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Topology and geometry of real singularities

Nicolas Dutertre

Abstract.

The aim of these lecture notes is to provide the students with tools and techniques used in the theory of real singularities and to apply them in order to get interesting results on the topology and geometry of real singularities.

§1. Introduction

This mini-course is aimed at young researchers and graduate students who want to learn basic tools and techniques of real singularity theory.

It starts with well-known notions and results of differential topology: the Brouwer degree, the index of a vector-field, the Poincaré-Hopf theorem, the Gauss-Bonnet theorem. Although theses notions may be very familiar to any researcher experienced in singularity theory, we believe it is worth recalling them here.

Then in the next chapter, we apply these techniques of differential topology to some real analytic or semi-analytic sets and we get several nice formulas for topological invariants of these sets.

We end with a chapter about semi-algebraic sets. After a brief introduction on semi-algebraic sets and maps, we give several semi-algebraic singular versions of the Poincaré-Hopf theorem and the Gauss-Bonnet theorem.

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§2. Tools of differential topology

In this chapter, we give the main tools and results in differential topology that we will need and apply in the study of real singularities. Our main references are [35], [37], [42] and [43].

2.1. The Brouwer degree

Let M and N be two oriented n-dimensional manifolds (without boundary) and let $f : M \to N$ be a smooth (i.e. C^∞) map. We assume that M is compact and N is connected. Let x be a regular point of f. This means that $Df(x) : T_xM \to T_{f(x)}N$ is a linear isomorphism between oriented vector spaces. We define the "sign" of $Df(x)$ to be $+1$ (resp. -1) if $Df(x)$ preserves (resp. reverses) the orientation.

Definition 2.1. *For any regular value $y \in N$, we define:*

$$\deg(f, y) = \sum_{x \in f^{-1}(y)} \text{sign } Df(x).$$

Remark 2.1. *Since y is a regular value of f, $f^{-1}(y)$ consists of regular points. Since M is compact, $f^{-1}(y)$ is a finite collection of points since it is a 0-dimensional manifold.*

Theorem 2.1. *The integer* $\deg(f, y)$ *does not depend on the choice of the regular value y.*

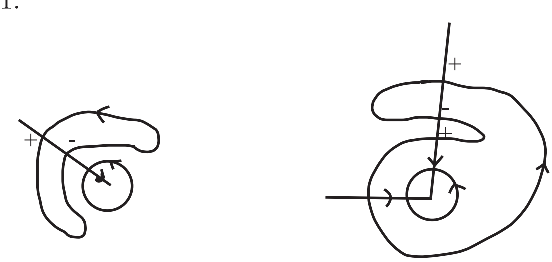
Definition 2.2. *It is called the (Brouwer) degree of f, denoted by* $\deg f$.

Theorem 2.2. *If f is smoothly homotopic to g then*

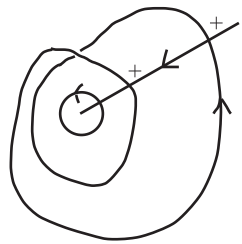
$$\deg f = \deg g.$$

For the proof of these two theorems, the reader is referred to Milnor's book [42].

<u>Examples:</u> 1) The map on the left has degree 0 and the map on the right has degree 1.



2) The map pictured below has degree 2.



3) Let $r_i : S^n \to S^n$ be defined by:

$$r_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, -x_i, \ldots, x_{n+1}).$$

The degree of r_i is -1.

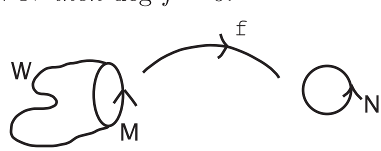
4) The map $S^1 \subset \mathbb{C} \to S^1 \subset \mathbb{C}$, $z \mapsto z^m$, $m \in \mathbb{Z}^*$ has degree m. Actually each point has exactly m preimages and the map is regular and preserves orientation if $m > 0$ and reverses it if $m < 0$.

Proposition 2.1. *If M, N and L are three smooth compact oriented manifolds of the same dimension and $f : M \to N$ and $g : N \to L$ are two smooth maps then* $\deg(g \circ f) = \deg g \times \deg f$.

Proof. It is easy from the definition. □

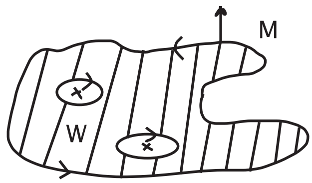
Example: The antipodal map $\sigma : S^n \to S^n$, $x \mapsto -x$ has degree $(-1)^{n+1}$ because $\sigma = r_1 \circ r_2 \circ \cdots \circ r_{n+1}$. Hence if n is even, σ is not smoothly homotopic to the identity.

Theorem 2.3. *If $M = \partial W$, W is compact, and $f : M \to N$ extends to $F : W \to N$ then* $\deg f = 0$.



Application: Assume that M is a smooth compact connected hypersurface in $\mathbb{R}^n$. By the Jordan-Brouwer Separation Theorem, it bounds a connected open set $D \subset \mathbb{R}^n$, i.e. $M = \partial \overline{D}$. This induces a natural orientation on M. Let $F : \overline{D} \to \mathbb{R}^n$ be a map that does not vanish on $\partial \overline{D} = M$. We assume that F inside D has finite number of zeroes $p_1, \ldots, p_m$ that are all regular points of F. We can define $\overline{F} = \frac{F}{|F|} : M \to S^{n-1}$. Then we have:

$$\deg \overline{F} = \sum_{i=1}^{m} \text{sign } \det[DF(p_i)].$$



Let us explain this equality. We remove a small open ball $B(p_i, \varepsilon_i)$ around each p_i. Let $W = \overline{D} \setminus \cup_{i=1}^m B(p_i, \varepsilon_i)$, it is a manifold with boundary:

$$\partial W = M \bigcup \cup_{i=1}^m S(p_i, \varepsilon_i).$$

The submanifold ∂W is oriented by the canonical orientation of the boundary. Let us consider $\overline{F}_{\partial W} : \partial W \to S^{n-1}, x \mapsto \frac{F}{|F|}$. Then $\overline{F}_{\partial W}$ extends to W, so $\deg \overline{F}_{\partial W} = 0$. On the other hand, we have:

$$\deg \overline{F}_{\partial W} = \deg \overline{F} - \sum_{i=1}^{m} \deg \overline{F}_i,$$

where $\overline{F}_i = \frac{F}{|F|} : S(p_i, \varepsilon_i) \to S^{n-1}$. A minus sign appears here because the orientation of $S(p_i, \varepsilon_i)$ as a component of the boundary of W is the opposite of the orientation of $S(p_i, \varepsilon_i)$ as the boundary of $B(p_i, \varepsilon_i)$.

Since p_i is a regular point of F, the degree of $\overline{F}_i$ is equal to the sign of $\det[DF(p_i)]$, because $\frac{F}{|F|} : S(p_i, \varepsilon_i) \to S^{n-1}$ is homotopic to the map:

$$S(p_i, \varepsilon_i) \to S^{n-1}, p \mapsto \frac{DF(p_i)(p - p_i)}{|DF(p_i)(p - p_i)]|},$$

and therefore has the same degree as the map:

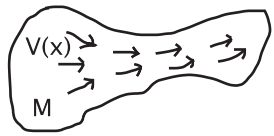
$$S(0, \varepsilon_i) \to S^{n-1}, h \mapsto \frac{DF(p_i)(h)}{|DF(p_i)(h)|}.$$

This last map has degree equal to sign det $[DF(p_i)]$, because the map $h \to DF(p_i)(h)$ is homotopic to $\pm Id_{\mathbb{R}^n}$, depending on the sign of the determinant of $DF(p_i)$ since $GL(n, \mathbb{R})$ has two connected components.

2.2. Vector fields and the Poincaré-Hopf theorem

Definition 2.3. *Let M be a smooth manifold. A vector field V on M is a smooth mapping $V : M \to TM$ such that for all $x \in M$, $pr(V(x)) \in T_x M$, where $pr : TM \to M$ is the natural projection.*

Remark 2.2. *If $M \subset \mathbb{R}^n$ then a vector field is just a smooth mapping $V : M \to \mathbb{R}^n$ such that for all $x \in M$, $V(x) \in T_x M$.*



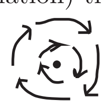
Definition 2.4. *Let p be an isolated zero of a vector field V on a manifold M of dimension n. In local coordinates, we can view V as a mapping from a small open set $U \subset \mathbb{R}^n$ to a small open set $U' \subset \mathbb{R}^n$ where $0 \in U$ and $0 \in U'$ and such that 0 is the only zero of V in U. We define the Poincaré-Hopf index of V at p by:*

$$\mathrm{Ind}(V, p) = \text{degree of } \frac{V}{|V|} : S_\varepsilon^{n-1} \to S^{n-1},$$

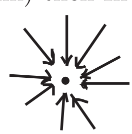
where S_ε^{n-1} is a small sphere included in U.

Examples in $\mathbb{R}^2$:

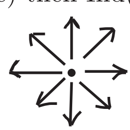
(1) If $V(x, y) = (y, -x)$ (circulation) then $\mathrm{Ind}(V, 0) = +1$.



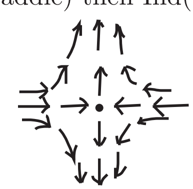
(2) If $V(x, y) = (-x, -y)$ (sink) then $\mathrm{Ind}(V, 0) = +1$.



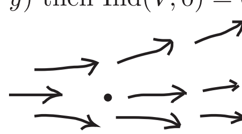
(3) If $V(x, y) = (x, y)$ (source) then $\mathrm{Ind}(V, 0) = +1$.



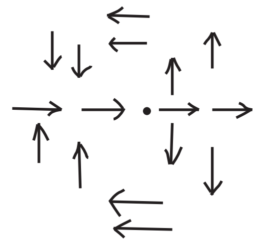
(4) If $V(x, y) = (-x, y)$ (saddle) then $\mathrm{Ind}(V, 0) = -1$.



(5) If $V(x, y) = (x^2, x + y)$ then $\mathrm{Ind}(V, 0) = 0$.



(6) If $V(x,y) = (x^2 - y^2, 2xy)$ ($z \mapsto z^2$ in complex coordinates) then $\mathrm{Ind}(V,0) = +2$.



Remark 2.3. *One may check that this definition does not depend on the local coordinates (see* [42] *and* [35] *for example).*

Theorem 2.4. *(Poincaré-Hopf theorem) Let M be a smooth compact manifold. Let V be a smooth vector field on M, with a finite number of zeroes $p_1, \ldots, p_k$. Then we have:*

$$\chi(M) = \sum_{i=1}^{k} \mathrm{Ind}(V, p_i).$$

Proof. See [42], [35] or [37]. □

2.3. Morse functions

Definition 2.5. *Let M be a smooth manifold of dimension n, let $p \in M$ and $f : M \to \mathbb{R}$ be a smooth function. Let $(x_1, \ldots, x_n)$ be local coordinates around p in M. We say that p is a non-degenerate critical point of f if p is a critical point of f (i.e. $\frac{\partial f}{\partial x_1}(p) = \ldots = \frac{\partial f}{\partial x_n}(p) = 0$) and the matrix:*

$$\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right]_{1 \le i,j \le n}$$

is non-singular.

Remark 2.4. *One may check that this definition does not depend on the choice of the local coordinate system (see* [35]*).*

Proposition 2.2. *(Morse lemma) Let $p \in M$, $\dim M = n$, be a non-degenerate critical point of a smooth function $f : M \to \mathbb{R}$. There exists a local coordinate system $(u_1, \ldots, u_n)$ around p such that:*

$$f = f(p) - u_1^2 - \cdots - u_\lambda^2 + u_{\lambda+1}^2 + \cdots + u_n^2.$$

Proof. See [43]. □

Definition 2.6. *The integer λ is called the index of f at p.*

Corollary 2.1. *Non-degenerate critical points are isolated (in the set of critical points).*

Definition 2.7. *Let M be a smooth manifold. A function $f : M \to \mathbb{R}$ is a Morse function if it admits only non-degenerate critical points.*

Theorem 2.5. *(Openness and density) For any manifold M, Morse functions form a dense open set in $C_s^\infty(M, \mathbb{R})$ (Whitney strong topology).*

Proof. It is a consequence of Thom's transversality theorem (See [37], [33] or [5]). □

For our applications, we will be mainly interested in fibres of analytic or polynomial mappings, so from now on, we shall assume that $M \subset \mathbb{R}^N$ and that dim $M = n$. Let $f : M \to \mathbb{R}$ be a smooth function. This defines a vector field $\nabla_M f$ (the gradient vector field of f on M) by:

$$\forall p \in M, \forall v \in T_p M, Df(p)(v) = \langle \nabla_M f(p), v \rangle.$$

Hence p is a critical point of f if and only if $\nabla_M f(p) = 0$. If p is a Morse critical point of f of index λ, then there is a local coordinate system such that:

$$f = f(p) - u_1^2 - \cdots - u_\lambda^2 + u_{\lambda+1}^2 + \cdots + u_n^2,$$

and so:

$$\nabla_M f = (-2u_1, \ldots, -2u_\lambda, 2u_{\lambda+1}, \ldots, 2u_n).$$

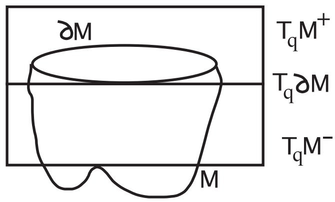
Hence the Poincaré-Hopf index $\mathrm{Ind}(\nabla_M f, p)$ is equal to $(-1)^\lambda$ because, as already explained above, the mapping $\frac{\nabla_M f}{|\nabla_M f|} : S(p, \varepsilon) \to S^{n-1}$ has degree equal to sign $\det[D(\nabla_M f)(p)]$. We can state:

Theorem 2.6. *Let $M \subset \mathbb{R}^N$ be a smooth compact manifold and let $f : M \to \mathbb{R}$ be a Morse function with critical points $p_1, \ldots, p_k$. Then, we have:*

$$\chi(M) = \sum_{i=1}^{k} (-1)^{\lambda(p_i)},$$

where $\lambda(p_i)$ is the Morse index of p_i.

We will also consider the case of manifolds with boundary. Let $(M, \partial M) \subset \mathbb{R}^N$ be a manifold with boundary. Let $q \in \partial M$, then $T_q \partial M$ is a hyperplane in $T_q M$ and $T_q M = T_q \partial M \sqcup T_q M^+ \sqcup T_q M^-$ where $T_q M^+$ consists of outwards pointing vectors (outward vectors for short) and $T_q M^-$ consists of inwards pointing vectors.



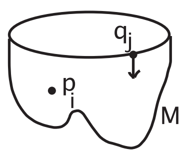
Definition 2.8. *Let $q \in \partial M$ and let $f : (M, \partial M) \to \mathbb{R}$ be a smooth function. We say that q is a correct critical point of f if q is a critical point of $f_{|\partial M} : \partial M \to \mathbb{R}$ and $Df(q)_{|T_qM}$ is not identically zero.*

Definition 2.9. *We say that $f : (M, \partial M) \to \mathbb{R}$ is a correct Morse function if f admits only Morse critical points on $M \setminus \partial M$ and $f_{|\partial M}$ admits only Morse correct critical points.*

Theorem 2.7. *Let $(M, \partial M) \subset \mathbb{R}^N$ be a compact manifold with boundary and let $f : M \to \mathbb{R}$ be a correct Morse function. Denote by $p_1, \ldots, p_k$ the critical points of $f_{|M\setminus\partial M}$ and by $q_1, \ldots, q_l$ those of $f_{|\partial M}$. Then we have:*

$$\chi(M) = \sum_{i=1}^{k} (-1)^{\lambda(p_i)} + \sum_{j \ \mid \ \nabla_M f(q_j) \ inward} (-1)^{\mu(q_j)},$$

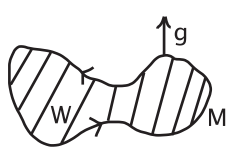
where $\lambda(p_i)$ is the Morse index of f at p_i and $\mu(q_j)$ is the Morse index of $f_{|\partial M}$ at q_j.



The following result is due to Haefliger [36] and Samelson [46].

Application: Let $M^n \subset \mathbb{R}^{n+1}$ be a compact hypersurface canonically oriented. Then M is the boundary of a compact manifold W of dimension $n+1$. Let $g : M \to S^n$ be the outwards pointing unit normal vector field. We have:

$$\deg g = \chi(W).$$



Proof. By Sard's theorem, we can find $a \in S^n$ such that a and $-a$ are regular value of g. Let us write $\{q_1, \ldots, q_l\} = g^{-1}(\pm a)$. Let

$a^* : \mathbb{R}^{n+1} \to \mathbb{R}$ be the function defined by $a^*(x) = \langle a, x\rangle$ and let us consider $a^*_{|M} : M \to \mathbb{R}$, $x \mapsto \langle a, x\rangle$ its restriction to M. The critical points of $a^*_{|M}$ are exactly the q_j's and furthermore, by a determinant computation, there are non-degenerate. Hence $a^*_{|M}$ is a Morse function having no critical points in $W \setminus M$. It is straightforward to see that $a^*_{|W}$ is a correct Morse function. By the previous theorem, we know that:

$$\chi(W) = \sum_{j \ | \ \nabla a^*(q_j) \ inward} (-1)^{\mu(q_j)},$$

where $\mu(q_j)$ is the Morse index of $a^*_{|M}$ at q_j. But $\nabla a^* = a$ so $\nabla a^*(q)$ is inward if and only if $g(q) = -a$. Therefore,

$$\chi(W) = \sum_{j \ | \ g(q_j)=-a} (-1)^{\mu(q_j)}.$$

It remains to relate $(-1)^{\mu(q_j)}$ to the local degree of g at q_j. We use the following lemma.

Lemma 2.1. *We have:*

$$\deg(g, q_j) = (-1)^n \mathrm{sign}\langle g(q_j), a\rangle^n (-1)^{\mu(q_j)}.$$

Proof. See [20], Lemma 2.3. □

If $g(q_j) = -a$ then

$$\deg(g, q_j) = (-1)^n \big(\mathrm{sign}(-|a|^2)\big)^n (-1)^{\mu(q_j)} = (-1)^{\mu(q_j)}.$$

Finally, we find that:

$$\chi(W) = \sum_{j \ | \ g(q_j)=-a} \deg(g, q_j) = \deg\, g.$$

□

In the following chapters of this mini-course, we will use relative versions of the previous two theorems on Morse theory.

Theorem 2.8. *Let $M \subset \mathbb{R}^N$ be a smooth compact manifold and let $f : M \to \mathbb{R}$ be a Morse function with critical points $p_1, \ldots, p_k$. For any $\alpha \in \mathbb{R}$, we have:*

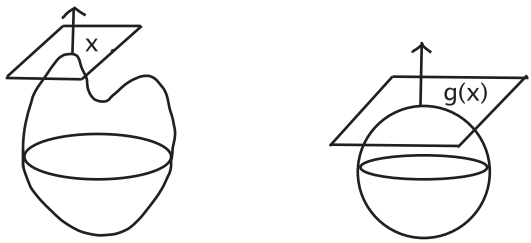
$$\chi(M \cap \{f \geq \alpha\}, M \cap \{f = \alpha\}) = \sum_{i \ | \ f(p_i)>\alpha} (-1)^{\lambda(p_i)},$$

where $\lambda(p_i)$ is the Morse index of p_i.

Theorem 2.7 has a similar relative version.

2.4. The Gauss-Bonnet theorem

Let $M \subset \mathbb{R}^{n+1}$ be a compact hypersurface, canonically oriented as the boundary of a compact manifold with boundary W. Let $g : M \to S^n$ be the Gauss map. Its Jacobian $J_g(x) = k(x)$ is called the curvature of M at x. It is the determinant of the differential $Dg(x) : T_xM \to T_{g(x)}S^n = T_xM$.



Theorem 2.9. *We have:*

$$\int_M k(x)dx = \mathrm{vol}(S^n)\chi(W).$$

Proof. We denote by dv the volume form on S^n. By integral calculus on manifolds, we can write:

$$\int_M k(x)dx = \int_M J_g(x)dx = \int_M g^*(dv) = \deg\, g \int_{S^n} dv =$$

$$\deg\, g \times \mathrm{vol}(S^n).$$

But we know that $\deg\, g = \chi(W)$. □

The following corollary is due to Hopf [38].

Corollary 2.2. *If M is even-dimensional, we have:*

$$\int_M k(x)dx = \frac{1}{2}\mathrm{vol}(S^n)\chi(M).$$

Proof. Use the equality $\chi(M) = 2\chi(W)$. □

§3. The Eisenbud-Levine formula, the Khimshiashvili formula and applications

3.1. The Eisenbud-Levine formula

As seen in the second chapter, the Poincaré-Hopf index of a vector field plays an important role in the topology of manifolds. Here we present an algebraic formula for this index.

Let $f = (f_1, \ldots, f_n) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a C^∞ map-germ (this is exactly the local expression of a vector field on a smooth manifold). We assume that:

$$Q(f) = \frac{C^\infty(\mathbb{R}^n, 0)}{(f_1, \ldots, f_n)},$$

is a finite dimensional vector space over $\mathbb{R}$. Here $C^\infty(\mathbb{R}^n, 0)$ is the algebra of germs at $0 \in \mathbb{R}^n$ of C^∞ real valued functions and $(f_1, \ldots, f_n)$ is the ideal generated by the components $f_1, \ldots, f_n$ of f. We write $\dim_{\mathbb{R}} Q(f) < +\infty$. We denote by J_f the jacobian of the map-germ f. Namely, we have:

$$J_f = \frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}.$$

Theorem 3.1. *(The Eisenbud-Levine formula) Let* $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *be a* C^∞ *map-germ such that* $\dim_{\mathbb{R}} Q(f) < +\infty$. *Then we have:*

(1) 0 *is isolated in* $f^{-1}(0)$,
(2) $J_f \neq 0$ *in* $Q(f)$,
(3) $\forall g \in Q(f)$, $gJ_f = g(0)J_f$ *in* $Q(f)$,
(4) *let* $\varphi : Q(f) \to \mathbb{R}$ *be a linear form such that* $\varphi(J_f) > 0$ *and let* $\Phi : Q(f) \times Q(f) \to \mathbb{R}$ *be the bilinear symmetric form defined by* $\Phi(g, h) = \varphi(gh)$. *Then* Φ *is non-degenerate and* signature $\Phi = \mathrm{Ind}(f, 0)$.

Proof. See [28], [5] or [8]. For a first approach, see [27]. □

<u>Example:</u> Let f be the map-germ defined by:

$$\begin{array}{rcccc} f & : & (\mathbb{R}^2, 0) & \to & (\mathbb{R}^2, 0) \\ & & (x, y) & \mapsto & (x^2 - y^2, 2xy). \end{array}$$

We have: $Q(f) = \frac{C^\infty(\mathbb{R}^2, 0)}{(x^2 - y^2, 2xy)}$. We see that $\dim_{\mathbb{R}} Q(f) = 4$ and that $\bar{1}$, $\bar{x}$, $\bar{y}$ and $\overline{x^2 + y^2}$ form a basis of $Q(f)$. It is clear that 0 is isolated in $f^{-1}(0)$. Let us compute J_f:

$$J_f(x, y) = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2).$$

Let $\varphi : Q(f) \to \mathbb{R}$ be the linear form given by:

$$\varphi(\bar{1}) = \varphi(\bar{x}) = \varphi(\bar{y}) = 0 \text{ and } \varphi(\overline{x^2 + y^2}) = \frac{1}{4}.$$

Then $\varphi(J_f) = 1$. Let Φ be the bilinear symmetric form defined by $\Phi(P, Q) = \varphi(PQ)$. Let us find its matrix in the basis $(\bar{1}, \bar{x}, \bar{y}, \overline{x^2 + y^2})$.

We have:

$$\Phi(\bar{1},\bar{1}) = \varphi(\bar{1}) = 0, \Phi(\bar{1},\bar{x}) = \Phi(\bar{x},\bar{1}) = 0, \Phi(\bar{1},\bar{y}) = \Phi(\bar{y},\bar{1}) = 0,$$

$$\Phi(\bar{x},\bar{x}) = \varphi(\bar{x}^2) = \varphi(\frac{1}{2}\overline{x^2+y^2}) = \frac{1}{8},$$

$$\Phi(\bar{y},\bar{y}) = \varphi(\bar{y}^2) = \varphi(\frac{1}{2}\overline{x^2+y^2}) = \frac{1}{8},$$

$$\Phi(\bar{x},\bar{y}) = \varphi(\overline{xy}) = \varphi(\bar{0}) = 0 = \Phi(\bar{y},\bar{x}),$$

$$\Phi(\bar{1},\overline{x^2+y^2}) = \frac{1}{4}, \Phi(\bar{x},\overline{x^2+y^2}) = \varphi(\bar{x}^3 + \overline{xy^2}) = \varphi(\bar{0}) = 0,$$

$$\Phi(\bar{y},\overline{x^2+y^2}) = 0, \Phi(\overline{x^2+y^2},\overline{x^2+y^2}) = \varphi\big(\overline{(x^2+y^2)(x^2+y^2)}\big) = 0.$$

So this matrix is:

$$\begin{bmatrix} 0 & 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{8} & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 \\ \frac{1}{4} & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues are $\frac{1}{8}$ with multiplicity 2, $\frac{1}{4}$ with multplicity 1 and $-\frac{1}{4}$ with multiplicity 1. So the signature of Φ is $3-1=2=\mathrm{Ind}(f,0)$.

The Eisenbud-Levine formula gives an algebraic formula for the index of a vector field, hence an algebraic and "effective" way to compute a topological data. In the sequel, using technics introduced in the second chapter, we will present several formulas relating topological invariants to indices of vector fields. Thanks to the Eisenbud-Levine formula, these topological invariants become algebraically computable.

3.2. The Khimshiashvili formula

From now on, we will restrict ourselves to the analytic or polynomial case.

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic-map germ with an isolated critical point at 0. The Khimshiashvili formula (see [39]) relates the Poincaré-Hopf index of the gradient vector field ∇f of f to the topology of a small regular level of f.

Theorem 3.2. *We have:*

$$\chi\big(f^{-1}(\delta) \cap B_\varepsilon^n\big) = 1 - \mathrm{sign}(-\delta)^n \mathrm{Ind}(\nabla f, 0), \qquad (1)$$

where δ is a regular value of f, $0 < |\delta| \ll \varepsilon \ll 1$, and:

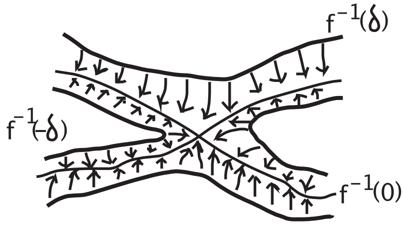
$$\chi(\{f \geq \delta\} \cap B_\varepsilon^n) - \chi(\{f \leq \delta\} \cap B_\varepsilon^n) = \mathrm{sign}(-\delta)^{n+1}\mathrm{Ind}(\nabla f, 0). \qquad (2)$$

Proof. Let U be a small open subset of $\mathbb{R}^n$ such that $0 \in U$, and f is defined in U. We pertub f in a Morse function $\tilde{f} : U \to \mathbb{R}$. Let $p_1, \ldots, p_k$ be the critical points of $\tilde{f}$, with respective indices $\lambda_1, \ldots, \lambda_k$. Let $\delta > 0$, by Morse theory we have:

$$\chi\big(f^{-1}([-\delta, \delta]) \cap B_\varepsilon^n\big) - \chi\big(f^{-1}(-\delta) \cap B_\varepsilon^n\big) = \sum_{i=1}^{k} (-1)^{\lambda_i}.$$

Actually we can choose $\tilde{f}$ sufficiently close to f so that the p_i's lie in $f^{-1}([-\frac{\delta}{4}, \frac{\delta}{4}])$. Now, $f^{-1}([-\delta, \delta]) \cap B_\varepsilon^n$ retracts to the central fibre $f^{-1}(0) \cap B_\varepsilon^n$ and $f^{-1}(0) \cap B_\varepsilon$ is the cone over $f^{-1}(0) \cap S_\varepsilon^{n-1}$ (see [44]) so:

$$\chi\big(f^{-1}([-\delta, \delta]) \cap B_\varepsilon^n\big) = 1.$$



Moreover, we have:

$$\sum_{i=1}^{k} (-1)^{\lambda_i} = \sum_{i=1}^{k} \text{sign det}[D(\nabla \tilde{f})(p_i)].$$

The sum on the right hand-side is the degree of the map $\frac{\nabla \tilde{f}}{|\nabla \tilde{f}|} : S_\varepsilon^{n-1} \to S^{n-1}$ which is equal, by homotopy, to the degree of $\frac{\nabla f}{|\nabla f|} : S_\varepsilon^{n-1} \to S^{n-1}$. By definition, this last degree is $\text{Ind}(\nabla f, 0)$. This gives the result for a negative regular value. For a positive regular value, we apply the result to $-f$ and use the relation $\text{Ind}(-\nabla f, 0) = (-1)^n \text{Ind}(\nabla f, 0)$. This proves formula (1). Formula (2) is proved with similar arguments. $\square$

We will call $f^{-1}(\delta) \cap B_\varepsilon^n$ the (positive or negative) real Milnor fibre. The following formulas are due to Arnol'd [6] and Wall [55].

Corollary 3.1. *With the same hypothesis on f, we have:*

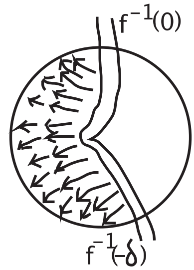
$$\chi(\{f \leq 0\} \cap S_\varepsilon^{n-1}) = 1 - \text{Ind}(\nabla f, 0),$$

$$\chi(\{f \geq 0\} \cap S_\varepsilon^{n-1}) = 1 + (-1)^{n-1} \text{Ind}(\nabla f, 0).$$

If n is even, we have:

$$\chi(\{f = 0\} \cap S_\varepsilon^{n-1}) = 2 - 2 \ \text{Ind}(\nabla f, 0).$$

Proof. By a deformation argument due to Milnor [44], $f(-\delta) \cap B_\varepsilon^n$, $\delta > 0$, is homeomorphic to $\{f \leq -\delta\} \cap S_\varepsilon^{n-1}$, which is homeomorphic to $\{f \leq 0\} \cap S_\varepsilon^{n-1}$ if δ is very small.



□

3.3. The Fukui formula

The above real Milnor fibre can be also written as $f_t^{-1}(0) \cap B_\varepsilon^n$, $0 < |t| \ll \varepsilon \ll 1$, where $f_t(x) = f(x) - t$. In this section, we will present a method for the computation of the Euler characteristic of $f_t^{-1}(0) \cap B_\varepsilon^n$, $0 < |t| \ll \varepsilon \ll 1$, where f_t is a one-parameter deformation of $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$. It is interesting to study such deformations because the topology of the fibre is somehow richer than the one of the real Milnor fibre and contains more information about the singularity. The setting is described below.

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic function-germ with an isolated critical point at 0. Let $F : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$, $(t, x) \mapsto F(t, x)$ be a one-parameter deformation of f, i.e. $F_0(x) = F(0, x) = f(x)$. Let $H : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^{n+1}, 0)$ be defined by $H(t, x) = (F, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n})$. We assume that H has an isolated critical zero at 0, so that $\mathrm{Ind}(H, 0)$ is well-defined.

Lemma 3.1. *The function F has an isolated critical point at the origin.*

Proof. By the Curve Selection Lemma [44], $\nabla F^{-1}(0)$ is included in $F^{-1}(0)$. Hence $\nabla F^{-1}(0) \subset H^{-1}(0)$. □

Lemma 3.2. *For $t \neq 0$ small, the fibre*

$$f_t^{-1}(0) = \{x \in \mathbb{R}^n \mid F(t, x) = 0\},$$

is smooth in a neighborhood of the origin.

Proof. A point $x \in f_t^{-1}(0)$ is a critical point of f_t if and only if $\forall i \in \{1, \ldots, n\}$, $\frac{\partial f_t}{\partial x_i}(x) = 0$. This implies that $\frac{\partial F}{\partial x_i}(t, x) = 0$ and that $H(t, x) = 0$.

□

The following theorem was proved by Fukui [31].

Theorem 3.3. *For* $0 < |\delta| \ll \varepsilon \ll 1$*, we have:*

$$\chi\big(F^{-1}(\delta) \cap \{t \geq 0\} \cap B_\varepsilon^{n+1}\big) - \chi\big(F^{-1}(\delta) \cap \{t \leq 0\} \cap B_\varepsilon^{n+1}\big) = -\mathrm{sign}(-\delta)^{n+1}\mathrm{Ind}(H, 0).$$

Proof. We work in a small open subset U of $\mathbb{R}^{n+1}$ that contains 0. Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}, (t, x) \mapsto t$ be the projection on the first coordinate. After a small perturbation of F, we can assume that $\pi_{|F^{-1}(\delta)\cap \mathring{B}_\varepsilon^{n+1}} : F^{-1}(\delta) \cap \mathring{B}_\varepsilon^{n+1} \to \mathbb{R}$ admits only Morse critical points $p_1, \ldots, p_k$ with respective indices $\lambda_1, \ldots, \lambda_k$. By Morse theory, we have:

$$\chi\big(F^{-1}(\delta) \cap \{t \geq 0\} \cap B_\varepsilon^{n+1}\big) - \chi\big(F^{-1}(\delta) \cap \{t = 0\} \cap B_\varepsilon^{n+1}\big) = \sum_{j \ | \ \pi(p_j)>0} (-1)^{\lambda_j},$$

$$\chi\big(F^{-1}(\delta) \cap \{t \leq 0\} \cap B_\varepsilon^{n+1}\big) - \chi\big(F^{-1}(\delta) \cap \{t = 0\} \cap B_\varepsilon^{n+1}\big) = (-1)^n \sum_{j \ | \ \pi(p_j)<0} (-1)^{\lambda_j}.$$

Here we have to notice that $F^{-1}(\delta) \cap B_\varepsilon^{n+1}$ is a manifold with boundary. By the Curve Selection Lemma, we can prove that the critical points of $\pi_{|F^{-1}(\delta)\cap S_\varepsilon^n}$ in $\{t \geq 0\}$ point outwards, hence they do not appear in the above equality. Now it is easy to see that the critical points of $\pi_{|F^{-1}(\delta)\cap \mathring{B}_\varepsilon^{n+1}}$ are exactly the zeros of $H_\delta = (F - \delta, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n})$. A determinant computation shows that the p_j's are non-degenerate zeros of H_δ and that:

$$(-1)^{\lambda_j} = -\mathrm{sign}\ \pi(p_j)^{n+1} \times\ \mathrm{sign}(-\delta)^{n+1} \times \mathrm{sign}\ \det[DH_\delta(p_j)].$$

Thus we obtain:

$$\chi\big(F^{-1}(\delta) \cap B_\varepsilon^{n+1} \cap \{t \geq 0\}\big) - \chi\big(F^{-1}(\delta) \cap B_\varepsilon^{n+1} \cap \{t \leq 0\}\big) = -\sum_{j=1}^{k} \mathrm{sign}\ \pi(p_j)^{n+1} \times (-1)^{\lambda_j} = -\mathrm{sign}(-\delta)^{n+1} \sum_{j=1}^{k} \mathrm{sign}\ \det[DH_\delta(p_j)] =$$

$$-\text{sign}(-\delta)^{n+1}\Big(\text{degree of } \tfrac{H_\delta}{|H_\delta|} : S^n_\varepsilon \to S^n\Big) = -\text{sign}(-\delta)^{n+1}\text{Ind}(H,0).$$

The last equality is explained by the fact that the maps $\frac{H_\delta}{|H_\delta|}$ and $\frac{H}{|H|}$ are homotopic if δ is small enough. □

Corollary 3.2. *If n is even, then we have:*

$$\chi\big(f_t^{-1}(0)\cap B^n_\varepsilon\big) = 1 - \text{Ind}(\nabla f, 0),$$

$$\chi(\{f_t \geq 0\}\cap B^n_\varepsilon) - \chi(\{f_t \leq 0\}\cap B^n_\varepsilon) = \text{Ind}(\nabla F, 0) + \text{sign}(t)\ \text{Ind}(H,0).$$

If n is odd, then we have:

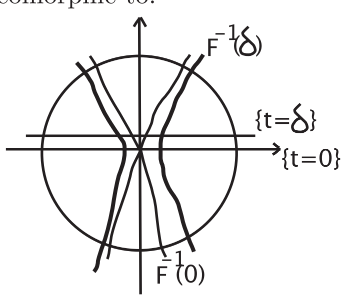
$$\chi\big(f_t^{-1}(0)\cap B^n_\varepsilon\big) = 1 - \text{Ind}(\nabla F, 0) - \text{sign}(t)\ \text{Ind}(H,0),$$

$$\chi(\{f_t \geq 0\}\cap B^n_\varepsilon) - \chi(\{f_t \leq 0\}\cap B^n_\varepsilon) = \text{Ind}(\nabla f, 0).$$

Proof. By a deformation argument, we have for $\delta > 0$:

$$F^{-1}(\delta)\cap\{t\geq 0\}\cap B^{n+1}_\varepsilon \simeq \{F\geq 0\}\cap\{t\geq 0\}\cap S^n_\varepsilon \simeq \{F\geq 0\}\cap\{t=\delta\}\cap B^{n+1}_\varepsilon,$$

where $\simeq$ means homeomorphic to.



Similarly, we can write:

$$F^{-1}(\delta)\cap\{t\leq 0\}\cap B^{n+1}_\varepsilon \simeq \{F\geq 0\}\cap\{t=-\delta\}\cap B^{n+1}_\varepsilon,$$

$$F^{-1}(-\delta)\cap\{t\geq 0\}\cap B^{n+1}_\varepsilon \simeq \{F\leq 0\}\cap\{t=\delta\}\cap B^{n+1}_\varepsilon,$$

$$F^{-1}(-\delta)\cap\{t\leq 0\}\cap B^{n+1}_\varepsilon \simeq \{F\leq 0\}\cap\{t=-\delta\}\cap B^{n+1}_\varepsilon.$$

By Khimshiashvili's formula, we get:

$$\chi\big(F^{-1}(\delta)\cap B^{n+1}_\varepsilon\big) = 1 + (-1)^n\text{Ind}(\nabla F, 0),$$

$$\chi\big(F^{-1}(-\delta)\cap B^{n+1}_\varepsilon\big) = 1 - \text{Ind}(\nabla F, 0),$$

$$\chi\big(F^{-1}(\delta)\cap B_\varepsilon^{n+1}\cap\{t=0\}\big)=1+(-1)^{n-1}\mathrm{Ind}(\nabla f,0),$$
$$\chi\big(F^{-1}(-\delta)\cap B_\varepsilon^{n+1}\cap\{t=0\}\big)=1-\mathrm{Ind}(\nabla f,0).$$

By the Mayer-Vietoris sequence, we have:

$$\chi\big(F^{-1}(\delta)\cap B_\varepsilon^{n+1}\big)+\chi\big(F^{-1}(\delta)\cap B_\varepsilon^{n+1}\cap\{t=0\}\big)=$$
$$\chi\big(F^{-1}(\delta)\cap\{t\geq 0\}\cap B_\varepsilon^{n+1}\big)+\chi\big(F^{-1}(\delta)\cap\{t\leq 0\}\cap B_\varepsilon^{n+1}\big)=$$
$$2+(-1)^n\mathrm{Ind}(\nabla F,0)+(-1)^{n-1}\mathrm{Ind}(\nabla f,0),$$

and:

$$\chi\big(F^{-1}(-\delta)\cap B_\varepsilon^{n+1}\big)+\chi\big(F^{-1}(-\delta)\cap B_\varepsilon^{n+1}\cap\{t=0\}\big)=$$
$$\chi\big(F^{-1}(-\delta)\cap\{t\geq 0\}\cap B_\varepsilon^{n+1}\big)+\chi\big(F^{-1}(-\delta)\cap\{t\leq 0\}\cap B_\varepsilon^{n+1}\big)=$$
$$2-\mathrm{Ind}(\nabla F,0)-\mathrm{Ind}(\nabla f,0).$$

Applying the previous theorem, we get:

$$\chi\big(\{F\geq 0\}\cap\{t=\delta\}\cap B_\varepsilon^{n+1}\big)=\chi\big(F^{-1}(\delta)\cap\{t\geq 0\}\cap B_\varepsilon^{n+1}\big)=$$
$$1+\frac{(-1)^n}{2}\big(\mathrm{Ind}(\nabla F,0)-\mathrm{Ind}(\nabla f,0)+\mathrm{Ind}(H,0)\big).$$

Similarly, we have:

$$\chi\big(\{F\geq 0\}\cap\{t=-\delta\}\cap B_\varepsilon^{n+1}\big)=$$
$$1+\frac{(-1)^n}{2}\big(\mathrm{Ind}(\nabla F,0)-\mathrm{Ind}(\nabla f,0)-\mathrm{Ind}(H,0)\big),$$

$$\chi\big(\{F\leq 0\}\cap\{t=\delta\}\cap B_\varepsilon^{n+1}\big)=$$
$$1-\frac{1}{2}\big(\mathrm{Ind}(\nabla F,0)+\mathrm{Ind}(\nabla f,0)+\mathrm{Ind}(H,0)\big),$$

$$\chi\big(\{F\leq 0\}\cap\{t=-\delta\}\cap B_\varepsilon^{n+1}\big)=$$
$$1-\frac{1}{2}\big(\mathrm{Ind}(\nabla F,0)+\mathrm{Ind}(\nabla f,0)-\mathrm{Ind}(H,0)\big).$$

We conclude with:

$$\chi(B_\varepsilon^n)=1=\chi(\{f_t\geq 0\}\cap B_\varepsilon^n)+\chi(\{f_t\leq 0\}\cap B_\varepsilon^n)-\chi(\{f_t=0\}\cap B_\varepsilon^n).$$

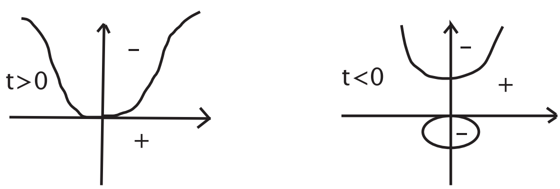
$\square$

Example: Let $f(x,y)=x^2-y^3$ and let $F(t,x,y)=x^2-y^3-ty$. If $t>0$ then:

$$\chi\big(f_t^{-1}(0)\cap B_\varepsilon^n\big)=1 \text{ and } \chi(\{f_t\geq 0\}\cap B_\varepsilon^n)-\chi(\{f_t\leq 0\}\cap B_\varepsilon^n)=0.$$

If $t<0$ then:

$$\chi\big(f_t^{-1}(0)\cap B_\varepsilon^n\big)=1 \text{ and } \chi(\{f_t\geq 0\}\cap B_\varepsilon^n)-\chi(\{f_t\leq 0\}\cap B_\varepsilon^n)=-2.$$



Let us check that the above formulas hold in this example. We have $\nabla f(x,y)=(2x,-3y^2)$, hence $\mathrm{Ind}(\nabla f,0)=0$. Let us compute $\mathrm{Ind}(\nabla F,0)$. We have $\nabla F(t,x,y)=(-y,2x,-3y^2-t)$. The matrix of the differential of ∇F is:

$$D(\nabla F)(t,x,y)=\begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & -6y \end{pmatrix}.$$

It is easy to see that $\nabla F^{-1}(0,0,\varepsilon)=(-\varepsilon,0,0)$ and that

$$\det[D(\nabla F)(-\varepsilon,0,0)]<0.$$

This implies that $\mathrm{Ind}(\nabla F,0)=-1$. Let us compute now $\mathrm{Ind}(H,0)$. We have $H(t,x,y)=(x^2-y^3-ty,2x,-3y^2-t)$ and

$$DH(t,x,y)=\begin{pmatrix} -y & 2x & -3y^2-t \\ 0 & 2 & 0 \\ -1 & 0 & -6y \end{pmatrix}.$$

Let us search for the preimages of $(0,0,\varepsilon)$ where $\varepsilon>0$. If (t,x,y) is such a preimage, then $x=0$, $y^3+ty=y(y^2+t)=0$ and $3y^2+t=-\varepsilon$. If $y^2+t=0$ then $3y^2-y^2=2y^2=-\varepsilon$, which is impossible. Therefore $y=0$ and $t=-\varepsilon$. It is easy to see that $DH(-\varepsilon,0,0)=2\varepsilon>0$. We conclude that $\mathrm{Ind}(H,0)=+1$. Applying Fukui's formula, we recover the above values for $\chi\big(f_t^{-1}(0)\cap B_\varepsilon^n\big)$ and $\chi(\{f_t\geq 0\}\cap B_\varepsilon^n)-\chi(\{f_t\leq 0\}\cap B_\varepsilon^n)$.

3.4. Applications to map-germs with an isolated critical point

Let $\psi = (f_1, \ldots, f_k) : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$, $2 \leq k \leq n$, be a real analytic map-germ with an isolated critical point at the origin. This means that 0 is isolated in

$$\Sigma(\psi) = \{x \in \mathbb{R}^n \mid \operatorname{rank}(\nabla f_1(x), \ldots, \nabla f_k(x)) < k\}.$$

This implies that for any $l \in \{1, \ldots, k\}$ and for any l-tuple $(i_1, \ldots, i_l)$ of pairwise distinct elements of $\{1, \ldots, k\}$, the mapping $(f_{i_1}, \ldots, f_{i_l}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^l, 0)$ has an isolated critical point at the origin as well, for otherwise 0 would not be isolated in $\Sigma(\psi)$. Let $\phi = (f_1, \ldots, f_{k-1}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^{k-1}, 0)$. The following result was proved by Araújo dos Santos, Dreibelbis and the author [7].

Proposition 3.1. *For $0 \ll |\delta| \ll \varepsilon \ll 1$, the following holds:*

(i) *if n is even, we have:*

$$\chi\big(\phi^{-1}(\delta) \cap {f_k}^{-1}(0) \cap B^n_\varepsilon\big) = 1 - \operatorname{Ind}(\nabla f_1, 0);$$

(ii) *if n is odd, we have* $\operatorname{Ind}(\nabla f_1, 0) = 0$ *and:*

$$\chi\big(\phi^{-1}(\delta) \cap {f_k}^{-1}(0) \cap B^n_\varepsilon\big) = 1.$$

Proof. Applying Morse theory for manifolds with boundary to the function $f_{k|\phi^{-1}(\delta) \cap B^n_\varepsilon}$, we have:

$$\chi\big(\phi^{-1}(\delta) \cap \{f_k \geq 0\} \cap B^n_\varepsilon\big) - \chi\big(\phi^{-1}(\delta) \cap f_k^{-1}(0) \cap B^n_\varepsilon\big) = 0,$$

because $f_{k|\phi^{-1}(\delta)}$ has no critical point as $\Sigma(\psi) = \{0\}$, and because the gradient vector field $\nabla f_{k|\phi^{-1}(\delta) \cap B^n_\varepsilon}$ points outwards at the critical points of $f_{k|\phi^{-1}(\delta) \cap S^{n-1}_\varepsilon}$ lying in $\{f_k > 0\}$. Similarly, we have:

$$\chi\big(\phi^{-1}(\delta) \cap \{f_k \leq 0\} \cap B^n_\varepsilon\big) - \chi\big(\phi^{-1}(\delta) \cap {f_k}^{-1}(0) \cap B^n_\varepsilon\big) = 0.$$

Summing these two equalities and using the Mayer-Vietoris sequence, we obtain that:

$$\chi\big(\phi^{-1}(\delta) \cap B^n_\varepsilon\big) = \chi\big(\phi^{-1}(\delta) \cap {f_k}^{-1}(0) \cap B^n_\varepsilon\big).$$

Applying this procedure $k-1$ times, we obtain that:

$$\chi\big(\phi^{-1}(\delta) \cap {f_k}^{-1}(0) \cap B^n_\varepsilon\big) = \chi\big({f_1}^{-1}(\alpha_1) \cap B^n_\varepsilon\big),$$

where α_1 is a small regular value of f_1.

By Khimshiashvili's formula, we know that:

$$\chi\big({f_1}^{-1}(\alpha_1)\cap B^n_\varepsilon\big)=1-\operatorname{sign}(-\alpha_1)^n\operatorname{Ind}(\nabla f_1,0).$$

Hence, if n is even, we find that:

$$\chi\big(\phi^{-1}(\delta)\cap {f_k}^{-1}(0)\cap B^n_\varepsilon\big)=1-\operatorname{Ind}(\nabla f_1,0).$$

If n is odd, just changing α_1 by $-\alpha_1$, we get that $\operatorname{Ind}(\nabla f_1,0)=0$ and $\chi\big(\phi^{-1}(\delta)\cap {f_k}^{-1}(0)\cap B^n_\varepsilon\big)=1$. ☐

Corollary 3.3. *Let γ be a small regular value of ψ. If n is even, we have:*

$$\chi\big(\psi^{-1}(\gamma)\cap B^n_\varepsilon\big)=1-\operatorname{Ind}(\nabla f_1,0)=\cdots=1-\operatorname{Ind}(\nabla f_k,0),$$

and $\operatorname{Ind}(\nabla f_1,0)=\cdots=\operatorname{Ind}(\nabla f_k,0)$. *If n is odd, we have* $\operatorname{Ind}(\nabla f_1,0)=\cdots=\operatorname{Ind}(\nabla f_k,0)=0$ *and* $\chi\big(\psi^{-1}(\gamma)\cap B^n_\varepsilon\big)=1$.

3.5. Real versions of the Lê-Greuel formula

In the previous sections, we studied map-germs from $(\mathbb{R}^n,0)$ to $(\mathbb{R}^k,0)$ when $k\in\{1,n\}$ or when $2\le k\le n$ and the map-germ has an isolated critical point at the origin. Here we will investigate the general case.

Let $1\le k<n$ and let $f=(f_1,\ldots,f_k):(\mathbb{R}^n,0)\to(\mathbb{R}^k,0)$ be an analytic map-germ such that 0 is an isolated singular point of $f^{-1}(0)$. This means that 0 is isolated in $\{x\in\mathbb{R}^n\mid \operatorname{rank}[Df(x)]<k\}\cap f^{-1}(0)$. Let $g:(\mathbb{R}^n,0)\to(\mathbb{R},0)$ be an analytic function-germ. Let I be the ideal in $\mathcal{O}_{\mathbb{R}^n,0}$, the algebra of analytic function-germs at the origin, generated by $f_1,\ldots,f_k$ and the minors $\frac{\partial(f_1,\ldots,f_k,g)}{\partial(x_{i_1},\ldots,x_{i_{k+1}})}$ and let $A_\mathbb{R}=\frac{\mathcal{O}_{\mathbb{R}^n,0}}{I}$. We will denote by C^ε_δ the real Milnor fibre $f^{-1}(\delta)\cap B^n_\varepsilon$, $0<|\delta|\ll\varepsilon\ll1$.

The following theorem appeared in [19].

Theorem 3.4. *If* $\dim_\mathbb{R} A_\mathbb{R}<+\infty$ *then we have:*

$$\chi(C^\varepsilon_\delta\cap\{g\ge\alpha\})-\chi(C^\varepsilon_\delta\cap\{g\le\alpha\})\equiv$$

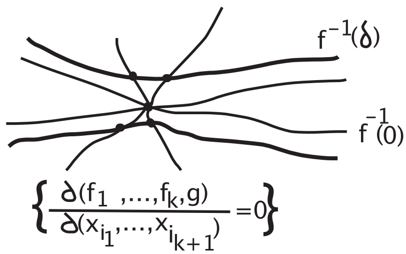
$$\chi(C^\varepsilon_\delta)-\chi(C^\varepsilon_\delta\cap\{g=\alpha\})\equiv\dim_\mathbb{R} A_\mathbb{R}\bmod 2,$$

where (δ,α) is a regular value of (f,g) such that $0\le|\delta|\ll|\varepsilon|\ll1$.

Proof. We pertub g in $\tilde g$ such that $\tilde g_{|C^\varepsilon_\delta}$ is Morse. By Morse theory, we have:

$$\begin{aligned}&\chi(C^\varepsilon_\delta\cap\{g\ge\alpha\})-\chi(C^\varepsilon_\delta\cap\{g=\alpha\})=\\&\qquad\#\{\text{critical points of }\tilde g_{|C^\varepsilon_\delta}\text{ such that }\tilde g>\alpha\}\bmod 2,\end{aligned}$$

$$\chi(C_\delta^\varepsilon \cap \{g \leq \alpha\}) - \chi(C_\delta^\varepsilon \cap \{g = \alpha\}) = \\ \#\{\text{critical points of } \tilde{g}_{|C_\delta^\varepsilon} \text{ such that } \tilde{g} < \alpha\} \bmod 2.$$



Hence we find that:

$$\chi(C_\delta^\varepsilon \cap \{g \geq \alpha\}) - \chi(C_\delta^\varepsilon \cap \{g \leq \alpha\}) = \\ \#\{\text{critical points of } \tilde{g}_{|C_\delta^\varepsilon}\} \bmod 2.$$

By intersection theory, the right-hand side of this last equality is equal to $\dim_{\mathbb{R}} A_{\mathbb{R}} \bmod 2$. □

Remark 3.1. *In the complex case, the Lê-Greuel formula (*[34], [41]*) states that:*

$$\mu(f) + \mu(f,g) = \dim_{\mathbb{C}} A_{\mathbb{C}},$$

where $\mu(f)$ and $\mu(f,g)$ are the Milnor numbers of f and (f,g) and $A_{\mathbb{C}}$ is defined as in the real case.

The natural question that arises after this theorem is to ask if it is possible to get rid of the mod2 in the equality, namely to express:

$$\chi(C_\delta^\varepsilon \cap \{g \geq \alpha\}) - \chi(C_\delta^\varepsilon \cap \{g \leq \alpha\}),$$

or:

$$\chi(C_\delta^\varepsilon) \pm \chi(C_\delta^\varepsilon \cap \{g = \alpha\}),$$

in terms of the signature of a bilinear symmetric form defined on $A_{\mathbb{R}}$. In general, as far as we know, this is still unknown and the question remains open. However, in some cases, the problem is solved. The strategy used is to find $n-k$ analytic function-germs such that $I = \langle f, \ldots, f_k; m_1, \ldots, m_{n-k}\rangle$ and to relate:

$$\chi(C_\delta^\varepsilon \cap \{g \geq \alpha\}) - \chi(C_\delta^\varepsilon \cap \{g \leq \alpha\}),$$

or:

$$\chi(C_\delta^\varepsilon) \pm \chi(C_\delta^\varepsilon \cap \{g = \alpha\}),$$

to the Poincaré-Hopf index at 0 of the following map H:

$$\begin{array}{cccc} H & : & (\mathbb{R}^n,0) & \to & (\mathbb{R}^n,0) \\ & & x & \mapsto & (f_1(x),\ldots,f_k(x);m_1(x),\ldots,m_{n-k}(x)). \end{array}$$

Let us list the cases where this strategy works. If $k=n-1$ then $f^{-1}(0)$ is a curve (or a point). Let $H=(f_1,\ldots,f_{n-1},\frac{\partial(g,f_1,\ldots,f_{n-1})}{\partial(x_1,\ldots,x_n)})$. Then we have:

$$\#\left\{\text{branches of } f^{-1}(0) \mid g>0\right\}-\#\left\{\text{branches of } f^{-1}(0) \mid g<0\right\} = 2(-1)^{n-1}\mathrm{Ind}(H,0).$$

Here a branch is a connected component of $f^{-1}(0)\setminus\{0\}$. This equality was proved in some cases by Aoki, Fukuda, Nishimura and Sun ([2], [3], [4]) and in full generality by Szafraniec [50].

If $k=1$ then f is a function-germ with an isolated critical point at the origin. As already explained above, when $g=x_1$, Fukui [31] proved that:

$$\chi(C_\delta^\varepsilon\cap\{x_1\geq 0\})-\chi(C_\delta^\varepsilon\cap\{x_1\leq 0\})=-\mathrm{sign}(-\delta)^n\mathrm{Ind}(H,0),$$

where $H=(f,\frac{\partial f}{\partial x_2},\ldots,\frac{\partial f}{\partial x_n})$. Note that $(\delta,0)$ is always a regular value of (f,x_1).

In [16], when $n=2,4$ or 8, we were able to construct explicitly a map $H=(f,m_2,\ldots,m_n)$ such that:

$$\chi(C_\delta^\varepsilon\cap\{g\geq\alpha\})-\chi(C_\delta^\varepsilon\cap\{g\leq\alpha\})=-\mathrm{Ind}(H,0).$$

The main idea that we used there is that the fibre $f^{-1}(\delta)$ is parallelizable (like the spheres $S^1\subset\mathbb{R}^2$, $S^3\subset\mathbb{R}^4$ and $S^7\subset\mathbb{R}^8$). This last formula was extended by Fukui and Khovanskii in [32]. They assume that g satisfies the following Condition (P): there exist C^∞ vector fields $v_2,\ldots,v_n$ defined in a neighborhood U of the origin such that:

(1) $v_2(x),\ldots,v_n(x)$ span $T_xg^{-1}(g(x))$ whenever $\nabla g(x)\neq 0$,
(2) $\det[\nabla g(x),v_2(x),\ldots,v_n(x)]>0$.

Let H be defined by:

$$\begin{array}{cccc} H & : & (\mathbb{R}^n,0) & \to & (\mathbb{R}^n,0) \\ & & x & \mapsto & (f(x),v_1f(x),\ldots,v_nf(x)). \end{array}$$

If 0 is isolated in $H^{-1}(0)$, then we have:

$$\chi(C_\delta^\varepsilon\cap\{g\geq 0\})-\chi(C_\delta^\varepsilon\cap\{g\leq 0\})=-\mathrm{sign}(-\delta)^n\mathrm{Ind}(H,0),$$

if $(\delta, 0)$ is a regular value of (f, g). If n is even, we can replace $(\delta, 0)$ with a regular value (δ, α) where $0 \leq |\alpha| \ll |\delta| \ll \varepsilon$. Furthermore, they gave situations where Condition (P) is satisfied:

(1) $n = 2, 4$ or 8 (see [16]),
(2) when $\frac{\partial g}{\partial x_1} \geq 0$,
(3) if $\nabla g^{-1}(0) \cap B_\varepsilon^n \subset \{0\}$ then Condition (P) is satisfied if:
  (a) $n = 2, 4$ or 8,
  (b) or n is even and $n \notin \{2, 4, 8\}$ and $\mathrm{Ind}(\nabla g, 0)$ is even,
  (c) or n is odd and $\mathrm{Ind}(\nabla g, 0) = 0$.

In [24], we continued this work and made some improvements.

3.6. Global versions

In this section, we briefly report on global versions of the previous results.

Let $F = (F_1, \ldots, F_k) : \mathbb{R}^n \to \mathbb{R}^k$ be a polynomial map and let $W = F^{-1}(0)$. Let $G_1, \ldots, G_l : \mathbb{R}^n \to \mathbb{R}$ be polynomials. The problem is to compute the Euler-Poincaré characteristic of $W \cap \{G_1 ?_1 0, \ldots, G_l ?_l 0\}$ where $?_j \in \{<, \leq, \geq, >\}$ for $j \in \{1, \ldots, l\}$, i.e. to express it as a mapping degree or a signature.

If the dimension of the algebra $A = \frac{\mathbb{R}[x_1, \ldots, x_n]}{(F_1, \ldots, F_k)}$ is finite, then W is a finite collection of points. It is possible to express

$$\# W \cap \{G_1 ?_1 0, \ldots, G_l ?_l 0\},$$

in terms of signatures of bilinear symmetric forms defined on A (see [45] and [9]).

In the case where W is a compact algebraic set, Szafraniec [49, 52] and Bruce [13] discovered a signature formula for $\chi(W)$. In [21], we extended it to semi-algebraic sets of the form $W \cap \{x_1 ?_1 0, \ldots, x_k ?_k 0\}$ where W is compact, $k \in \{1, \ldots, n\}$ and $?_j \in \{\leq, \geq\}$.

The Bruce-Szafraniec method does not work if W is not compact. In [51, 53], Szafraniec proved several degree or signature formulas when $F = (F_1, \ldots, F_k)$ with $1 \leq k \leq n - 1$ and $W = F^{-1}(0)$ is a smooth $(n-k)$-dimensional manifold (not necessarily compact). In [15, 18] we gave formulas for some semi-algebraic sets of the form $W \cap \{G ? 0\}$, where $? \in \{\leq, \geq\}$, and of the form $W \cap \{G_1 ?_1 0, G_2 ?_2 0\}$, where $?_1$ and $?_2$ lie in $\{\leq, \geq\}$. In [17, 22] we gave generalizations in some cases where W admits isolated singularities. However, in general when W is not compact, we do not have any signature formula neither for $\chi(W)$ nor for $\chi(W \cap \{G_1 ?_1 0, \ldots, G_l ?_l 0\})$. This seems to be a very difficult problem.

§4. Results on the topology and geometry of semi-algebraic sets

In this chapter, we recall briefly the definition and the main properties of semi-algebraic sets and semi-algebraic maps. Then we give results on the topology and the geometry of semi-algebraic sets, that are singular versions of the results presented in Chapter 2. Our main references for the results of the first section of this chapter are [10] and [11].

4.1. Definitions and important properties

Definition 4.1. *A subset $V \subset \mathbb{R}^n$ is called semi-algebraic if its admits a representation of the form:*

$$V = \bigcup_{i=1}^{s} \bigcap_{j=1}^{r_j} \{x \in \mathbb{R}^n \mid P_{i,j}(x)\sigma_{i,j}0\},$$

where, for each $i = 1, \ldots, s$ and $j = 1, \ldots, r_j$:

$$\sigma_{i,j} \in \{<, =, >\} \text{ and } P_{i,j} \in \mathbb{R}[x_1, \ldots, x_n].$$

Examples:

- Real algebraic sets are semi-algebraic.
- A semi-algebraic subset of $\mathbb{R}$ is either empty or a finite union of intervals (eventually reduced to a point or unbounded).

Let us list some important properties of semi-algebraic sets.

Proposition 4.1. (1) *The family of semi-algebraic sets is closed with respect to the set-theoretic operations of finite union, finite intersection and complementation.*

(2) *A semi-algebraic set has a finite number of connected components and is locally connected.*

(3) *(The Tarski-Seidenberg theorem) The projection of a semi-algebraic set is semi-algebraic.*

(4) *The closure $\bar{X}$ of a semi-algebraic set X, its interior $\mathring{X}$ and its frontier $\bar{X} \setminus \mathring{X}$ are semi-algebraic.*

Definition 4.2. *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be semi-algebraic sets. A map $f : X \to Y$ is called semi-algebraic if its graph is a semi-algebraic set of $\mathbb{R}^{n+m}$.*

Proposition 4.2. *Let $f : X \to Y$ be a semi-algebraic map. Then the image $f(X) \subset Y$ is a semi-algebraic set.*

Theorem 4.1. *(Hardt's theorem) Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be two semi-algebraic sets and let $f : X \to Y$ be a semi-algebraic continuous map. There exists a finite partition of Y into semi-algebraic sets $Y = \sqcup_{j=1}^r Y_j$ such that f is semi-algebraically trivial over each Y_j. This means that there exists a semi-algebraic set F_j and a semi-algebraic homeomorphism $h_j : f^{-1}(Y_j) \to Y_j \times F_j$ such that the following diagram commutes:*

$$\begin{array}{ccc} f^{-1}(Y_j) \subset X & \overset{h_j}{\longrightarrow} & Y_j \times F_j \\ f \searrow & & \swarrow \text{projection} \\ & Y_j \subset Y & \end{array}$$

Moreover if $Z_1, \ldots, Z_q$ are finitely many semi-algebraic subsets of X, we can ask that each trivialization $h_j : f^{-1}(Y_j) \to Y_j \times F_j$ is compatible with all the Z_k's.

Theorem 4.2. *Every semi-algebraic set admits a semi-algebraic and finite Whitney stratification. This means that if $X \subset \mathbb{R}^n$ is a semi-algebraic set then there exists a finite semi-algebraic partition of X, $X = \sqcup_{j=1}^l S_j$, such that each S_j is a smooth semi-algebraic manifold and this partition is a Whitney stratification of X.*

We end this section with an important result on Whitney stratified sets (not necessarily semi-algebraic). Let $X \subset \mathbb{R}^n$ be a closed Whitney stratified set and let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map such that:

(1) $f_{|X}$ is proper,
(2) for each stratum S of X, the restriction $f_{|S} : S \to \mathbb{R}^m$ is a submersion.

Definition 4.3. *We call $f_{|X}$ a proper stratified submersion.*

Theorem 4.3. *(Thom's first isotopy lemma) Let $f_{|X} : X \to \mathbb{R}^m$ be a proper stratified submersion. Then $f_{|X}$ is trivial i.e. there exists a homeomorphism $h : X \to \mathbb{R}^m \times f_{|X}^{-1}(0)$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \overset{h}{\longrightarrow} & \mathbb{R}^m \times f_{|X}^{-1}(0) \\ f_{|X} \searrow & & \swarrow \text{projection} \\ & \mathbb{R}^m & \end{array}$$

4.2. Integration with respect to the Euler characteristic and Poincaré-Hopf type theorems

In this section, we present Viro's method of integration with respect to the Euler characteristic with compact support (see [54]). We derive a Morse type theorem for semi-algebraic functions on semi-algebraic sets.

We first give the definition of the Euler characteristic with compact support, denoted by χ_c. Our definition is specific to the semi-algebraic case and there are more general definitions. If $X \subset \mathbb{R}^n$ is a semi-algebraic set then it is possible to write it in the following way (see [11], Theorem 2.3.6):

$$X = \sqcup_{j=1}^{l} C_j,$$

where C_j is semi-algebraically homeomorphic to $]-1,1[^{d_j}$ (C_j is called a cell of dimension d_j). We set $\chi_c(X) = \sum_{j=1}^{l}(-1)^{d_j}$.

Remark 4.1. *This definition of χ_c does not depend on the cell decomposition.*

Proposition 4.3.
- *If X is compact, then $\chi_c(X) = \chi(X)$,*
- *χ_c is multiplicative: $\chi_c(X \times Y) = \chi_c(X) \times \chi_c(Y)$,*
- *χ_c is additive: $\chi_c(X \sqcup Y) = \chi_c(X) + \chi_c(Y)$,*
- *χ_c is invariant by (semi-algebraic) homeomorphism.*

Examples: $\chi_c(\{*\}) = 1$, $\chi_c(\mathbb{R}) = -1$, $\chi_c([0,+\infty[) = 0$, $\chi_c(\mathbb{R}^2) = 1$,

$\chi(\{(x,y) \in \mathbb{R}^2 \mid x > 0, y > 0\}) = 1$ because the open first quadrant is the product $]0,+\infty[\times]0,+\infty[$.

Remark 4.2. *The Euler characteristic with compact support is not invariant by homotopy.*

Definition 4.4. *Let $X \subset \mathbb{R}^n$ be a semi-algebraic set. A constructible function $\varphi : X \to \mathbb{Z}$ is a $\mathbb{Z}$-valued function that can be written as a finite sum:*

$$\varphi = \sum_{i \in I} m_i 1_{X_i},$$

where X_i is a semi-algebraic subset of X.

The sum and the product of two constructible functions on X are again constructible. The set of constructible functions on X is thus a commutative ring, denoted by $F(X)$.

Definition 4.5. *If $\varphi \in F(X)$ then we set:*

$$\int_X \varphi d\chi_c = \sum_{i \in I} m_i \chi_c(X_i),$$

where $\varphi = \sum_{i \in I} m_i 1_{X_i}$. The integral $\int_X \varphi d\chi_c$ is called the Euler integral of φ.

Definition 4.6. *Let $f : X \to Y$ be a continuous semi-algebraic map and let $\varphi : X \to \mathbb{Z}$ be a constructible function. The pushforward $f_*\varphi$ of φ along f is the function $f_*\varphi : Y \to \mathbb{Z}$ defined by:*

$$f_*\varphi(y) = \int_{f^{-1}(y)} \varphi d\chi_c.$$

Proposition 4.4. *The pushforward of a constructible function is a constructible function.*

Proof. Let us write $\varphi = \sum_{i\in I} m_i 1_{X_i}$. By Hardt's theorem, there is a finite semi-algebraic partition $Y = \sqcup_{j\in J} Y_j$ such that, over each Y_j, there is a semi-algebraic trivialization of f compatible with the X_i's. Hence, since for any $y \in Y$, $f_*\varphi(y)$ is equal to $\sum_{i\in I} m_i \chi_c(X_i \cap f^{-1}(y))$, we see that $f_*\varphi$ is constant on each Y_j. □

Theorem 4.4. *(Fubini's theorem) Let $f : X \to Y$ be a continuous semi-algebraic map and let φ be a constructible function on X. Then we have:*

$$\int_Y f_*\varphi d\chi_c = \int_X \varphi d\chi_c.$$

Proof. We keep the notations of the previous proof. Let $j \in J$ and $y_j \in Y_j$. Then, for every $i \in I$, $f^{-1}(Y_j) \cap X_i$ is semi-algebraically homeomorphic to $Y_j \times (f^{-1}(y_j) \cap X_i)$. Therefore, we have:

$$\int_Y f_*\varphi d\chi_c = \sum_{j\in J} \chi_c(Y_j) f_*\varphi(y_j) = \sum_{j\in J} \chi_c(Y_j) \sum_{i\in I} m_i \chi_c(f^{-1}(y_j) \cap X_i) =$$

$$\sum_{i\in I} m_i \sum_{j\in J} \chi_c(Y_j)\chi_c(f^{-1}(y_j) \cap X_i) = \sum_{i\in I} m_i \sum_{j\in J} \chi_c(f^{-1}(Y_j) \cap X_i) =$$

$$\sum_{i\in I} m_i \chi_c(X_i) = \int_X \varphi d\chi_c.$$

□

Let us give a nice application of this theory. Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set equipped with a finite semi-algebraic Whitney stratification: $X = \sqcup_{\alpha\in\Lambda} S_\alpha$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2-semi-algebraic function.

Definition 4.7. *A point $p \in X$ is a critical point of $f_{|X}$ if it is a critical point of $f_{|S(p)}$, where $S(p)$ is the stratum that contains p.*

Definition 4.8. *If p is an isolated critical point of $f_{|X}$, we define the index of f at p by:*

$$\mathrm{ind}(f, X, p) = 1 - \chi\big(X \cap \{f = f(p) - \delta\} \cap B^n_\varepsilon(p)\big),$$

where $0 < \delta \ll \varepsilon \ll 1$.

Theorem 4.5. *If X is compact and $f_{|X}$ has a finite number of critical points $p_1, \ldots, p_k$ then:*

$$\chi(X) = \sum_{i=1}^{k} \mathrm{ind}(f, X, p_i).$$

Proof. For all $x \in X$, let $\varphi(x) = \chi_c\big(X \cap f^{-1}(x^-) \cap B_\varepsilon^n(x)\big)$ where x^- is a regular value of f close to $f(x)$ with $x^- \leq f(x)$. Note that φ is constructible because $\varphi(x) = 1$ if $x \notin \{p_1, \ldots, p_k\}$.

Applying Fubini's theorem, we get:

$$\int_X \varphi(x) d\chi_c = \int_{\mathbb{R}} \left(\int_{f^{-1}(y)} \varphi(x) d\chi_c \right) d\chi_c.$$

For any $y \in \mathbb{R}$, let y^- be a regular value of $f_{|X}$ close to y with $y^- \leq y$. Let us denote by $q_1, \ldots, q_s$ the critical points of $f_{|X}$ lying in $f^{-1}(y)$. We have:

$$\chi_c\big(X \cap f^{-1}(y^-)\big) = \chi_c\big(X \cap f^{-1}(y^-) \setminus \cup_{i=1}^s B_\varepsilon^n(q_i)\big) + \sum_{i=1}^{s} \chi_c\big(X \cap f^{-1}(y^-) \cap B_\varepsilon^n(q_i)\big) =$$

$$\chi_c\big(X \cap f^{-1}(y) \setminus \cup_{i=1}^s B_\varepsilon^n(q_i)\big) + \sum_{i=1}^{s} \varphi(q_i) =$$

$$\chi_c\big(X \cap f^{-1}(y) \setminus \{q_1, \ldots, q_s\}\big) + \sum_{i=1}^{s} \varphi(q_i) =$$

$$\textstyle\int_{X \cap f^{-1}(y) \setminus \{q_1, \ldots, q_s\}} \varphi(x) d\chi_c(x) + \sum_{i=1}^s \varphi(q_i) = \int_{f^{-1}(y)} \varphi(x) d\chi_c(x).$$

Since X is compact, $f(X)$ is a compact subset of $\mathbb{R}$. Let us choose $]A, B]$ such that $f(X) \subsetneq]A, B]$. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_l$ be the critical values of f. Let us write:

$$]A, B] =]\alpha_0, \alpha_1] \cup]\alpha_1, \alpha_2] \cup \ldots \cup]\alpha_l, \alpha_{l+1}].$$

Since $\chi_c(]a, b]) = 0$ and $f_{|X \cap]\alpha_j, \alpha_{j+1}[}$ is a trivial fibration, we obtain that:

$$\int_{\mathbb{R}} \Big(\int_{f^{-1}(y)} \varphi(x) d\chi_c \Big) d\chi_c = \int_{]A,B]} \chi_c\big(X \cap f^{-1}(y^-)\big) d\chi_c = 0,$$

and so,

$$\int_X \varphi(x) d\chi_c = 0.$$

But:

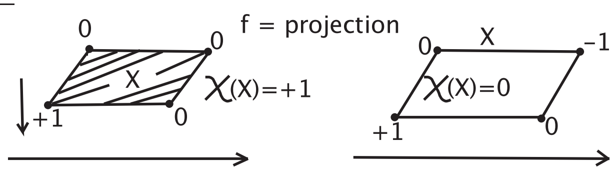
$$\int_X \varphi(x) d\chi_c = \chi_c(X \setminus \{p_1, \ldots, p_k\}) + \sum_{i=1}^{k} 1 - \operatorname{ind}(f, X, p_i),$$

and we find that:

$$0 = \chi(X) - \sum_{i=1}^{k} \operatorname{ind}(f, X, p_i).$$

$\square$

Examples:



In a recent paper [26], we generalized this result to the case of closed semi-algebraic sets. Let us present these results now. Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set equipped with a semi-algebraic finite Whitney stratification $(S_\alpha)_{\alpha \in \Lambda}$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2 semi-algebraic function such that $f_{|X} : X \to \mathbb{R}$ has a finite number of critical points $p_1, \ldots, p_k$.

Definition 4.9. *Let $* \in \{\leq, =, \geq\}$. We define Λ_f^* by:*

$$\Lambda_f^* = \Big\{\alpha \in \mathbb{R} \mid \beta \mapsto \chi\big(\mathrm{Lk}^\infty(X \cap \{f * \beta\})\big) \textit{ is not constant } \textit{in a neighborhood of } \alpha\Big\}.$$

Here $\mathrm{Lk}^\infty(Y) = Y \cap S_R^{n-1}$, $R \gg 1$, for any semi-algebraic set Y of $\mathbb{R}^n$.

Lemma 4.1. *The sets $\Lambda_f^{\leq}$, $\Lambda_f^{=}$ and $\Lambda_f^{\geq}$ are finite.*

We can write $\Lambda_f^{\leq} = \{a_1, \ldots, a_r\}$ where $a_1 < a_2 < \ldots < a_r$ and:

$$\mathbb{R} \setminus \Lambda_f^{\leq} =]-\infty, a_1[\cup]a_1, a_2[\cup \cdots \cup]a_{r-1}, a_r[\cup]a_r, +\infty[.$$

On each connected component of $\mathbb{R} \setminus \Lambda_f^{\leq}$, the function $\beta \mapsto \chi\big(\mathrm{Lk}^\infty(X \cap \{f \leq \beta\}\big)$ is constant. For each $j \in \{0, \ldots, r\}$, let a_j^+ be an element of $]a_j, a_{j+1}[$ where $a_0 = -\infty$ and $a_{r+1} = +\infty$.

Theorem 4.6. *We have:*

$$\chi(X) = \sum_{i=1}^{k} \operatorname{ind}(f, X, p_i) + \sum_{j=0}^{r} \chi\big(\mathrm{Lk}^\infty(X \cap \{f \leq a_j^+\})\big)$$

$$-\sum_{j=1}^{r} \chi\big(\mathrm{Lk}^\infty(X \cap \{f \leq a_j\})\big).$$

Similarly, we can write $\Lambda_f^{\geq} = \{b_1, \ldots, b_s\}$ with $b_1 < b_2 < \cdots < b_s$ and:

$$\mathbb{R} \setminus \Lambda_f^{\geq} =]-\infty, b_1[\cup]b_1, b_2[\cup \cdots \cup]b_{s-1}, b_s[\cup]b_s, +\infty[.$$

For each $i \in \{0, \ldots, s\}$, let b_i^+ be an element in $]b_i, b_{i+1}[$ with $b_0 = -\infty$ and $b_{s+1} = +\infty$.

Theorem 4.7. *We have:*

$$\chi(X) = \sum_{i=1}^{k} \mathrm{ind}(-f, X, p_i) + \sum_{j=0}^{s} \chi\big(\mathrm{Lk}^\infty(X \cap \{f \geq b_j^+\})\big) - \sum_{j=1}^{s} \chi\big(\mathrm{Lk}^\infty(X \cap \{f \geq b_j\})\big).$$

Let us write $\Lambda_f^{=} = \{c_1, \ldots, c_t\}$ with $c_1 < c_2 < \ldots < c_t$ and:

$$\mathbb{R} \setminus \Lambda_f^{=} =]-\infty, c_1[\cup]c_1, c_2[\cup \cdots \cup]c_{t-1}, c_t[\cup]c_t, +\infty[.$$

For each $i \in \{0, \ldots, t\}$, let c_i^+ be an element in $]c_i, c_{i+1}[$.

Theorem 4.8. *We have:*

$$2\chi(X) - \chi\big(\mathrm{Lk}^\infty(X)\big) = \sum_{i=1}^{k} \mathrm{ind}(f, X, p_i) + \sum_{i=1}^{k} \mathrm{ind}(-f, X, p_i) + \sum_{j=0}^{t} \chi\big(\mathrm{Lk}^\infty(X \cap \{f = c_j^+\})\big) - \sum_{j=1}^{t} \chi\big(\mathrm{Lk}^\infty(X \cap \{f = c_j\})\big).$$

When $X = \mathbb{R}^n$, we find global versions of the Khimshiashvili formula and the Arnol'd-Wall formula presented in Chapter 3.

Theorem 4.9. *We have:*

$$1 = \deg_\infty \nabla f + \sum_{j=0}^{r} \chi\big(\mathrm{Lk}^\infty(\{f \leq a_j^+\})\big) - \sum_{j=1}^{r} \chi\big(\mathrm{Lk}^\infty(\{f \leq a_j\})\big) =$$

$$(-1)^n \deg_\infty \nabla f + \sum_{j=0}^{s} \chi\big(\mathrm{Lk}^\infty(\{f \geq b_j^+\})\big) - \sum_{j=1}^{s} \chi\big(\mathrm{Lk}^\infty(\{f \geq b_j\})\big).$$

If n is even then we have:

$$2 = 2\deg_\infty \nabla f + \sum_{j=0}^{t} \chi\big(\mathrm{Lk}^\infty(\{f = c_j^+\})\big) - \sum_{j=1}^{t} \chi\big(\mathrm{Lk}^\infty(\{f = c_j\})\big).$$

Here $\deg_\infty \nabla f$ is the degree of the map $\frac{\nabla f}{|\nabla f|} : S_R^{n-1} \to S^{n-1}$ where S_R^{n-1} is a sufficiently big sphere.

Remark 4.3. *The third equality of this last theorem was discovered by Sekalski* [47] *for $n = 2$.*

4.3. Gauss-Bonnet type theorems

In Chapter 2 of this mini-course, we gave a Gauss-Bonnet formula for a smooth compact hypersurface $M \subset \mathbb{R}^n$. Here we present a version for smooth submanifolds of $\mathbb{R}^n$ and afterwards we give semi-algebraic versions of the Gauss-Bonnet theorem.

4.4. Smooth case

Let $M \subset \mathbb{R}^n$ be a smooth submanifold of dimension d ($1 \leq d \leq n-1$). Let $x \in M$ and let S_x denote the unit sphere in $(T_x M)^\perp$. Let $v \in S_x$ and let $II_{x,v}$ be the second fundamental form of M at x along the vector v. It is defined as follows:

$$II_{x,v}(x_1, x_2) = -\langle DV(x)(x_1), x_2 \rangle,$$

where:

- V is a vector field in $\mathbb{R}^n$ normal to M such that $V(x) = v$,
- $x_1, x_2 \in T_x M$.

The form $II_{x,v}$ is bilinear and symmetric.

Definition 4.10. *For $i \in \{0, \ldots, d\}$ and for $x \in M$, we define $K_i(x)$ by:*

$$K_i(x) = \int_{S_x} \sigma_i(II_{x,v}) dv,$$

where σ_i is the i-th elementary symmetric function of the eigenvalues of $II_{x,v}$. We call K_i the i-th Lipschitz-Killing curvature.

Remark 4.4.

- *If i is odd then $K_i(x) = 0$.*
- *The quantity $\frac{1}{s_{n-d+i-1}} K_i$ is intrinsic (here s_k is the volume of S^k).*

The following Gauss-Bonnet theorem is due to Fenchel [29] and Allendoerfer [1].

Theorem 4.10. *(Gauss-Bonnet theorem) If M is compact then:*

$$\chi(M) = \frac{1}{s_{n-1}} \int_M K_d(x)dx.$$

Remark 4.5. *This theorem is trivial if M is odd-dimensional because both sides of the equality vanish.*

The following theorem is due to Weyl [56].

Theorem 4.11. *(Volume of the tube) If $r > 0$ is small enough, then:*

$$\mathrm{vol}(\mathrm{Tub}_r(M)) = \sum_{i=0}^{d} \frac{1}{n-d+i} \int_M K_i(x)dx \cdot r^{n-d+i}.$$

Here $\mathrm{Tub}_r(M)$ *is the tubular neighborhood of radius r around M.*

Example: Let $\mathcal{C} \subset \mathbb{R}^3$ be the circle centered at the origin and of radius R. Then $\mathrm{Tub}_r(\mathcal{C})$ is a torus. Applying the previous theorem, we obtain:

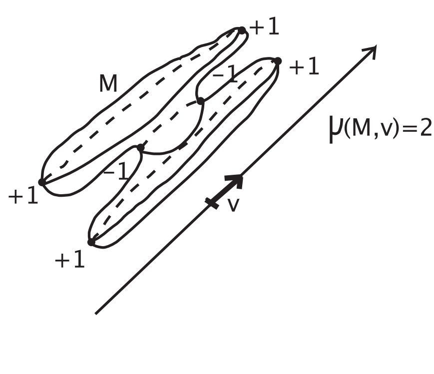
$$\mathrm{vol}(\mathrm{Tub}_r(\mathcal{C})) = \frac{1}{3-1} \int_{\mathcal{C}} \mathrm{vol}(S^1)dxr^2 = \frac{1}{2}(2\pi)(2\pi R)r^2.$$

Hence we recover the well-known result:

$$\mathrm{vol}(\mathrm{Tub}_r(\mathcal{C})) = 2\pi^2 R r^2.$$

4.4.1. *Exchange formulas* In this subsection, we explain how to give a topological proof of the Gauss-Bonnet theorem using Morse theory.

Let $M \subset \mathbb{R}^n$ be a smooth submanifold of dimension d ($1 \le d \le n-1$). For almost all $v \in S^{n-1}$, the function $v^*_{|M} : M \to \mathbb{R}$, $x \mapsto \langle v, x \rangle$ is a Morse function and hence admits isolated non-degenerate critical points $\{p_i\}_{i \in I}$, with respective indices $\{\lambda_i\}_{i \in I}$. Let $U \subset M$ be a bounded borelian set. We set $\mu(U, v) = \sum_{i \mid p_i \in U} (-1)^{\lambda_i}$ (this sum is finite since U is bounded).



The following proposition is proved in [40].

Proposition 4.5. *(Exchange formula) We have:*

$$\int_U K_d(x)dx = \int_{S^{n-1}} \mu(U,v)dv.$$

As a corollary, we recover the above Gauss-Bonnet formula.

Corollary 4.1. *If M is compact, then we have:*

$$\int_M K_d(x)dx = s_{n-1}\chi(M).$$

4.4.2. *Singular semi-algebraic case* Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set equipped with a finite and semi-algebraic Whitney stratification: $X = \sqcup_{\alpha\in\Lambda} S_\alpha$.

Lemma 4.2. *There exists a semi-algebraic set $\Gamma_1(X) \subset S^{n-1}$ of dimension strictly less than $n-1$ such that if $v \notin \Gamma_1(X)$, then $v^*_{|X}$ has a finite number of critical points $p_1^v, \ldots, p_{l_v}^v$.*

Definition 4.11. *Let U be a bounded borelian set of X. We set:*

$$\Lambda_0(X,U) = \frac{1}{s_{n-1}} \int_{S^{n-1}} \sum_{x\in U} \mathrm{ind}(v^*, X, x)dv,$$

where $\mathrm{ind}(v^*, X, x) = 0$ *if x is not a critical point of $v^*_{|X}$. The measure $\Lambda_0(X,-)$ is called the Gauss-Bonnet measure.*

The following results are due to Broecker and Kuppe [12] and Fu [30].

Proposition 4.6. (1) *(Gauss-Bonnet theorem) If X is compact, then we have:*

$$\Lambda_0(X,X) = \chi(X).$$

(2) *The measure $\Lambda_0(X,-)$ is invariant by semi-algebraic isometries.*

Now we explain how to generalize the above Gauss-Bonnet theorem when X is only closed (see [23] and [26]).

Lemma 4.3. *There exists a semi-algebraic set $\Gamma_2(X)$ of S^{n-1} of dimension strictly less than $n-1$ such that if $v \notin \Gamma_2(X)$, $\Lambda^{\geq}_{v^*} = \Lambda^{\leq}_{v^*} = \Lambda^{=}_{v^*} = \emptyset$.*

Corollary 4.2. *If $v \notin \Gamma_1(X) \cup \Gamma_2(X)$ then for all $\alpha \in \mathbb{R}$, we have:*

$$\chi\big(\mathrm{Lk}^\infty(X \cap \{v^* = \alpha\})\big) = 2\chi(X) - \chi\big(\mathrm{Lk}^\infty(X)\big) - \sum_{i=1}^{l_v} \mathrm{ind}(v^*, X, p_i) - \sum_{i=1}^{l_v} \mathrm{ind}(-v^*, X, p_i).$$

Proof. Apply Theorem 4.8. □

Let $(K_R)_{R>0}$ be an exhaustive family of compact sets of X, that is a family $(K_R)_{R>0}$ of compact sets of X such that $\cup_{R>0} K_R = X$ and $K_R \subseteqq K_{R'}$ if $R \leq R'$. For every $R > 0$, we have:

$$\Lambda_0(X, X \cap K_R) = \frac{1}{s_{n-1}} \int_{S^{n-1}} \sum_{x \in X \cap K_R} \mathrm{ind}(v^*, X, x) dv.$$

Moreover the following limit:

$$\lim_{R \to +\infty} \sum_{x \in X \cap K_R} \mathrm{ind}(v^*, X, x),$$

is equal to $\sum_{x \in X} \mathrm{ind}(v^*, X, x)$, which is uniformly bounded by Hardt's theorem. Applying Lebesgue's theorem, we obtain:

$$\lim_{R \to +\infty} \Lambda_0(X, X \cap K_R) = \frac{1}{s_{n-1}} \int_{S^{n-1}} \lim_{R \to +\infty} \sum_{x \in X \cap K_R} \mathrm{ind}(v^*, X, x) dv = \frac{1}{s_{n-1}} \int_{S^{n-1}} \sum_{x \in X} \mathrm{ind}(v^*, X, x) dv.$$

Definition 4.12. *We set:*

$$\Lambda_0(X, X) = \lim_{R \to +\infty} \Lambda_0(X, X \cap K_R),$$

where $(K_R)_{R>0}$ is an exhaustive family of compact sets of X.

Theorem 4.12. *If X is a closed semi-algebraic set then:*

$$\Lambda_0(X, X) = \chi(X) - \frac{1}{2}\chi\big(\mathrm{Lk}^\infty(X)\big) - \frac{1}{2s_{n-1}} \int_{S^{n-1}} \chi\big(\mathrm{Lk}^\infty(X \cap \{v^* = 0\})\big) dv.$$

Proof. We have:

$$\Lambda_0(X, X) = \frac{1}{s_{n-1}} \int_{S^{n-1}} \sum_{x \in X} \mathrm{ind}(v^*, X, x) dv =$$

$$\frac{1}{2s_{n-1}}\int_{S^{n-1}}\sum_{x\in X}\operatorname{ind}(v^*,X,x)+\operatorname{ind}(-v^*,X,x)dv=$$

$$\frac{1}{2s_{n-1}}\int_{S^{n-1}}2\chi(X)-\chi\big(\operatorname{Lk}^\infty(X)\big)-\chi\big(\operatorname{Lk}^\infty(X\cap\{v^*=0\})\big)dv,$$

by Corollary 4.2. □

If X is smooth of dimension d, $1\leq d\leq n-1$, then:

$$\Lambda_0(X,X)=\frac{1}{s_{n-1}}\int_X K_d(x)dx.$$

If d is even then $\chi\big(\operatorname{Lk}^\infty(X)\big)=0$ because $\operatorname{Lk}^\infty(X)$ is a compact odd-dimensional manifold. Furthermore, $\operatorname{Lk}^\infty(X\cap\{v^*=0\})$ is equal to $X\cap\{v^*=0\}\cap S_R^{n-1}$ where $R\gg 1$; it is thus the boundary of a compact odd-dimensional manifold with boundary and therefore its Euler characteristic is equal to $2\chi(X\cap\{v^*=0\}\cap B_R^n)$, which is actually $2\chi(X\cap\{v^*=0\})$. Hence, if d is even, the Gauss-Bonnet formula takes the following form:

$$\frac{1}{s_{n-1}}\int_X K_d(x)dx=\chi(X)-\frac{1}{s_{n-1}}\int_{S^{n-1}}\chi(X\cap\{v^*=0\})dv.$$

<u>Examples:</u>

(1) Let $V_1=\{(x_1,x_2,x_3)\in\mathbb{R}^3\mid x_1^2+x_2^2-x_3^2-1=0\}$ be the one-sheeted hyperboloid. We have $\int_{V_1}K_2(x)dx=-4\pi\sqrt{2}$.

(2) Let $V_2=\{(x_1,x_2,x_3)\in\mathbb{R}^3\mid x_1^2-x_2^2-x_3^2-1=0\}$ be the two-sheeted hyperboloid. We have $\int_{V_2}K_2(x)dx=4\pi(2-\sqrt{2})$.

(3) Let $V_3=\{(x_1,x_2,x_3)\in\mathbb{R}^3\mid x_1^2+x_2^2-x_3=0\}$ be the elliptic paraboloid. We have $\int_{V_3}K_2(x)dx=4\pi$.

(4) Let $V_4=\{(x_1,x_2,x_3)\in\mathbb{R}^3\mid x_1^2-x_2^2-x_3=0\}$ be the hyperbolic paraboloid. We have $\int_{V_4}K_2(x)dx=-4\pi$.

If d is odd then:

$$\Lambda_0(X,X)=\frac{1}{s_{n-1}}\int_X K_d(x)dx=0,$$

and,

$$\chi(\operatorname{Lk}^\infty(X\cap\{v^*=0\})=0,$$

because $\operatorname{Lk}^\infty(X\cap\{v^*=0\})$ is an odd-dimensional compact manifold. Furthermore for the same reasons as above, $\chi(X)=\frac{1}{2}\chi(\operatorname{Lk}^\infty(X))$. So, in case of an odd-dimensional closed semi-algebraic manifold, the above Gauss-Bonnet theorem is trivial, as in the compact case. However, the Euler characteristic of such a manifold is not necessarily zero and one can ask if it is possible to express it in terms of curvatures. This is actually the aim of the following theorem that we proved in [25].

Theorem 4.13. *Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set which is a smooth submanifold of dimension d, $1 \le d \le n-1$. If d is even, we have:*

$$\chi(X) = \frac{1}{s_{n-1}} \int_X K_d(x)dx + \sum_{i=0}^{\frac{d-2}{2}} \lim_{R\to+\infty} \frac{1}{s_{n-d+2i-1} b_{d-2i} R^{d-2i}} \int_{X\cap B_R} K_{2i} dx,$$

where b_i denotes the volume of the unit ball of dimension i. If d is odd, we have:

$$\chi(X) = \sum_{i=0}^{\frac{d-1}{2}} \lim_{R\to+\infty} \frac{1}{s_{n-d+2i-1} b_{d-2i} R^{d-2i}} \int_{X\cap B_R} K_{2i} dx.$$

Examples:

- If $d = 1$ then X is a smooth semi-algebraic curve. The above formula just states that the number of non-compact connected components of X is equal to $\lim_{R\to+\infty} \frac{\text{length}(X\cap B_R^n)}{2R}$.
- If X is of dimension 3, then the formula relates $\chi(X)$ to the volume form and the scalar curvature K_2. Namely, we have:

$$\chi(X) = \lim_{R\to+\infty} \frac{1}{s_{n-2} b_1 R} \int_{X\cap B_R} K_2 dx + \lim_{R\to+\infty} \frac{\text{vol}(X\cap B_R^n)}{b_3 R^3}.$$

  Let us give an application of this equality. If $K_2 > 0$ then $\chi(X) > 0$ and $\chi(\text{Lk}^\infty(X)) > 0$. If the link $\text{Lk}^\infty(X)$ is orientable then we can conclude that $\text{Lk}^\infty(X)$ has at least one connected component homeomorphic to S^2.

Let us end with some remarks and questions.

(1) A version of Theorem 4.12 was proved by Dillen and Kuehnel for submanifolds with finitely many cone-like ends in [14].
(2) A version of Theorem 4.13 was proved by Shiohama in [48] for a class of riemaninan surfaces (i.e. $d = 2$).
(3) Is it possible to enlarge the class of riemannian manifolds for which a similar formula is valid ?
(4) Is it possible to replace in Theorem 4.13 the distance on $\mathbb{R}^n$ with the intrinsic distance on X, in order to get a fully intrinsic formula ?

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Aix-Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France.
E-mail address: nicolas.dutertre@univ-amu.fr