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# Smooth double subvarieties on singular varieties. II

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***Dedicated to Professor H. Hironaka on the occasion of his 80th birthday and to Professor S. Ishii on the occasion of her 60th birthday***

**Abstract.**

Let k be an algebraically closed field of characteristic 0. We give a brief survey on multiplicity-2 structures on varieties. Let $Z$ be a reduced irreducible nonsingular $(n-1)$-dimensional variety such that $2Z = X \cap F$, where $X$ is a normal $n$-fold with canonical singularities, $F$ is an $(N-1)$-fold in $\mathbb{P}^N$, such that $Z \cap \mathrm{Sing}(X) \neq \emptyset$. Assume that $\mathrm{Sing}(X)$ is equidimensional and $\mathrm{codim}_X(\mathrm{Sing}(X)) = 3$. We study the singularities of $X$ through which $Z$ passes. We also consider Fano cones. We discuss the construction of some vector bundles and the resolution property of a variety.

## §1. Introduction

Multiplicity-2 structures on nonsingular varieties appear in several instances; for example, when studying nonsingular curves on a Kummer surface in $\mathbb{P}^3$, passing through some of its nodes [3]. In [1, p. 43], W. Barth gave a construction of the Horrocks–Mumford bundle assuming the existence of a nonsingular irreducible curve with certain properties. The Horrocks–Mumford bundle is a stable indecomposable rank 2 vector bundle over $\mathbb{P}^4$. A generic irreducible nonsingular curve of degree 8 and genus 5 on a Kummer surface satisfies all but one of Barth's conditions [5, Proposition 3.5] to be the variety of jumping lines of the Horrocks–Mumford bundle in $\mathbb{P}^4$.

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To define a multiplicity-2 structure $\tilde{Y}$ on a codimension 2 nonsingular variety $Y$ is, under some conditions, equivalent to defining a subbundle $L \subset N_{Y|\mathbb{P}^n}$.

Hulek, Okonek and Van de Ven [8] studied multiplicity-2 structures on Castelnuovo and Bordiga surfaces in $\mathbb{P}^4$ as well as on codimension-2 Castelnuovo manifolds. They also studied locally free resolutions on them as well as the stability of the normal bundle on Castelnuovo and Bordiga surfaces. Let $Y$ denote a Castelnuovo surface in $\mathbb{P}^4$ and $\tilde{Y}$ a multiplicity-2 structure on $Y$. Under suitable conditions one can construct a rank 2 vector bundle, $E$, in $\mathbb{P}^4$ with the non-reduced structure $\tilde{Y}$ as the zero-set of a section of $E$, [9].

Vogelaar [17] proved that any local complete intersection subscheme of codimension 2 of a nonsingular variety $F$ can be obtained as the dependency locus of $r-1$ sections of a rank $r$ vector bundle over $F$ of determinant $L$ if and only if the determinant of its normal bundle twisted with $L^*$ is generated by $r-1$ global sections, provided the vanishing of the second order cohomology of $L^*$.

Schneider [15] gave a list of problems about vector bundles and low odimension subvarieties in projective spaces.

We believe that our study of varieties which are complete intersections with a non reduced structure on them could be used in the construction of vector bundles in $\mathbb{P}^n$. These multiplicity-2 structures passing through the singular locus of another variety provide a better understanding of the geometry. They could also be of interest in answering Totaro's Question: Does every algebraic variety $Y$ have the resolution property, i.e. every coherent sheaf on $Y$ is a quotient of a locally free sheaf of finite rank? [16]. If $Y$ has the resolution property, one could construct a resolution of any coherent sheaf $F$ on $Y$ by vector bundles. The question has an affirmative answer for quasiprojective varieties [10]. The answer is also affirmative for smooth and $\mathbb{Q}$-factorial varieties, since every coherent sheaf has a resolution by sums of line bundles. Payne [12] studied the question for threefolds and observed that, for a complete toric variety $X$, the resolution property implies the existence of nontrivial toric vector bundles. These are vector bundles for the dense torus $T \subset X$ whose underlying vector bundles are nontrivial. In general, there is not known way of constructing a nontrivial toric vector bundle on an arbitrary complete toric variety [12, p. 3].

All varieties are reduced and irreducible unless stated otherwise.

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## §2. Curves on Kummer surfaces, Multiplicity-2 structures and the Horrocks–Mumford bundle

Kummer surfaces appear in many different contexts: they are related to abelian surfaces and to the quadric line complex. The minimal desingularization of a Kummer surface is a K3 surface.

**Definition 2.1.** A $(16, 6)$ *configuration* is a set of 16 planes and 16 points in $\mathbb{P}^3$ such that every plane contains exactly 6 of the 16 points and every point lies on exactly 6 of the 16 planes.

A $(16, 6)$ configuration is *non-degenerate* if every two planes share exactly two points of the configuration and every pair of points is contained in exactly two planes.

**Definition 2.2.** A *Kummer surface* $S$ in $\mathbb{P}^3$ is a reduced irreducible quartic surface having 16 nodes, $P_i$, $1 \leq i \leq 16$, and no other singularities.

**Definition 2.3.** The lines $\overline{P_1P_i}$, $2 \leq i \leq 16$, are called *special lines.* The planes forming irreducible components of the sixteen enveloping cones of $S$ at the nodes are called *special planes.* The section of $S$ by one of the special planes is a non-singular conic, counted twice; we call this conic a *special conic.*

**Proposition 2.4.** *The union of the* 16 *enveloping cones at the* 16 *nodes of* $S$ *consists of* 16 *planes. Each plane cuts out a conic on* $S$ *containing* 6 *nodes. Each node lies on exactly* 6 *of the* 16 *conics. Together the nodes of* $S$ *and the* 16 *special planes form a non-degenerate* $(16, 6)$ *configuration. We call this the* $(16, 6)$ *configuration associated to the Kummer surface* $S$.

*Proof.* [4, Proposition 2.16, Corollary 2.18]. Q.E.D.

Barth's construction [1] relates nonsingular curves of degree 8 and genus 5 to the variety of jumping lines of a stable rank 2 vector bundle in $\mathbb{P}^4$ through a fixed point $P \in \mathbb{P}^4$ (the Horrocks–Mumford bundle). According to Barth's construction of the Horrocks–Mumford bundle, $E$, [1, p. 43], the nonsingular curve $C$ which would be the variety of jumping lines of $E$, has to satisfy 5 properties; we can prove that it satisfies the following four:

- Set-theoretically, $C$ is the complete intersection of a Kummer surface $S_1$ and a quartic surface $S_2$ in $\mathbb{P}^3$, since $2C \simeq 4H$, [5, (2.91)].
- $C$ is the curve of contact of these surfaces [5, (2.74), (2.93)].

- The exact sequence

$$0 \to \omega_C(\sum_{i=1}^{16} P_i) \to N_C \to O_C(4)(-\sum_{i=1}^{16} P_i) \to 0$$

  splits. [5, Theorem 3.17].
- $C$ is linearly normal [5, (3.15)],

but it does not satisfy the required fifth property as we show in the following proposition.

**Proposition 2.5.** *Let $C$ be a generic irreducible nonsingular curve of degree 8 and genus 5 on a Kummer surface $S$, passing through its 16 nodes $P_i$, $1 \le i \le 16$. If $L = \omega_C(\sum_{i=1}^{16} P_i)$ and $M = O_C(4)(-\sum_{i=1}^{16} P_i)$, then $M \not\cong L(-1)$.*

*Proof.* [5, (3.18)]. Q.E.D.

**Definition 2.6.** Let $Y$ be a smooth variety in $\mathbb{P}^n$, with ideal sheaf $I_Y$. A non-reduced structure $\tilde{Y}$ is a *multiplicity-2 structure* on $Y$ if

(a) the ideal $I_{\tilde{Y}}$ is such that $I_{\tilde{Y}} \subset I_Y$,
(b) $\tilde{Y}$ is locally a complete intersection,
(c) $\tilde{Y}$ has multiplicity 2, i.e. for each point $P \in Y$ and a general hyperplane $H$ through $P$ the local intersection multiplicity is

$$i(P;\tilde{Y},H) = \dim_{\mathrm{k}} O_{P|I(\tilde{Y}\cap H)} = 2.$$

**Lemma 2.7.** *To define a multiplicity-2 structure $\tilde{Y}$ on a codimension 2 nonsingular variety $Y$ is equivalent to defining a subbundle $L \subset N_{Y|\mathbb{P}^n}$, assuming that $I_Y/I_{\tilde{Y}}$ is locally free.*

*Proof.* Generalization of [8, Lemma 2]. Q.E.D.

**Example A.** Let $X$ be the quadric cone in $\mathbb{P}^3$ defined by $xy - z^2$. $X$ is normal. The line $L$, defined by $x = z = 0$, is a Weil divisor on $X$ but not a Cartier divisor because it cannot be defined near the origin by one equation (the ideal $(x, z)$ is not principal in the local ring of $X$ at the origin). $2L$ is a Cartier divisor.

**Definition 2.8.** A codimension 2 variety $Y \subset \mathbb{P}^{n+2}$ is a *Castelnuovo variety* of dimension $n$ if Y has a resolution

$$0 \to {O^2}_{\mathbb{P}^{n+2}} \to O_{\mathbb{P}^{n+2}}(1) \oplus O_{\mathbb{P}^{n+2}}(b) \to I_Y(b+2) \to 0,$$

[8, p. 442].

A *Bordiga surface* is a rational surface in $\mathbb{P}^4$ of degree 6, [8, p. 445].

**Proposition 2.9.** *Let $Y$ be a nonsingular Castelnuovo surface in $\mathbb{P}^4$ of degree $2b+1$. If $Y$ has a multiplicity-2 structure $\tilde{Y}$ with induced canonical bundle $\omega_{\tilde{Y}}$ the this structure is given by a quotient ${N_{Y|\mathbb{P}^n}}^* \to \omega_Y(2-2b)$. In this case $\tilde{Y}$ is a complete intersection of type $(2, 2b+1)$. The hyperquadric through $\tilde{Y}$ is unique and is singular along a line $L_0 \subset Y$.*

*Proof.* [8, Prop. 12]. Q.E.D.

**Proposition 2.10.** *The only Castelnuovo manifold of dimension $n \geq 3$ which admits a multiplicity-2 structure $\tilde{Y}$ such that $\tilde{Y}$ is a complete intersection is $\mathbb{P}^n$ embedded linearly.*

*Proof.* [8, Prop. 15]. Q.E.D.

## §3. On smooth double subvarieties on singular varieties

**Notation.** Let $X$ be a normal variety. Let $f : V \to X$ be a proper birational morphism where $V$ is a nonsingular variety. Let $D$ be a $\mathbb{Q}$-Gorenstein divisor. The pullback $f^*D$ is the divisor $f^*D = {f_*}^{-1}D + \sum d_i E_i$, $d_i \in \mathbb{Q}$, satisfying $E_j.({f_*}^{-1}D + \sum d_i E_i) = 0$, for all $E_j \in \mathrm{Exc}f$, where ${f_*}^{-1}D$ is the strict transform of $D$, [11, 4-6-3].

**Definition 3.1.** A normal variety $X$ of dimension $n$ has only *canonical singularities* (resp. *terminal singularities*, resp. *log terminal singularities*, resp. *log canonical singularities*) if

(a) the canonical divisor $K_X$ is $\mathbb{Q}$-Cartier, that is, there exists $e \in \mathbb{N}$ such that $eK_X$ is a Cartier divisor. The *index of the singularity* is

$$\mathrm{index}\,(K_X) = \min\{e \in \mathbb{N} : eK_X \text{is a Cartier divisor}\}.$$

(b) Consider a projective divisorial resolution $f : V \to X$, where $V$ is a nonsingular variety. In the ramification formula

$$K_V = f^*K_X + \sum a_i E_i$$

all the coefficients for the exceptional divisors are nonnegative, that is $a_i \geq 0$, (resp. $a_i > 0$, resp. $a_i > -1$, resp. $a_i \geq -1$) for all $i$.

**Definition 3.2.** (a) Let $(O_{X,P}, M_P)$ be the local ring of a point $P \in X$ of a k-scheme. Let $V \subset M_P$ be a finite dimensional k-vector space which generates $M_P$ as an ideal of

$O_{X,P}$. By a *general hyperplane through* $P$ we mean the subscheme $H \subset U$ defined in a suitable pen neighbourhood $U$ of $P$ by the ideal $(v)O_X$, where $v \in V$ is a k-point of a certain dense Zariski open set in $V$, [13, (2.5)]. By a *general linear variety of codimension* $r$ *through* $P$ we mean the subscheme $L \subset U$ defined in a suitable open neighbourhood $U$ of $P$ by the ideal $(v_1, \cdots, v_r)O_X$, where $v_1, \cdots, v_r \in V$ are k-points of a certain dense Zariski open set in $V$.

(b) Let $X$ be a singular $n$-fold. We say that a point $Q \in \mathrm{Sing}(X)$ is *a general point of* $\mathrm{Sing}(X)$ if, for a general hyperplane $H$ such that $Q \in H$ and for some a divisorial resolution $f : V \to X$, the preimage $f^{-1}(Q)$ of $Q$ and the strict transform $f_*^{-1}(X \cap H)$ satisfy that $f^{-1}(Q) \subset f_*^{-1}(X \cap H)$.

**Remark B.** Saying that $P \in X$ Cohen–Macaulay and canonical of index 1 is equivalent to saying that $P \in X$ rational Gorenstein, [13, p. 286].

**Definition 3.3.** (a) Let $X$ be a threefold. A point $P \in X$ is called a *compound Du Val singularity or a cDV point* if, for some hyperplane section $H$ through $P$, $P \in H$ is a Du Val singularity. Equivalently, $P \in X$ is cDV if it is locally analytically isomorphic to the hypersurface singularity given by $f + tg$, where $g \in \mathrm{k}[x, y, z, t]$ is arbitrary and $f \in \mathrm{k}[x, y, z]$ represents a Du Val singularity, [13, (2.1)].

(b) Let $X$ be an $n$-dimensional normal variety and $P$ a point of $X$. Let $P$ be an $n$-fold isolated singularity (that is, the spectrum of an equicharacteristic local noetherian complete ring of Krull dimension $n$, without zero divisors, whose closed point $P$ is singular). Let $\pi : \tilde{X} \to X$ be the minimal desingularization of $X$ at $P$. The *genus* of a normal singularity $P$ is defined to be $\dim_{\mathrm{k}} (R^{n-1}\pi_* O_{\tilde{X}})_P$. If the genus is 0, the singularity is said to be *rational.* If the genus is 1, it is *elliptic.*

**Proposition 3.4.** *Let $X$ be an $n$-dimensional variety, $n \geq 2$.*

(a) *If $P \in X$ is a rational Gorenstein point then, for a general hyperplane section $H$ through $P$, $P \in H$ is elliptic or rational Gorenstein.*

(b) *If there exists a hyperplane section $H$ through $P$ such that $P \in H$ is a rational Gorenstein then $P \in X$ is a rational Gorenstein. In particular, cDV points are canonical.*

*Proof.* [13, (2.6)]. Q.E.D.

*Note* C (Generalized Reid's Method). Let $X$ be a normal variety of dimension $n$ in $\mathbb{P}^N$. To study canonical and terminal singularities of the $n$-fold $X$, we reduce by one its dimension by taking a general hyperplane section meeting $\mathrm{Sing}(X)$. We use the information on the hyperplane section to analyze the original singularity of $X$, [11, p. 198], [14]. We keep repeating this procedure as follows:

Let $H_0$ be a general hyperplane through $\mathrm{Sing}(X)$.

Let $H_{r+1}$, $0 \leq r \leq n-3$, be a general hyperplane through the singular locus of $X_r = X \cap H_0 \cap \cdots \cap H_r$.

$\dim(X_r) = n - r - 1$.

Let $L_{k+1}$ be a general linear variety of codimension $k+1$ in $\mathbb{P}^N$, $0 \leq k \leq n-3$ such that $\mathrm{Sing}(X) \cap L_{k+1} \neq \emptyset$. Let $W_k = X \cap L_{k+1}$.

Note that, if $L_{k+1} = H_0 \cap \cdots \cap H_k$, $X_k = W_k$, [7, Note 3.3].

This method of studying singularities by taking hyperplane sections encounter serious problems when studying isolated singularities. Note that, by Proposition 3.4, if $P \in X$ is a rational Gorenstein point then, for a general hyperplane section $H$ through $P$, $P \in H$ is elliptic or rational Gorenstein.

**Remark D.** Note that to study canonical terminal singularities, log-terminal and log-canonical of the $n$-fold $X$, we could reduce the problem to study $X \cap Y$, where $Y$ is a general nonsingular variety [7, (3.8)].

**Proposition 3.5.** *Let $X$ be a normal singular $n$-fold with only canonical singularities. Let $W_r$ be as in Note C. Assume that*

$$\mathrm{codim}_{W_r}(\mathrm{Sing}(W_r)) = 2,$$

*for all $r$, $0 \leq r \leq n-3$. Every point of $X$ has an analytic neighbourhood which is (nonsingular or) isomorphic to $P \times A^{n-2}$, where $P$ is a Du Val surface singularity.*

*Proof.* [7, (5.2)]. Q.E.D.

*Note* E. Let $C$ be an irreducible nonsingular curve $2C = V \cap W$, where $V$ and $W$ are two surfaces and $W$ has at most rational double points. Let us suppose that $C$ passes through a rational double point $P$ of $W$. Let $\tilde{W}$ be the minimal desingularization of $W$ at $P$, $\pi : \tilde{W} \to W$. Let $E_k$, $1 \leq k \leq n$, be the irreducible components of the exceptional divisor. The total transform $\pi^*(2C) = \sum_{j=1}^{n} \beta_j E_j + 2E$, where $E$ is the strict transform of $C$, $\beta_j \in \mathbb{N}$.

**Proposition 3.6.** *Let $C$ be an irreducible nonsingular curve $2C = V \cap W$, where $V$ and $W$ are two surfaces and $W$ has only rational double* *points as singularities. Assume that $C$ passes through a rational double point $P$ of $W$. $P$ cannot be either of type $A_{2r}$, $r \in \mathbb{N}$, or type $E_6$, or $E_8$. For $C$ to pass only through one singularity of type $A_{2r+1}$, $r \in \mathbb{N}$, we must have $\left(\sum_{j=1}^{2r+1} \beta_j E_j\right)^2 = -(2r+2)$. For $C$ to pass only through one singularity of type $E_7$, we must have $\left(\sum_{j=1}^{7} \beta_j E_j\right)^2 = -6$. For $C$ to pass only through one singularity of type $D_n$, $n \geq 4$, we must have that either $\left(\sum_{j=1}^{n} \beta_j E_j\right)^2 = -4$, or, for $n = 2k$, $k \in \mathbb{N}$, $k \geq 3$, $\left(\sum_{j=1}^{n} \beta_j E_j\right)^2 = -n$.*

*Proof.* [6, Theorem 0.9]. Q.E.D.

**Proposition 3.7.** *Let $Z$ be a reduced irreducible nonsingular $(n-1)$-dimensional variety such that $2Z = X \cap Y$, where $X$ is an $n$-fold and $Y$ is an $(N-1)$-fold in $\mathbb{P}^N$, $X$ normal with canonical singularities and such that $Z \cap \operatorname{Sing}(X) \neq \emptyset$. Let $W_r$ be as in Note C. Assume that* $\operatorname{codim}_{W_r}(\operatorname{Sing}(W_r)) = 2$, *for all $r$, $0 \leq r \leq n-4$. Then $Z$ has empty intersection with canonical singularities of $X$ which have analytical neighbourhoods isomorphic to $P \times A^{n-2}$, where $P$ is a rational surface singularity of types $A_{2k}$, $k \in \mathbb{N}$, $E_6$ and $E_8$. For $Z$ to have non-empty intersection with canonical singularities of $X$ which have analytical neighbourhoods isomorphic to $P \times A^{n-2}$, where $P$ is a rational surface singularity of type $A_{2k+1}$, $k \in \mathbb{N}$ we must have $\left(\sum_{j=1}^{2k+1} \beta_j E_j\right)^2 = -(2k+2)$, where $E_j$, $1 \leq j \leq 2k+1$, are the irreducible components of the exceptional divisor supported on $\pi^{-1}(P)$ for $\pi : \tilde{W}_{n-3} \to W_{n-3}$ the minimal resolution of $P \in W_{n-3} \cap Y$. For $P$ to be of type $E_7$, we must have $\left(\sum_{j=1}^{7} \beta_j E_j\right)^2 = -6$, where $E_k$, $1 \leq k \leq 7$, are the irreducible components of the exceptional divisor as above. For $P$ to be of type $D_n$, $n \geq 4$, we must have that either $\left(\sum_{j=1}^{n} \beta_j E_j\right)^2 = -4$, or, for $n = 2k$, $k \in \mathbb{N}$, $k \geq 3$, $\left(\sum_{j=1}^{n} \beta_j E_j\right)^2 = -n$, where $E_k$, $1 \leq k \leq n$, are the irreducible components of the exceptional divisor as above.*

*Proof.* [7, Corollary 7.2]. Q.E.D.

**Proposition 3.8.** *Let $Z$ be a reduced irreducible nonsingular $(n-1)$-dimensional variety such that $2Z = X \cap Y$, where $X$ is an $n$-fold and $Y$ is an $(N-1)$-fold in $\mathbb{P}^N$, $X$ normal with canonical singularities and such that $Z \cap \operatorname{Sing}(X) \neq \emptyset$. Assume that* $\operatorname{codim}_X(\operatorname{Sing}(X)) = 3$. *Let $W_r$ be as in Note C, for all $r$, $0 \leq r \leq n-4$. Then,* $\operatorname{Sing}(W_{n-4})$ *is a union of canonical isolated singularities $P$'s. Let us assume that there exists a hyperplane section $H'$ through $P$ such that $W_{n-4} \cap H'$ is a normal surface with rational double points. Then $Z$ has empty intersection with canonical singularities of $X$ which have analytical neighbourhoods isomorphic to $P \times A^{n-3}$, where $P$ is a rational surface singularity in*

$\mathrm{Sing}(W_{n-4} \cap H')$ *of types* $A_{2k}$, $k \in \mathbb{N}$, $E_6$ *and* $E_8$. *For* $Z$ *to have non-empty intersection with canonical singularities of* $X$ *which have analytical neighbourhoods isomorphic to* $P \times A^{n-3}$, *where* $P$ *is a rational surface singularity in* $\mathrm{Sing}(W_{n-4} \cap H')$ *of type* $A_{2k+1}$, $k \in \mathbb{N}$, *we must have* $\left(\sum_{j=1}^{2k+1} \beta_j E_j\right)^2 = -(2k+2)$, *where* $E_j$, $1 \leq j \leq 2k+1$, *are the irreducible components of the exceptional divisor supported on* $(\pi_{W_{n-4}\cap H'})^{-1}(P)$ *for* $\pi_{W_{n-4}\cap H'} : (\widetilde{W_{n-4} \cap H'}) \to W_{n-4} \cap H'$ *the minimal resolution of* $P$, $P \in W_{n-4} \cap H' \cap Y$, *or* $P$ *to be of type* $E_7$, *we must have* $\left(\sum_{j=1}^{7} \beta_j E_j\right)^2 = -6$, *where* $E_k$, $1 \leq k \leq 7$, *are the irreducible components of the exceptional divisor as above. For* $P$ *to be of type* $D_n$, $n \geq 4$, *we must have that either* $\left(\sum_{j=1}^{n} \beta_j E_j\right)^2 = -4$, *or, for* $n = 2k$, $k \in \mathbb{N}$, $k \geq 3$, $\left(\sum_{j=1}^{n} \beta_j E_j\right)^2 = -n$, *where* $E_k$, $1 \leq k \leq n$, *are the irreducible components of the exceptional divisor as above.*

*Proof.* Since $\dim(W_{n-4}) = 3$, $\dim(\mathrm{Sing}(W_{n-4})) = 0$.

Thus, $\mathrm{Sing}(W_{n-4})$ is a union of isolated canonical singularities $P$'s. We assume that there exists a hyperplane section $H'$ through $P$ such that $W_{n-4} \cap H'$ is a normal surface with rational double points. Given $2Z = X \cap Y$ we intersect it with $H_0$, $H_r$, $0 \leq r \leq n-4$, as follows: $2Z \cap H_0 \cap \cdots \cap H_{n-4} \cap H' = Y \cap X \cap H_0 \cap \cdots \cap H_{n-4} \cap H'$. We obtain a nonsingular curve $C$ such that $2C = Y \cap X \cap H_0 \cap \cdots \cap H_{n-4} \cap H'$ and that $C \cap \mathrm{Sing}(W_{n-4} \cap H') \neq \emptyset$. We apply Proposition 3.6 to obtain the result. Q.E.D.

**Definition 3.9.** A *Fano variety* $X$ is a normal projective variety with log terminal singularities such that the anticanonical divisor $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor. Let $H \in Pic(X)$ be a primitive ample divisor class. The *Fano index* $s = i(X)$ is defined by $K_X = -sH$; $s \leq \dim X + 1$.

**Lemma 3.10.** *Let* $Y$ *be a smooth projectively normal subvariety of* $\mathbb{P}^N$, *with hyperplane divisor* $H$ *such that* $K_Y$ *linearly equivalent to* $qH$, *for* $q \in \mathbb{Q}$. *Let* $X$ *be the cone in* $\mathbb{P}^{N+1}$ *over* $Y$. *Let* $\tilde{X}$ *be the* $\mathbb{P}^1$*-bundle* $\pi : \mathbb{P}(O_Y \oplus O_Y(H)) \to Y$. *Let* $Y_0$ *be the section corresponding to the quotient* $O_Y(H)$ *of* $O_Y \oplus O_Y(H)$, *such that* $Y_0|Y \simeq -H$. *Let* $f : \tilde{X} \to X$ *the contraction of* $Y_0$. *We have that*

$$K_{\tilde{X}} = f^* K_X + (-1-q)H.$$

*Thus, the singularities of* $X$ *are log terminal if and only if* $q < 0$. $X$ *is a Fano variety if and only if* $Y$ *is a Fano variety.*

*Proof.* [2, p. 95]. Q.E.D.

**Corollary 3.11.** *Let $Y$ be a smooth projectively normal subvariety of $\mathbb{P}^N$, with hyperplane divisor $H$ such that $K_Y$ linearly equivalent to $qH$, for $q \in \mathbb{Q}$. Let $X$ be the cone in $\mathbb{P}^{N+1}$ over $Y$. Thus, the singularities of $X$ are terminal (resp. canonical, resp. log canonical) if and only if $q < -1$ (resp. $q \leq -1$, resp. $q \leq 0$).*

*Proof.* Immediate from Lemma 3.10 and Definition 3.1. Q.E.D.

**Example F.** Let us consider the canonical Fano 4-fold $X$ obtained as follows. Let us embed $\mathbb{P}^1 \times \mathbb{P}^3$ into $\mathbb{P}^{19}$ by the line bundle $H = O(1,2)$. Let $Y$ be a hyperplane section of $\mathbb{P}^1 \times \mathbb{P}^3$. Let $X$ be the projective cone over $Y$. $K_{\mathbb{P}^1 \times \mathbb{P}^3} = -2H$, $K_Y = -H$, $K_X = -2H$. $X$ is a canonical Fano 4-fold, with a canonical singularity at the vertex of the cone. Let $Z$ be a reduced irreducible nonsingular threefold such that $2Z = X \cap Y$, where $X$ is the 4-fold and $Y$ is a hypersurface in $\mathbb{P}^{19}$, $X$ normal with canonical singularities and such that $Z \cap \mathrm{Sing}(X) \neq \emptyset$. We consider a linear variety of dimension 2, $W$, through $P \in Z \cap \mathrm{Sing}(X)$, $W$ sufficiently general. $P' \in W \cap Z \cap \mathrm{Sing}(X)$ is an elliptic surface singularity. Note that the multiplicity of the vertex of the cone is greater than 2.

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