, B)$, and $\Gamma = \mathbb{R}^6$. Since perturbations naturally take place in the corresponding family (4.2) the full parameter space of interest is $\Gamma_\lambda := \mathbb{R} \times \Gamma$.

Definition 4.1.1. *For parameters* (λ, γ) *let* $n((\lambda, \gamma), \epsilon))$ *denote the number of limit cycles of the corresponding system* (4.2) *that lie wholly within an* ϵ*-neighborhood of the origin. The singularity at the origin for the system* (4.2) *that corresponds to the fixed choice of parameters* $(\lambda^*, \gamma^*) \in \Gamma_\lambda$ *has* cyclicity c with respect to the space Γ_λ *if there exist positive constants* δ_0 *and* ϵ_0 *such that for every pair* ϵ *and* δ *satisfying* $0 < \epsilon < \epsilon_0$ *and* $0 < \delta < \delta_0$

$$\max\{n((\lambda, \gamma), \epsilon)) : |(\lambda, \gamma) - (\lambda^*, \gamma^*)| < \delta\} = c\,.$$

The concept of cyclicity and the idea of the method presented below are due to N. N. Bautin [7]. In our exposition we follow mainly [47, 70]. The reader can also consult [18, 40, 41, 72] for application of the same methods to other problems and the relation of the cyclicity problem to the second half of Hilbert's 16th problem. In our exposition we will restrict our attention to the problem of finding an upper bound on the cyclicity.

If system (4.2) is written in polar coordinates then as described in Subsection 2.1 a "first return" mapping $\mathscr{R}$ is defined from a short segment of the positive u-axis back into the positive u-axis; it is the first intersection in positive time of the orbit through a point with the positive u-axis. The mapping $\mathscr{R}$ is analytic and extends analytically to a neighborhood of 0 in the x-axis. Small cycles surrounding the origin correspond to small positive zeros of the real analytic function

$$\mathscr{D}(r) = \mathscr{R}(r) - r = \eta_1 r + \eta_2 r^2 + \eta_3 r^3 + \cdots\,. \tag{4.3}$$

This is the function of (2.8). It is apparent from the analyticity of solutions of (4.2) that the coefficients η_j, the *Lyapunov quantities*, are real analytic functions of λ and the coefficients of $\widetilde{P}$ and $\widetilde{Q}$. For fixed values of λ they are polynomials in the coefficients of $\widetilde{P}$ and $\widetilde{Q}$ (for example, Proposition 6.2.2 of [70]). Recall from Subsection 2.1 that they can be computed by recursively solving a collection of initial value problems that arise in connection with the polar coordinate expression of the system.

It follows from the geometry of the first return map that the first non-zero Lyapunov quantity, when it exists, has odd index, say $2k + 1$, $k \in \mathbb{N}_0$. In this case the singularity is called a kth order fine focus, and

no matter what the analytic family that contains $\widetilde{P}$ and $\widetilde{Q}$ (the set of allowable perturbations) at most k small zeros of $\mathscr{D}(r)$, hence at most k small limit cycles surrounding the origin, can be produced (see, for example, Theorem 6.2.7 of [70]).

When the origin is a center, so that $\mathscr{D}(r) \equiv 0$, the solution of the cyclicity problem hinges on identifying a special basis of the ideal generated by the Lyapunov quantities when they are viewed as elements of a relevant ring of functions or germs of functions in the parameters of the family. The general definition is as follows.

Definition 4.1.2. *Let R be a Noetherian ring and let $\{f_0, f_1, f_2, \ldots\}$ be an ordered set of elements of R. The* minimal basis *of the ideal $\langle f_0, f_1, f_2, \ldots\rangle$ in R is the set M generated in the following recursive fashion:*

(a) initially set $M = \{f_J\}$, where J is the smallest index j for which f_j is not the zero of R;

(b) successively check elements f_j, $j \geq J+1$, adjoining f_j to M if and only if $f_j \notin \langle M\rangle$.

The expression "the minimal basis of $I = \langle f_1, f_2, \ldots\rangle$" will always mean the minimal basis of this ideal with the generators ordered as specified by the order in which they are listed. Note that the minimal basis need not be the basis of minimal cardinality. As an extreme example (and an illustration of the notational convention just described), in the principal ideal domain $\mathbb{R}[x]$ the minimal basis of $\langle x^3, x^2, x\rangle$ is not $\{x\}$, but $\{x^3, x^2, x\}$.

Following [41] we make the following definition.

Definition 4.1.3. *The* Bautin depth *of the ideal $I = \langle f_1, f_2, \ldots\rangle$ (with generators ordered as specified by the order in which they are listed) is the cardinality of the minimal basis of I.*

The proof of the following theorem can be found, for instance, in [70].

Theorem 4.1.4. *Let $\mathscr{F}(z,\theta) = \sum_{j=0}^{\infty} f_j(\theta) z^j$ be an analytic function that converges on $U = \{(z,\theta) : |z| < \epsilon$ and $|\theta - \theta^*| < \delta\} \subset \mathbb{R} \times \mathbb{R}^n$ and let $\mathbf{f}_j$ denote the germ of f_j at $\boldsymbol{\theta}^*$ in the ring of germs $\mathscr{G}_{\boldsymbol{\theta}^*}$ of complex analytic functions at $\boldsymbol{\theta}^*$ when $\boldsymbol{\theta}^*$ is regarded as an element of $\mathbb{C}^n$. Suppose the minimal basis of the ideal $\langle \mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2, \ldots\rangle$ in $\mathscr{G}_{\boldsymbol{\theta}^*}$ is $\{\mathbf{f}_{j_1}, \ldots, \mathbf{f}_{j_m}\}$, $j_1 < \cdots < j_m$. Then there exist positive numbers $\widetilde{\epsilon} \leq \epsilon$ and $\widetilde{\delta} \leq \delta$ such that for each fixed $\boldsymbol{\theta}$ satisfying $|\boldsymbol{\theta} - \boldsymbol{\theta}^*| < \widetilde{\delta}$ the equation $\mathscr{F}(z, \boldsymbol{\theta}) = 0$, regarded as an equation in z alone, has at most $m-1$ isolated solutions in the interval $(0, \widetilde{\epsilon})$.*

The hypotheses insure that there exist positive real numbers $\widetilde{\epsilon} \leq \epsilon$ and $\widetilde{\delta} \leq \delta$ and m analytic functions $\psi_{j_s}(z,\theta)$ on $\widetilde{U} = \{(z,\theta) : |z| < \widetilde{\epsilon} \text{ and } |\theta - \theta^*| < \widetilde{\delta}\}$ that satisfy $\psi_{j_s}(0,\theta^*) = 0$ and are such that $\mathscr{F}(z,\theta)$ can be expressed on $\widetilde{U}$ as

$$(4.4)\quad \mathscr{F}(z,\theta) = f_{j_1}(\theta)(1+\psi_{j_1}(z,\theta))z^{j_1} + \cdots + f_{j_m}(\theta)(1+\psi_{j_m}(z,\theta))z^{j_m}$$

(e.g., Lemma 6.1.6 of [70]). But a function of the form (4.4), when regarded (for any θ within distance $\widetilde{\delta}$ of θ^*) as a function of z alone, has at most $m-1$ isolated zeros in the interval $a < z < \widetilde{\epsilon}$ (e.g., Proposition 6.1.2 of [70]).

In the setting of interest to us Theorem 4.1.4 implies that we can obtain an upper bound on the cyclicity of the origin by finding the Bautin depth (Definition 4.1.3) of the ideal generated by the Lyapunov quantities (always regarded as ordered in the natural way by their indices).

4.2. The Focus Quantities and an Upper Bound on Cyclicity

As pointed out at the end of Section 2.1, from the point of view of computations the Lyapunov quantities η_j in (4.3) are difficult to work with. Since both they and the focus quantities, which are readily computed, identify by their vanishing the centers in family (4.1), there must be a connection between them. In this section we derive this relationship, which allows us to use the focus quantities to treat the cyclicity problem. The key to connecting these two sets of objects, one of which arises directly from the original real system and the other of which pertains to the complexification of the original system, is to work on the invariant plane $x = \bar{x}$ in $\mathbb{C}^2$, which contains a copy of the phase portrait of the real system. A complication that also arises, which will be addressed later in this section, is that whereas the focus quantities arise from the complexification of system (4.1), perturbations occur within the larger family (4.2).

Suppose that in (4.1)

$$\widetilde{P}(u,v) = \sum_{(j,k)\in T} A_{jk}u^j v^k \qquad \widetilde{Q}(u,v) = \sum_{(j,k)\in T} B_{jk}u^j v^k$$

for some set $T \subset \mathbb{N}_0 \times \mathbb{N}_0$ of allowable indices. Any element of the set of allowable perturbations of the original system is given by a string of coefficients, which we abbreviate (A,B). The set of all such coefficient strings is our parameter space, which we will denote $E(A,B)$ when perturbations are restricted to (4.1) and $E(\lambda,(A,B))$ when perturbations

are made in (4.2). Any complexification corresponds to a string of complex coefficients which we abbreviate (a, b), which lies in corresponding parameter space $E(a, b)$. Since the coefficients (a, b) satisfy $b = \bar{a}$ and $g_{kk}(a, \bar{a}) \in \mathbb{R}$ for all admissible a, and because Re a_{jk} and Im a_{jk} are polynomials with rational coefficients in the original coefficients (A, B),

$$g^{\mathbb{R}}_{kk}(A, B) \stackrel{\text{def}}{=} g_{kk}(a(A, B), \bar{a}(A, B)) \tag{4.5}$$

is a polynomial in (A, B) with rational coefficients.

Theorem 4.2.1. *For the Lyapunov quantities η_j as defined by* (4.3) *with respect to* (4.1) *and the focus quantities $g^{\mathbb{R}}_{kk}$ as defined by* (4.5),
a. $\eta_1 = \eta_2 = 0$,
b. $\eta_3 = \pi g^{\mathbb{R}}_{11}$,
and for $k \in \mathbb{N}$, $k \geq 2$,
c. $\eta_{2k} \in \langle g^{\mathbb{R}}_{11}, \ldots, g^{\mathbb{R}}_{k-1,k-1} \rangle$ *and* $\eta_{2k+1} - \pi g^{\mathbb{R}}_{kk} \in \langle g^{\mathbb{R}}_{11}, \ldots, g^{\mathbb{R}}_{k-1,k-1} \rangle$ *in* $\mathbb{R}[A, B]$.

Proof. The first Lyapunov quantity for (4.2) satisfies $\eta_1 = e^{2\pi\lambda} - 1$, hence vanishes when $\lambda = 0$. Since the first non-zero Lyapunov quantity has odd index (or by a simple computation) $\eta_2 = 0$.

Let $\mathscr{X}$ denote the vector field on a neighborhood of the origin in $\mathbb{C}^2$ that corresponds to the complexification

$$\dot{x} = ix + X(x, y), \qquad \dot{y} = -iy + Y(x, y)$$

of (4.1) (compare (2.4) and (2.38)). For the remaining conclusions, first recall from the definition of the focus quantities that for any $N \in \mathbb{N}$ there exists a polynomial

$$\Psi_N(x, y) = xy + \sum_{j+k=3}^{2N+1} v_{jk} x^j y^k \tag{4.6}$$

such that

$$\mathscr{X}\Psi(x, y) = g_{11}(xy)^2 + g_{22}(xy)^3 + \cdots + g_{NN}(xy)^{N+1}. \tag{4.7}$$

We have truncated the series for Ψ at $2N + 1$ since it need not be convergent. The idea of the proof (which comes from [90]) is to relate the change $\mathscr{D}(r)$ in position along the positive u-axis in one turn around the origin to the change in the value of Ψ_N for sufficiently large N, computing the change in Ψ_N by integrating its derivative along solutions, which naturally generates the focus quantities according to (4.7). In fact Ψ_N is defined for $(x, y) \in \mathbb{C}^2$, but we evaluate it on $(x, \bar{x})$, which as

already mentioned is an invariant plane that contains the phase portrait of (4.1). Moreover since Ψ_N and the focus quantities g_{kk} pertain to the complexification of (4.1), the focus quantities g_{kk} are actually the quantities $g_{kk}^{\mathbb{R}}$.

For every $r > 0$ we have a number $\xi = \Psi_N(r, r)$. Since we restrict to $r > 0$ the function $\xi = f(r) = \Psi_N(r, r)$ is invertible; writing the inverse as $r = g(\xi)$, if ϵ is the change in ξ in one turn about the origin then

$$\begin{aligned}\mathscr{D}(r) = \mathscr{R}(r) - r &= g(\xi + \epsilon) - g(\xi) = g'(\xi)\epsilon + \tfrac{1}{2}g''(\widetilde{\xi})\epsilon^2 \\ &= \frac{1}{f'(r)}\triangle\Psi_N(r, r) - \frac{1}{2}\frac{f''(\widetilde{r})}{f'(\widetilde{r})^3}\triangle\Psi_N(r, r)^2\end{aligned}$$

for some $\widetilde{\xi}$ between ξ and $\xi + \epsilon$, with corresponding $\widetilde{r}$ between r and $\mathscr{R}(r)$. Since $\Psi(x, y) = xy + \cdots$, $\xi = f(r) = \Psi_N(r, r) = r^2 + \psi_3 r^3 + \cdots$, so

$$\mathscr{D}(r) = \left(\frac{1}{2r} + \cdots\right)\triangle\Psi_N(r, r) - \left(\frac{1}{8\widetilde{r}^3} + \cdots\right)\triangle\Psi_N(r, r)^2. \tag{4.8}$$

In one turn about the origin (from the point $(u, v) = (r, 0)$ to the point $(u, v) = (\mathscr{R}(r), 0)$) the change in Ψ_N is (for some time $\tau > 0$)

$$\begin{aligned}\triangle\Psi_N(r, r) &= \int_0^\tau \tfrac{d}{dt}\left[\Psi_N(x(t), \bar{x}(t))\right] dt \\ &= \int_0^\tau \sum_{k=1}^{N} g_{kk}^{\mathbb{R}}(x(t)\bar{x}(t))^{k+1} + o(|x(t)|^{2N+2})\, dt \\ &= \int_0^\tau \sum_{k=1}^{N} g_{kk}^{\mathbb{R}}|x(t)|^{2k+2} + o(|x(t)|^{2N+2})\, dt.\end{aligned}$$

From (4.1) it is clear that the polar angle θ satisfies $\dot{\theta} = 1 + \cdots$, hence may be used as the variable of integration. Writing

$$|x(t)| = r + w_2(\theta)r^2 + w_3(\theta)r^3 + \cdots$$

and

$$\frac{dt}{d\theta} = 1 + u_1(\theta)r + u_2(\theta)r^2 + \cdots$$

and letting f_{kj} denote a function of θ whose exact identity is not important, we have

$$\begin{aligned}\triangle\Psi_N(r,r)&\\ &= \int_0^{2\pi} \sum_{k=1}^{N} g_{kk}^{\mathbb{R}}(r + w_2(\theta)r^2 + \cdots)^{2k+2}(1 + u_1(\theta)r + u_2(\theta)r^2 + \cdots)\, d\theta\\ &\qquad\qquad + o(r^{2N+2})\\ &= \sum_{k=1}^{N} \left[2\pi g_{kk}^{\mathbb{R}} r^{2k+2} + g_{kk}^{\mathbb{R}}(f_{k,1}r^{2k+3} + f_{k,2}r^{2k+4} + \cdots)\right] + o(r^{2N+2}).\end{aligned}$$

Since $\triangle\Psi_N$ is of order at least four in r it is apparent that $\widetilde{r}$ is of order r. Thus when we insert this expression for $\triangle\Psi_N(r,r)$ into (4.8) we obtain

(4.9)
$$\mathscr{D}(r) = \sum_{k=1}^{N} \left[\pi g_{kk}^{\mathbb{R}} r^{2k+1} + g_{kk}^{\mathbb{R}}(\tilde{f}_{k,1}r^{2k+2} + \tilde{f}_{k,2}r^{2k+3} + \cdots)\right] + o(r^{2N+1}).$$

Combining (4.3) and (4.9) yields

$$\begin{aligned}\eta_3 r^3 + \eta_4 r^4 + \eta_5 r^5 + \cdots &\\ &= \pi g_{11}^{\mathbb{R}} r^3 + g_{11}^{\mathbb{R}}(\tilde{f}_{1,1}r^4 + \tilde{f}_{1,2}r^5 + \cdots)\\ &\quad + \pi g_{22}^{\mathbb{R}} r^5 + g_{22}^{\mathbb{R}}(\tilde{f}_{2,1}r^6 + \tilde{f}_{2,2}r^7 + \cdots)\\ &\quad + \pi g_{33}^{\mathbb{R}} r^7 + g_{33}^{\mathbb{R}}(\tilde{f}_{3,1}r^8 + \tilde{f}_{3,2}r^9 + \cdots)\\ &\quad + \cdots\\ &\quad + \pi g_{NN}^{\mathbb{R}} r^{2N+1} + g_{NN}^{\mathbb{R}}(\tilde{f}_{N,1}r^{2N+2} + \tilde{f}_{N,2}r^{2N+3} + \cdots)\\ &\quad + o(r^{2N+1}).\end{aligned}$$

Thus $\eta_3 = \pi g_{11}^{\mathbb{R}}$ and given $k \in \mathbb{N}$, the choice $N = k$ shows that the last pair of assertions of the proposition holds for η_4 through η_{2k+1}. Q.E.D.

An immediate consequence of Theorem 4.2.1 is the equality of the ideals

$$\langle g_{kk}^{\mathbb{R}} : k \in \mathbb{N}\rangle = \langle \eta_k : k \in \mathbb{N}\rangle = \langle \eta_{2k+1} : k \in \mathbb{N}\rangle$$

in $\mathbb{R}[A, B]$, and of the corresponding ideals in the ring $\mathscr{G}_{(A^*,B^*)}$ of germs at (A^*, B^*) of real analytic functions of (A, B). This in turn implies the following result.

Theorem 4.2.2. *Let η_k be the Lyapunov quantities for the singularity of* (4.1) *at the origin, let g_{kk} be the focus quantities for its*

complexification, and let $g_{kk}^{\mathbb{R}}$ *denote the polynomial function defined by* (4.5). *Suppose* $\{\boldsymbol{\eta}_{k_1}, \ldots, \boldsymbol{\eta}_{k_m}\}$ *and* $\{\mathbf{g}_{j_1,j_1}, \ldots, \mathbf{g}_{j_n,j_n}\}$ *are the minimal bases for the ideal* $\langle \boldsymbol{\eta}_{2k+1} : k \in \mathbb{N}\rangle = \langle \mathbf{g}_{kk} : k \in \mathbb{N}\rangle$ *in* $\mathscr{G}_{(A^*,B^*)}$ *with respect to the ordered sets* $\{\boldsymbol{\eta}_3, \boldsymbol{\eta}_5, \boldsymbol{\eta}_7, \ldots\}$ *and* $\{\mathbf{g}_{11}, \mathbf{g}_{22}, \ldots\}$, *respectively. Then* $m = n$ *and for* $q = 1, 2, \ldots, m$, $k_q = 2j_q + 1$.

Theorem 4.2.2 relates the Lyapunov quantities generated by an element of the family (4.1) to the focus quantities of its complexification. However, bifurcations to produce limit cycles naturally take place in the larger family (4.2). In order to use the focus quantities to treat the cyclicity of the origin in (4.1) we must establish the relationship between the minimal basis of the ideal generated by the Lyapunov quantities for the restricted family (4.1) and the minimal basis of the ideal generated by the Lyapunov quantities of larger family (4.2). Henceforth we will write just η_k for the Lyapunov quantities that depend on just the parameters (A, B) and $\eta_k(\lambda)$ for those that depend on the parameters $(\lambda, (A, B))$, although of course $\eta_k(0, (A, B)) = \eta_k(A, B)$. Because the functions $\eta_k(\lambda)$ are not polynomials in the parameters $(\lambda, (A, B))$ we must work in the ring of germs of analytic functions at $(0, (A^*, B^*))$.

Theorem 4.2.3. *Fix families* (4.1) *and* (4.2) *for which the nonlinearities* $\widetilde{P}$ *and* $\widetilde{Q}$ *are restricted to the same set of polynomials. Let* $\{\eta_k(\lambda) : k \in \mathbb{N}\}$ *be the Lyapunov quantities for family* (4.2) *and let* $\{\eta_k : k \in \mathbb{N}\}$ *be the Lyapunov quantities for family* (4.1). *Fix* (A^*, B^*) *and suppose that the minimal basis of the ideal* $\langle \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \ldots\rangle$ *in* $\mathscr{G}_{(A^*,B^*)}$ *is* $\{\boldsymbol{\eta}_{k_1}, \ldots, \boldsymbol{\eta}_{k_m}\}$, $k_1 < \ldots < k_m$. *Then* $\{\boldsymbol{\eta}_1(\lambda), \boldsymbol{\eta}_{k_1}, \ldots, \boldsymbol{\eta}_{k_m}\}$ *is the minimal basis of the ideal* $\langle \boldsymbol{\eta}_1(\lambda), \boldsymbol{\eta}_2(\lambda), \boldsymbol{\eta}_3(\lambda), \ldots\rangle$ *in* $\mathscr{G}_{(0,(A^*,B^*))}$.

Proof. Rearranging the terms in the power series expansion of $\eta_k(\lambda, (A, B))$ we may write

$$\eta_k(\lambda, (A, B)) = \breve{\eta}_k(\lambda, (A, B)) + \breve{\eta}_k(A, B),$$

where $\breve{\eta}_k(0, (A, B)) \equiv 0$. But since $\eta_k(0, (A, B)) \equiv \eta_k(A, B)$ in fact

$$\eta_k(\lambda, (A, B)) = \breve{\eta}_k(\lambda, (A, B)) + \eta_k(A, B). \tag{4.10}$$

Since

$$\eta_1(\lambda, (A, B)) = e^{2\pi\lambda} - 1 = 2\pi\lambda(1 + \tfrac{1}{2!}(2\pi\lambda) + \cdots)$$

there exists a function $u_k(\lambda, (A, B))$ that is real analytic on a neighborhood of $(0, (A^*, B^*))$ in the parameter space such that

$$\breve{\eta}_k(\lambda, (A, B)) = u_k(\lambda, (A, B))\eta_1(\lambda, (A, B)).$$

Thus (4.10) becomes, suppressing the (A, B) dependence in the notation,

$$\eta_k(\lambda) = u_k(\lambda)\eta_1(\lambda) + \eta_k. \tag{4.11}$$

Let L denote the set $\{\boldsymbol{\eta}_{k_1}, \dots, \boldsymbol{\eta}_{k_m}\}$. Because L is the minimal basis of the ideal $\langle \boldsymbol{\eta}_k : k \in \mathbb{N}\rangle$ in $\mathscr{G}_{(A^*,B^*)}$, (4.11) implies that for all $k \in \mathbb{N}$ the identity

$$\begin{aligned}\eta_k(\lambda,(A,B))&\\ &= u_k(\lambda,(A,B))\eta_1(\lambda,(A,B)) + h_{k,1}(A,B)\eta_{k_1}(A,B)\\ &\qquad + \cdots + h_{k,m}(A,B)\eta_{k_m}(A,B)\end{aligned}$$

holds on a neighborhood of $(0,(A^*,B^*))$ in $E(\lambda,(A^*,B^*))$ for functions $h_{k,q}$ that are defined and real analytic on that neighborhood, though without λ dependence. The same equation is therefore true at the level of germs in $\mathscr{G}_{(0,(A^*,B^*))}$. Thus

$$M = \{\boldsymbol{\eta}_1(\lambda), \boldsymbol{\eta}_{k_1}, \dots, \boldsymbol{\eta}_{k_m}\}$$

is a basis of the ideal $\langle \boldsymbol{\eta}_1(\lambda), \boldsymbol{\eta}_2(\lambda), \dots\rangle \subset \mathscr{G}_{(0,(A^*,B^*))}$. We must show that it is the minimal basis. Hence let

$$N = \{\boldsymbol{\eta}_1(\lambda), \boldsymbol{\eta}_{j_1}(\lambda), \dots, \boldsymbol{\eta}_{j_n}(\lambda)\}$$

be the unique minimal basis (which must contain $\boldsymbol{\eta}_1(\lambda)$, since $\boldsymbol{\eta}_1(\lambda)$ is first on the list and is not $\mathbf{0}$). There are four ways in which M could fail to be the minimal basis N:

i. There exists $p \in \{1, 2, \dots, \min\{m, n\}\}$ such that for $q \le p-1$, $k_q = j_q$ and $\boldsymbol{\eta}_{k_q} = \boldsymbol{\eta}_{j_q}(\lambda)$ but $\boldsymbol{\eta}_{k_p} \ne \boldsymbol{\eta}_{j_p}(\lambda)$ and $j_p < k_p$.
ii. There exists $p \in \{1, 2, \dots, \min\{m, n\}\}$ such that for $q \le p-1$, $k_q = j_q$ and $\boldsymbol{\eta}_{k_q} = \boldsymbol{\eta}_{j_q}(\lambda)$ but $\boldsymbol{\eta}_{k_p} \ne \boldsymbol{\eta}_{j_p}(\lambda)$ and $j_p > k_p$.
iii. $n < m$ and for $q \in \{1, \dots, n\}$: $k_q = j_q$ and $\boldsymbol{\eta}_{k_q} = \boldsymbol{\eta}_{j_q}(\lambda)$.
iv. $n > m$ and for $q \in \{1, \dots, m\}$: $k_q = j_q$ and $\boldsymbol{\eta}_{k_q} = \boldsymbol{\eta}_{j_q}(\lambda)$.

We will show that the first case is impossible, and leave the exclusion of the remaining cases to the reader (or see Lemma 6.2.8 of [70]). Thus suppose that there exists $p \in \{1, 2, \dots, \min\{m, n\}\}$ as in point (i). Then $k_{p-1} = j_{p-1} < j_p < k_p$ so because L is minimal

$$\boldsymbol{\eta}_{j_p} = \mathbf{h}_1\boldsymbol{\eta}_{k_1} + \cdots + \mathbf{h}_{p-1}\boldsymbol{\eta}_{k_{p-1}}$$

for $\mathbf{h}_1, \dots, \mathbf{h}_{p-1} \in \mathscr{G}_{(A^*,B^*)}$. Applying the corresponding equality of functions that holds on a neighborhood of (A^*, B^*) to (4.11) implies

that the identity

$$\begin{aligned}\eta_{j_p}(\lambda) &= u_{j_p}(\lambda)\eta_1(\lambda) + \eta_{j_p}\\ &= u_{j_p}(\lambda)\eta_1(\lambda) + h_1\eta_{k_1} + \cdots + h_{p-1}\eta_{k_{p-1}}\\ &= u_{j_p}(\lambda)\eta_1(\lambda) + h_1\eta_{j_1}(\lambda) + \cdots + h_{p-1}\eta_{j_{p-1}}(\lambda)\end{aligned}$$

is valid on a neighborhood of $(0,(A,B))$ in $E(\lambda,(A,B))$ (although h_q is independent of λ), so the corresponding equality of germs contradicts the fact that N is minimal. Q.E.D.

Theorem 4.2.4. *Suppose that for a coefficient string* (A^*,B^*) *the Bautin depth (Definition 4.1.3) of the ideal* $\langle \mathbf{g}_{11}^{\mathbb{R}}, \mathbf{g}_{22}^{\mathbb{R}}, \ldots\rangle$ *in* $\mathscr{G}_{(A^*,B^*)}$ *for the corresponding system of the form* (4.1) *is* m*. Then the cyclicity of the origin of the system* (4.1) *with respect to perturbation in* (4.2) *is at most* m*.*

Proof. The cyclicity of the origin of an element of family (4.1) with respect to perturbation within the family (4.2) is equal to the maximum number of small positive zeros of the function

$$\mathscr{D}(r) = \mathscr{R}(r) - r = \eta_1(\lambda)r + \eta_2(\lambda)r^2 + \eta_3(\lambda)r^3 + \cdots$$

that can be made to bifurcate from the origin under arbitrarily small perturbation of the parameters $(\lambda,(A,B))$. By the hypothesis and Theorem 4.2.2 the minimal basis of the ideal $\langle \boldsymbol{\eta}_3, \boldsymbol{\eta}_5, \boldsymbol{\eta}_7, \ldots\rangle$ in $\mathscr{G}_{(A^*,B^*)}$ has m elements, hence by Theorem 4.2.3 the minimal basis of the ideal

$$\langle \boldsymbol{\eta}_1(\lambda), \boldsymbol{\eta}_2(\lambda), \boldsymbol{\eta}_3(\lambda), \ldots\rangle$$

in $\mathscr{G}_{(0,(A^*,B^*))}$ has $m+1$ elements. Then by Theorem 4.1.4 the maximum possible number of small positive zeros of the function $\mathscr{D}(r)$ is m. Q.E.D.

4.3. Radical Bautin Ideal

In order to apply Theorem 4.2.4 we need a computationally feasible method for determining the minimal basis of the ideal $\langle g_{kk}^{\mathbb{R}} : k \in \mathbb{N}\rangle$, of which only the first few generators are ever explicitly known. Since an upper bound for the cyclicity of the origin is at issue only when the origin is a center (in the sense that if it is a fine focus of order k then the cyclicity is at most k), we suppose that the center problem has been solved: we know the minimum K such that $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_K)$. (Note that this does not say the same thing as $\mathcal{B} = \mathcal{B}_K$, but only that $\sqrt{\mathcal{B}} = \sqrt{\mathcal{B}_K}$, provided the ground field is $\mathbb{C}$.) The key issue now proves to be whether or not the ideal $\mathcal{B}_K$ is a radical ideal.

Theorem 4.3.1. *Suppose K is such that*
(1) $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_K)$ *and*
(2) $\mathcal{B}_K$ *is a radical ideal.*
Then the cyclicity of the origin of system (4.1) *with respect to perturbation in* (4.2) *is at most the Bautin depth (Definition 4.1.3) of* $\mathcal{B}_K$.

Proof. By the first hypothesis g_{kk} vanishes on $\mathbf{V}(\mathcal{B}_K)$ for every k, so $g_{kk} \in \mathbf{I}(\mathbf{V}(\mathcal{B}_K))$. But the Strong Hilbert Nullstellensatz states that $\mathbf{I}(\mathbf{V}(\mathcal{B}_K)) = \sqrt{\mathcal{B}_K}$, which by the second hypothesis is $\mathcal{B}_K$, so $\mathcal{B} \subset \mathcal{B}_K$, hence $\mathcal{B} = \mathcal{B}_K$. Then the minimal basis of $\mathcal{B}$ is the minimal basis $\{g_{k_1k_1}, \ldots, g_{k_mk_m}\}$ of $\mathcal{B}_K$, which is computable. Thus for any $k \in \mathbb{N}$ there exist $f_{k,1}, \ldots, f_{k,m} \in \mathbb{C}[a, b]$ such that

$$g_{kk} = f_1 g_{k_1,k_1} + \cdots + f_m g_{k_m,k_m}. \tag{4.12}$$

But since $g_{kk}(a, \bar{a}) = g^{\mathbb{R}}_{kk}(A(a, \bar{b}), B(a, \bar{b})) \in \mathbb{R}$ for all $k \in \mathbb{N}$, this implies that

$$g^{\mathbb{R}}_{kk} = (\operatorname{Re} f_1) g^{\mathbb{R}}_{j_1,j_1} + \cdots + (\operatorname{Re} f_m) g^{\mathbb{R}}_{j_m,j_m}.$$

Thus for any (A^*, B^*) in $E(A, B)$, $L := \{\mathbf{g}^{\mathbb{R}}_{k_1,k_1}, \ldots, \mathbf{g}^{\mathbb{R}}_{k_m,k_m}\}$ is a basis of the ideal $I = \langle \mathbf{g}^{\mathbb{R}}_{kk} : k \in \mathbb{N} \rangle$ in $\mathscr{G}_{(A^*,B^*)}$. It is apparent that even if L were not the minimal basis of the ideal $I = \langle \mathbf{g}^{\mathbb{R}}_{11}, \mathbf{g}^{\mathbb{R}}_{22}, \ldots \rangle$ in $\mathscr{G}_{(A^*,B^*)}$ (because of possible collapsing of g_{k_q,k_q} to $\mathbf{g}^{\mathbb{R}}_{k_q,k_q} = \mathbf{0}$), it nevertheless contains the minimal basis, which therefore can have at most m elements. The conclusion of the theorem then follows from Theorem 4.2.4. Q.E.D.

In Subsection 6.2 we will use this theorem to prove the important result that the cyclicity of a center or focus in a quadratic system is at most three. (A quadratic system is a system ordinary differential equations in which the right hand sides are polynomials of degree at most two.) The reader can examine the proof now, but we defer the result in order to present it in one context with the solution of the center problem for quadratic systems, to which it is intimately connected and on which it depends.

4.4. Nonradical Ideal $\mathcal{B}_K$

Suppose that the center problem has been solved for a family of interest, so that we know a value of K for which $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_K)$, but that the ideal $\mathcal{B}_K$ is not a radical ideal. In this case Theorem 4.3.1 does not apply, but it is sometimes possible to still obtain an upper bound on the cyclicity of centers in family (4.1) by using a structure possessed by the focus quantities to move the ideals in question to a different ring. In order to more easily describe this structure we will write the family of

complexifications of elements of the family (4.1) of polynomial systems in a form like that of (2.42),

$$\begin{aligned}\dot{x} &= \quad ix - \sum_{(p,q)\in S} a_{pq}x^{p+1}y^q = \quad ix + \widetilde{P}(x,y)\\ \dot{y} &= -iy + \sum_{(p,q)\in S} b_{qp}x^q y^{p+1} = -iy + \widetilde{Q}(x,y)\end{aligned} \tag{4.13}$$

for a fixed finite set $S \subset (\{-1\}\cup\mathbb{N}_0)\times\mathbb{N}_0$ each element of which satisfies $p+q \geq 1$. (Here $\mathbb{N}_0 = \{0,1,2,\ldots\}$.) We let ℓ denote the cardinality of S, which we order in some manner and write as

$$S = \{(p_1,q_1),\ldots,(p_\ell,q_\ell)\}.$$

Consistent with this ordering we order the coefficients as

$$(a_{p_1q_1},\ldots,a_{p_\ell q_\ell},b_{q_\ell,p_\ell},\ldots,b_{q_1p_1}),$$

which as always we shorten to (a,b). We write $\mathbb{C}[a,b]$ for the polynomial ring in the indeterminates $(a_{p_1q_1},\ldots,b_{q_1p_1})$ over $\mathbb{C}$. Any monomial in $\mathbb{C}[a,b]$ will be written in the abbreviated form $[\nu]$ where

$$[\nu] = a_{p_1q_1}^{\nu_1}\ldots a_{p_\ell q_\ell}^{\nu_\ell} b_{q_\ell,p_\ell}^{\nu_{\ell+1}}\ldots b_{q_1p_1}^{\nu_{2\ell}}, \qquad \nu = (\nu_1,\ldots,\nu_{2\ell}).$$

Define a mapping $L : \mathbb{N}_0^{2\ell} \to \mathbb{Z}^2$ by

$$L(\nu) = \nu_1(p_1,q_1)+\cdots\nu_\ell(p_\ell,q_\ell)+\nu_{\ell+1}(q_\ell,p_\ell)+\cdots+\nu_{2\ell}(q_1,p_1). \tag{4.14}$$

Then the focus quantities g_{kk} for family (4.13) have the form

$$g_{kk} = \frac{1}{2}\sum_{\{\nu:L(\nu)=(k,k)\}} g_{kk}^{(\nu)}([\nu]-[\widehat{\nu}]) \tag{4.15}$$

where L is the mapping defined by (4.14) and for $\nu = (\nu_1,\ldots,\nu_{2\ell}) \in \mathbb{N}_0^{2\ell}$, $\widehat{\nu} = (\nu_{2\ell},\ldots,\nu_1)$. (See Corollary 3.4.6 of [70]. Similar properties of the focus quantities were also obtained in [25, 50].)

To exploit this structure, let

$$\mathscr{M} = \{\nu \in \mathbb{N}_0^{2\ell} : L(\nu) = (j,j) \text{ for some } j \in \mathbb{N}_0\}.$$

$\mathscr{M}$ has the structure of an abelian monoid under addition (an abelian group except for the existence of inverses). The fact that the monomials in the polynomials g_{kk} have exponent strings in $\mathscr{M}$ implies that they are invariants for group actions (3.16). Indeed, the algorithm in [67] (also [70, Table 5.1]), given here in Table 5, produces a Hilbert basis

$\mathscr{H} = \{\mu_1, \ldots, \mu_M\}$ of $\mathscr{M}$ by computing a Gröbner basis G of the ideal H of (3.30) corresponding to (3.26), with the appropriate modifications so as to apply to the two-dimensional system (4.13). (To say that $\mathscr{H}$ is a Hilbert basis of $\mathscr{M}$ means that each element of $\mathscr{M}$ is a finite sum of elements of $\mathscr{H}$ but no element of $\mathscr{H}$ can be expressed as a sum of other elements of $\mathscr{H}$ (where repetition of summands is allowed in each case).) For each element μ_j of the Hilbert basis $\mathscr{H}$ so constructed let $h_j = [\mu_j]$ be the corresponding monomial in $\mathbb{C}[a, b]$ and define the mapping

$$F : \mathbb{C}^{2\ell} \to \mathbb{C}^M : (a, b) \mapsto (c_1, \ldots, c_M) = (h_1(a, b), \ldots, h_M(a, b)) \tag{4.16}$$

which, letting $c = (c_1, \ldots, c_M)$, induces the homomorphism of $\mathbb{C}$-algebras

$$\begin{aligned} F^\sharp &: \mathbb{C}[c] \to \mathbb{C}[a, b] \\ &: \sum d_{(\alpha)} c_1^{\alpha_1} \cdots c_M^{\alpha_M} \mapsto \sum d_{(\alpha)} h_1^{\alpha_1}(a, b) \cdots h_M^{\alpha_M}(a, b). \end{aligned} \tag{4.17}$$

Let $W \subset \mathbb{C}^M$ denote the image of F and $\overline{W} \subset \mathbb{C}^M$ its Zariski closure, the smallest affine subvariety of $\mathbb{C}^M$ that contains it. Let $\mathbb{C}[\overline{W}]$ denote the coordinate ring of $\overline{W}$, which can be regarded as the set of mappings from $\mathbb{C}^M$ (c-space) into $\mathbb{C}$, each of which agrees on $\overline{W}$ with a polynomial function (see [26] or [70]). Since $\nu \in \mathscr{M}$ if and only if $\widehat{\nu} \in \mathscr{M}$, by (4.15) the kth focus quantity has the form $g_{kk} = \sum \widetilde{g}_{kk}^{(\alpha)} h_1^{\alpha_1} \cdots h_M^{\alpha_M}$, hence is in Image($F^\sharp$) for all k. Let g_{kk}^c denote any pre-image in $\mathbb{C}[c]$ of g_{kk}. Then each g_{kk}^c lies in $\mathbb{C}[\overline{W}]$, which has the structure of a commutative ring, so for any $K \in \mathbb{N}$ we may view $\langle g_{11}^c, \ldots, g_{KK}^c \rangle$ as an ideal in $\mathbb{C}[\overline{W}]$.

Theorem 4.3.1 carries over to the coordinate ring because the Strong Hilbert Nullstellensatz is valid there: let H be any ideal in $\mathbb{C}[\overline{W}]$, let S be any subset of $\overline{W}$, and define

$$\mathbf{V}_{\overline{W}}(H) = \{\mathbf{x} \in \overline{W} : h(\mathbf{x}) = 0 \text{ for all } h \in H\}$$

and

$$\mathbf{I}_{\overline{W}}(S) = \{f \in \mathbb{C}[\overline{W}] : f(\mathbf{x}) = 0 \text{ for all } \mathbf{x} \in S\}.$$

Then

$$\mathbf{I}_{\overline{W}}(\mathbf{V}_{\overline{W}}(H)) = \sqrt{H}.$$

The next result is the analogue of Theorem 4.3.1.

Theorem 4.4.1. *Suppose that for the complexification* (4.13) *of the family* (4.1) *with focus quantities* $\{g_{kk} : k \in \mathbb{N}\}$ *the polynomials* g_{kk}^c *as just described satisfy*

(1) $\mathbf{V}_{\overline{W}}(g_{11}^c, g_{22}^c, \ldots) = \mathbf{V}_{\overline{W}}(g_{11}^c, \ldots, g_{KK}^c)$ *and*

(2) $\langle g_{11}^c, \ldots, g_{KK}^c \rangle$ *is radical in* $\mathbb{C}[\overline{W}]$,

Algorithm for computing a Hilbert basis of $\mathscr{M}$

Input:

An ordered index set $\widetilde{S} = \{(p_1, q_1), \ldots, (p_\ell, q_\ell)\}$ specifying a family of systems (4.13).

Output:

A Hilbert basis $\mathscr{H}$ for the monoid $\mathscr{M}$ for family (4.13).

Procedure:

1. Compute the reduced Gröbner basis G_H for H defined by (3.30) with respect to lexicographic order with $\{y, \gamma\} > \{a, b, c\}$.
2. $G := G_H \cap \mathbb{C}[a, b]$.
3. Writing $\mathbf{e}_j = (0, \ldots, 0, \overset{j}{1}, 0, \ldots, 0)$,

$$\begin{aligned}\mathscr{H} = &\{\mu, \hat{\mu} : [\mu] - [\hat{\mu}] \in G\} \\ &\cup \{\mathbf{e}_j + \mathbf{e}_{2\ell-j+1} : 1 \le j \le \ell \\ &\qquad \text{and } \pm([\mathbf{e}_j] - [\mathbf{e}_{2\ell-j+1}]) \notin G\}.\end{aligned}$$

Table 5. Algorithm for computing a Hilbert basis of $\mathscr{M}$

for some $K \in \mathbb{N}$. Then the cyclicity of the singularity at the origin of (4.1) *with respect to perturbation in* (4.2) *is at most the cardinality of the minimal basis of $\mathcal{B}_K$.*

Proof. The proof is practically identical to the proof of Theorem 4.3.1. Writing $\mathcal{B}^c = \langle g^c_{11}, g^c_{22}, \ldots \rangle$ and $\mathcal{B}^c_K = \langle g^c_{11}, \ldots, g^c_{KK} \rangle$ for these ideals in $\mathbb{C}[\overline{W}]$, for any $k \in \mathbb{N}$,

$$g^c_{kk} \in \mathbf{I}_{\overline{W}}(\mathbf{V}_{\overline{W}}(\mathcal{B}^c)) = \mathbf{I}_{\overline{W}}(\mathbf{V}_{\overline{W}}(\mathcal{B}^c_K)) = \sqrt{\mathcal{B}^c_K} = \mathcal{B}^c_K,$$

so $\mathcal{B}^c = \mathcal{B}^c_K$. Thus for every $k \in \mathbb{N}$ there exist polynomials $f_{j,k} \in \mathbb{C}[a, b]$ such that, as polynomial mappings,

$$g^c_{kk}(\boldsymbol{c}) = f_{1,k}(\boldsymbol{c}) g^c_{11}(\boldsymbol{c}) + \cdots + f_{K,k}(\boldsymbol{c}) g^c_{KK}(\boldsymbol{c})$$

holds for every $\boldsymbol{c} \in \overline{W}$. Since $\mathbb{C}[\overline{W}] \cong \mathbb{C}[c]/\ker F^\sharp$, applying $F^\sharp$ we obtain

$$g_{kk} = (F^\sharp f_{1,k})g_{11} + \cdots + (F^\sharp f_{K,k})g_{KK}$$

in $\mathbb{C}[a,b]$, from which (4.12) follows. The remainder of the proof of Theorem 4.3.1 from (4.12) onward gives the result. Q.E.D.

The implementation of Theorem 4.4.1 is as practical as that of Theorem 4.3.1. To describe it we need the following definitions and observations.

Let $J = \langle c_1 - h_1(a,b), \ldots, c_M - h_M(a,b)\rangle \subset \mathbb{C}[a,b,c]$.

Let $R = J \cap \mathbb{C}[c]$. Then $R = \ker F^\sharp$ (Theorem 2.4.2 of [1]) and is a radical ideal because it is the kernel of a ring homomorphism into an integral domain, hence is prime.

Let $V_C = F(\mathbf{V}(\mathcal{B}))$, the image under F of the center variety.

It is readily verified that if $f \in \mathbb{C}[a,b]$ and $f^c \in \mathbb{C}[c]$ are any two polynomials that satisfy $F^\sharp(f^c) = f$ and if $(\hat{a},\hat{b}) \in \mathbb{C}^N$ and $\hat{c} \in \mathbb{C}^M$ are any two coordinate strings that satisfy $F(\hat{a},\hat{b}) = \hat{c}$ then $f^c(\hat{c}) = f(\hat{a},\hat{b})$. That is,

$$F^\sharp(f^c) = f \text{ and } F(\hat{a},\hat{b}) = \hat{c} \text{ imply } f^c(\hat{c}) = f(\hat{a},\hat{b}),$$

which implies that $V_C \subset \mathbf{V}(\mathcal{B}^c)$. Thus because $\overline{V}_C$ is the smallest variety that contains V_C, we obtain the first inclusion in

$$\overline{V}_C \subset \mathbf{V}(\mathcal{B}^c) \subset \mathbf{V}(\mathcal{B}^c_K). \tag{4.18}$$

Let R' denote the ideal in $\mathbb{C}[a,b,c]$ generated by any set of generators of the ideal R in $\mathbb{C}[c]$.

Let $N = R' + \mathcal{B}_K + J$ in $\mathbb{C}[a,b,c]$.

Let $H = N \cap \mathbb{C}[c]$.

The ideas in §1.8.3 of [37] imply that $\overline{V}_C = \mathbf{V}(H)$, which yields the implication

$$(R' + \mathcal{B}_K + J) \cap \mathbb{C}[c] = \mathcal{B}^c_K \text{ implies } \overline{V}_C = \mathbf{V}(\mathcal{B}^c_K). \tag{4.19}$$

If the antecedent in (4.19) is true then it together with (4.18) implies Condition (1) of Theorem 4.4.1.

Since $W = \text{Image}(F)$, $\overline{W} = \mathbf{V}(R)$ (Theorem 1.4.2).

Letting $\simeq$ denote isomorphism of rings, $\mathbb{C}[\overline{W}] \simeq \mathbb{C}[c]/\mathbf{I}(\overline{W})$ (Theorem 7 in §5.2 of [26]), so by Theorem 1.1.4 and the fact that R is a radical ideal,

$$\mathbb{C}[\overline{W}] \simeq \mathbb{C}[c]/\mathbf{I}(\overline{W}) = \mathbb{C}[c]/\mathbf{I}(\mathbf{V}(R)) = \mathbb{C}[c]/\sqrt{R} = \mathbb{C}[c]/R. \tag{4.20}$$

Let $\prec_{(a,b)}$ be a monomial ordering on $\mathbb{C}[a,b,c]$, for instance, a lexicographic ordering, that possesses an elimination property for (a,b). That is, with respect to $\prec_{(a,b)}$ any monomial containing only the variables of (c) is smaller than every element that contains a variable of (a,b).

Let J_G be a Gröbner basis of J with respect to $\prec_{(a,b)}$.

Then $R_G = \{r_1, \ldots, r_T\} = J_G \cap \mathbb{C}[c]$ is a Gröbner basis of R by the Elimination Theorem (Theorem 1.4.1).

By the isomorphism (4.20)

$$\langle g_{11}^c, \ldots, g_{KK}^c \rangle \text{ is radical in } \mathbb{C}[\overline{W}]$$

if and only if

$$\langle g_{11}^c + \ker F^\sharp, \ldots, g_{KK}^c + \ker F^\sharp \rangle \text{ is radical in } \mathbb{C}[c]/\ker F^\sharp,$$

which in turn holds if

$$\langle g_{11}^c, \ldots, g_{KK}^c, r_1, \ldots, r_T \rangle \text{ is radical in } \mathbb{C}[c]. \tag{4.21}$$

Therefore condition (4.21) implies Condition (2) of Theorem 4.4.1.

Thus in summary, Theorem 4.4.1 can be implemented in the following steps.

1. Use the algorithm that is given in Table 5 to compute a Hilbert basis $\mathscr{M}_H = \{\nu_1, \ldots, \nu_M\} \subset \mathbb{N}_0^N$ of $\mathscr{M}$. For $1 \le j \le M$ let h_j denote the monomial $[\nu_j]$.
2. Compute a Gröbner basis $J_G = \{f_1, \ldots, f_U\}$ of the ideal
$$J = \langle c_1 - h_1(a,b), \ldots, c_M - h_M(a,b) \rangle \subset \mathbb{C}[a,b,c]$$
with respect to any elimination ordering $\prec_{(a,b)}$. Form the Gröbner basis $R_G = J_G \cap \mathbb{C}[c] = \{r_1, \ldots, r_T\}$ of $R = J \cap \mathbb{C}[c] = \ker F^\sharp$.
3. Compute, with respect to $\prec_{(a,b)}$, the reduced Gröbner basis N_G of the ideal $N = \langle r_1, \ldots, r_T, g_{11}, \ldots, g_{KK}, f_1, \ldots, f_U \rangle$ in $\mathbb{C}[a,b,c]$. Form $H_G = N_G \cap \mathbb{C}[c]$, the reduced Gröbner basis of $N \cap \mathbb{C}[c]$.
4. For $1 \le k \le K$ compute g_{kk}^c as the remainder when g_{kk} is divided by the Gröbner basis J_G of J in $\mathbb{C}[a,b,c]$ (Proposition 7 of §7.3 of [26]). The computation is in $\mathbb{C}[a,b,c]$ but because $g_{kk} \in \text{Image}(F^\sharp)$, $g_{kk}^c \in \mathbb{C}[c]$.
5. Compute, with respect to the monomial order on $\mathbb{C}[c]$ induced by the order $\prec_{(a,b)}$ on $\mathbb{C}[a,b,c]$, the unique reduced Gröbner basis $(\mathcal{B}_K^c)_G$ of $\mathcal{B}_K^c$ in $\mathbb{C}[c]$. If $(\mathcal{B}_K^c)_G = H_G$ then Condition (1) of Theorem 4.4.1 holds.
6. Check whether $\langle g_{11}^c, \ldots, g_{KK}^c, r_1, \ldots, r_T \rangle$ is a radical ideal in $\mathbb{C}[c]$. If so then Condition (2) in Theorem 4.4.1 holds.

We illustrate the method by using it to prove the following theorem, first proved in [47] using the same procedure.

Theorem 4.4.2. *The cyclicity of a center at the origin in the family of real systems whose expression in complex form (using the indexing scheme of* (4.13)*) is*

$$\dot{x} = i(x - a_{-12}\bar{x}^2 - a_{20}x^3 - a_{02}x\bar{x}^2)$$

is at most four.

Proof. The complexification of the underlying real system is

$$\begin{aligned}\dot{x} &= \quad i(x - a_{-12}y^2 - a_{20}x^3 - a_{02}xy^2)\\ \dot{y} &= -i(y - b_{2,-1}x^2 - b_{20}x^2y - b_{02}y^3).\end{aligned}$$

The first six focus quantities, each reduced modulo the ideal generated by the previous ones, are

$$\begin{aligned}
g_{11} &= 0\\
g_{22} &= -i(3a_{20}a_{02} - 3b_{20}b_{02})\\
g_{33} &= 0\\
g_{44} &= -10i(216a_{20}^3a_{-12}^2 + 576a_{20}^2b_{20}a_{-12}^2 + 216a_{20}b_{20}^2a_{-12}^2 - 144b_{20}^3a_{-12}^2\\
&\quad + 144a_{02}^3b_{2,-1}^2 - 216a_{02}^2b_{02}b_{2,-1}^2 - 576a_{02}b_{02}^2b_{2,-1}^2 - 216b_{02}^3b_{2,-1}^2)\\
g_{55} &= -100i(-3402a_{20}^2b_{20}a_{12}^3b_{2,-1} - 2268a_{20}b_{20}^2a_{-12}^3b_{2,-1}\\
&\quad + 1134b_{20}^3a_{-12}^3b_{2,-1} - 1134a_{02}^3a_{-12}b_{2,-1}^3 + 2268a_{02}^2b_{02}a_{-12}b_{2,-1}^3\\
&\quad + 3402a_{02}b_{02}^2a_{-12}b_{2,-1}^3)\\
g_{66} &= -10000i(10206a_{20}^2b_{20}^2b_{02}a_{-12}^2 + 6804a_{20}b_{20}^3b_{02}a_{-12}^2\\
&\quad - 3402b_{20}^4b_{02}a_{-12}^2 + 3402a_{02}^3b_{20}b_{02}b_{2,-1}^2\\
&\quad - 6804a_{02}^2b_{20}b_{02}^2b_{2,-1}^2 - 10206a_{02}b_{20}b_{02}^3b_{2,-1}^2).
\end{aligned}$$

It follows from the results of [49] that $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_6)$, so Condition (1) of Theorem 4.3.1 holds with $K = 6$. Calculations (e.g. with SINGULAR) show that $\mathcal{B}_6$ is not a radical ideal, so that Condition (2) fails. We seek therefore to apply Theorem 4.4.1.

Applying the algorithm mentioned above for computing a Hilbert basis of $\mathscr{M}$ we compute the reduced Gröbner basis of the ideal

$$\mathcal{J} = \langle 1 - w\alpha, a_{-12} - t_1, \alpha^3 b_{2,-1} - t_1, a_{20} - t_2, b_{02} - \alpha^2 t_2, a_{02} - t_3, \alpha^2 b_{20} - t_3\rangle$$

with respect to lexicographic order with $w > \alpha > t_1 > t_2 > t_3 > a_{-12} > a_{20} > a_{11} > a_{02} > b_{20} > b_{11} > b_{02} > b_{2,-1}$. The polynomials of the output list that do not depend on w, α, t_1, t_2, t_3, or t_4 are

$$\begin{array}{ll} a_{02}a_{20} - b_{02}b_{20} & -a_{-12}^2 b_{20}^3 + a_{02}^3 b_{2,-1}^2 \\ -a_{-12}^2 a_{20} b_{20}^2 + a_{02}^2 b_{02} b_{2,-1}^2 & -a_{-12}^2 a_{20}^2 b_{20} + a_{02} b_{02}^2 b_{2,-1}^2. \\ -a_{-12}^2 a_{20}^3 + b_{02}^3 b_{2,-1}^2 & \end{array}$$

A Hilbert basis of $\mathscr{M}$ is thus the 13-element set

(100 001)	(010 010)	(000 032)	(002 012)	(200 300)
(011 000)	(001 100)	(001 022)	(210 200)	
(000 110)	(230 000)	(220 100)	(003 002)	

so $M = 13$. We denote the j-th element of this list by ν_j and let $h_j = [\nu_j] \in \mathbb{C}[a,b]$. For example $h_2 = a_{20}a_{02}$ and $h_{12} = a_{02}^3 b_{2,-1}^2$.

For the elimination ordering $\prec_{(a,b)}$ choose lex with the ordering of the variables

$$a_{-12} > a_{20} > a_{02} > b_{20} > b_{02} > b_{2,-1} > c_1 > \cdots > c_{13}.$$

Computing a Gröbner basis J_G of the ideal $J = \langle c_1 - h_1(a,b), \ldots, c_{13} - h_{13}(a,b)\rangle$ in $\mathbb{C}[a,b,c]$ with respect to $\prec_{(a,b)}$ and forming the Gröbner basis $R_G = J_G \cap \mathbb{C}[c]$ of the ideal R yields a 44-element set $\{r_1, \ldots, r_{44}\}$. We then compute the reduced Gröbner basis N_G, with respect to $\prec_{(a,b)}$, of the ideal $N = \langle r_1, \ldots, r_{44}\rangle + \langle g_{11}, \ldots, g_{66}\rangle + J$ in $\mathbb{C}[a,b,c]$ and form the reduced Gröbner basis $H_G = N_G \cap \mathbb{C}[c]$ of $N \cap \mathbb{C}[c]$.

Division of each g_{kk}, $1 \le k \le 6$, by the Gröbner basis J_G yields

$$\begin{aligned} g_{11}^c &= 0 \qquad g_{22}^c = -i(-3c_4 + 3c_9) \qquad g_{33}^c = 0 \\ g_{44}^c &= -10i(144c_5 - 216c_6 - 576c_7 - 216c_8 - 1440_{10} + 216c_{11} + 576c_{12} \\ &\qquad + 216c_{13}) \\ g_{55}^c &= -100i(-1134c_1c_5 + 2268c_1c_6 + 3402c_1c_7 + 1134c_1c_{10} - 2268c_1c_{11} \\ &\qquad - 3402c_1c_{12}) \\ g_{66}^c &= -10000i(3402c_4c_5 - 6804c_4c_6 - 10206c_4c_7 - 3402c_4c_{10} \\ &\qquad + 6804c_4c_{11} + 10206c_4c_{12}). \end{aligned}$$

When the unique reduced Gröbner basis of the ideal $\mathcal{B}_6^c$ is computed with respect to lex with $c_1 > \cdots > c_{13}$ it is the same as the set of polynomials H_G computed above, so we conclude that Condition (1) of Theorem 4.4.1 holds.

Finally, computing the radical of $\langle g_{11}^c, \ldots, g_{66}^c, r_1, \ldots, r_{44}\rangle$ with SINGULAR we find that this ideal is radical, so that Condition (2) in Theorem 4.4.1 holds.

Since the minimal basis of $\mathcal{B}_6$ is clearly $\{g_{22}, g_{44}, g_{55}, g_{66}\}$ we conclude by Theorem 4.4.1 that the cyclicity of a center at the origin in the original real family is at most four. Q.E.D.

§5. Centers on Center Manifolds

One of the most effective tools for studying the behavior of trajectories of high-dimensional systems of ordinary differential equations is the theory of center manifolds, which in some respects originates from the work of V. A. Pliss on the Reduction Principle ([64]; see also [63]) and which was further developed by many others (see, e.g. [19, 80] and references therein).

Consider an $(m + n)$-dimensional system of ordinary differential equations of the form

$$\begin{aligned} \dot{x} &= Ax + u(x, y) \\ \dot{y} &= By + v(x, y), \end{aligned} \tag{5.1}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, where A and B are square matrices whose spectra $\sigma(A)$ and $\sigma(B)$ satisfy $\operatorname{Re}\sigma(A) = 0$ and $\operatorname{Re}\sigma(B) \neq 0$, and where u, v are C^k-functions, $k \geq 1$, which vanish together with their first derivatives at the origin. A C^k-manifold $W^c \equiv W^c(\mathbf{0}, U)$ in a neighborhood U of $\mathbf{0}$ is said to be a *center manifold* of (5.1) if W^c is invariant under the flow induced by (5.1) as long as the solution remains in U and W^c is the graph of a C^k function $y = h(x)$ which is tangent at $(\mathbf{0}, \mathbf{0}) \in \mathbb{R}^m \times \mathbb{R}^n$ to the x-space.

A proof of the following theorem can be found in many places, for example in [19].

Theorem 5.0.3. *There exists a neighborhood U of $\mathbf{0} \in \mathbb{R}^m \times \mathbb{R}^n$ such that there exists a local center manifold $W^c(\mathbf{0}, U)$ of (5.1) which is the graph of a C^k-function $y = h(x)$.*

5.1. Center Manifolds and First Integrals

In Subsection 3.1 we considered completely integrable systems. However, complete integrability is a relatively rare phenomenon. Most systems are not completely integrable. Nevertheless if we know even just one integral of a system often we can say a great deal about the behavior of trajectories of the system. An important such case is that in which a first integral defines a surface with a family of periodic solutions on it. The surface is usually a center manifold. Under a small perturbation the family of periodic solutions can be destroyed, but the center manifold could very well persist for perturbations that are relevant to the context of the problem and might contain limit cycles. In this subsection we study center manifolds and their connection to the center problem, limiting the consideration to the case of three-dimensional systems, in large part as developed in [32].

Suppose U is an open neighborhood of the origin in $\mathbb{R}^3$, $\mathbf{f} : U \to \mathbb{R}^3$ is a real analytic mapping, and that $\mathrm{d}\mathbf{f}(\mathbf{0})$ has one non-zero and two purely imaginary eigenvalues. By an invertible linear change of coordinates and a possibly negative rescaling of time the system of differential equations $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u})$ can be written in the form

$$\begin{aligned} \dot{u} = &\quad -v + \widetilde{P}(u,v,w) = P(u,v,w) \\ \dot{v} = &\quad u + \widetilde{Q}(u,v,w) = Q(u,v,s) \\ \dot{w} = &-\lambda w + \widetilde{R}(u,v,w) = R(u,v,w) \end{aligned} \tag{5.2}$$

where λ is a positive real number. As usual we will let $\mathcal{X}$ denote the corresponding vector field

$$\mathcal{X} = P\tfrac{\partial}{\partial u} + Q\tfrac{\partial}{\partial v} + R\tfrac{\partial}{\partial w} \tag{5.3}$$

on a neighborhood of the origin.

For every $r \in \mathbb{N}$ there exists in a sufficiently small neighborhood of the origin a C^r center manifold W^c, which contains all the recurrent behavior of system (5.2) in a neighborhood of the origin in $\mathbb{R}^3$ ([17, §4.1], [61, §2], [80]). It is not necessarily unique, but the local flows near the origin on any two C^{r+1} center manifolds are C^r conjugate in a neighborhood of the origin ([14]). This fact justifies our abuse of language in speaking below of limit cycles on "the" center manifold.

The following example of V. I. Bulgakov and A. A. Grin' ([13]) illustrates something of the richness of limit cycle bifurcations possible even in low-degree polynomial systems in low dimension. Consider the quadratic three-dimensional system

$$\begin{aligned} \dot{u} &= \alpha u - \beta v + a_2 uv + a_3 v^2 + a_4 uw + a_5 vw \\ \dot{v} &= \beta u + \alpha v - a_2 u^2 - a_3 uv - a_5 uw + a_4 vw \\ \dot{w} &= 2\alpha w + 2a_4 w^2 \,. \end{aligned} \tag{5.4}$$

The system has the infinite family of algebraic invariant surfaces

$$w = \tfrac{1}{k}(u^2 + v^2), \qquad k \in \mathbb{R} \setminus \{0\}. \tag{5.5}$$

Passing to polar coordinates $u = r\cos\varphi$, $v = r\sin\varphi$ we obtain on each algebraic invariant surface the system

$$\begin{aligned} \dot{r} &= \alpha r + \frac{a_4}{k} r^3 \\ \dot{\varphi} &= \beta - \frac{a_5}{k} r^2 - a_2 r\cos\varphi - a_3 r\sin\varphi \,. \end{aligned} \tag{5.6}$$

If we start with system (5.6) with $\alpha = 0$ and $a_4 \neq 0$ and perturb it so as to make α non-zero and of the opposite sign from that of a_4/k then on every surface (5.5) there occurs a Hopf bifurcation that produces a limit cycle emanating from the origin. This process therefore produces a family of systems (5.4) with infinitely many distinct algebraic invariant surfaces with one small limit cycle on each one.

A natural generalization of the concept of cyclicity to higher dimensional systems would be to say that a singular point of an autonomous system of ordinary differential equations has cyclicity k if at most k limit cycles can appear from the singular point under small perturbation and there exist perturbations that yield k limit cycles. With this definition of cyclicity the example of Bulgakov and Grin' shows that the cyclicity of even a quadratic system could be infinite. (There remains, however, the following open problem: *what is an upper bound for the cyclicity of a singular point of a quadratic three-dimensional system restricted to a single center manifold?*)

The following theorem is a generalization of the Poincaré-Lyapunov Theorem (Theorem 2.2.1) given in Subsection 2.2.

Theorem 5.1.1 (Lyapunov Center Theorem)**.** *For system* (5.2) *with corresponding vector field* (5.3) *the origin is a center for* $\mathcal{X}|W^c$ *if and only if* $\mathcal{X}$ *admits a real analytic local first integral of the form* $\Phi(u, v, w) = u^2 + v^2 + \cdots$ *in a neighborhood of the origin in* $\mathbb{R}^3$*. Moreover when there exists a center the local center manifold* W^c *is unique and is analytic.*

We have formulated Theorem 5.1.1 for the three-dimensional case, but in fact Lyapunov studied n-dimensional systems (2.17) in the case that the matrix A has a pair of purely imaginary eigenvalues and the real parts of all the other eigenvalues are negative ([48]). Lyapunov showed that the zero solution of system is asymptotically stable or unstable depending on the sign of a certain constant. In the "transcendental" case ($X+Y \equiv 0$ for X and Y in the normal form (5.9) of the complexification (5.7) of (5.2)) Lyapunov proved the existence of a family of periodic solutions of (2.18) without inferring the analyticity of the corresponding invariant manifold. Yu. N. Bibikov ([9, §13]) did this step, but still in the case where eigenvalues of A with positive real parts are excluded (see [5] for the treatment of the general case).

By Theorem 5.1.1 existence of a center of $\mathcal{X}|W^c$ is equivalent to existence of a first integral for $\mathcal{X}$, so we can restrict our efforts to investigation of conditions for existence of an integral Φ, which can be assumed to have no constant term, hence must have the form $\Phi(u, v, w) =$

$u^2 + v^2 + \cdots$. We further suppose that each of P, Q, and R is a sum of homogeneous polynomials of degrees between 2 and some number N.

We begin by introducing the complex variable $x = u + iv$. Then the first two equations in (5.2) are equivalent to a single equation $\dot{x} = ix + X_1(x, \bar{x}, w)$, where X_1 is a sum of homogeneous polynomials of degrees between 2 and N. As in Subsection 2.4, adjoining to this equation its complex conjugate, replacing $\bar{x}$ everywhere by y, regarding y as an independent complex variable, and replacing w by z simply as a notational convenience we obtain the complexification of family (5.2),

$$\begin{aligned} \dot{x} &= ix + \sum_{p+q+r=2}^{N} a_{pqr} x^p y^q z^r \\ \dot{y} &= -iy + \sum_{p+q+r=2}^{N} b_{pqr} x^p y^q z^r \\ \dot{z} &= -\lambda z + \sum_{p+q+r=2}^{N} c_{pqr} x^p y^q z^r, \end{aligned} \tag{5.7}$$

where $b_{qpr} = \bar{a}_{pqr}$ and the c_{pqr} are such that $\sum_{p+q+r=2}^{N} c_{pqr} x^p \bar{x}^q w^r$ is real for all $x \in \mathbb{C}$, for all $w \in \mathbb{R}$. Let $\mathfrak{Z}$ denote the corresponding vector field on $\mathbb{C}^3$. Existence of a first integral $\Phi(u, v, w) = u^2 + v^2 + \cdots$ for a system in family (5.2) is equivalent to existence of a first integral

$$\Psi(x, y, z) = xy + \sum_{j+k+\ell=3} v_{jk\ell} x^j y^k z^\ell \tag{5.8}$$

for the corresponding system in family (5.7).

We first characterize existence of a formal first integral in terms of normal forms of systems in family (5.7). A direct computation shows that every normal form of a system in family (5.7) has the form

$$\begin{aligned} \dot{x}_1 &= ix_1 + x_1 X(x_1 y_1) \\ \dot{y}_1 &= -iy_1 + y_1 Y(x_1 y_1) \\ \dot{z}_1 &= -\lambda z_1 + z_1 Z(x_1 y_1). \end{aligned} \tag{5.9}$$

We do not assume here that (5.7) arises as the complexification of a real system, hence the coefficients in (5.7) and (5.9) are unrestricted.

The following theorem shows that existence of a first integral (5.8) is related to the fulfillment of the Pliss-Bryuno condition.

Theorem 5.1.2. *A system of the form* (5.7) *admits a formal first integral of the form* (5.8) *if and only if the functions X and Y in some*

normal form (5.9) *satisfy* $X+Y\equiv 0$ *(that is, the system of the first two equations in* (5.9) *satisfies the Pliss-Bryuno condition).*

Proof. Suppose system (5.7) has a formal first integral of the form $\Psi(x,y) = xy + \cdots$. Writing $\mathbf{x} = (x,y,z)$ and $\mathbf{x}_1 = (x_1,y_1,z_1)$, if $\mathbf{x} = \mathbf{H}(\mathbf{x_1})$ is the distinguished normalizing transformation that converts (5.7) into a normal form (5.9) then $F = \Psi \circ \mathbf{H}$ is a formal first integral for the normal form, and from the proof of Lemma 2.3.7 we see that $F(x_1,y_1,z_1) = f(x_1y_1)$, so

$$x_1\frac{\partial F}{\partial x_1}(x_1,y_1,z_1) = x_1y_1f'(x_1y_1)$$

and

$$y_1\frac{\partial F}{\partial y_1}(x_1,y_1,z_1) = x_1y_1f'(x_1y_1).$$

Thus, letting $\zeta = x_1y_1$, we obtain

$$0 \equiv \zeta f'(\zeta)(X(\zeta)+Y(\zeta))\,.$$

But because F is a formal first integral it is not a constant, so we conclude $X(\zeta)+Y(\zeta)\equiv 0$.

Conversely, if $X+Y\equiv 0$ then $\widehat{\Psi}(x_1,y_1,z_1) = x_1y_1$ is a first integral of (5.9). Since the coordinate transformation $\mathbf{x} = \mathbf{x}_1 + \mathbf{h}(\mathbf{x}_1)$ has an inverse of the form $\mathbf{x}_1 = \mathbf{x} + \widehat{\mathbf{h}}(\mathbf{x})$, system (5.7) therefore admits a formal first integral of the form $\Psi(x,y,z) = xy + \cdots$. Q.E.D.

The following statement follows almost immediately from this theorem.

Theorem 5.1.3. *Fix a system* (5.2) *in which the functions* P, Q, *and* R *are real analytic on a neighborhood of the origin. Let* $\mathcal{X}$ *denote the corresponding vector field* (5.3)*. The following statements are equivalent.*

1. *The origin is a center for* $\mathcal{X}|W^c$, W^c *the local center manifold at the origin.*
2. *System* (5.2) *admits a formal first integral.*
3. *System* (5.2) *admits a local analytic first integral.*

Proof. The equivalence of the first and third statements is Theorem 5.1.1. The third statement implies the second. If the second statement holds, then by Theorem 5.1.2 the functions X and Y in any normal form (5.9) of the complexification (5.7) of (5.2) satisfy $X+Y\equiv 0$. In such a case $\Psi(x_1,y_1,z_1) = x_1y_1$ is an analytic first integral of (5.9). In a manner similar to the proof of Lemma 2.3.9 (or see §5 of [9] for a detailed proof) one can show that for family (5.2) the condition $X+Y\equiv 0$

implies that the distinguished normalizing transformation $\mathbf{x} = \mathbf{x}_1 + \mathbf{h}(\mathbf{x}_1)$ that transforms (5.7) into (5.9) is real analytic. Since the normalizing transformation has an analytic local inverse, the analytic integral Ψ yields an analytic integral $\Phi(u, v, w) = u^2 + v^2 + \cdots$ of (5.2). Q.E.D.

Recalling that $\mathfrak{Z}$ denotes the vector field associated to (5.7), we now investigate the existence of a first integral Ψ for a system in family (5.7) by computing the coefficients of $\mathfrak{Z}\Psi$ and equating them to zero. When Ψ has the form (5.8) the coefficient $g_{k_1k_2k_3}$ of $x^{k_1}y^{k_2}z^{k_3}$ in $\mathfrak{Z}\Psi$ is

(5.10)

$$
\begin{aligned}
&(-\lambda k_3 + (k_1 - k_2)i)v_{k_1k_2k_3} \\
&\qquad + a_{k_1,k_2-1,k_3} + b_{k_1-1,k_2,k_3} \\
&\qquad + \sum_{r=0}^{\min\{k_3,N\}} \left[\sum_{\substack{j+k=3+r-k_3 \\ j\geq 1, k\geq 0}}^{k_1+k_2+r-1} j\, a_{k_1-j+1,k_2-k,r}\, v_{j,k,k_3-r} \right] \\
&\qquad + \sum_{r=0}^{\min\{k_3,N\}} \left[\sum_{\substack{j+k=3+r-k_3 \\ j\geq 0, k\geq 1}}^{k_1+k_2+r-1} k\, b_{k_1-j,k_2-k+1,r}\, v_{j,k,k_3-r} \right] \\
&\qquad + \sum_{r=0}^{\min\{k_3,N\}} \left[\sum_{\substack{j+k=2+r-k_3 \\ j\geq 0, k\geq 0}}^{k_1+k_2+r-2} (k_3 - r + 1)\, c_{k_1-j,k_2-k,r}\, v_{j,k,k_3-r+1} \right].
\end{aligned}
$$

The maximum of the sum of the subscripts on $v_{\alpha\beta\gamma}$ in the sums is $k_1 + k_2 + k_3 - 1$. Thus except when $(k_1, k_2, k_3) = (K, K, 0)$ for $K \in \mathbb{N}$, the equation $g_{k_1k_2k_3} = 0$ can be solved uniquely for $v_{k_1k_2k_3}$ in terms of the known quantities $v_{\alpha\beta\gamma}$ with $\alpha + \beta + \gamma < k_1 + k_2 + k_3$. A formal first integral Ψ thus exists if $g_{kk0} = 0$ for all $k \in \mathbb{N}$. An obstruction to the existence of the formal series Ψ occurs when the coefficient g_{KK0} is non-zero. This coefficient is the Kth *focus quantity*,

(5.11)

$$
\begin{aligned}
g_{KK0} = &\sum_{\substack{j+k=2 \\ j\geq 0, k\geq 0}}^{2K-1} [j\, a_{K-j+1,K-k,0} + k\, b_{K-j,K-k+1,0}]\, v_{j,k,0} \\
&+ \sum_{\substack{j+k=2 \\ j\geq 0, k\geq 0}}^{2K-2} c_{K-j,K-k,0}\, v_{j,k,1},
\end{aligned}
$$

where we have incorporated the summands in the second line in (5.10) into the sums by making the natural assignments $v_{110} = 1$ and $v_{\alpha\beta\gamma} = 0$ for $\alpha + \beta + \gamma = 2$ but $(\alpha, \beta, \gamma) \neq (1, 1, 0)$.

Remark. The focus quantity g_{KK0} is obviously a polynomial in the coefficients $a_{\alpha\beta\gamma}$, $b_{\alpha\beta\gamma}$, $c_{\alpha\beta\gamma}$ of (5.7), but contains λ in the denominator of its coefficients. A similar sequence of such quantities could be found based on Theorem 5.1.2 by zeroing coefficients of $X + Y$.

The focus quantities g_{110} and g_{220} are uniquely determined, but the remaining ones depend on the choices made for v_{KK0}, $K \in \mathbb{N}$, $K \geq 2$. Once such an assignment is made Ψ is determined and satisfies

$$\mathfrak{Z}\Psi(x, y, z) = g_{110}\, xy + g_{220}\, (xy)^2 + g_{330}\, (xy)^3 + \cdots . \tag{5.12}$$

The vanishing of all the focus quantities is sufficient for existence of the formal first integral. Using reasoning similar to that used in the proof of Theorem 2.4.4 one can show that it is also necessary by proving that if for one choice of the v_{KK0} at least one focus quantity is non-zero then the same is true for every other choice of the v_{KK0}. Thus the set of all systems in family (5.7) (with a fixed choice of λ) admitting a first integral of the form (5.12) is the variety V_C of the ideal $\langle g_{110}, g_{220}, \ldots \rangle$.

5.2. Periodic Solutions of the Moon-Rand System

We now apply the theory presented in the previous subsection to study a particular family of three-dimensional systems. In [62] (see also Exercise 5 of §5.5 of [46]) Moon and Rand introduced the following system of differential equations, which we call the *Moon-Rand system*, in the context of modelling the control of flexible structures:

$$\begin{aligned} \dot{u} &= \quad v \\ \dot{v} &= -u - uw \\ \dot{w} &= -\lambda w + f(u, v) \end{aligned} \tag{5.13}$$

where

$$f(u, v) = c_{20}u^2 + c_{11}uv + c_{02}v^2 \tag{5.14}$$

or

$$f(u, v) = \frac{c_{11}uv}{1 + \eta u^2}\,.$$

We will limit our consideration to the case of polynomial Moon-Rand systems so in the following when speaking about system (5.13) we mean the system with f defined by (5.14). (A full treatment of both cases can be found in [60].) In (5.13) λ, c_{20}, c_{11}, and c_{02} are real numbers, $\lambda > 0$.

Moon and Rand showed that the origin is asymptotically stable for the flow restricted to the center manifold if

$$2c_{20} - 2c_{02} - \lambda c_{11} < 0.$$

This condition was found by approximating the local center manifold W^c of (5.13), transforming the system restricted to W^c to a normal form by means of an unspecified near-identity transformation, and going over to polar coordinates.

We will give a complete stability analysis of the flow restricted to a neighborhood of the origin in any center manifold. We allow negative values of λ, requiring only that λ be nonzero so that the singularity at the origin be isolated. For the flow on any center manifold the origin is either a center or a fine focus of order up to three (up to four in the rational case). We derive discriminant quantities which specify the order and stability of foci.

We begin with a computation of the lowest order terms of any center manifold. The first three were already found in [62]. The higher order coefficients will be needed only after simplifying conditions apply, so only the simpler versions are listed.

Lemma 5.2.1. *Let a center manifold at the origin of the Moon-Rand system* (5.13) *be expressed as*

$$w = h(u, v) = p_{20}u^2 + p_{11}uv + p_{02}v^2 + \cdots .$$

Then $p_{jk} = 0$ *if* $j + k$ *is odd. In general*

$$\begin{aligned} p_{20} &= \frac{1}{\lambda(\lambda^2+4)}(2c_{02} + 2c_{20} + \lambda c_{11} + \lambda^2 c_{20}) \\ p_{11} &= \frac{1}{(\lambda^2+4)}(2c_{02} - 2c_{20} + \lambda c_{11}) \\ p_{02} &= \frac{1}{\lambda(\lambda^2+4)}(2c_{02} + 2c_{20} - \lambda c_{11} + \lambda^2 c_{02}). \end{aligned} \tag{5.15}$$

When $c_{02} = c_{20} - (\lambda/2)c_{11}$

(5.16)

$$p_{13} = \frac{1}{2\lambda(\lambda^2+4)(\lambda^2+16)}(\lambda c_{11} - 2c_{20})(\lambda(\lambda^2+10)c_{11} - 2(\lambda^2+16)c_{20})$$

$$p_{31} = \frac{1}{\lambda(\lambda^2+4)(\lambda^2+16)}(\lambda c_{11} - 2c_{20})(3\lambda c_{11} - (\lambda^2+16)c_{20}).$$

When $c_{20} = (\lambda/4)c_{11}$ *and* $c_{02} = c_{20} - (\lambda/2)c_{11} = -(\lambda/4)c_{11}$

$$
\begin{aligned}
p_{51} &= \frac{\lambda(2\lambda^4 + 53\lambda^2 + 216)}{2(\lambda^2+16)^2(\lambda^4+40\lambda^2+144)}c_{11}^3 \\
p_{33} &= -\frac{3\lambda(3\lambda^2+8)}{4(\lambda^2+16)(\lambda^4+40\lambda^2+144)}c_{11}^3 \\
p_{15} &= \frac{\lambda(\lambda^4+13\lambda^2+72)}{2(\lambda^2+16)^2(\lambda^4+40\lambda^2+144)}c_{11}^3.
\end{aligned} \tag{5.17}
$$

Proof. The coefficients p_{jk} are found by equating coefficients in the expression that determines the center manifold,

$$h_u\dot{u} + h_v\dot{v} = -\lambda h + c_{20}u^2 + c_{11}uv + c_{02}v^2.$$

When this expression is written out with homogeneous terms in h collected the assertion that $p_{jk} = 0$ if $j + k$ is odd follows by induction.

The first three coefficients listed in the present lemma can be found manually, but the computer algebra system MATHEMATICA was used for the remaining ones. A MATHEMATICA code for this computation can be found in the appendix of [60]. Q.E.D.

Theorem 5.2.2. *Let* $\mathcal{X}$ *denote the vector field determined on* $\mathbb{R}^3$ *by the Moon-Rand system* (5.13)*. Define the discriminant quantities*

$$
\begin{aligned}
W_1 &= 2c_{20} - 2c_{02} - \lambda c_{11} \\
W_2 &= -\lambda(2c_{20} - \lambda c_{11})(4c_{20} - \lambda c_{11}) \\
W_3 &= -\lambda c_{02}^2 c_{11}
\end{aligned}
$$

For any center manifold W^c *of* (5.13) *at the origin of* $\mathbb{R}^3$*, with regard to* $\mathcal{X}|W^c$*:*

a. if $W_1 \neq 0$ *then the origin is a first order fine focus whose stability is determined by* $\operatorname{sgn} W_1$ *(i.e., is asymptotically stable iff* $W_1 < 0$*);*

b. if $W_1 = 0$ *but* $W_2 \neq 0$ *then the origin is a second order fine focus whose stability is determined by* $\operatorname{sgn} W_2$*;*

c. if $W_1 = W_2 = 0$ *but* $W_3 \neq 0$ *then the origin is a third order fine focus whose stability is determined by* $\operatorname{sgn} W_3$*;*

d. if $W_1 = W_2 = W_3 = 0$ *then the origin is a center;*

e. the origin is a center iff $c_{02} = 2c_{20} - \lambda c_{11} = 0$*.*

Proof. We prove part (e) first. The first four nonzero focus quantities were computed by means of the method described above, first complexifying (5.13) and then computing as described earlier. The first

two nonzero focus quantities, for example, are

$$g_{220} = \frac{2c_{20} - 2c_{02} - c_{11}\lambda}{4 + \lambda^2}$$
$$g_{330} = \frac{(c_{20} + c_{02})\big[c_{11}\lambda(12 + \lambda^2) - 2c_{02}(-4 + \lambda^2) - 2c_{20}(12 + \lambda^2)\big]}{4\lambda(4 + \lambda^2)^2}.$$

All were computed using MATHEMATICA. The code is given in the appendix of [60].

Let $\widetilde{g}_{kk0}$ be the numerator of g_{kk0} and $\mathcal{B}_k = \langle \widetilde{g}_{220}, \widetilde{g}_{330}, \ldots, \widetilde{g}_{kk0} \rangle$. Using Theorem 1.4.3 we find that $\sqrt{\mathcal{B}_4} \subsetneqq \sqrt{\mathcal{B}_5}$ but $\sqrt{\mathcal{B}_5} = \sqrt{\mathcal{B}_6}$. Since as indicated in Subsections 1.1 and 1.3 it is the radical $\sqrt{I}$ of an ideal I that in fact determines the corresponding variety $\mathbf{V}(I)$, this computation suggests that $V_C = \mathbf{V}(\mathcal{B}_5)$. Using the routine `minAssGTZ` ([28]) of the computer algebra system SINGULAR to decompose the radical of $\mathcal{B}_5$ into a unique intersection of prime ideals, we obtain the irreducible decomposition of the variety $\mathbf{V}(\mathcal{B}_5)$ as the union of three components $\mathbf{V}(J_j)$, where the ideals J_j are:

$$J_1 = \langle c_{02}, -\lambda c_{11} + 2c_{20} - 2c_{02} \rangle$$
$$J_2 = \langle c_{11}^2 + 16c_{02}^2, 4\lambda c_{02} - c_{11}, \lambda c_{11} + 4c_{02}, \lambda^2 + 1, -\lambda c_{11} + 2c_{20} - 2c_{02} \rangle$$
$$J_3 = \langle \lambda^2 + 4, -\lambda c_{11} + 2c_{20} - 2c_{02} \rangle.$$

Since system (5.13) is real the components $\mathbf{V}(J_2)$ and $\mathbf{V}(J_3)$ are irrelevant; we get the necessary conditions

$$c_{02} = 2c_{20} - \lambda c_{11} = 0$$

for the origin to be a center for $\mathcal{X}|W^c$.

When these conditions hold it is not too difficult to find the algebraic surface

$$F(u, v, w) := c_{20}u^2 - \lambda w = 0$$

as an invariant surface (with cofactor $-\lambda$) for the flow associated to system (5.13). Since it is tangent to the plane $w = 0$ at the origin it is in fact a center manifold for this system. Using this explicit expression for a center manifold W^c we find that the dynamics on W^c are given by

$$\dot{u} = \quad v$$
$$\dot{v} = -u - \frac{c_{20}}{\lambda}u^3.$$

This system is Hamiltonian with the Hamiltonian function

$$H(u, v) = \frac{1}{2}(u^2 + v^2) + \frac{c_{20}}{4\lambda}u^4$$

so it admits a center at the origin. Thus the condition in part (e) is also sufficient, and part (e) is established.

Now we turn to parts (a) through (d), in order. The system $\mathcal{X}|W^c$ is

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= -u - uh(u,v) \end{aligned} \tag{5.18}$$

for $h(u,v) = \sum p_{jk} u^j v^k$ whose first few coefficients are given in Lemma 5.2.1. We find the first few Lyapunov quantities as described in Subsection 2.1. In polar coordinates $u = r\cos\theta$, $v = r\sin\theta$ system (5.18) is

$$\begin{aligned} \dot{r} &= -r\cos\theta\sin\theta\, h(r\cos\theta, r\sin\theta) = -\sum \alpha_j r^j \\ \dot{\theta} &= -1 - \cos^2\theta\, h(r\cos\theta, r\sin\theta) = -1 - \sum \beta_j r^j, \end{aligned} \tag{5.19}$$

where

$$\begin{aligned} \alpha_j(\theta) &= \cos\theta\sin\theta \left(\sum\nolimits_{k+\ell=j-1} p_{k\ell}\cos^k\theta\sin^\ell\theta\right), \quad j \geq 3 \\ \beta_j(\theta) &= \cos^2\theta \left(\sum\nolimits_{k+\ell=j} p_{k\ell}\cos^k\theta\sin^\ell\theta\right), \quad j \geq 2. \end{aligned}$$

Since $p_{k\ell} = 0$ when $k+\ell$ is odd, $\alpha_{2n} = \beta_{2n+1} = 0$. Then

$$\frac{dr}{d\theta} = \sum R_j(\theta) r^j \tag{5.20}$$

where

$$\begin{aligned} &R_1 = 0 && R_2 = 0 && R_3 = \alpha_3 \\ &R_4 = 0 && R_5 = \alpha_5 - \alpha_3\beta_2 && R_6 = 0 \\ &R_7 = \alpha_7 - \alpha_5\beta_2 - \alpha_3\beta_4 + \alpha_3\beta_2^2 && && R_8 = 0 \\ &R_9 = \alpha_9 - \alpha_7\beta_2 + \alpha_5\beta_2^2 - \alpha_3\beta_2^3 - \alpha_5\beta_4 + 2\alpha_3\beta_2\beta_4 - \alpha_3\beta_6 \end{aligned}$$

It is important to note that since $\dot{\theta} < 0$ near 0 (arising from the noncanonical location of the minus sign in the linear part of (5.13) that gives rise to the complex eigenvalues; compare with (2.11)), in passing from (5.19) to (5.20) the direction of time is essentially reversed: as θ increases from 0 to 2π time t decreases. Therefore the usual polar coordinate procedure for computing the Lyapunov quantities gives the negatives of the correct values. The *negative* of the first Lyapunov quantity, $-\eta_1$, is $w_1(2\pi) - 1$ and for $j \geq 2$ the *negative* of the jth Lyapunov quantity, $-\eta_j$, is $w_j(2\pi)$, where the w_j are the solutions of the

differential equations that arise from

$$\sum w_j'(\theta) r_0^j = \sum R_j(\theta)\,(w_1(\theta)\,r_0 + w_2(\theta)\,r_0^2 + \cdots)^j \tag{5.21}$$

with initial values

$$w_1(0) = 1, \quad w_j(0) = 0 \quad \text{for} \quad j > 1.$$

In particular,

$$w_1' = R_1 w_1 = 0, \quad w_1(0) = 1$$

yields $w_1(\theta) \equiv 1$, so that $\eta_1 = 0$, and

$$w_2' = R_2 w_1^2 = 0, \quad w_2(0) = 0$$

yields $w_2(\theta) \equiv 0$, so that $\eta_2 = 0$. Then

$$w_3' = R_3 = \alpha_3, \quad w_3(0) = 0$$

yields

$$w_3(\theta) = \tfrac{1}{4}p_{20} + \tfrac{1}{8}p_{11}\theta - \tfrac{1}{4}p_{20}\cos^4\theta + \tfrac{1}{4}p_{02}\sin^4\theta - \tfrac{1}{32}p_{11}\sin 4\theta$$

so that

$$\eta_3 \sim -p_{11} \sim 2c_{20} - 2c_{02} - \lambda c_{11}$$

where $a \sim b$ means that a is a positive constant times b. This establishes point (a).

From (5.21) $w_4' = 0$ so $w_4 \equiv 0$, hence $\eta_4 \equiv 0$. It is apparent that $w_4 = 0$ in turn implies that $w_6' = 0$ so $w_6 \equiv 0$, hence $\eta_6 \equiv 0$ (and $w_4 = w_6 = 0$ imply that $w_8' = 0$ so $w_8 \equiv 0$, hence $\eta_8 \equiv 0$).

From (5.21)

$$w_5' = 3R_3 w_1(w_2^2 + w_1 w_3) + 4R_4 w_1^3 w_2 + R_5 w_1^5 = 3R_3 w_3 + R_5.$$

At this point hand computations are infeasible and intermediate results too long to copy here. Proceeding on the assumption that $\eta_3 = 0$, so that $p_{11} = 0$, which we implement by the substitution $c_{02} = c_{20} - (\lambda/2)c_{11}$, we use MATHEMATICA to compute w_5 and obtain $\eta_5 \sim p_{31} + p_{13}$, which by (5.16) gives

$$\eta_5 = -\frac{(2c_{20} - \lambda c_{11})(4c_{20} - \lambda c_{11})}{2\lambda(4 + \lambda^2)},$$

which has the sign of W_2. Since $\eta_4 \equiv 0$ (or appealing to the fact that the first non-zero Lyapunov quantity has odd index) this establishes part (b).

If $\eta_5 = 0$ because $2c_{20} - \lambda c_{11} = 0$ then $\eta_3 = 0$ implies that $c_{02} = 0$ as well, and by part (e) of the theorem the origin is a center. The factor c_{02}^2 in W_3 makes $W_3 = 0$ in this case. We proceed on the assumption that $\eta_5 = 0$ but $2c_{20} - \lambda c_{11} \neq 0$. Thus $4c_{20} - \lambda c_{11} = 0$ and (from $\eta_3 = 0$) $c_{02} = c_{20} - (\lambda/2)c_{11} \neq 0$. From (5.21) and what we already know we have

$$w_7' = 3R_3(w_3^2 + w_5) + 4R_4 w_4 + 5R_5 w_3 + R_7,$$

which with the initial condition gives

$$\eta_7 = w_7(2\pi) \sim -5p_{15} - 5p_{51} - 3p_{33} + p_{02}p_{31} + 5p_{20}p_{31} \sim -\lambda c_{11}^3.$$

Since $c_{02} \neq 0$, η_7 is zero iff $W_3 = -\lambda c_{02}^2 c_{11}$ is zero, and has the same sign as W_3 when they are nonzero. Since $\eta_6 \equiv 0$ this establishes (c). (In fact, in the case at hand $\eta_7 \neq 0$, for since we have assumed $W_1 = W_2 = 0$, $c_{11} = 0$ would ultimately imply that $c_{02} = 0$. What is shown in the last display is not η_7, but η_7 under certain restrictive conditions.) Recalling the comment above that if η_5 vanishes because $2c_{20} - \lambda c_{11} = 0$ then $W_3 = 0$ is forced, point (d) holds as well. Q.E.D.

Since the first non-zero Lyapunov quantity is known under every circumstance, bifurcation of limit cycles from the origin can also be discussed. Uncertainty as to the analyticity of the center manifold prevents us from being able to assert sharpness of the bounds, hence the cyclicity of the singularity in the center manifold, in the case of bifurcations from a center. On the other hand, there always exists a local center manifold of arbitrarily high smoothness ([80]) and, as mentioned earlier, the flows on any two C^{k+1} center manifolds are C^k conjugate on a neighborhood of the origin ([14]) so that all contain the same number of small cycles. Thus the statements in the following theorem do not depend on the center manifold selected, if there is more than one. We also remark that, even though center manifolds may not be analytic, if there are infinitely many cycles in a neighborhood of the origin on a center manifold, then the origin is a center on the manifold ([5]).

Theorem 5.2.3. *For family* (5.13) *restrict attention to the flow on a center manifold at the origin.*

a. A first order fine focus at the origin has cyclicity zero: no limit cycles bifurcate from it under small perturbation within (5.13).

b. For $k = 2$ and $k = 3$, a fine focus of order k at the origin has cyclicity $k - 1$: up to $k - 1$ limit cycles can be made to bifurcate under small perturbation within the family (5.13).

c. In the case $c_{20} = c_{11} = c_{02} = 0$ the center on the center manifold $w = 0$ can be made to bifurcate two limit cycles. Otherwise the

center on the center manifold can be made to bifurcate one limit cycle.

Proof. A first order fine focus at the origin has cyclicity zero because η_1 is always zero.

To abbreviate the notation write $\xi = (\lambda, c_{20}, c_{11}, c_{02}) \in \mathbb{R}^4$ for the parameters. To obtain an upper bound on the number of limit cycles that can bifurcate from a fine focus of order two or three let us suppose to be specific that for some ξ^* the origin is a fine focus of order three and consider the system restricted to a center manifold that is C^r for $r \geq 8$. Fix a neighborhood N of ξ^* on which $\eta_7 = \eta_7(\xi) \neq 0$ and an interval $I = [0, \epsilon)$ so that the difference map $\mathscr{D} = \mathscr{D}(r, \xi)$ is defined on $I \times N$. Then since as seen in the proof of Theorem 5.2.2 η_4 and η_6 are identically zero

$$\mathscr{D}(r, \xi) = \eta_3(\xi) r^3 + \eta_5(\xi) r^5 + \eta_7(\xi) r^7 + R(r, \xi),$$

where R is at least C^7 and $R^{(j)}(0, \xi) = 0$ for $0 \leq j \leq 7$.

Divide $\mathscr{D}$ by r^3 (defined at zero by the limit) and differentiate with respect to r to obtain the function

$$\mathscr{D}_1(r, \xi) = 2\eta_5(\xi) r + 4\eta_7(\xi) r^3 + R_1(r, \xi),$$

where R_1 is at least C^3 and $R_1^{(j)}(0, \xi) = 0$ for $0 \leq j \leq 3$. Either $\mathscr{D}$ and $\mathscr{D}_1$ both have infinitely many zeros on $(0, \epsilon)$ or $\mathscr{D}$ has at most one more zero on $(0, \epsilon)$ than $\mathscr{D}$ does.

Divide $\mathscr{D}_1$ by r (defined at zero by the limit) and differentiate with respect to r to obtain the function

$$\mathscr{D}_2(r, \xi) = 8\eta_7(\xi) r + R_2(r, \xi)$$

where R_2 is at least C^1 and $R_2^{(j)}(0, \xi) = 0$ for $0 \leq j \leq 1$. Either $\mathscr{D}_1$ and $\mathscr{D}_2$ both have infinitely many zeros on $(0, \epsilon)$ or $\mathscr{D}_1$ has at most one more zero on $(0, \epsilon)$ than $\mathscr{D}_2$ does.

Divide $\mathscr{D}_2$ by r (defined at zero by the limit) to obtain the function

$$\mathscr{D}_3(r, \xi) = 8\eta_7(\xi) + R_3(r, \xi)$$

where R_3 is continuous and $R_3(0, \xi) = 0$. For ξ in a neighborhood $N' \subset N$ of ξ^* and an $\epsilon' \in (0, \epsilon)$, $\mathscr{D}_3(r, \xi)$ has no zeros in $(0, \epsilon')$. Thus $\mathscr{D}(r, \xi)$ has at most two zeros in $(0, \epsilon')$ for all $\xi \in N'$.

In the same way a second order fine focus can be made to bifurcate at most one limit cycle.

The bounds are sharp because, as shown in the proof of Theorem 5.2.2, the Lyapunov number η_3, η_5, and η_7 can be adjusted independently. To see what we mean, suppose the origin is a third order fine focus, so that $\eta_j = 0$ for $1 \leq j \leq 6$ but $\eta_7 \neq 0$. We must have $4c_{20} - \lambda c_{11} = 0$ but $2c_{20} - \lambda c_{11} \neq 0$ else by $\eta_3 = 0$, $c_{02} = 0$ is forced and the singularity is a center. Moving c_{11} by an arbitrarily small amount in the correct direction, leaving c_{20} unchanged, but maintaining $c_{02} = c_{20} - \lambda c_{11}/2$ makes the sign of η_5 change to the opposite sign of that of η_7 but maintains $\eta_3 = 0$. A zero of the difference map on a section of the flow near the origin is produced, corresponding to a limit cycle. Then c_{20} or c_{02} can be moved by an arbitrarily small amount to create a second limit cycle. A single limit cycle can be produced similarly from a second order fine focus.

If the origin is a center, then unless $c_{20} = c_{11} = c_{02} = 0$ there is no third order fine focus near it and the same technique produces one limit cycle. When $c_{20} = c_{11} = c_{02} = 0$ one can first make an arbitrarily small perturbation to a third order fine focus, and from there produce two limit cycles. Q.E.D.

§6. Epilogue: Planar Quadratic Systems

The discussion in previous sections illustrate the fact that in problems of the type that we have been considering computations quickly become intractable, even when we apply such techniques as performing special coordinate transformations to reduce the number of parameters and computing using modular arithmetic followed by rational reconstruction. All too frequently, unless the system at hand is a restricted one that arises in an application, such as the Moon-Rand family treated in Subsection 5.2, resort must be made to restricting the study to special cases. Even so the results obtained can be excessively complicated, with a number of individual cases, each of which can involve many large polynomials.

The one "naturally occurring" general family of system of ordinary differential equations on the plane for which the center problem and the problem of the cyclicity of a singularity of focus or center type have a fully satisfactory answer is the family of quadratic systems, systems $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u})$ of the form

$$\dot{u} = P(u, v), \quad \dot{v} = Q(u, v),$$

where P and Q are polynomial functions (usually restricted to being coprime) for which $\max\{\deg P, \deg Q\} = 2$. In this section we draw together many of the ideas that we have discussed to give a complete

solution of the center problem and to derive a sharp upper bound on the cyclicity of a singularity of focus or center type. Our goal in this final section is both to highlight a major accomplishment of the theory and to bring together in one section a number of ideas that have been developed separately in earlier sections, in the process giving a concrete demonstration of how all the pieces are implemented and how they fit together into one cohesive whole.

An isolated singularity of a quadratic system, which by a translation of the coordinate system we may assume to be located at the origin, is of focus or center type only if it is *simple* or *nondegenerate*, meaning that the Jacobian determinant $\det \mathrm{d}\mathbf{f}(\mathbf{0}) \neq 0$ (e.g., Lemma 6.3.2 of [70]). Thus by a linear change of coordinates and a time rescaling the system can be placed in the form

$$\begin{aligned} \dot{u} &= \lambda u - v + A_{20}u^2 + A_{11}uv + A_{02}v^2 \\ \dot{v} &= u + \lambda v + B_{20}u^2 + B_{11}uv + B_{02}v^2 \,. \end{aligned} \tag{6.1}$$

We regard (6.1) as a family of systems parametrized by the coefficients A_{jk}, B_{jk}.

The complexification of (6.1) is

$$\begin{aligned} \dot{x} &= (\lambda + i)x - a_{20}x^2 - a_{11}xy - a_{02}y^2 \\ \dot{y} &= (\lambda - i)y + b_{20}x^2 + b_{11}xy + b_{02}y^2 \,, \end{aligned} \tag{6.2}$$

where we have rescaled by $-i$ and where the a_{jk} and b_{jk} are complex numbers that satisfy $b_{kj} = \bar{a}_{jk}$.

6.1. The Center Problem

To solve the center problem for (6.1) we first solve it for family (6.2), with $\lambda = 0$ of course, and then use the linear isomorphism between the parameters

$$(A, B) := (A_{20}, A_{11}, A_{02}, B_{20}, B_{11}, B_{02})$$

and the parameters

$$(a, b) := (a_{20}, a_{11}, a_{02}, b_{20}, b_{11}, b_{02})$$

to obtain necessary and sufficient conditions for a center at the origin for family (6.1). In fact we solve the center problem for the general family (6.2) without the conditions $b_{kj} = \bar{a}_{jk}$ that are satisfied by any member of the family that is the complexification of a real system of the form (6.1). The characterization of centers for members of (6.1) is then obtained by first applying the conditions $b_{kj} = \bar{a}_{jk}$ and then the linear isomorphism just mentioned.

The first step in the process is to compute the first few focus quantities g_{kk}, $k \in \mathbb{N}$ for (6.2) (Definition 2.4.2 and (2.48) with $p = q = 1$), and at each stage use the Radical Membership Test (Theorem 1.4.3) to check whether or not g_{kk} is in the ideal generated by the previous focus quantities. The computation of the focus quantities and the test are both completely algorithmic. (A MATHEMATICA code for computing the focus quantities for two-dimensional systems appears in the appendix of [70].) The first three focus quantities are, after a reduction of the second modulo the first and a reduction of the third modulo the first two,

$$\begin{aligned}
g_{11} &= -i(a_{20}a_{11} - b_{02}b_{11}) \\
g_{22} &= -i(a_{20}a_{02}b_{11}^2 - b_{02}b_{20}a_{11}^2 - \tfrac{2}{3}(a_{02}b_{11}^3 - b_{20}a_{11}^3) \\
&\quad - \tfrac{2}{3}(a_{11}b_{02}^2 b_{20} - b_{11}a_{20}^2 a_{02})) \\
g_{33} &= i\tfrac{5}{8}(-a_{11}a_{02}b_{11}^4 + 2a_{02}b_{02}b_{11}^4 + a_{11}^4 b_{11}b_{20} \\
&\quad - 2a_{11}^3 b_{02}b_{11}b_{20} - 2\,a_{20}\,a_{02}^2\,b_{11}^2\,b_{20} + a_{02}^2\,b_{11}^3\,b_{20} \\
&\quad - a_{11}^3\,a_{02}\,b_{20}^2 + 2\,a_{11}^2\,a_{02}\,b_{02}\,b_{20}^2).
\end{aligned}$$

We find that $g_{22} \notin \sqrt{\langle g_{11}\rangle}$ and $g_{33} \notin \sqrt{\langle g_{11}, g_{22}\rangle}$ but that, for as large a $k \geq 4$ as we care to check, $g_{kk} \in \sqrt{\langle g_{11}, \ldots, g_{33}\rangle}$. Since for any ideal I, $\mathbf{V}(I) = \mathbf{V}(\sqrt{I})$, this suggests that

$$\mathbf{V}(\langle g_{kk} : k \in \mathbb{N}\rangle) = \mathbf{V}(g_{11}, g_{22}, g_{33}),$$

that is, that

$$\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_3),$$

meaning that the center variety is the variety of the ideal $\mathcal{B}_3$. We know that the inclusion

$$\mathbf{V}(\langle g_{kk} : k \in \mathbb{N}\rangle) \subset \mathbf{V}(g_{11}, g_{22}, g_{33})$$

holds. To establish the reverse inclusion we must show that for any $(a^*, b^*) \in \mathbf{V}(g_{11}, g_{22}, g_{33})$, the corresponding system (6.2) has a center at the origin, that is, that it admits a local first integral of the form $\Psi(x, y) = xy + \cdots$. This task is greatly simplified, one would even say made feasible, by decomposing the variety $\mathbf{V}(g_{11}, g_{22}, g_{33})$ into its irreducible components. To do so we use one of the SINGULAR routines `primdecGTZ` or `primdecSY` in the primdec library to decompose the ideal $\langle g_{11}, g_{22}, g_{33}\rangle$ into an intersection of primary (or perhaps prime) ideals. Doing so we obtain

$$\mathcal{B}_3 = J_1 \cap J_2 \cap J_3 \cap J_4,$$

where

$$
\begin{aligned}
J_1 &= \langle a_{11},\ b_{11} \rangle\,, \\
J_2 &= \langle 2a_{11} + b_{02},\ a_{20} + 2b_{11},\ a_{11}b_{11} - a_{02}b_{20} \rangle\,, \\
J_3 &= \langle 2a_{20} - b_{11},\ 2b_{02} - a_{11} \rangle\,, \\
J_4 &= \langle f_1, f_2, f_3, f_4, f_5 \rangle\,,
\end{aligned}
\tag{6.3}
$$

where

$$
\begin{aligned}
f_1 &= a_{11}^3 b_{20} - a_{02} b_{11}^3\,, \\
f_2 &= a_{20}a_{11} - b_{02}b_{11}\,, \\
f_3 &= a_{20}^3 a_{02} - b_{20} b_{02}^3\,, \\
f_4 &= a_{20}a_{02}b_{11}^2 - a_{11}^2 b_{20}b_{02}\,, \\
f_5 &= a_{20}^2 a_{02} b_{11} - a_{11} b_{20} b_{02}^2.
\end{aligned}
\tag{6.4}
$$

and from the output learn as well that each ideal J_k is prime. Therefore $\mathcal{B}_3$ is a radical ideal and the irreducible decomposition of the center variety of family (6.2) is

$$
\mathbf{V}(B) = \mathbf{V}(J_1) \cup \mathbf{V}(J_2) \cup \mathbf{V}(J_3) \cup \mathbf{V}(J_4). \tag{6.5}
$$

Now we attempt to prove that for each k, for every $(a^*, b^*) \in \mathbf{V}(J_k)$, the system in family (6.2) with parameters (a^*, b^*) has a center at the origin, that is, has a local first integral of the form $\Psi(x, y) = xy + \cdots$.

For every element of $\mathbf{V}(J_3)$ the corresponding system (6.2) is hamiltonian, hence has a center.

Making the obvious modifications in Subsection 3.3 for the two-dimensional situation, the ideal J_4 is the Sibirsky ideal I_S of Theorem 3.3.2, hence by Corollary 3.3.3 for every element of $\mathbf{V}(J_4)$ the corresponding system (6.2) has a center.

For the remaining two components of $\mathbf{V}(\mathcal{B}_3)$ we look for algebraic partial integrals in order to construct either a Darboux first integral or a Darboux integrating factor.

Suppose $(a^*, b^*) \in \mathbf{V}(J_1)$. By a straightforward computation searching for an invariant line and its cofactor, we find that when $b_{20} \neq 0$ (an artifact of the method of proof) a line $f(x, y) = 1 + rx + sy$ not passing through the origin is invariant if r is a root of the cubic equation

$$
r^3 + 2a_{20}r^2 + (a_{20}^2 + b_{20}b_{02})r + (a_{20}b_{20}b_{02} - a_{02}b_{20}^2) = 0
$$

and $s = (r^2 + a_{20}r)/b_{20}$. If $C(a, b)$ is the constant term of the cubic and $D(a, b)$ is the discriminant of the cubic (which is a homogeneous

polynomial of degree four in a_{20}, a_{02}, b_{20}, and b_{02}) then for $(a,b) \notin \mathbf{V}(D)$ the cubic has three distinct roots r_1, r_2, r_3 ([82]), and for $(a,b) \notin \mathbf{V}(C)$ none of them is zero. This gives three invariant lines f_k, whose cofactors are

$$K_k = r_k x - s_k y.$$

The condition in Theorem 3.2.5 is the pair of linear equations

$$\begin{aligned} r_1\alpha_1 + r_2\alpha_2 + r_3\alpha_3 &= 0 \\ s_1\alpha_1 + s_2\alpha_2 + s_3\alpha_3 &= 0 \end{aligned}$$

in the three unknowns α_1, α_2, and α_3, and has infinitely many non-trivial solutions. By Theorem 3.2.5 there exists a Darboux first integral $H(x,y) = f_1^{\alpha_1} f_1^{\alpha_1} f_1^{\alpha_1}$. With some extra work ([70, §3.7]) one can show that $r_1\alpha_1 + r_2\alpha_2 + r_3\alpha_3$ is zero only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$ so that

$$\Psi(x,y) = (r_1\alpha_1 + r_2\alpha_2 + r_3\alpha_3)^{-1}(1 - H(x,y))$$

is a first integral of the required form $\Psi(x,y) = xy + \cdots$.

Any system that corresponds to an element of $\mathbf{V}(J_1) \setminus \mathbf{V}(C\,D\,b_{20})$ is therefore integrable because of the presence of three invariant lines in its phase portrait. Irreducibility of $\mathbf{V}(J_1)$ and the fact that

$$\mathbf{V}(J_1) \cap \mathbf{V}(C\,D\,b_{20})$$

is a proper subset of $\mathbf{V}(J_1)$ imply (Exercise 1.45 in [70]) that

$$\overline{\mathbf{V}(J_1) \setminus \mathbf{V}(C\,D\,b_{20})} = \mathbf{V}(J_1),$$

so that by Lemma 2.5.1 for every element of $\mathbf{V}(J_1)$ the corresponding system in (6.2) is integrable.

Suppose $(a^*, b^*) \in \mathbf{V}(J_2)$. Here a straightforward computation with any popular computer algebra system leads to the algebraic partial integrals

$$\begin{aligned} f_1 =& (2\,b_{11}\,b_{20}^2)^{-1}\Big[2\,b_{11}\,b_{20}^2 + 6\,b_{11}^2\,b_{20}^2\,x + 6\,a_{11}\,b_{11}\,b_{20}^2\,y \\ &+ 3\,b_{11}\,b_{20}^2(b_{11}^2 - a_{11}\,b_{20})\,x^2 \\ &+ 3\,b_{20}(2\,a_{11}\,b_{11}^2\,b_{20} - b_{11}^4 - a_{11}^2\,b_{20}^2)\,x\,y \\ &+ 3\,a_{11}\,b_{11}\,b_{20}(a_{11}\,b_{20} - b_{11}^2)\,y^2 \\ &+ a_{11}\,b_{20}^3(a_{11}\,b_{20} - b_{11}^2)\,x^3 + 3\,a_{11}\,b_{11}\,b_{20}^2(b_{11}^2 - a_{11}\,b_{20})\,x^2\,y \\ &+ 3\,a_{11}\,b_{11}^2\,b_{20}(a_{11}\,b_{20} - b_{11}^2)\,x\,y^2 + a_{11}\,b_{11}^3(b_{11}^2 - a_{11}\,b_{20})\,y^3\Big] \end{aligned}$$

and

$$f_2 = 1 + 2\,b_{11}\,x + 2\,a_{11}\,y - a_{11}\,b_{20}\,x^2 + 2\,a_{11}\,b_{11}\,x\,y - \frac{a_{11}\,{b_{11}}^2}{b_{20}}\,y^2$$

with respective cofactors

$$K_1(x,y) = 3\,(b_{11}\,x - a_{11}\,y)$$

and

$$K_2 = 2\,(b_{11}\,x - a_{11}\,y)\,.$$

By inspection $2\,K_1 - 3\,K_2 \equiv 0$, hence by Theorem 3.2.5 a Darboux first integral is $H(x,y) = f_1^2 f_2^{-3}$ which exists for the system that corresponds to any element of $\mathbf{V}(J_2) \setminus \mathbf{V}(b_{11}\,b_{20})$. For $h = 2a_{11}b_{11}^2b_{20} + b_{11}^4 + a_{11}^2b_{20}^2$,

$$H(x,y) = 1 - 3(b_{11}\,b_{20})^{-1}hxy + \cdots$$

so that

$$\Psi = b_{20}b_{11}(3h)^{-1}(1 - H) = xy + \cdots$$

is a first integral of the required form for systems corresponding to elements of $\mathbf{V}(J_2) \setminus \mathbf{V}(b_{20}b_{11}\,h)$. The exact same reasoning as in the conclusion of the previous case shows that for every element of $\mathbf{V}(J_2)$ the corresponding system in (6.2) is integrable.

We have thus proved the following theorem, which is the solution of the center problem for quadratic systems.

Theorem 6.1.1. *An element of the family* (6.2) *with* $\lambda = 0$ *has a center at the origin if and only if the coefficients* (a,b) *lie in at least one of the irreducible varieties* $\mathbf{V}(J_1)$, $\mathbf{V}(J_2)$, $\mathbf{V}(J_3)$, *and* $\mathbf{V}(J_4)$, *where the ideals* J_1, J_2, J_3, *and* J_4 *are given by* (6.3) *and* (6.4).

It is important to note that not only have we found the center conditions, but the proof shows the mechanism by which the center arises in each of the four sets of conditions. The variety $\mathbf{V}(J_1)$ is the Zariski closure of the set of systems that have three invariant lines in their phase portraits, $\mathbf{V}(J_2)$ is the Zariski closure of the set of systems that have an invariant conic and an invariant cubic in their phase portraits, $\mathbf{V}(J_4)$ is the Zariski closure of the set of systems that are time-reversible, and the variety $\mathbf{V}(J_1)$ corresponds to systems that are hamiltonian. The Zariski closure operation is essential. In fact there are systems in $\mathbf{V}(J_1)$ that do not have three invariant lines in their phase portraits, and similarly in the case of $\mathbf{V}(J_2)$ and $\mathbf{V}(J_4)$.

Using the linear isomorphism that connects the coefficients (a,b) of the complexification (6.2) (which, we recall, as the complexification of a

real system satisfy $b_{kj} = \bar{a}_{jk}$) with the coefficients (A, B) of the original real system (6.1) yields the center conditions for (6.1), of course with $\lambda = 0$. It was observed by Kapteyn that there exists a rotation that places (6.1) in the form (using notation that is traditional)

$$\begin{aligned} \dot{u} &= -v - bu^2 - Cuv - dv^2 \\ \dot{v} &= \quad u + au^2 + Auv - av^2. \end{aligned}$$

Setting $\alpha = A - 2b$ and $\beta = C + 2a$, in these coordinates a necessary and sufficient condition that there be a center at the origin is that at least one of the following three conditions holds:

(1) $b + d = 0$
(2) $a\,\alpha = \beta = 0$
(3) $\alpha + 5(b + d) = a^2 + bd + 2d^2 = \beta = 0.$

6.2. The Cyclicity Problem

As mentioned at the beginning of this section a focus or a center of any quadratic system must be simple, hence the system has the form (6.1). If $\lambda \neq 0$ then the singularity is hyperbolic, hence the system is locally structurally stable and no bifurcation can occur: the cyclicity is zero. Alternatively, we can compute the first Lyapunov quantity as $\eta_1 = e^{2\pi\lambda} - 1$ which, if the original value of λ was nonzero, is nonzero originally and remains nonzero under sufficiently small perturbation of the coefficients in (6.1). The following theorem covers the remaining cases. The work done in the previous subsection in solving the center problem makes the result almost immediate.

Theorem 6.2.1. *The cyclicity of a focus or center of a quadratic system is at most three.*

Proof. As was just pointed out the system must be capable of being written in the form (6.1), and the only case of interest is $\lambda = 0$.

Because each of the first three focus quantities listed in the previous subsection has been reduced modulo the previous ones, the three of them must form the minimal basis of $\mathcal{B}_3$. We also saw in the previous subsection that $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_3)$ and that the ideal $\mathcal{B}_3$, as the intersection of the four prime ideals J_k, is a radical ideal. The result then follows from Theorem 4.3.1. Q.E.D.

Heretofore we have not addressed the issue of the sharpness of the upper bounds on the cyclicity obtained by the methods that we have described. In general this question must be addressed on a case by case basis. In the situation of a fine focus of order k, so that (2.9) holds, the usual procedure is to attempt to adjust the coefficients of the system

in question so as to successively make the Lyapunov quantities of odd index nonzero one by one, starting with η_{2k-1}, continuing with η_{2k-3}, η_{2k-5}, and so on, and with alternating signs, thus successively changing the stability of the origin and creating limit cycles by an exchange of stability. The procedure is illustrated in the next to last paragraph of the proof of Theorem 5.2.3. In the case of a center one can first perturb to a nearby system with a fine focus and apply the same procedure, as was illustrated in the last paragraph of the proof of Theorem 5.2.3. In this manner it can be shown, for example, that the bound in Theorem 6.2.1 is sharp: there exist both centers and (third order) fine foci in quadratic systems from which three small cycles can be made to bifurcate under arbitrarily small perturbation of the coefficients. A proof can be found in many places; one expressed in the complex form of the real system (6.1) with $\lambda = 0$ is given in Section 6.3 of [70] (proof of Theorem 6.3.3).

Another method that in some cases allows one to obtain the precise number of limit cycles bifurcating from generic points of the components of the center variety is given by Theorem 2.1 of [21].

Acknowledgments

Valery Romanovski acknowledges the support by the Slovenian Research Agency and by a Marie Curie International Research Staff Exchange Scheme Fellowship within the 7th European Community Framework Programme, FP7-PEOPLE-2012- IRSES-316338.

References

[1] W. W. Adams and P. Loustaunau. *An Introduction to Gröbner Bases*. Graduate Studies in Mathematics, Vol. 3. Providence, RI: American Mathematical Society, 1994.

[2] V. V. Amel'kin, N. A. Lukashevich, and A. P. Sadovskii. *Nonlinear Oscillations in Second Order Systems*. Minsk: Belarusian State University, 1982.

[3] E. A. Arnold. Modular algorithms for computing Gröbner bases. *J. Symbolic Comput.* **35** (2003) 403–419.

[4] V. I. Arnold. *Geometrical Methods in the Theory of Ordinary Differential Equations*. Grundlehren der mathematischen Wissenschaften, Vol. 250. New York: Springer-Verlag, 1983.

[5] B. Aulbach. A classical approach to the analyticity problem of center manifolds. *Z. Angew. Math. Phys.* **36** (1985) 1—23.

[6] W. Aziz and C. Christopher. Local integrability and linearizability of three-dimensional Lotka-Volterra systems. *Appl. Math. Comput.* **219** (2012) 4067–4081.

[7] N. N. Bautin. On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type. *Mat. Sb.* **30** (1952) 181–196; *Amer. Math. Soc. Transl.* **100** (1954) 181–196.

[8] T. Becker and V. Weispfenning. *Gröbner Bases: A computational Approach to Commutative Algebra.* Graduate Texts in Mathematics, Vol. 141. New York: Springer–Verlag, 1993.

[9] Y. N. Bibikov. *Local Theory of Nonlinear Analytic Ordinary Differential Equations*. Lecture Notes in Mathematics, Vol. 702. New York: Springer-Verlag, 1979.

[10] A. D. Brjuno. Analytic form of differential equations. I, II. (Russian) *Tr. Mosk. Mat. Obs.* **25** (1971) 119–262; **26** (1972) 199–239.

[11] A. D. Brjuno. *A Local Method of Nonlinear Analysis for Differential Equations*. Moscow: Nauka, 1979; *Local Methods in Nonlinear Differential Equations.* Translated from the Russian by William Hovingh and Courtney S. Coleman. Berlin: Springer-Verlag, 1989.

[12] B. Buchberger. Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal. Ph. D. Thesis, Mathematical Institute, University of Innsbruck, Austria, 1965; An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal. *J. Symbolic Comput.* **41** (2006) 475–511.

[13] V. I. Bulgakov and A. A. Grin'. On a bifurcation of a non-rough focus of a third-order autonomous system. (Russian) *Differ. Uravn.* **32** (1996) 1703, 1727; *Differential Equations* **32** (1996) 1697-698.

[14] A. Burchard, B. Deng, and K. Lu. Smooth conjugacy of center manifolds. *Proc. Roy. Soc. Edinburgh Sect. A* **120** (1992) 61–77.

[15] J. Chen, Y. Yi, and X. Zhang. First integrals and normal forms for germs of analytic vector fields. *to appear.*

[16] X. Chen, J. Giné, V. G. Romanovski, and D. S. Shafer. The $1 : -q$ resonant center problem for certain cubic Lotka-Volterra systems. *Appl. Math. Comput.* **218** (2012) 11620-1633.

[17] C. Chicone. *Ordinary Differential Equations with Applications.* New York: Springer-Verlag, 1999.

[18] C. Chicone and M. Jacobs. Bifurcation of critical periods for plane vector fields. *Trans. Amer. Math. Soc.* **312** (1989) 433–486.

[19] S. Chow and J. K. Hale. *Methods of Bifurcation Theory.* New York: Springer-Verlag, 1982.

[20] C. Christopher. Invariant algebraic curves and conditions for a centre. *Proc. Roy. Soc. Edinburgh Sect. A* **124** (1994) 1209–1229.

[21] C. Christopher. Estimating limit cycles bifurcations from centers. In: *Trends in Mathematics, Differential Equations with Symbolic Computations*, D. Wang and Z. Zheng, Eds., 23–36. Basel: Birkhäuser-Verlag, 2005.

[22] C. Christopher and C. Li. *Limit Cycles of Differential Equations*. Basel: Birkhäuser-Verlag, 2007.

[23] C. Christopher, P. Mardešić, and C. Rousseau. Normalizable, integrable, and linearizable saddle points for complex quadratic systems in $\mathbb{C}^2$. *J. Dyn. Control Sys.* **9** (2003) 311–363.

[24] C. Christopher and C. Rousseau. Nondegenerate linearizable centres of complex planar quadratic and symmetric cubic systems in $\mathbb{C}^2$. *Publ. Mat.* **45** (2001) 95–123

[25] A. Cima, A. Gasull, V. Mañosa, and F. Manosas. Algebraic properties of the Liapunov and periodic constants. *Rocky Mountain J. Math.* **27** (1997) 471–501.

[26] D. Cox, J. Little, D. O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. New York: Springer, 1997.

[27] G. Darboux. Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré. *Bull. Sci. Math. Sér.* 2 **2** (1878) 60–96, 123–144, 151–200.

[28] W. Decker, G. Pfister, H. Schönemann, and S. Laplagne. A SINGULAR 3.0 library for computing the primary decomposition and radical of ideals, `primdec.lib`, 2005.

[29] W. Decker, G.-M. Greuel, G. Pfister, H. Shönemann. SINGULAR 3-1-6—A Computer Algebra System for Polynomial Computations. http://www.singular.uni-kl.de (2012).

[30] D. Doliánin, J. Giné, R. Oliveira, V. G. Romanovski. The center problem for a 2:-3 resonant cubic Lotka-Volterra system. *Appl. Math. Comp.* **220** (2013) 12–19.

[31] H. Dulac. Détermination et intégration d'une certaine classe d'équations différentielles ayant pour point singulier un centre. *Bull. Sci. Math.* (2) **32** (1908) 230–252.

[32] V. F. Edneral, A. Mahdi, V. G. Romanovski, and D. S. Shafer. The center problem on a center manifold in R^3. *Nonlinear Anal.* **75** (2012) 2614–2622.

[33] M. Frommer. Über das Auftreten von Wirbeln und Strudeln (geschlossener und spiraliger Integralkurven) in der Umgebung rationaler Unbestimmtheitsstellen. *Math. Ann.* **109** (1934) 395–424.

[34] I. A. Garciá and M. Grau. Survey on the inverse integrating factor. *Qual. Theory Dyn. Syst.* **9** (2010) 115–166.

[35] P. Gianni, B. Trager, and G. Zacharias. Gröbner bases and primary decomposition of polynomials, *J. Symbolic Comput.* **6** (1988) 146–167.

[36] J. Giné, C. Christopher, M. Prešern, V. G. Romanovski, and N. L. Shcheglova. The Resonant Center Problem for a $2 : -3$ Resonant Cubic Lotka-Volterra System. *Computer algebra in scientific computing. Lecture Notes in Comput. Sci.* **7442** (2012) 129–142.

[37] G.-M. Greuel and G. Pfister. *A SINGULAR Introduction to Commutative Algebra*. New York: Springer-Verlag, 2002.

[38] R. Hartshorne. *Algebraic Geometry*. New York: Springer-Verlag, 1977.

[39] Z. Hu, M. Han, and V. G. Romanovski. Local integrability of a family of three-dimensional quadratic systems. *Physica D: Nonlinear Phenomena* **265** (2013) 78–86.

[40] Yu. Ilyashenko, S. Yakovenko, editors. *Concerning the Hilbert 16th Problem*. American Mathematical Society Translations, Ser. 2, Advances in the Mathematical Sciences, 165. Providence, RI: American Mathematical Society, 1995.

[41] Yu. Ilyashenko, S. Yakovenko. *Lectures on analytic differential equations*. Graduate Studies in Mathematics, 86. Providence, RI: American Mathematical Society, 2008.

[42] H. Ito. Convergence of Birkhoff normal forms for integrable systems. *Comment. Math. Helv.* **64** (1989) 412–461.

[43] J. P. Jouanolou. Equations de Pfaff algébriques. Lectures Notes in Mathematics vol. 708. New York: Springer-Verlag, 1979.

[44] W. Kapteyn. On the centra of the integral curves which satisfy differential equations of the first order and the first degree. *Proc. Kon. Akad. Wet., Amsterdam* **13** (1911) 1241–1252.

[45] W. Kapteyn. New researches upon the centra of the integrals which satisfy differential equations of the first order and the first degree. *Proc. Kon. Acad. Wet., Amsterdam* **14** (1912) 1185–1185; **15** (1912) 46–52.

[46] Y. A. Kuznetsov. *Elements of applied bifurcation theory*, Third edition. New York: Springer-Verlag, 2004.

[47] V. Levandovskyy, V. G. Romanovski, and D. S. Shafer. The cyclicity of a cubic system with nonradical Bautin ideal. *J. Differential Equations* **246** (2009) 1274–1287.

[48] A. Liapounoff. Problème général de la stabilité du mouvement. *Annales de la Faculté des Sciences de Toulouse Sér. 2* **9** (1907) 204–474. Reproduction in *Annals of Mathematics Studies* **17**, Princeton: Princeton University Press, 1947; reprinted 1965, Kraus Reprint Corporation, New York.

[49] Y.-R. Liu. Formulas of values of singular point and the integrability conditions for a class of cubic system, $M(3) \geq 7$. *Chinese Sci. Bull.* **35** (1990) 1241–1245.

[50] Y.-R. Liu and J.-B. Li. Theory of values of singular point in complex autonomous differential systems. *Sci. China Ser. A* **33** (1990) 10–23.

[51] Y.-R. Liu, J.-B. Li, W. T. Huang. *Singular point values, center problem and bifurcations of limit cycles of two-dimensional differential autonomous systems*. Beijing: Science Press, 2008.

[52] J. Llibre. On the integrability of the differential systems in dimension two and of the polynomial differential systems in arbitrary dimension, *J. Appl. Anal. Comput.* **1** (2011) 33–52.

[53] J. Llibre, C. Pantazi, and S. Walcher. First integrals of local analytic differential systems. *Bull. Sci. Math.* **136** (2012) 342–359.

[54] J. Llibre and X. Zhang. Rational first integrals in the Darboux theory of integrability in $\mathbb{C}^n$. *Bull. Sci. Math.* **134** (2010) 189–195.

[55] J. Llibre and X. Zhang. On the Darboux integrability of polynomial differential systems. *Qual. Theory Dyn. Syst.* **11** (2012) 129–144.

[56] *Macaulay2*, a computer algebra system supporting research in algebraic geometry and commutative algebra, http://www.math.uiuc.edu/Macaulay2/ .

[57] Maple (trademark of Waterloo Maple, Inc.), a general purpose computer algebra system, http://www.maplesoft.com .

[58] S. MacLane and G. Birkhoff. *Algebra.* New York: Macmillan, 1967.

[59] *Mathematica*, a general purpose computer algebra system. Wolfram Research, Inc., Champaign, IL, http://www.wolfram.com .

[60] A. Mahdi, V. G. Romanovski, and D. S. Shafer. Stability and periodic oscillations in the Moon-Rand systems. *Nonlinear Analysis: Real World Applications* **14** (2013) 294-313.

[61] J. E. Marsden and M. M. McCracken. *The Hopf Bifurcation and Its Applications.* New York: Springer-Verlag, 1976.

[62] F. C. Moon and R. H. Rand. *Parametric stiffness control of flexible structures*, Jet Propulsion Laboratory Publication 85-29, Vol. II, California Institute of Technology (1985) 329–342.

[63] S. Yu. Pilyugian and G. R. Sell. A Biographical Sketch of Victor A. Pliss. *J. Dynam. Differential Equations* **15** (2003) 225–235.

[64] V. A. Pliss. A reduction principle in the theory of stability of motion. *Izv. Akad. Nauk SSSR. Ser. Mat.* **28** (1964) 1297–1324.

[65] V. A. Pliss. On the reduction of an analytic system of differential equations to linear form. *Differ. Uravn.* **1** (1965) 153–161; *Differ. Equ.* **1** (1965) 111–118.

[66] REDUCE, a general purpose computer algebra system. http://reduce-algebra.com .

[67] V. G. Romanovski. Time-Reversibility in 2-Dimensional Systems. *Open Syst. Inf. Dynamics* **15** (2008) 359–370.

[68] V. G. Romanovski, X. Chen, and Z. Hu. Linearizability of linear systems perturbed by fifth degree homogeneous polynomials. *J. Phys. A* **40** (2007) 5905–5919.

[69] V. G. Romanovski and M. Prešern. An approach to solving systems of polynomials via modular arithmetics with applications. *J. Comput. Appl. Math.* **236** (2011) 196–208.

[70] V. G. Romanovski and D. S. Shafer. *The Center and Cyclicity Problems: A Computational Algebra Approach.* Boston: Birkhäuser, 2009.

[71] V. G. Romanovski, Y. Xia, and X. Zhang. Varieties of local integrability of analytic differential systems and their applications. *J. Diff. Eq.* **257** (2014) 3079–3101.

[72] R. Roussarie. *Bifurcations of planar vector fields and Hilbert's sixteenth problem.* Progress in Mathematics **164**. Basel: Birkhäuser, 1998.

[73] C. Rousseau and D. Schlomiuk. Cubic systems symmetric with respect to a center. *J. Diff. Eq.* **123** (1995) 388–436.

[74] D. Schlomiuk. Algebraic particular integrals, integrability and the problem of the center. *Trans. Amer. Math. Soc* **338** (1993) 799–841.

[75] D. Schlomiuk. Algebraic and Geometric Aspects of the Theory of Polynomial Vector Fields. In *Bifurcations and Periodic Orbits of Vector Fields*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **408**. Dordrecht: Kluwer Academic, 1993.

[76] T. Shimoyama and K. Yokoyama. Localization and primary decomposition of polynomial ideals. *J. Symbolic Comput.* **22** (1996) 247–277.

[77] K. S. Sibirskii. *Algebraic Invariants of Differential Equations and Matrices.* (Russian) Kishinev: Shtiintsa, 1976.

[78] K. S. Sibirsky. *Introduction to the Algebraic Theory of Invariants of Differential Equations.* Kishinev: Shtiintsa, 1982; Nonlinear Science: Theory and Applications. Manchester: Manchester University Press, 1988.

[79] C. L. Siegel. Über die normalform analytischer Differentialgleichungen in der Nähe einer Gleichgewichtslösung. *Nachr. der Akad. Wiss. Göttingen Math.–Phys. Kl. IIa* (1952) 21–30.

[80] Sijbrand, J. Properties of center manifolds. *Trans. Amer. Math. Soc.* **289** (1985) 431–469.

[81] L. Stolovitch. Singular complete integrability, *Institut des Hautes Études Scientifiques. Publications Mathématiques* **91** (2000) 133–210.

[82] J. V. Uspensky. *Theory of Equations*. New York: McGraw-Hill, 1948.

[83] B. L. van der Waerden. *Modern Algebra*. New York: Frederick Ungar Publishing, 1969.

[84] S. Walcher. On Convergent Normal Form Transformations in Presence of Symmetries. *J. Math. Anal. App.* **244** (2000) 17–26.

[85] S. Walcher. Symmetries and Convergence of normal form transformations, *Monografías de la Real Academia de Ciencias de Zaragoza* **25** (2004) 251–268.

[86] P. S. Wang, M. J. T. Guy, and J. H. Davenport. P-adic reconstruction of rational numbers. *ACM SIGSAM Bull.* **16** (1982) 2–3.

[87] X. Zhang. Analytic normalization of analytic integrable systems and the embedding flows. *J. Differential Equations* **244** (2008) 1080–1092.

[88] X. Zhang. Analytic integrable systems: Analytic normalization and embedding flows. *J. Differential Equations* **254** (2013) 3000–3022.

[89] S. L. Ziglin. Branching of solutions and nonexistence of first integrals in Hamiltonian mechanics, I. *Funct. Anal. Appl.* **16** (1983) 181–189.

[90] H. Żołądek. Eleven small limit cycles in a cubic vector field. *Nonlinearity* **8** (1995) 843–860.

[91] H. Żołądek. The problem of center for resonant singular points of polynomial vector fields. *J. Differential Equations* **135** (1997) 94–118.

Center for Applied Mathematics and Theoretical Physics, University of Maribor and
Faculty of Natural Science and Mathematics, University of Maribor, Maribor, Slovenia
E-mail address: valery.romanovsky@uni-mb.si

Mathemtics Department, University of North Carolina at Charlotte, Charlotte, NC, 28223, USA
E-mail address: dsshafer@uncc.edu