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Survey of apparent contours of stable maps between surfaces

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Dedicated to Professor Masahiko Suzuki on his 60th birthday

Abstract.

This is a survey paper about studies of the simplest shape of the apparent contour for stable maps between surfaces. Such studies first appeared in [10] then in [1], [3], [6], [20], [22]. Let M be a connected and closed surface, N a connected surface. For a stable map $\varphi\colon M \to N$, denote by $c(\varphi)$, $n(\varphi)$ and $i(\varphi)$ the numbers of cusps, nodes and singular set components of φ, respectively. For a C^∞ map $\varphi_0\colon M \to S^2$ into the sphere, we study the minimal pair $(i, c+n)$ and triples (i, c, n), (c, i, n), (n, c, i) and (i, n, c) among stable maps $M \to S^2$ homotopic to φ_0 with respect to the lexicographic order.

§1. Introduction

Let M be a connected and closed surface, N a connected surface. For a C^∞ map $\varphi\colon M \to N$, $S(\varphi)$ denotes the set of singular points of φ. Call $\varphi(S(\varphi))$ the *apparent contour* (*contour* for short), and denote it by $\gamma(\varphi)$.

A C^∞ map $\varphi\colon M \to N$ is said to be *stable* if it satisfies the following two properties.

(1) For each $p \in M$, the map germ at $p \in M$ is C^∞ right-left equivalent to one of the map germs at $0 \in \mathbb{R}^2$ below:
$(a, x) \mapsto (a, x)$: a *regular point*,
$(a, x) \mapsto (a, x^2)$: a *fold point*,
$(a, x) \mapsto (a, x^3 + ax)$: a *cusp point*.
Hence, $S(\varphi)$ is a finite disjoint union of circles.

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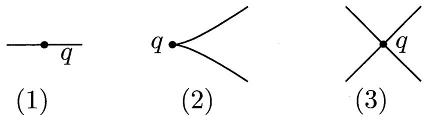

Fig. 1. The multi-germs of $\varphi|_{S(\varphi)}$

(2) For each $q \in \gamma(\varphi)$, $\varphi^{-1}(q) \cap S(\varphi)$ consists of at most two points and the multi-germ $(\varphi|_{S(\varphi)}, \varphi^{-1}(q) \cap S(\varphi))$ is right-left equivalent to one of the three multi-germs as depicted in Fig. 1: (1) corresponds to a single fold point, (2) corresponds to a cusp point, and (3) represents a normal crossing of two immersion germs, each of which corresponds to a fold point. The normal crossing point in the target as in (3) is called a *node.*

According to a classical result of Whitney [17], stable maps form an open dense subset in the space of all C^∞ maps $M \to N$ with respect to the Whitney C^∞ topology.

For a stable map $\varphi\colon M \to N$, denote by $c(\varphi)$, $n(\varphi)$ and $i(\varphi)$ the numbers of cusps, nodes and connected components of $S(\varphi)$, respectively.

In this paper, we study stable maps with non-empty set of singular points.

Let $\mathbb{A}$ be an ordered pair or triple consisting of some elements of $\{c, i, n, c+n\}$. For a stable map $\varphi\colon M \to N$, denote by $\mathbb{A}(\varphi)$ the ordered pair or triple consisting of the corresponding elements of $\{c(\varphi), i(\varphi), n(\varphi), c(\varphi) + n(\varphi)\}$. For a C^∞ map $\varphi_0\colon M \to N$, we say that a stable map $\varphi\colon M \to N$ has an $\mathbb{A}$-*minimal contour for* φ_0 if $\mathbb{A}(\varphi)$ is minimal with respect to the lexicographic order among those stable maps which are homotopic to φ_0. In this case, we also say that the contour $\gamma(\varphi)$ is $\mathbb{A}$-*minimal.* Pignoni [10] introduced the notion of a minimal contour, which corresponds to that of an $(i, c+n)$-minimal contour in our terminology, and studied such minimal contours for C^∞ maps $M \to \mathbb{R}^2$ of closed surfaces into the plane.

In this paper, $(i, c+n)$-minimal contours, (i, c, n)-minimal contours, (c, i, n)-minimal contours, (n, c, i)-minimal contours, and (i, n, c)-minimal contours for C^∞ maps $M \to S^2$ of closed surfaces into the sphere are studied.

This paper is organized as follows. In §2, $(i, c+n)$-minimal contours are studied. In §3, (i, c, n)-minimal contours are studied. In §4, (c, i, n)-minimal contours, (n, c, i)-minimal contours and (i, n, c)-minimal contours are studied. In §5, some problems about the topology of stable maps between manifolds are posed. In §6, some inductive constructions of stable maps between surfaces are introduced.

Throughout this paper, all surfaces and manifolds, and maps between them are of class C^∞. Furthermore, all surfaces and manifolds are assumed to be connected. The symbols $d, g \geq 0$ and $h \geq 0$ denote integers unless otherwise stated. For a topological space X, id_X denotes the identity map of X. The orientable (resp. non-orientable) and closed surface of genus g, that is the connected sum of g copies of the 2-dimensional torus T^2 (resp. the projective plane $\mathbb{R}P^2$) is denoted by Σ_g (resp. F_g). The 2-dimensional sphere and the plane are denoted by S^2 and $\mathbb{R}^2$ respectively. For two manifolds M_1 and M_2, the symbol $M_1 \# M_2$ denotes the connected sum of M_1 and M_2. Each orientable surface is given an orientation, although it will not be explicitly mentioned.

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§2. $(i, c+n)$-Minimal contours

Pignoni [10] introduced the notion of an $(i, c{+}n)$-minimal contour for a C^∞ map between surfaces and studied that for a C^∞ map $M \to \mathbb{R}^2$ of a closed surface. Then, Demoto [1] studied an $(i, c+n)$-minimal contour for a C^∞ map between S^2. Kamenosono and the author [6] studied an $(i, c+n)$-minimal contour for a C^∞ map of a closed surface into the sphere (Theorems 2.1 and 2.3 below). Note that for a C^∞ map $\varphi_0\colon M \to S^2$ (or a C^∞ map $\varphi_0\colon M \to \mathbb{R}^2$), there exists a stable map φ homotopic to φ_0 such that $S(\varphi)$ consists of one component, see [2, Theorem 4.8] for the details.

If two C^∞ maps f_1 and $f_2\colon \Sigma_g \to N$ into an oriented surface N are homotopic, then their mapping degrees coincide. Furthermore, f_1 and $f_2\colon \Sigma_g \to S^2$ are homotopic if and only if their degrees coincide, see [9] for the details. Thus, a homotopy class of a C^∞ map $\Sigma_g \to S^2$ is characterized by the pair of the mapping degree and the genus g.

Theorem 2.1 ([1], [6]). *Let $f\colon \Sigma_g \to S^2$ be a degree $d \geq 0$ stable map. The contour $\gamma(f)$ is $(i, c+n)$-minimal if and only if $i(f) = 1$ and the pair (c, n) for $\gamma(f)$ is one of the following:*

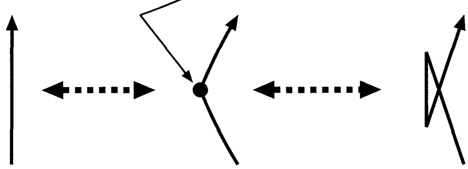


Fig. 2. Swallow-tail singularity

$$(c,n) = \begin{cases} (2d,0) & \text{if } g=0, \\ (2(d-1),4) \text{ or } (2d+2,0) & \text{if } d \neq 0 \text{ and } g=1, \\ (2,4) \text{ or } (6,0) & \text{if } (d,g)=(1,2), \\ (2(d-g),2g+2) & \text{if } d \geq g > 1, \\ (0,d+g+2) & \text{if } g \neq 0,\ d \leq g,\ d \equiv g \pmod 2, \\ & \qquad \text{and } (d,g) \neq (1,1), \\ (2,d+g+1) & \text{if } g \neq 0,\ d < g,\ d \not\equiv g \pmod 2, \\ & \qquad \text{and } (d,g) \neq (1,2). \end{cases}$$

Note that for a stable map $\varphi\colon M \to S^2$ of a closed surface, the number of cusps of φ and the Euler characteristic of M have the same parity, see [15] for the details.

Theorem 2.1 implies the following.

Corollary 2.2. *For a stable map $f\colon \Sigma_g \to S^2$, if the contour $\gamma(f)$ is $(i, c+n)$-minimal, then the number of nodes $n(f)$ is even.*

Note that there exists a stable map $\Sigma_g \to S^2$ whose number of nodes is odd. Fig. 2 shows the idea for constructing such a stable map.

It is known that two C^∞ maps h_1 and $h_2\colon F_g \to S^2$ are homotopic if and only if their modulo two degrees coincide.

Theorem 2.3 ([6]). *Let $h\colon F_g \to S^2$ $(g \geq 1)$ be a modulo two degree d_2 stable map. The contour $\gamma(h)$ is $(i, c+n)$-minimal if and only* *if* $i(h) = 1$ *and the pair* (c, n) *for* $\gamma(h)$ *is one of the following*:

$$(c,n) = \begin{cases} (3,0) & \textit{if } (d_2, g) = (1,1), \\ (4,0) \textit{ or } (0,4) & \textit{if } (d_2, g) = (1,2), \\ (1, (g+5)/2) & \textit{if } d_2 = 1,\ g \textit{ is odd, and } (d_2, g) \neq (1,1), \\ (0, (g+6)/2) & \textit{if } d_2 = 1,\ g \textit{ is even, and } (d_2, g) \neq (1,2), \\ (3, (g+1)/2) & \textit{if } d_2 = 0 \textit{ and } g \textit{ is odd}, \\ (0, (g+4)/2) & \textit{if } d_2 = 0,\ g \textit{ is even, and } g/2 \textit{ is even}, \\ (2, (g+2)/2) & \textit{if } d_2 = 0,\ g \textit{ is even, and } g/2 \textit{ is odd}. \end{cases}$$

In the following, we give the outline of a proof of Theorem 2.1, see [6] for the details of the proof.

Let us introduce some notations concerning the apparent contour of a stable map $M \to S^2$ of a closed surface.

Let M be a closed surface and $\varphi\colon M \to S^2$ a stable map whose contour is non-empty. Let $S(\varphi) = S_1 \cup \cdots \cup S_\ell$ be the decomposition of $S(\varphi)$ into the connected components and set $\gamma_i = \varphi(S_i)$ $(i = 1, \ldots, \ell)$. Note that $\gamma(\varphi) = \gamma_1 \cup \cdots \cup \gamma_\ell$. Let $m(\varphi)$ be the smallest number of elements in the set $\varphi^{-1}(y)$, where $y \in S^2$ runs over all regular values of φ. Fix a regular value ∞ such that $\varphi^{-1}(\infty)$ consists of $m(\varphi)$ points. For each γ_i, denote by U_i the component of $S^2 \setminus \gamma_i$ which contains ∞. Note that $\partial U_i \subset \gamma_i$.

Orient γ_i so that at each fold point image, the surface is "folded to the left hand side". More precisely, for a point $y \in \gamma_i$ which is not a cusp or a node, choose a normal vector v of γ_i at y such that $\varphi^{-1}(y')$ contains more elements than $\varphi^{-1}(y)$, where y' is a regular value of φ close to y in the direction of v. Let τ be a tangent vector of γ_i at y such that the ordered pair (τ, v) is compatible with the given orientation of S^2. It is easy to see that τ gives a well-defined orientation for γ_i.

Definition 2.4. A point $y \in \partial U_i \setminus \{\text{cusps, nodes}\}$ is said to be *positive* if the normal vector v at y points toward U_i. Otherwise, it is said to be *negative.*

A component γ_i is said to be *positive* if all points of $\partial U_i \setminus \{\text{cusps, nodes}\}$ are positive; otherwise, γ_i is said to be *negative.* The number of positive (or negative) components is denoted by i^+ (resp. i^-). Note that there is at least one negative component unless $S(\varphi) = \emptyset$.

Definition 2.5. A point $y \in \partial U_i \setminus \{\text{cusps, nodes}\}$ is called an *admissible starting point* if y is a positive (or negative) point of a positive (resp. negative) component γ_i. Note that for each γ_i, there always exists an admissible starting point on it.

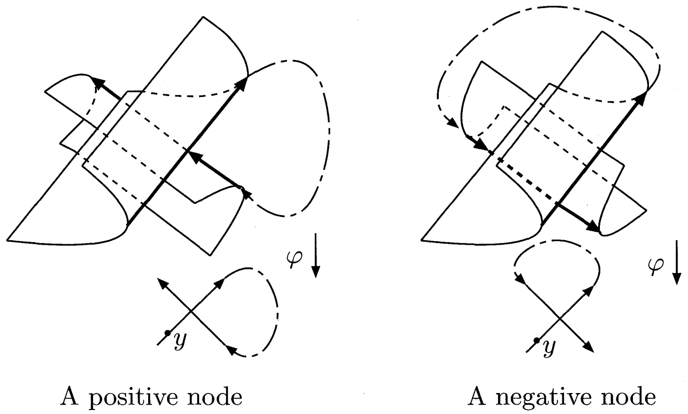


Fig. 3. A positive node and a negative node

Definition 2.6. Let $y \in \gamma_i$ be an admissible starting point and $Q \in \gamma_i$ a node. Let $\alpha\colon [0,1] \to \gamma_i$ be a C^∞ parameterization consistent with the orientation which is singular only when the image is a cusp such that $\alpha^{-1}(y) = \{0,1\}$. Then, there are two numbers $0 < t_1 < t_2 < 1$ satisfying $\alpha(t_1) = \alpha(t_2) = Q$. We say that Q is *positive* if the orientation of S^2 at Q defined by the ordered pair $(\alpha'(t_1), \alpha'(t_2))$ coincides with that of S^2 at Q; *negative*, otherwise. See Fig. 3 for the details.

The number of positive (or negative) nodes on γ_i is denoted by N_i^+ (resp. N_i^-). The definition of a positive (or negative) node on γ_i depends on the choice of an admissible starting point y. However, it is known that the difference $N_i^+ - N_i^-$ does not depend on the choice of y, see [16] for the details. Thus, the number $N^+ - N^- = \sum_{i=1}^{\ell}(N_i^+ - N_i^-)$ is well defined. Note that nodes arising from $\gamma_i \cap \gamma_j$ $(i \neq j)$ play no role in the computation.

Then, we obtain the following formula as an easy application of Pignoni's one [10].

Proposition 2.7 ([6]). *For a stable map $\varphi\colon M \to S^2$ of a closed surface of genus g, we have*

$$(1) \qquad g = \varepsilon(M)\left((N^+ - N^-) + \frac{c(\varphi)}{2} + (1 + i^+ - i^-) - m(\varphi)\right),$$

where $\varepsilon(M)$ is equal to one if M is orientable and two otherwise.

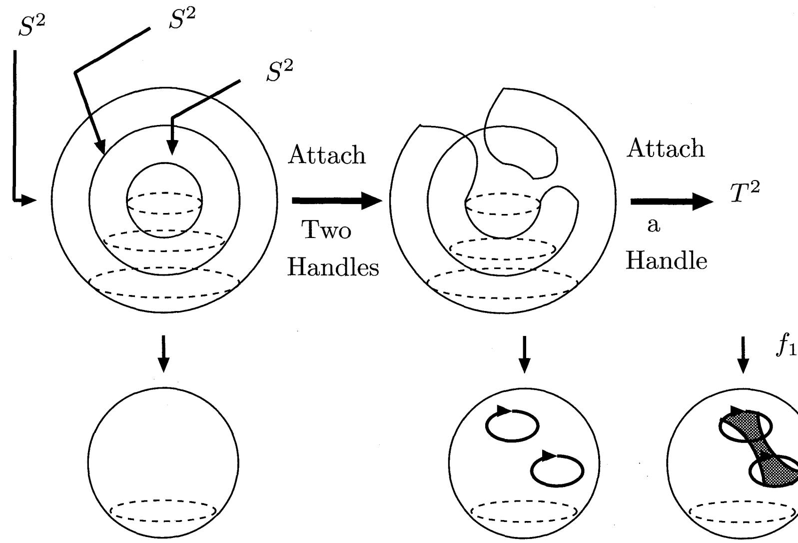


Fig. 4. Stable map $f_1\colon T^2 \to S^2$ of degree one

Let us consider an $(i, c+n)$-minimal contour for a degree one C^∞ map $f_0\colon T^2 \to S^2$. To prove Theorem 2.1, we need the following lemma.

Lemma 2.8 ([6]). *Let $f\colon M \to S^2$ be a stable map such that $S(f)$ consists of one component. If $\gamma(f)$ has a node, then $N^- \geq 1$.*

Let $f\colon T^2 \to S^2$ be a degree one stable map such that $S(f)$ consists of one component. Then, formula (1) implies that

$$1 = (N^+ - N^-) + \frac{c(f)}{2} - m(f). \tag{2}$$

Thus, if $\gamma(f)$ has a node, then Lemma 2.8 implies

$$c(f) + n(f) = 1 + \frac{c(f)}{2} + 2N^- + m(f) \geq 1 + 0 + 2 + 1 = 4.$$

If $\gamma(f)$ has no nodes, then we have $c(f) \geq 4$. Hence, f satisfies $c(f) + n(f) \geq 4$.

Note that equation (2) shows that there is no degree one stable map $f\colon T^2 \to S^2$ whose triple (i, c, n) is equal to $(1, 2, 2)$.

Thus, the contours of degree one stable maps f_1 and $f_2\colon T^2 \to S^2$ in Figs. 4 and 5, respectively, are $(i, c+n)$-minimal.

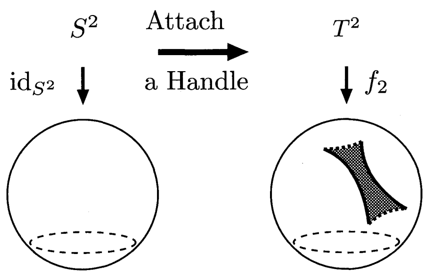


Fig. 5. Stable map $f_2\colon T^2 \to S^2$ of degree one

The other cases of Theorems 2.1 and 2.3 are treated similarly. We omit the proofs here.

Note that to study the simplest contour for stable maps $M \to N$, constructing explicit stable maps $M \to N$ is important. Some inductive constructions of stable maps between surfaces will be given in §6.

§3. (i, c, n)-Minimal contours

The notion of an (i, c, n)-minimal contour was introduced and studied by Pignoni [10], where it was called an *essential contour.*

A formula of Quine [11] implies the following lemma, see [6] for the details.

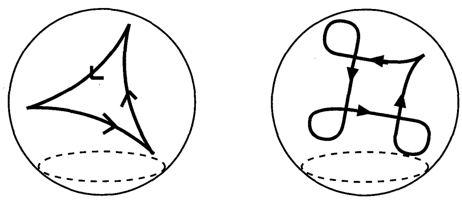
Lemma 3.1 ([6]). *Let $f\colon \Sigma_g \to S^2$ be a degree $d \geq 0$ stable map such that $S(f)$ consists of one component.*

(1) *The contour has at least two cusps if the number $d + g$ is odd.*
(2) *The contour has at least $2(d - g)$ cusps if $d \geq g$.*

Theorem 2.1 and Lemma 3.1 yield the following corollary.

Corollary 3.2. *Let $f\colon \Sigma_g \to S^2$ be a stable map. If the contour $\gamma(f)$ is (i, c, n)-minimal, then the contour is $(i, c + n)$-minimal.*

Fig. 6 shows the contours of stable maps $\mathbb{R}P^2 \to S^2$ of modulo two degree one. Fig. 6(a) shows an $(i, c + n)$-minimal contour and Fig. 6(b) shows an (i, c, n)-minimal contour. This shows that even if the contour of a stable map $h\colon \mathbb{R}P^2 \to S^2$ is (i, c, n)-minimal (or $(i, c + n)$-minimal), $\gamma(h)$ may not necessarily be $(i, c + n)$-minimal (resp. (i, c, n)-minimal). Note that Pignoni [10] observed the same type of difference between the $(i, c + n)$-minimality and the (i, c, n)-minimality for maps $\mathbb{R}P^2 \to \mathbb{R}^2$.



(a) (b)

Fig. 6. Apparent contours of stable maps $\mathbb{R}P^2 \to S^2$

§4. (c,i,n)-Minimal, (n,c,i)-minimal, and (i,n,c)-minimal contours

The notions of (c,i,n)-minimal, (n,c,i)-minimal, and (i,n,c)-minimal contours were introduced and the following theorems were obtained in [20]. The following three theorems are proved by using formula (1) and some lemmas, see [20] for the details. We omit the proofs here.

Theorem 4.1 ([20]). (1) *Let $f\colon \Sigma_g \to S^2$ be a stable map of degree $d \geq 0$. Then, $\gamma(f)$ is (c,i,n)-minimal if and only if the triple (i,c,n) for $\gamma(f)$ is one of the following:*

$$(c,i,n) = \begin{cases} (0, d+1, 0) & \text{if } g = 0, \\ (0,2,0) & \text{if } (d,g) = (0,1), \\ (0,1,d+g+2) & \text{if } g \neq 0,\ d \leq g, \text{ and } d \equiv g \pmod 2, \\ (0,2,d+g+1) & \text{if } g \neq 0,\ d < g,\ d \not\equiv g \pmod 2, \\ & \qquad\qquad \text{and } (d,g) \neq (0,1), \\ (0, d-g+1, 2g+2) & \text{if } g \neq 0 \text{ and } d \geq g. \end{cases}$$

(2) *Let $h\colon F_g \to S^2$ be a stable map of modulo two degree d_2. Then, $\gamma(h)$ is (c,i,n)-minimal if and only if the triple (i,c,n) for $\gamma(h)$ is one of the following:*

$$(c,i,n) = \begin{cases} (1,1,(g+5)/2) & \text{if } d_2 = 1 \text{ and } g \text{ is odd}, \\ (0,1,(g+6)/2) & \text{if } d_2 = 1 \text{ and } g \text{ is even}, \\ (1,1,(g+7)/2) & \text{if } d_2 = 0 \text{ and } g \text{ is odd}, \\ (0,1,(g+8)/2) & \text{if } d_2 = 0,\ g \text{ is even, and } g/2 \text{ is odd}, \\ (0,1,(g+4)/2) & \text{if } d_2 = 0,\ g \text{ is even, and } g/2 \text{ is even}. \end{cases}$$

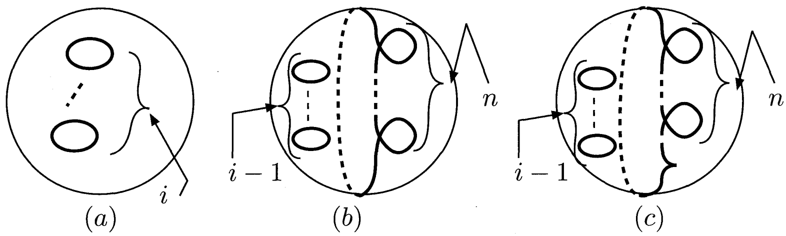


Fig. 7. (c, i, n)-minimal contours of stable maps $M \to S^2$ of closed surfeces

Fig. 7(a), (b), and (c) show examples of (c, i, n)-minimal contours of stable maps $M \to S^2$ of closed and orientable (or non-orientable) surfaces for the cases (a) $c = n = 0$, (b) $c = 0$ and $n > 0$, and (c) $c = 1$ and $n \geq 0$, respectively.

Theorem 4.1 implies the following corollary.

Corollary 4.2. *Let $f\colon \Sigma_g \to S^2$ be a degree d stable map whose contour is (c, i, n)-minimal. Then, we have the following.*

(1) *The number of nodes $n(f)$ is even.*

(2) *The numbers $i(f)$ and $(\chi(\Sigma_g)/2) + d$ have the same parity, where $\chi(\Sigma_g)$ denotes the Euler characteristic of Σ_g.*

Minoru Yamamoto [18] determined the minimal number of connected components of the set of singular points for fold maps $\Sigma_g \to \Sigma_h$, where a *fold map* between manifolds is a C^∞ map having only fold singularities. Theorem 4.1(1) gives the minimal number of nodes among fold maps $\Sigma_g \to S^2$ such that the number of connected components of the set of singular points is minimal.

Theorem 4.3 ([20]). (1) *Let $f\colon \Sigma_g \to S^2$ be a stable map of degree $d \geq 0$. Then, $\gamma(f)$ is (n, c, i)-minimal if and only if the triple (n, c, i) for $\gamma(f)$ satisfies*

$$(n, c, i) = (0, 0, d + g + 1).$$

(2) *Let $h\colon F_g \to S^2$ be a stable map of modulo two degree one. Then, $\gamma(h)$ is (n, c, i)-minimal if and only if the triple (n, c, i) for $\gamma(h)$ is one of the following:*

$$(n, c, i) = \begin{cases} (0, 0, (g+4)/2) & \text{if } g \text{ is even,} \\ (0, 1, (g+3)/2) & \text{if } g \text{ is odd.} \end{cases}$$

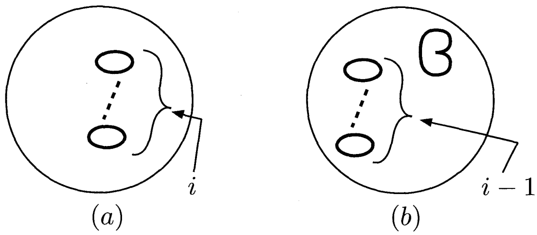


Fig. 8. (n, c, i)-minimal contours of stable maps $M \to S^2$ of closed surfaces

Fig. 8(a) and (b) show examples of (n, c, i)-minimal contours of stable maps $M \to S^2$ of closed and orientable (or non-orientable) surfaces for the cases (a) $c = 0$ and (b) $c = 1$, respectively, except the cases of modulo two degree zero stable maps $F_g \to S^2$.

Note that the study of (n, c, i)-minimal contours of a modulo two degree zero stable map $F_g \to S^2$ has some difficulties and the problem is still open, as far as the author knows.

Theorem 4.4 ([20]). (1) *Let $f\colon \Sigma_g \to S^2$ be a stable map of degree $d \geq 0$. Then, $\gamma(f)$ is (i, n, c)-minimal if and only if the triple (i, n, c) for $\gamma(f)$ is one of the following:*

$$(i, n, c) = \begin{cases} (1, 0, 2(g+2)) & \text{if } d = 0 \text{ and } g \geq 1, \\ (1, 0, 2(d+g)) & \text{otherewise.} \end{cases}$$

(2) *Let $h\colon F_g \to S^2$ be a stable map of modulo two degree d_2. Then, $\gamma(h)$ is (i, n, c)-minimal if and only if the triple (i, n, c) for $\gamma(h)$ satisfies*

$$(i, n, c) = (1, 0, -2\delta + g + 4),$$

where δ is equal to 1 if the modulo two degree d_2 of h is equal to one, and 0 otherwise.

Fig. 9 shows an example of an (i, n, c)-minimal contour of stable maps $M \to S^2$ of closed and orientable (or non-orientable) surfaces.

§5. Problems

In this section we pose some problems concerning the topology of stable maps between manifolds. For two manifolds M and N, denote

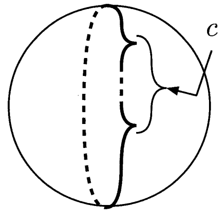


Fig. 9. An (i, n, c)-minimal contour of a stable map $M \to S^2$ of closed surfaces

by $C^\infty(M, N)$ the space of all C^∞ maps $M \to N$ equipped with the Whitney C^∞ topology. In general, we say that $f \in C^\infty(M, N)$ is C^∞ *stable* (or *stable* for short) if the $\mathcal{A}$-orbit of f is open in $C^\infty(M, N)$. The $\mathcal{A}$-orbit of $f \in C^\infty(M, N)$ is defined as follows. Denote by Diff(M) and Diff(N) the groups of self-diffeomorphisms of M and N respectively. Then, the group Diff(M)$\times$ Diff(N) acts on $C^\infty(M, N)$ by $(\Phi, \Psi)f = \Psi \circ f \circ \Phi^{-1}$, where $(\Phi, \Psi) \in \mathrm{Diff}(M) \times \mathrm{Diff}(N)$ and $f \in C^\infty(M, N)$. Then, the $\mathcal{A}$-orbit of $f \in C^\infty(M, N)$ is the orbit through f with respect to this action. Throughout this section, we assume that the dimension pair (dim$M$, dim$N$) for a C^∞ map $M \to N$ is in the nice range in the sense of Mather [8]. Thus, if M is a closed manifold, then the set of stable maps $M \to N$ forms an open and dense subset in $C^\infty(M, N)$. Note that in the case of C^∞ maps between surfaces, the notion of a stable map which has been introduced in §1 coincides with that introduced in this paragraph.

Note that the notions of $(i, c+n)$-minimal, (i, c, n)-minimal, (c, i, n)-minimal, and (i, n, c)-minimal contours are generalized to C^∞ maps φ: $M \to N$ of closed m-dimensional manifolds with $m \geq 2$ into surfaces in a straightforward way.

Problem 5.1. For a closed m-dimensional manifold M, study $(i, c+n)$-minimal, (i, c, n)-minimal, (c, i, n)-minimal, and (i, n, c)-minimal contours for C^∞ maps $M \to \mathbb{R}^2$.

Taishi Fukuda and the author [3] studied stable maps $\Sigma_g \to S^2$ whose numbers $c + n$ are minimal among stable maps which are homotopic to a given C^∞ map and whose singular point set consists of i components for each integer $i \geq 2$.

Let M be a closed surface and M_1 denote M with an open disk removed. A C^∞ map φ: $M_1 \to \mathbb{R}^2$ is an *admissible* C^∞ *map* if φ is an immersion on some neighborhood of the boundary component of

M_1. Admissible C^∞ maps φ_1 and $\varphi_2\colon M_1 \to \mathbb{R}^2$ are *admissibly homotopic* if there is a C^∞ map $H\colon M_1 \times [0,1] \to \mathbb{R}^2$ such that the map $h_t = H(\cdot, t)\colon M_1 \to \mathbb{R}^2$ is admissible for each $t \in [0,1]$, and $h_0 = \varphi_1$ and $h_1 = \varphi_2$. The contour of an admissible stable map $\varphi\colon M_1 \to \mathbb{R}^2$ is an *admissible* $(i, c+n)$*-minimal contour* for an admissible C^∞ map $\varphi_0\colon M_1 \to \mathbb{R}^2$ if the pair $(i(\varphi), c(\varphi)+n(\varphi))$ is minimal among admissible stable maps $M_1 \to \mathbb{R}^2$ which are admissibly homotopic to φ_0 with respect to the lexicographic order. The author [22] introduced the notion of an admissible $(i, c+n)$-minimal contour for an admissible C^∞ map $M_1 \to \mathbb{R}^2$ and studied such minimal contours of admissible C^∞ maps $(\Sigma_g)_1 \to \mathbb{R}^2$.

Saeki [12] showed that a closed orientable 3-manifold M is a graph manifold[1] if and only if there exists a stable map $g\colon M \to \mathbb{R}^2$ such that $g|_{S(g)}$ is a C^∞ embedding, see [12, Theorem 3.1] for the details. This theorem implies that for a closed and orientable 3-manifold M which is not a graph manifold, each stable map $g\colon M \to \mathbb{R}^2$ has a cusp or a node. Note that a hyperbolic 3-manifold is not a graph manifold.

Problem 5.2. For a closed m-dimensional manifold M and a surface N, characterize those numbers i, c and n which are realized by stable maps $M \to N$.

Recently, the author [21] studied the numbers i, c and n which are realized by stable maps $\Sigma_g \to S^2$ and stable maps $\Sigma_g \to \mathbb{R}^2$.

Let M and N be smooth manifolds such that the dimension pair $(\dim M, \dim N)$ is in the nice range in the sense of Mather [8] and that M is compact. Let $\mathbb{A}$ be a certain ordered set consisting of some numerical invariants for stable maps $M \to N$: for example, the number of singular points of a certain type, the number of singular fibers[2] of a certain type in the sense of [13], the number of connected components of the set of singular points, etc. For a stable map $\varphi\colon M \to N$, we denote by $\mathbb{A}(\varphi)$ the ordered set consisting of the corresponding numerical invariants for φ. Then, for a given C^∞ map $\varphi_0\colon M \to N$, a stable map $\varphi\colon M \to N$ is said to be $\mathbb{A}$*-minimal for* φ_0 if $\mathbb{A}(\varphi)$ is minimal among the stable maps homotopic to φ_0, with respect to the lexicographic order. When $N = \mathbb{R}^n$, an $\mathbb{A}$-minimal stable map is also said to be $\mathbb{A}$*-minimal for* M.

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[1]A *graph manifold* is a 3-manifold which is built up of S^1-bundles over surfaces attached along their torus boundaries.

[2]For a C^∞ map $\varphi\colon M \to N$, the *fiber* over $q \in N$ is the map germ $\varphi\colon (M, \varphi^{-1}(q)) \to (N, q)$ along the inverse image $\varphi^{-1}(q)$. The fiber over q is a *singular fiber* of φ if q is a singular value. The singular fibers of stable maps of closed 4-dimensional manifolds into 3-dimensional manifolds were classified in [13] and [19].

Problem 5.3. Let $\mathbb{A}$ be as above. Study $\mathbb{A}$-minimal stable maps for a given m-dimensional manifold M. Then, study $\mathbb{A}$-minimal stable maps for a C^∞ map $\varphi_0 \colon M \to N$ for a general manifold N.

It is known that the following characterization of a stable map $M \to N$ of a closed m-dimensional manifold ($m \geq 3$) into a 3-manifold holds: A C^∞ map $\varphi \colon M \to N$ is stable if and only if it satisfies the following conditions.

(1) For each $p \in M$, the germ (φ, p) is a submersion, a fold singularity, a cusp singularity, or a swallow-tail singularity. Then, it is known that $S(\varphi) \subset M$ is a submanifold of codimension $m - 2$.

(2) For each $q \in \varphi(S(\varphi))$, the map germ $(\varphi|_{S(\varphi)}, \varphi^{-1}(q) \cap S(\varphi))$ is an embedding, an immersion with normal crossings (a double point or a triple point), a cuspidal edge, a transverse crossing of a cuspidal edge and a fold sheet, or a swallow-tail.

This characterization of the stable map is proved by using the transversality theorem and the multi-transversality theorem, since the dimension pair $(m, 3)$ is in the nice range in the sense of Mather [8] (see [4], for details).

For a stable map $\varphi \colon M \to N$ of a closed m-dimensional manifold ($m \geq 3$) into a 3-manifold, denote by $T(\varphi)$ the number of triple points of $\varphi|_{S(\varphi)}$. Thus, the notions of a *T-minimal stable map for a C^∞ map $M \to N$* and a *T-minimal stable map for a manifold M* make sense.

Saeki and the author [14] obtained the following signature formula for an oriented closed 4-manifold. For a stable map $f \colon M \to N$ of a closed and oriented 4-manifold into a 3-manifold, the signature of M coincides with the algebraic number of singular fibers of type III^8. For a stable map $\varphi \colon M \to N$ of a closed and orientable 4-manifold into a 3-manifold, denote by $\mathrm{III}^8(\varphi)$ the geometric number of singular fibers of type III^8 of φ. Thus, the notions of a *III^8-minimal stable map for a C^∞ map $M \to N$* and a *III^8-minimal stable map for a closed and orientable 4-manifold M* make sense. Note that a singular fiber of type III^8 appears over a triple point. A stable map $f \colon 2\mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{R}^3$ such that $f|_{S(f)}$ has only one triple point over which lies a singular fiber of type III^8 was constructed in [13]. Hence, the stable map f is III^8-minimal for $2\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Furthermore, the stable map f is T-minimal for $2\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, since $f|_{S(f)}$ has exactly one triple point. Kobayashi [7] constructed two stable maps $f_1, f_2 \colon \mathbb{C}P^2 \to \mathbb{R}^3$. The map f_1 has two triple points. The singular fiber over one of the triple points is of type III^8. The map f_2 has only one triple point over which lies a singular fiber of type III^8, see [7] for the details. Both of the stable maps f_1 and

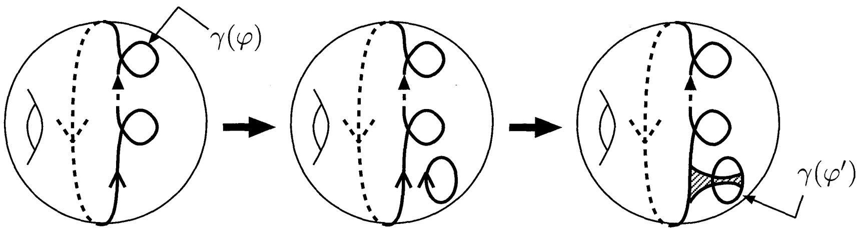

Fig. 10. Attaching a pair of handles

f_2 are III^8-minimal for $\mathbb{C}P^2$. The stable map f_2 is T-minimal for $\mathbb{C}P^2$, while the stable map f_1 is not.

Problem 5.4. For a general m-dimensional manifold M ($m \geq 3$), study T-minimal stable maps for M. Furthermore, study III^8-minimal stable maps for a closed and orientable 4-manifold.

Problem 5.5. Count the right-left equivalence classes of stable maps $M \to N$ which are $\mathbb{A}$-minimal for a C^∞ map.

Pignoni [10] and Demoto [1] counted the numbers of right-left equivalence classes of $(i, c+n)$-minimal contours for C^∞ maps $M \to \mathbb{R}^2$ of closed surfaces, and for C^∞ maps $S^2 \to S^2$, respectively.

§6. Appendix

In this section, some inductive constructions of stable maps between closed surfaces are given. For an ordered pair or triple $\mathbb{A}$ consisting of the numbers i, c, n or $c+n$, a stable map $\Sigma_g \to S^2$ whose contour is $\mathbb{A}$-minimal is obtained by applying the following constructions inductively to a stable map $T^2 \to S^2$ whose contour is $\mathbb{A}$-minimal, see [6], [20] for the details.

Let M be a closed surface and $\varphi\colon M \to \Sigma_h$ ($h \geq 0$) be a stable map on M.

Let us attach a pair of handles to M as shown in Fig. 10, where we attach a handle vertically to the source surface first and then attach another handle horizontally to the source surface. Then, we obtain a stable map $\varphi'\colon M\#2T^2 \to \Sigma_h$ whose triple (c, n, i) is equal to $(c(\varphi), n(\varphi)+2, i(\varphi))$ and whose degree is equal to that of φ.

The operation of attaching a "vertical" handle is called a *vertical surgery* in [5].

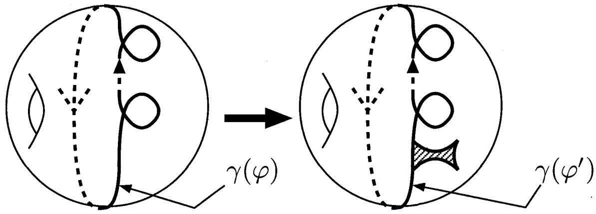

Fig. 11. Attaching a handle horizontally

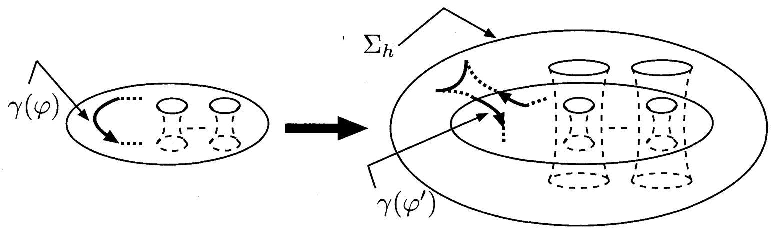

Fig. 12. Attach Σ_h horizontally

By attaching a handle horizontally to M as shown in Fig. 11, we obtain a stable map $\varphi'\colon M\#T^2 \to \Sigma_h$ whose triple (c, n, i) is equal to $(c(\varphi)+2, n(\varphi), i(\varphi))$ and whose degree is equal to that of φ.

By attaching a Σ_h horizontally to M, and by connecting Σ_h and M by a horizontal handle, as shown in Fig. 12, we obtain a stable map $\varphi'\colon M\#\Sigma_h \to \Sigma_h$ whose triple (c, n, i) is equal to $(c(\varphi)+2, n(\varphi), i(\varphi))$ and whose degree is equal to that of φ plus or minus one.

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