Advanced Studies in Pure Mathematics 65, 2015
Algebraic Geometry in East Asia — Taipei 2011
pp. 117–122

Singularities of pluri-theta divisors in Char $p > 0$

Christopher D. Hacon

Abstract.

We show that if (X, Θ) is a PPAV over an algebraically closed field of characteristic $p > 0$ and $D \in |m\Theta|$, then $(X, \frac{1}{m}D)$ is a limit of strongly F-regular pairs and in particular $\operatorname{mult}_x(D) \leq m \cdot \dim X$ for any $x \in X$.

§1. Introduction

Let (X, Θ) be a principally polarized abelian variety (PPAV) so that X is a connected projective algebraic group and Θ is an ample divisor with $h^0(X, \mathcal{O}_X(\Theta)) = 1$. The geometry of X is often studied in terms of the singularities of the theta divisor Θ (or more genereally of the singularities of pluri-theta divisors i.e. divisors in $|m\Theta|$). For PPAVs over $\mathbb{C}$ there are a number of well known results saying that the singularities of pluri-theta divisors are mild (see for example [Kollar95], [EL97] and [Hacon99]). According to a result of Ein and Lazarsfeld (cf. [EL97, 3.5]), if $D \in |m\Theta|$, then $(X, \frac{1}{m}D)$ is log canonical. Since X is smooth, this is equivalent to saying that $(X, \frac{1-\epsilon}{m}D)$ is Kawamata log terminal for any $0 < \epsilon < 1$. The purpose of this brief note is to prove the analogous result in characteristic $p > 0$.

Theorem 1. *Let (X, Θ) be a PPAV over an algebraically closed field of characteristic $p > 0$. If $D \in |m\Theta|$, then $(X, \frac{1-\epsilon}{m}D)$ is strongly F-regular for any rational number $0 < \epsilon < 1$.*

Remark 2. In particular it follows that $(X, \frac{1}{m}D)$ is log canonical in the sense that all log-discrepancies are ≥ 0 and therefore $\operatorname{mult}_x(D) \leq m \cdot \dim X$ for any $x \in X$. Note that in characteristic 0 it is known

Received December 10, 2011.
Revised March 24, 2012.
2010 *Mathematics Subject Classification.* Primary 14K05; Secondary 14F17, 13A35.
Key words and phrases. Theta divisor.

that if $\mathrm{mult}_x(D) = m \cdot \dim X$, then X is a product of elliptic curves [EL97] and [Hacon99]. We do not know if the analogous result holds in characteristic $p > 0$. By [Hernandez11, 4.1] it follows that (X, D) is F-pure, and if p and m are coprime, then (X, D) is sharply F-pure.

Remark 3. Since in characteristic 0 it is known that irreducible Θ divisors are normal with rational singularities, it is natural to wonder if over an algebraically closed field of characteristic $p > 0$, irreducible Θ divisors are normal with F-rational singularities. Note that a related result appears in [BBE07, 6.7].

The characteristic 0 argument of Ein and Lazarsfeld relies on the theory of multiplier ideal sheaves, Kawamata–Viehweg vanishing and the Fourier–Mukai functor. In characteristic $p > 0$, the theory of Fourier–Mukai functors still applies and multiplier ideal sheaves can be replaced by test ideals. However, there is no good substitute for Kawamata–Viehweg vanishing (which is known to fail in this context). Instead, inspired by some ideas contained in [Schwede11], we use the "generic vanishing results" from [CH03], [Hacon04] and [PP08] to show that $h^0(X, \sigma(X, \frac{1-\epsilon}{m}D) \otimes \mathcal{O}_X(\Theta) \otimes P_{\hat{x}}) > 0$ for a general $\hat{x} \in \widehat{X}$. But then a general translate of Θ vanishes along the cosupport of $\sigma(X, \frac{1-\epsilon}{m}D)$. This is only possible if $\sigma(X, \frac{1-\epsilon}{m}D) = \mathcal{O}_X$ and so $(X, \frac{1-\epsilon}{m}D)$ is strongly F-regular and (1) follows.

§2. Preliminaries

Throughout this paper we work over an algebraically closed field k of characteristic $p > 0$. Recall that the ring homomorphism $F : k \to k$ defined by $F(x) = x^p$ endows k with a non-trivial k-module structure.

2.1. Test ideals

Here we recall the definition of test ideals and some related results that will be needed in this paper. We refer the reader to [BST11] and [Schwede11] (and the references therein) for a more complete treatment. Let $(X, \Delta = \sum d_i D_i)$ be a log pair so that X is a normal variety and $\Delta \geq 0$ is a $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $F : X \to X$ be the Frobenius morphism and for any integer $e > 0$, let F^e be its e-th iterate. The **parameter test submodule** of (X, Δ) denoted by $\tau(\omega_X, \Delta)$ is locally defined as the unique smallest non-zero $\mathcal{O}_X$-submodule M of ω_X such that $\phi(F^e_* M) \subset M$ for any $e > 0$ and any $\phi \in \mathrm{Hom}_{\mathcal{O}_X}(F^e_*\omega_X(\lceil (p^e - 1)\Delta \rceil), \omega_X)$. The **test ideal** $\tau(X, \Delta)$ is defined by $\tau(\omega_X, K_X + \Delta)$. It is known that $\tau(X, \Delta) \subset \mathcal{O}_X$ is an ideal sheaf such that

$$\tau(X, \Delta + A) = \tau(X, \Delta) \otimes \mathcal{O}_X(-A), \quad \text{and} \quad \tau(X, \Delta + eA) = \tau(X, \Delta)$$

for any Cartier divisor A and any rational number $0 \leq e \ll 1$. We also have that test ideals are contained in multiplier ideals in the sense that if $\pi : Y \to X$ is a proper birational morphism, then $\tau(X, \Delta) \subset \pi_* \mathcal{O}_Y(K_Y - \lfloor \pi^*(K_X + \Delta) \rfloor)$. (Recall that in characteristic 0, if π is a log resolution, then the multiplier ideal is defined by $\mathcal{J}(X, \Delta) = \pi_* \mathcal{O}_Y(K_Y - \lfloor \pi^*(K_X + \Delta) \rfloor)$.) In particular, if X is a smooth variety and $\mathrm{mult}_x(\Delta) \geq \dim X$, then $\tau(X, \Delta) \subset \mathfrak{m}_x$ where $\mathfrak{m}_x$ is the maximal ideal of x in X.

Suppose now that p does not divide the index of $K_X + \Delta$ so that $(p^e - 1)(K_X + \Delta)$ is Cartier for some integer $e > 0$. Let $\mathcal{L}_{e,\Delta} = \mathcal{O}_X((1 - p^e)(K_X + \Delta))$. There is a canonically determined (up to unit) homomorphism of line bundles $\phi_\Delta : F^e_* \mathcal{L}_{e,\Delta} \to \mathcal{O}_X$. We have that $\tau(X, \Delta)$ is the smallest non-zero ideal $J \subset \mathcal{O}_X$ such that $\phi_\Delta(F^e_*(J \cdot \mathcal{L}_{e,\Delta})) = J$. Similarly, we define $\sigma(X, \Delta)$ to be the largest ideal $J \subset \mathcal{O}_X$ such that $\phi_\Delta(F^e_*(J \cdot \mathcal{L}_{e,\Delta})) = J$. By definition (X, Δ) is **strongly F-regular** if $\tau(X, \Delta) = \mathcal{O}_X$, and **sharply F-pure** if $\sigma(X, \Delta) = \mathcal{O}_X$. We have that 1) if (X, Δ) is strongly F-regular then it is also sharply F-pure, and 2) if (X, Δ) is sharply F-pure and X is strongly F-regular, then $(X, (1-\epsilon)\Delta)$ is strongly F-regular for any $0 < \epsilon < 1$ (cf. [TW04, 2.2]).

2.2. Abelian varieties and the Fourier–Mukai transform

Here we recall some facts about the Fourier–Mukai transform introduced in [Mukai81]. Let $\widehat{X}$ be the dual abelian variety and P be the normalized Poincaré line bundle on $X \times \widehat{X}$. We denote by $\mathbf{R}\hat{S} : \mathbf{D}(X) \to \mathbf{D}(\widehat{X})$ the usual Fourier–Mukai functor given by $\mathbf{R}\hat{S}(\mathcal{F}) = \mathbf{R}p_{\widehat{X}*}(p_X^* \mathcal{F} \otimes P)$. There is a corresponding functor $\mathbf{R}S : \mathbf{D}(\widehat{X}) \to \mathbf{D}(X)$ such that

$$\mathbf{R}S \circ \mathbf{R}\hat{S} = (-1_X)^*[-g] \qquad \text{and} \qquad \mathbf{R}\hat{S} \circ \mathbf{R}S = (-1_{\widehat{X}})^*[-g].$$

Let A be any ample line bundle on $\widehat{X}$, then $\mathbf{R}^0 S(A) = \mathbf{R}S(A)$ is a vector bundle on X of rank $h^0(A)$ which we denote by $\widehat{A}$. For any $x \in X$, let $t_x : X \to X$ be the translation by x and let $\phi_A : \widehat{X} \to X$ be the isogeny determined by $\phi_A(\hat{x}) = t^*_{\hat{x}} A - A$, then $\phi_A^*(\widehat{A}) = \bigoplus_{h^0(A)} A^\vee$.

If (X, Θ) is a PPAV, then $\phi_\Theta : (X, \Theta) \to (\widehat{X}, \widehat{\Theta} = \phi_\Theta(\Theta))$ is an isomorphism of PPAVs. If $A = \mathcal{O}_X(m\widehat{\Theta})$ for some positive integer $m > 0$, then $\phi_A : \widehat{X} \to X$ can be identified with $m_X : X \to X$ (multiplication by m) and so it has degree $m^{2 \dim X}$ and $\phi_A^* \Theta \equiv m^2 \widehat{\Theta}$. (Note that the above notation is customary, but somewhat confusing as $\mathcal{O}_{\widehat{X}}(-\widehat{\Theta}) = \mathbf{R}^0 \hat{S}(\mathcal{O}_X(\Theta))$.)

We will need the following result.

Proposition 4. *Let $\mathcal{F}$ be a non-zero coherent sheaf on X such that $H^i(\mathcal{F}\otimes P_{\hat{x}}) = 0$ for all $i > 0$ and all $\hat{x} \in \widehat{X}$ (where for any $\hat{x} \in \hat{X}$, we let $P_{\hat{x}} = P|_{X\times\hat{x}}$). If $\mathcal{F} \to k(x)$ is a surjective morphism for some $x \in X$, then the induced map $H^0(\mathcal{F}\otimes P_{\hat{x}}) \to H^0(k(x)\otimes P_{\hat{x}}) \cong k(x)$ is surjective for general $\hat{x} \in \widehat{X}$.*

Proof. (Cf. [CH03, 2.3] or [PP08].) By cohomology and base change, one sees that $\hat{\mathcal{F}} = \mathbf{R}^0 S(\mathcal{F}) = \mathbf{R}S(\mathcal{F})$ is a sheaf and since $\mathbf{R}\hat{S}(\hat{\mathcal{F}}) = (-1_X)^*\mathcal{F}[-g] \neq 0$, we have that $\hat{\mathcal{F}} \neq 0$. Let $P_x = P|_{x\times\hat{X}}$, then $P_x = \mathbf{R}^0 S(k(x)) = \mathbf{R}S(k(x))$ and the homomorphism $\phi : \hat{\mathcal{F}} \to P_x$ is non-zero. However, as P_x is a line bundle (and hence torsion free of rank 1), it follows that ϕ is generically surjective. The proposition now follows since for any $\hat{x} \in \hat{X}$, the corresponding fiber of $\hat{\mathcal{F}}$ (resp. $\mathbf{R}S^0(k(x))$) is $H^0(\mathcal{F}\otimes P_{\hat{x}})$ (resp. $H^0(k(x)\otimes P_{\hat{x}}) = H^0(k(x))$). Q.E.D.

§3. Main result

Proof of (1). Let $0 < \epsilon < 1$ be a rational number such that the index of $\Delta = \frac{1-\epsilon}{m}D$ is not divisible by p. We will show that (X, Δ) is sharply F-pure. Note that for any given m, we may find a sequence of ϵ_i such that the index of $\Delta = \frac{1-\epsilon_i}{m}D$ is not divisible by p and $0 = \lim_{i\to\infty}\epsilon_i$. Since X is regular, it follows by what we have observed in Subsection 2.1, that $(X, (1-\epsilon)\Delta)$ is strongly F-regular for all rational numbers $0 < \epsilon \leq 1$.

We now fix $e > 0$ such that $(p^e - 1)\Delta$ is Cartier. For any $n > 0$ we have that

$$\phi^n_\Delta\left(F^{ne}_*(\sigma(X,\Delta)\cdot\mathcal{O}_X((1-p^{ne})(K_X+\Delta)))\right) = \sigma(X,\Delta).$$

We will show the following.

Claim 5. *For any sufficiently big integer $n \gg 0$, we have*

$$H^i(X, F^{ne}_*(\sigma(X,\Delta)\cdot\mathcal{O}_X((1-p^{ne})(K_X+\Delta)))\otimes\mathcal{O}_X(\Theta)\otimes P_{\hat{x}}) = 0$$

for all $i > 0$ and all $\hat{x} \in \widehat{X}$.

Granting the claim for the time being, we will now conclude the proof of the theorem. Let $x \in X$ be a general point, so that in particular x is not contained in the co-support of $\sigma(X,\Delta)$. We have a surjection

$$F^{ne}_*(\sigma(X,\Delta)\cdot\mathcal{O}_X((1-p^{ne})(K_X+\Delta)))\otimes\mathcal{O}_X(\Theta) \to k(x),$$

which factors through $\sigma(X,\Delta)\otimes\mathcal{O}_X(\Theta) \to k(x)$. By (4) and (5), we have that

$$H^0(F^{ne}_*(\sigma(X,\Delta)\cdot\mathcal{O}_X((1-p^{ne})(K_X+\Delta)))\otimes\mathcal{O}_X(\Theta)\otimes P_{\hat{x}}) \to k(x)$$

is surjective for general $\hat{x} \in \widehat{X}$. Since this map factors through $H^0(\sigma(X, \Delta) \otimes \mathcal{O}_X(\Theta) \otimes P_{\hat{x}})$, we have that the induced homomorphism $H^0(\sigma(X, \Delta) \otimes \mathcal{O}_X(\Theta) \otimes P_{\hat{x}}) \to k(x)$ is surjective. In particular $H^0(\sigma(X, \Delta) \otimes \mathcal{O}_X(\Theta) \otimes P_{\hat{x}}) \neq 0$, i.e. the corresponding translate of Θ vanishes along the co-support of $\sigma(X, \Delta)$. But then this co-support is empty so that $\sigma(X, \Delta) = \mathcal{O}_X$. Q.E.D.

Proof of Claim 5. It suffices to show that for all $i > 0$ we have

$$H^i(X, \sigma(X, \Delta) \cdot \mathcal{O}_X((1 - p^{ne})(K_X + \Delta)) \otimes F^{ne*}(\mathcal{O}_X(\Theta) \otimes P_{\hat{x}})) = 0.$$

Note that $F^{ne*}P_{\hat{x}} \cong P_{\hat{x}}^{\otimes p^{ne}} \cong P_{p^{ne}\hat{x}}$. By Serre vanishing (applied to the projective morphism $p_{\widehat{X}} : X \times \widehat{X} \to \widehat{X}$ and the coherent sheaf $p_X^* \sigma(X, \Delta) \cdot (F^{ne} \times \mathrm{id}_{\widehat{X}})^* P$, we may fix $t > 0$ such that

$$H^i(X, \sigma(X, \Delta) \cdot F^{ne*} P_{\hat{x}} \otimes \mathcal{O}_X(t\Theta)) = 0$$

for $i > 0$. By [PP03, 2.9], it suffices to show that

$$H^i(X, \mathcal{O}_X((1 - p^{ne})(K_X + \Delta) + (p^{ne} - t)\Theta)) = 0$$

for $i > 0$. By assumption $(1 - p^{ne})(K_X + \Delta) + (p^{ne} - t)\Theta \sim_{\mathbb{Q}} ((p^{ne} - 1)\epsilon + 1 - t)\Theta$. The claim now follows since $(p^{ne} - 1)\epsilon + 1 - t > 0$ for $n \gg 0$. Q.E.D.

Acknowledgments. The author was partially supported by NSF research grant DMS 0757897. He would also like to thank Mihnea Popa, Karl Schwede and Kevin Tucker for some useful discussions.

References

[BBE07] P. Berthelot, S. Bloch and H. Esnault, On Witt vector cohomology for singular varieties, Compos. Math., **143** (2007), 363–392.

[BST11] M. Blickle, K. Schwede and K. Tucker, F-singularities via alterations, preprint, arXiv:1107.3807v2.

[CH03] J. A. Chen and C. D. Hacon, Linear series of irregular varieties, In: Algebraic Geometry in East Asia, World Scientific 2002, pp. 143–153.

[EL97] L. Ein and R. Lazarsfeld, Singularities of theta divisors, and the birational geometry of irregular varieties, J. Amer. Math. Soc., **10** (1997), 243–258.

[Hacon99] C. D. Hacon, Divisors on principally polarized abelian varieties, Compos. Math., **119** (1999), 321–329.

[Hacon04] C. D. Hacon, A derived category approach to generic vanishing, J. Reine Angew. Math., **575** (2004), 173–187.

[Kollar95] J. Kollár, Shafarevich Maps and Automorphic Forms, M. B. Porter Lectures, Princeton Univ. Press, 1995.

[Hernandez11] D. Hernandez, F-purity of hypersurfaces, preprint, 2011.

[Mukai81] S. Mukai, Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves, Nagoya Math. J., **81** (1981), 153–175.

[PP03] G. Pareschi and M. Popa, Regularity on abelian varieties. I, J. Amer. Math. Soc., **16** (2003), 285–302.

[PP08] G. Pareschi and M. Popa, Regularity on abelian varieties III: Relationship with generic vanishing and applications, In: Grassmannians, Moduli Spaces and Vector Bundles, Clay Math. Proc., **14**, Amer. Math. Soc., Providence, RI, 2011, pp. 141–167.

[Schwede11] K. Schwede, A canonical linear system associated to adjoint divisors in characteristic $p > 0$, preprint, arXiv:1107.3833v2.

[TW04] S. Takagi and K.-I. Watanabe, On F-pure thresholds, J. Algebra, **282** (2004), 278–297.

Department of Mathematics
University of Utah
155 South 1400 East
Salt Lake City
UT 48112-0090
U.S.A.
E-mail address: hacon@math.utah.edu