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$\mathbb{Q}$-homology projective planes with nodes or cusps

JongHae Keum

Abstract.

We classify all $\mathbb{Q}$-homology projective planes with A_1- or A_2-singularities (and with no worse singularities). It turns out that such a surface is isomorphic to a global quotient X/G, where X is a fake projective plane or the complex projective plane and G a finite abelian group of bi-holomorphic automorphisms. There are only finitely many such surfaces.

§1. Introduction

A $\mathbb{Q}$-homology projective plane is a normal projective complex surface with the Betti numbers of $\mathbb{CP}^2$. A nonsingular $\mathbb{Q}$-homology projective plane is a fake projective plane or $\mathbb{CP}^2$. The quotient of a $\mathbb{Q}$-homology projective plane by a finite group of bi-holomorphic automorphisms is again a $\mathbb{Q}$-homology projective plane, hence all weighted projective planes are $\mathbb{Q}$-homology projective planes. When a normal projective surface has rational singularities only, it is a $\mathbb{Q}$-homology projective plane if its second Betti number $b_2 = 1$. This can be seen by considering the Albanese fibration on a resolution of the surface. A minimal resolution of a $\mathbb{Q}$-homology projective plane with rational singularities always has $p_g = q = 0$.

It is known that a $\mathbb{Q}$-homology projective plane with quotient singularities (and no worse singularities) cannot have more than 5 singular points (cf. [7] Corollary 3.4). Recently, we have shown that the bound 5 is sharp, and moreover classified the case with 5 quotient singularities (Hwang–Keum [7]). As for the classification of the general case,

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there are infinitely many possible combinations of types of singularities and a full classification seems impossible. Other difficulty in classifying $\mathbb{Q}$-homology projective planes comes from the incompleteness of classification of surfaces of general type with $p_g = 0$. We refer the reader to our previous works [7], [8], [9], [10], [11] and [15], as for recent progress on $\mathbb{Q}$-homology projective planes and a related problem, the algebraic Montgomery–Yang Problem.

In this short note, we classify all $\mathbb{Q}$-homology projective planes with nodes or cusps (and no worse singularities). Here, a node and a cusp are cyclic singularities respectively of type $\frac{1}{2}(1,1)$ and $\frac{1}{3}(1,2)$.

Theorem 1. *Let Z be a $\mathbb{Q}$-homology projective plane with n nodes and c cusps and with no worse singularities. Then*

$$(n, c) = (0,0),\ (1,0),\ (1,1),\ (0,3),\ (0,4)$$

and in each case we have the following classification:

(1) *when $(n,c) = (0,0)$, $Z \cong \mathbb{CP}^2$ or a fake projective plane;*
(2) *when $(n,c) = (1,0)$, $Z \cong \mathbb{CP}(1,1,2)$ and the minimal resolution of Z is the rational ruled surface $\mathbb{F}_2$;*
(3) *when $(n,c) = (1,1)$, $Z \cong \mathbb{CP}(1,2,3)$ and the minimal resolution of Z is the double blow-up of $\mathbb{F}_2$ at a point away from the negative section;*
(4) *when $(n,c) = (0,3)$,*
  (a) *$Z \cong X/C_3$, where X is a fake projective plane with an order 3 automorphism; or*
  (b) *$Z \cong \mathbb{CP}^2/\langle\sigma\rangle$, where σ is the order 3 automorphism given by*

$$\sigma(x:y:z) = (x:\omega y:\omega^2 z), \ \ \textit{where } \omega = \exp(2\pi\sqrt{-1}/3);$$

(5) *when $(n,c) = (0,4)$,*
  (a) *$Z \cong X/C_3^2$, where X is a fake projective plane with $Aut(X) \cong C_3^2$; or*
  (b) *$Z \cong \mathbb{CP}^2/\langle\sigma,\tau\rangle$, where σ and τ are the commuting order 3 automorphisms given by*

$$\sigma(x:y:z) = (x:\omega y:\omega^2 z), \quad \tau(x:y:z) = (z:x:y).$$

In particular, Z is isomorphic to a global quotient X/G, where X is a fake projective plane or the complex projective plane. The moduli of such surfaces has dimension 0.

The existence of the surfaces appearing in Theorem 1 have been known, thus the main claim of the theorem is that there are no more.

Main tools in the proof are the orbifold Bogomolov–Miyaoka–Yau inequality (Sakai [26], Miyaoka [23], Kobayashi–Nakamura–Sakai [21], Megyesi [22]) and the results and methods given in our previous works, [7] and [8].

The case (2) of Theorem 1 is the only case where Z has nodes only, and is contained in a result of Dolgachev, Mendes Lopes and Pardini [5] (see also Keum [16], Corollary 1.2): they classified all smooth projective surfaces with $p_g = q = 0$ having $h^{1,1} - 1$ mutually disjoint (-2)-curves, which are exactly the minimal resolutions of $\mathbb{Q}$-homology projective planes with A_1-singularities only, where $h^{i,j}$ is the (i,j)-th Hodge number. They also classified all smooth rational surfaces with $h^{1,1} - 2$ mutually disjoint (-2)-curves, and then the author [16] classified all smooth projective surfaces, not necessarily minimal, with $h^{1,1}-2$ mutually disjoint (-2)-curves.

A fake projective plane is a smooth surface X of general type with $q(X) = p_g(X) = 0$, $c_2(X) = 3$ and $c_1(X)^2 = 9$. By Aubin [2] and Yau [27], its universal cover is the unit 2-ball $\mathbf{B} \subset \mathbb{C}^2$ and hence its fundamental group $\pi_1(X)$ is infinite. More precisely, $\pi_1(X)$ is a discrete torsion-free cocompact subgroup Π of $PU(2,1)$ having minimal Betti numbers and finite abelianization. By Mostow's rigidity theorem [24], such a ball quotient is strongly rigid, i.e., Π determines a fake projective plane up to holomorphic or anti-holomorphic isomorphism. By Kharlamov–Kulikov [19], no fake projective plane can be complex conjugate to itself. Thus the moduli space of fake projective planes consists of a finite number of points, and the number is the double of the number of distinct fundamental groups Π. By Hirzebruch's proportionality principle [6], Π has covolume 1 in $PU(2,1)$. Furthermore, Klingler [20] proved that the discrete torsion-free cocompact subgroups of $PU(2,1)$ having minimal Betti numbers are arithmetic (see also Yeung [28]). With these informations, Prasad and Yeung [25] carried out a classification of fundamental groups of fake projective planes. They exhibited 28 non-empty classes of fundamental groups ([25], Addendum). This classification has been refined by Cartwright and Steger [3]: using a computer-based group-theoretic computation they have shown that there are exactly 28 non-empty classes, yielding a complete list of fundamental groups of fake projective planes: there are exactly 50 of them, so the moduli space consists of exactly 100 points, corresponding to 50 pairs of complex conjugate fake projective planes. According to their result, the group $\mathrm{Aut}(X)$ of bi-holomorphic automorphisms of a fake projective plane X is isomorphic to one of the four groups:

$$\{1\},\ C_3,\ C_3^2,\ \text{or}\ 7:3,$$

where C_n denotes the cyclic group of order n, and $7:3$ the unique non-abelian group of order 21. This explains how the surfaces in the cases (4a) and (5a) of Theorem 1 arise. We refer the reader to our previous works [13], [14], [17] and [18], as for progress on geometric construction of fake projective planes.

Remark 2. (1) There are cubic surfaces with 3 cusps. By Theorem 1, such a surface is isomorphic to a quotient of $\mathbb{CP}^2$ by an order 3 automophism. This was known classically. In fact, a cubic surface with 3 cusps is isomorphic to the surface

$$Y := (w^3 - xyz = 0) \subset \mathbb{CP}^3.$$

The equation is exactly the relation among the invariants of the automorphism σ of $\mathbb{CP}^2$. The surface Y admits an order 3 automorphism

$$\mu(x:y:z:w) = (z:x:y:w).$$

It rotates the three cusps of Y,

$$\mu : (1:0:0:0) \to (0:1:0:0) \to (0:0:1:0),$$

and fixes the three points on Y,

$$(1:1:1:1),\ \ (\omega:\omega:\omega:1),\ \ (\omega^2:\omega^2:\omega^2:1).$$

Thus the quotient $Y/\langle\mu\rangle$ is a $\mathbb{Q}$-homology projective plane with 4 cusps (see [1], Proposition 5.1).

(2) Log del Pezzo surfaces of Picard number 1 with 3 or 4 cusps can be constructed in many ways. One way is to consider a rational elliptic surface V with 4 singular fibres of type I_3. Such an elliptic surface can be constructed by blowing up $\mathbb{P}^2$ at the 9 base points of the Hesse pencil. Every section is a (-1)-curve. Contracting a section, we get a nonsingular rational surface W with eight (-2)-curves forming a Dynkin diagram of type $4A_2$. Contracting these eight curves, we get a log del Pezzo surface of Picard number 1 with 4 cusps. On W, by contracting a string of two rational curves forming a diagram (-1)—(-2), we get a nonsingular rational surface with six (-2)-curves forming a Dynkin diagram of type $3A_2$. Contracting these six curves, we get a log del Pezzo surface of Picard number 1 with 3 cusps.

Notation

- $\mathbb{F}_e$:=Proj$(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$ the rational ruled surface
- C_n: a cyclic group of order n

- K_X: a canonical Weil divisor of a normal projective variety X
- $b_i(X) := \dim H^i(X, \mathbb{Q})$, the i-th Betti number of X
- $e(X)$: the topological Euler number of a complex variety X
- $p_g(X) := \dim H^2(X, \mathcal{O}_X)$, the geometric genus of a compact smooth surface X
- $q(X) := \dim H^1(X, \mathcal{O}_X)$, the irregularity of a compact smooth surface X

§2. Preliminaries

Let S be a normal projective surface with quotient singularities and

$$f : \tilde{S} \to S$$

be a minimal resolution of S. It is well-known (e.g., [12]) that quotient singularities are log-terminal singularities. Thus one can write the adjunction formula,

$$K_{\tilde{S}} \underset{num}{\equiv} f^*K_S - \sum_{p \in Sing(S)} \mathcal{D}_p,$$

where $\mathcal{D}_p = \sum(a_j A_j)$ is an effective $\mathbb{Q}$-divisor with $0 \leq a_j < 1$ supported on $f^{-1}(p) = \cup A_j$ for each singular point p. Here, note that the last term in the adjunction formula has negative sign so that the coefficients of $\mathcal{D}_p$ are nonnegative. It implies that

$$K_S^2 = K_{\tilde{S}}^2 - \sum_p \mathcal{D}_p^2 = K_{\tilde{S}}^2 + \sum_p \mathcal{D}_p K_{\tilde{S}}.$$

The coefficients of the $\mathbb{Q}$-divisor $\mathcal{D}_p$ can be obtained by solving the equations

$$\mathcal{D}_p A_j = -K_{\tilde{S}} A_j = 2 + A_j^2$$

given by the adjunction formula for each exceptional curve $A_j \subset f^{-1}(p)$.

The singular point p is a rational double point iff $\mathcal{D}_p = 0$.

Orbifold Bogomolov–Miyaoka–Yau Inequality. ([26], [23], [21], [22]) *Let S be a normal projective surface with quotient singularities such that K_S is nef. Then*

$$K_S^2 \leq 3e_{orb}(S),$$

where

$$e_{orb}(S) := e(S) - \sum_{p \in Sing(S)} \left(1 - \frac{1}{|G_p|}\right)$$

is the orbifold Euler characteristic, and G_p is the local fundamental group of p.

It is well known that the torsion-free part of the second cohomology group,

$$H^2(\tilde{S}, \mathbb{Z})_{free} := H^2(\tilde{S}, \mathbb{Z})/(torsion)$$

has a lattice structure which is unimodular. For a quotient singular point $p \in S$, let

$$R_p \subset H^2(\tilde{S}, \mathbb{Z})_{free}$$

be the sublattice spanned by the numerical classes of the components of $f^{-1}(p)$. It is a negative definite lattice, and its discriminant group

$$\mathrm{disc}(R_p) := \mathrm{Hom}(R_p, \mathbb{Z})/R_p$$

is isomorphic to the abelianization $G_p/[G_p, G_p]$ of the local fundamental group G_p. In particular, the absolute value $|\det(R_p)|$ of the determinant of the intersection matrix of R_p is equal to the order $|G_p/[G_p, G_p]|$. Let

$$R = \oplus_{p \in Sing(S)} R_p \subset H^2(\tilde{S}, \mathbb{Z})_{free}$$

be the sublattice spanned by the numerical classes of the exceptional curves of $f : \tilde{S} \to S$. We have

$$|\det(R)| = \prod |\det(R_p)| = \prod |G_p/[G_p, G_p]|.$$

We also consider the sublattice

$$R + \langle K_{\tilde{S}} \rangle \subset H^2(\tilde{S}, \mathbb{Z})_{free}$$

spanned by R and the canonical class $K_{\tilde{S}}$. Note that

$$\mathrm{rank}(R) \leq \mathrm{rank}(R + \langle K_{\tilde{S}} \rangle) \leq \mathrm{rank}(R) + 1.$$

We define

$$D := |\det(R + \langle K_{\tilde{S}} \rangle)|.$$

Lemma 3 ([7], Lemma 3.3). *Let S be a normal projective surface with quotient singularities and $f : \tilde{S} \to S$ be a minimal resolution of S. Then the following hold true.*

(1) $\mathrm{rank}(R + \langle K_{\tilde{S}} \rangle) = \mathrm{rank}(R)$ *if and only if K_S is numerically trivial.*

(2) $D = |\det(R + \langle K_{\tilde{S}} \rangle)| = |\det(R)| \cdot K_S^2$ *if K_S is not numerically trivial.*

(3) *If in addition* $b_2(S) = 1$ *and* K_S *is not numerically trivial, then* $R + \langle K_{\tilde{S}} \rangle$ *is a sublattice of finite index in the unimodular lattice* $H^2(\tilde{S}, \mathbb{Z})_{free}$, *in particular* $|\det(R + \langle K_{\tilde{S}} \rangle)|$ *is a nonzero square number.*

Proposition 4. *Let* Z *be a* $\mathbb{Q}$*-homology projective plane with* n *nodes and* c *cusps and no worse singularities. Assume that* Z *is singular. Then*

$$(n, c) = (1, 0),\ (1, 1),\ (0, 3),\ (0, 4).$$

Proof. Let

$$f : \tilde{Z} \to Z$$

be the minimal resolution.

A $\mathbb{Q}$-homology projective plane with quotient singularities can have at most 5 singular points, and the case with 5 quotient singularities was classified in [7]. According to this classification, a $\mathbb{Q}$-homology projective plane with quotient singularities has 5 singular points if and only if it has 3 singular points of type A_1 and 2 of type A_3, thus we have

$$n + c \leq 4.$$

A minimal resolution of a $\mathbb{Q}$-homology projective plane with rational singularities has $p_g = q = 0$. Note that

$$b_2(\tilde{Z}) = 1 + n + 2c.$$

Hence by the Noether formula,

$$K_{\tilde{Z}}^2 = 12 - e(\tilde{Z}) = 9 - n - 2c.$$

By the adjunction formula, $K_{\tilde{Z}} = f^* K_Z$, hence

$$K_Z^2 = K_{\tilde{Z}}^2 = 9 - n - 2c \geq 1.$$

In particular, $K_Z^2 \neq 0$, hence K_Z is not numerically trivial. By Lemma 3, the product D of the orders of abelianized local fundamental groups and K_Z^2 is a positive square number. In our situation,

$$D = |\det(R)| \cdot K_Z^2 = 2^n 3^c (9 - n - 2c),$$

which is a square only if $(n, c) = (1, 0),\ (1, 1),\ (0, 3),\ (0, 4)$. Q.E.D.

§3. The case with $(n, c) = (1, 0)$

This case was considered in [5] (see also [16], Corollary 1.2). They classified all $\mathbb{Q}$-homology projective planes with at most A_1-singularities. Such a surface, if not smooth, is isomorphic to the weighted projective plane $\mathbb{CP}(1, 1, 2)$, which is a global quotient of $\mathbb{CP}^2$.

§4. The case with $(n, c) = (1, 1)$

Let

$$f : \tilde{Z} \to Z$$

be the minimal resolution. In this case,

$$K_Z^2 = K_{\tilde{Z}}^2 = 6.$$

Since

$$e_{orb}(Z) = e(Z) - \sum \Big(1 - \frac{1}{|G_p|}\Big) = \frac{11}{6},$$

we have $K_Z^2 > 3e_{orb}(Z)$. Thus by the orbifold Bogomolov–Miyaoka–Yau Inequality, $-K_Z$ is ample and Z is a log del Pezzo surface.

Since $K_{\tilde{Z}}^2 < 8$, $\tilde{Z}$ is not minimal.

Lemma 5. *There is a (-1)-curve E on $\tilde{Z}$ that fits in the dual diagram*

$$(-2) — (-2) — E — (-2),$$

where the three (-2)-curves are exceptional for the resolution $f : \tilde{Z} \to Z$.

Proof. We use the technique from [8].

Let p_1 (resp. p_2) be the node (resp. the cusp) of Z. Note first that

$$D = |\det(R)| \cdot K_Z^2 = 36.$$

Let E be a (-1)-curve on $\tilde{Z}$. By Lemma 3.3 of [8], E can be written as

$$E = \frac{k}{\sqrt{D}} f^* K_Z + E(1) + E(2), \tag{4.1}$$

for some integer k, where $E(i)$ is a $\mathbb{Q}$-divisor supported on $f^{-1}(p_i)$. For the equality (4.1), we refer to Section 4 of [8].

Since $(f^*K_Z)E(i) = 0$ for all i, we have

$$(f^*K_Z)E = (f^*K_Z)(\frac{k}{6} f^*K_Z) = \frac{k}{6} K_Z^2 = k.$$

Since $(f^*K_Z)E = K_{\tilde{Z}}E = -1$,

$$k = -1.$$

From (4.1), we have

$$-1 = E^2 = (-\frac{1}{6} f^*K_Z)^2 + E(1)^2 + E(2)^2,$$

hence

$$E(1)^2 + E(2)^2 = -\frac{7}{6}.$$

Now by Lemma 7 of [8], we see that

$$E(1)^2 = -\frac{1}{2}, \quad E(2)^2 = -\frac{2}{3},$$

and moreover E meets with multiplicity 1 the component of $f^{-1}(p_1)$ and meets with multiplicity 1 exactly one of the two components of $f^{-1}(p_2)$. This completes the proof. Q.E.D.

Now, by contracting the (-1)-curve E and then contracting the terminal (-1)-curve, we get a smooth rational surface Y with a configuration of 2 smooth rational curves

$$(-2)\text{—}(0).$$

Note that $b_2(\tilde{Z}) = 4$. Thus

$$b_2(Y) = 2.$$

Contracting the (-2)-curve on Y, we get a $\mathbb{Q}$-homology projective plane X with one A_1-singularity. Then by [5] or [16],

$$X \cong \mathbb{CP}(1,1,2), \text{ hence } Y \cong \mathbb{F}_2.$$

This proves that the minimal resolution of Z is the double blow-up of $\mathbb{F}_2$ at a point away from the negative section. The choice of a point on $\mathbb{F}_2$ away from the negative section is unique up to an automorphism of $\mathbb{F}_2$, thus the uniqueness of Z follows. On the other hand, $\mathbb{CP}(1,2,3)$ is a $\mathbb{Q}$-homology projective plane with 1 node and 1 cusp. This gives the case (2) of Theorem 1.

§5. The cases with $(n, c) = (0, 3), (0, 4)$

Let

$$f : \tilde{Z} \to Z$$

be the minimal resolution. Since K_Z is not numerically trivial, either K_Z or $-K_Z$ is ample.

5.1. The case: K_Z is ample

In this case, $K_{\tilde{Z}} = f^*K_Z$ is nef, hence $\tilde{Z}$ is a minimal surface of general type. By ([18], Theorem 0.4), Z is the quotient of a fake projective plane by a group of order 3 if $(n, c) = (0, 3)$, and by a 3-elementary group of order 9 if $(n, c) = (0, 4)$.

5.2. The case: $-K_Z$ is ample

In this case, Z is a log del Pezzo surface of Picard number 1 with 3 or 4 cusps. In particular, $\tilde{Z}$ is a rational surface, hence $H^2(\tilde{Z},\mathbb{Z})$ is torsion free.

5.2.1. *The subcase: Z has 3 cusps* There is a C_3-cover

$$\pi : X \to Z$$

branched exactly at the 3 cusps, where X is a non-singular surface. The proof given in ([18], Section 2.1) for the existence of such a C_3-cover works for any $\mathbb{Q}$-homology projective plane Z with 3 cusps, not necessarily of general type. In fact, the proof in the log del Pezzo case is simpler, as $\tilde{Z}$ has no torsion, and goes as follows:

Consider the cohomology lattice $H^2(\tilde{Z},\mathbb{Z})$, which is unimodular of signature $(1,6)$ under intersection pairing. Since Z is a $\mathbb{Q}$-homology projective plane, $p_g(\tilde{Z}) = q(\tilde{Z}) = 0$ and hence

$$\mathrm{Pic}(\tilde{Z}) = H^2(\tilde{Z},\mathbb{Z}).$$

Let A_{i1} and A_{i2} be the two components of $f^{-1}(p_i)$, and

$$\mathcal{R}_i \subset H^2(\tilde{Z},\mathbb{Z})$$

be the sublattice spanned by the divisor classes of A_{i1} and A_{i2}. Consider the sublattice

$$\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \mathcal{R}_3 \subset H^2(\tilde{Z},\mathbb{Z}).$$

Its discriminant group $\mathcal{R}^*/\mathcal{R}$ is generated by three order 3 elements e_1, e_2, e_3, where e_i is the generator of $\mathcal{R}_i^*/\mathcal{R}_i$ of the form

$$e_i = \frac{A_{i1} + 2A_{i2}}{3}.$$

Since the orthogonal complement $\mathcal{R}^\perp \subset H^2(\tilde{Z},\mathbb{Z})$ is of rank 1, we see that $\bar{\mathcal{R}}/\mathcal{R}$ is a non-zero subgroup of $\mathcal{R}^*/\mathcal{R}$, where $\bar{\mathcal{R}}$ is the primitive closure of $\mathcal{R}$ in $H^2(\tilde{Z},\mathbb{Z})$. Thus there is an element $D \in \bar{\mathcal{R}} \setminus \mathcal{R}$ such that

$$D = a_1e_1 + a_2e_2 + a_3e_3 \text{ modulo } \mathcal{R}.$$

Since $e_i^2 = -\frac{2}{3}$, none of the a_i's is equal to 0 modulo 3; otherwise D^2 would not be an integer. Note that

$$-e_i = 2e_i = \frac{2A_{i1} + A_{i2}}{3} \text{ modulo } \mathcal{R}.$$

Thus up to a permutation of A_{i1} and A_{i2}, we may assume that $a_1 = a_2 = a_3 = 1$, and hence

$$D = \frac{A_{11} + 2A_{12}}{3} + \frac{A_{21} + 2A_{22}}{3} + \frac{A_{31} + 2A_{32}}{3} + R \text{ for some } R \in \mathcal{R}.$$

It follows that there is a divisor class $L \in \mathrm{Pic}(\tilde{Z})$ such that

$$3L = B,$$

where $B = A_{11} + 2A_{12} + A_{21} + 2A_{22} + A_{31} + 2A_{32}$ an integral divisor supported on the six (-2)-curves contracted to the points p_1, p_2, p_3 by the map $f : \tilde{Z} \to Z$. Now, L gives a C_3-cover of $\tilde{Z}$ branched along B and un-ramified outside B, hence yields a C_3-cover $X \to Z$ branched exactly at the three points p_1, p_2, p_3. Since the local fundamental group of the punctured germ of p_i is cyclic of order 3, the covering of the punctured germ is either trivial or the standard one. Since the C_3-cover $X \to Z$ is branched at each p_i, the latter case should occur. Thus X is a nonsingular surface.

Note that

$$K_Z^2 = 3.$$

Since $K_X = \pi^* K_Z$, we have

$$K_X^2 = 3K_Z^2 = 9.$$

Computing the topological Euler number of X, we have

$$e(X) = 3e(Z \setminus Sing(Z)) + 3 = 3.$$

Then by the Noether formula,

$$p_g(X) = q(X).$$

Since $-K_Z$ is ample, $-K_X$ is nef and not numerically trivial. Hence

$$p_g(X) = 0.$$

This shows that

$$X \cong \mathbb{CP}^2.$$

Up to a projective change of coordinates, we may assume that the covering automorphism of $\pi : \mathbb{CP}^2 \to Z$ fixes the three coordinate points $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)$. Thus it is given by the order 3 automorphism

$$\sigma(x : y : z) = (x : \omega y : \omega^2 z), \quad \omega = \exp(2\pi\sqrt{-1}/3).$$

5.2.2. *The subcase: Z has 4 cusps* Let p_1, p_2, p_3, p_4 be the four cusps of Z.

Lemma 6. *If there is a C_3-cover $\pi_1 : Y \to Z$ branched exactly at three of the four cusps of Z, then Y is a $\mathbb{Q}$-homology projective plane with 3 cusps such that $-K_Y$ is ample.*

Proof. We may assume that the three points are p_2, p_3, p_4. Note that Y has 3 cusps, the pre-image of p_1. Let $\tilde{Y} \to Y$ be the minimal resolution.

Note that

$$K_Z^2 = 1.$$

Since $K_Y = \pi_1^* K_Z$, we have

$$K_Y^2 = 3K_Z^2 = 3.$$

Computing the topological Euler number of Y, we have

$$e(Y) = 3e(Z \setminus \{p_2, p_3, p_4\}) + 3 = 3.$$

Hence

$$K_{\tilde{Y}}^2 = K_Y^2 = 3 \text{ and } e(\tilde{Y}) = e(Y) + 6 = 9.$$

Then by the Noether formula,

$$p_g(\tilde{Y}) = q(\tilde{Y}).$$

Since $-K_Z$ is ample, $K_Y = \pi_1^* K_Z$ is neither ample nor numerically trivial, hence it is anti-ample. It implies that

$$p_g(\tilde{Y}) = 0.$$

Thus

$$b_1(\tilde{Y}) = 0, \;\; b_2(\tilde{Y}) = 7,$$

hence $b_1(Y) = 0, b_2(Y) = 1$ and Y is a $\mathbb{Q}$-homology projective plane such that $-K_Y$ is ample. Q.E.D.

Lemma 7. *There is a C_3-cover $\pi_1 : Y \to Z$ branched exactly at three of the four cusps of Z, and a C_3-cover $\pi_2 : X \to Y$ branched exactly at the three cusps of Y.*

Proof. The existence of two C_3-covers can be proved by a lattice theoretic argument. Note that $\mathrm{Pic}(\tilde{Z}) = H^2(\tilde{Z}, \mathbb{Z})$. We know that $H^2(\tilde{Z}, \mathbb{Z})$ is a unimodular lattice of signature $(1, 8)$ under intersection pairing. Let $\mathcal{R}_i \subset H^2(\tilde{Z}, \mathbb{Z})$ be the sublattice spanned by the classes of the components A_{i1}, A_{i2} of $f^{-1}(p_i)$. Consider the sublattice

$$\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \mathcal{R}_3 \oplus \mathcal{R}_4 \subset H^2(\tilde{Z}, \mathbb{Z}).$$

Its discriminant group $\mathcal{R}^*/\mathcal{R}$ is 3-elementary of length 4, generated by four order 3 elements e_1, e_2, e_3, e_4, where e_i is the generator of $\mathcal{R}_i^*/\mathcal{R}_i$ of the form

$$e_i = \frac{A_{i1} + 2A_{i2}}{3}.$$

Since the orthogonal complement $\mathcal{R}^\perp \subset H^2(\tilde{Z}, \mathbb{Z})$ is of rank 1, we see that $\bar{\mathcal{R}}/\mathcal{R}$ is a subgroup of order 9 of $\mathcal{R}^*/\mathcal{R}$. Note that $e_i^2 = -\frac{2}{3}$. Every non-zero element of $\bar{\mathcal{R}}/\mathcal{R}$ has self-intersection number in $\mathbb{Z}$, hence must be of the form

$$\pm e_i \pm e_j \pm e_k.$$

Thus, up to a permutation of e_i's and modulo $\mathcal{R}$, $\bar{\mathcal{R}}/\mathcal{R}$ is generated by the two order 3 elements

$$e_2 + e_3 + e_4 \text{ and } e_1 - e_3 + e_4.$$

As in the proof of (3.2.1), we infer that there are two divisor classes $L_1, L_2 \in \mathrm{Pic}(\tilde{Z})$ such that

$$3L_1 = B_1, \quad 3L_2 = B_2,$$

where B_i is an integral divisor supported on the six (-2)-curves contained in $\cup_{j\neq i} f^{-1}(p_j)$ and each coefficient of B_i is 1 or 2.

Now L_1 gives a C_3-cover

$$\pi_1' : Y' \to \tilde{Z}$$

branched exactly at B_1, which induces a C_3-cover

$$\pi_1 : Y \to Z$$

branched exactly at the three points p_2, p_3, p_4. It is easy to see that the map

$$Y' \to Y$$

factors through

$$g : Y' \to \tilde{Y}.$$

By Lemma 6, Y is a $\mathbb{Q}$-homology projective plane with 3 cusps such that $-K_Y$ is ample. Note that

$$3g_*\pi_1'^* L_2 = g_*\pi_1'^* B_2,$$

where $g_*\pi_1'^*B_2$ is an integral divisor supported on the six (-2)-curves contracted to the 3 cusps by the map $\tilde{Y} \to Y$ and each coefficient in $g_*\pi_1'^*B_2$ is 1 or 2.

By the result of (5.2.1), there is a C_3-cover

$$\pi_2 : X \to Y$$

branched exactly at the three cusps of Y, where

$$X \cong \mathbb{CP}^2.$$

Q.E.D.

It remains to show that the composite map $X \to Z$ is a C_3^2-cover. Let σ be the order 3 automorphism of $\tilde{Y}$ corresponding to the C_3-cover $Y \to Z$. It preserves each of the two divisors, $g_*\pi_1'^*L_2$ and $g_*\pi_1'^*B_2$, hence lifts to an automorphism σ' of X, which normalizes the order 3 automorphism τ of X corresponding to the C_3-cover $X \to Y$. The fixed locus $X^{\sigma'}$ is not contained in the fixed locus X^τ. Thus $\tau \neq \sigma'^3$, hence the group generated by σ' and τ is isomorphic to C_3^2.

The Galois group of the covering $X \to Z$ is a conjugate of the group $\langle \sigma', \tau \rangle$. As we saw in (5.2.1), up to a change of coordinates σ' can be written as

$$\sigma'(x : y : z) = (x : \omega y : \omega^2 z), \quad \omega = \exp(2\pi\sqrt{-1}/3).$$

Then by a linear algebra we see that an order 3 automorphism commuting with σ', but not belonging to $\langle \sigma' \rangle$, must be of the form

$$\tau(x : y : z) = (z : ax : by),$$

where a and b are non-zero constants. By a diagonal change of coordinates we get the desired form for τ, without changing σ'.

Remark 8. In the case (3-a) and (4-a) of Theorem 1, the fundamental group $\pi_1(Z)$ is given by the list of Cartwright and Steger ([3], [4]):

$\pi_1(Z) \cong \{1\}$, C_2, C_3, C_4, C_6, C_7, C_{13}, C_{14}, C_2^2, $C_2 \times C_4$, S_3, D_8 or Q_8,

in the case (3-a);

$$\pi_1(Z) \cong \{1\} \text{ or } C_2$$

in the case (4-a).

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School of Mathematics
Korea Institute for Advanced Study
Seoul 130-722
Korea
E-mail address: `jhkeum@kias.re.kr`