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Solutions for some families of Fuchsian differential equations free from accessory parameters in terms of the integral of Euler type

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An ordinary differential equation of regular singular type defined on the Riemann sphere is called the *Fuchsian differential equation free from accessory parameters* or the *rigid Fuchsian differential equation*, if the equation is determined by the set of local data on monodromy, in particular, its spectral type.

About two decades ago, Yokoyama [29] classified such equations into eight types, I, II, III, IV, I*, II*, III*, and IV*, under some conditions from the viewpoint of the differential equation of Okubo type [23] (see also [8]). While the equation of type I is nothing but the generalized hypergeometric equation ${}_{n+1}E_n$ and that of type I* the Jordan–Pochhammer equation, the equations of the other types are new ones. Concerning the latter cases, very little has been understood: a restriction into one variable case of Appel's F_3 satisfies the equation II* of rank 4, the function satisfying the equation II of rank 4 is found in [16], and the functions satisfying the equation III* of rank 5 and of rank 7, the functions satisfying the equation II* of rank 4 and of rank 6, and the functions satisfying the equation II of rank 6 are found in [9].

The purpose of the present paper is to give solutions for the equations of types II, III, IV, II*, III*, and IV* in terms of the integral of Euler type.

In this paper, we frequantly use the symbol

$$e(A) = \exp(2\pi\sqrt{-1}A)$$

for abbreviation.

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§1. Fuchsian differential equations free from accessory parameters

For a Fuchsian differential equation with regular singular points $\{p_1, \dots, p_N, \infty\}$, let

$$\{\lambda_{j,1}, \lambda_{j,1}+1, \dots, \lambda_{j,1}+m_{j,1}, \lambda_{j,2}, \lambda_{j,2}+1, \dots, \lambda_{j,2}+m_{j,2}, \dots \\ \dots, \lambda_{j,n_j}, \lambda_{j,n_j}+1, \dots, \lambda_{j,n_j}+m_{j,n_j}\}$$

with $\lambda_{j,k}-\lambda_{j,l} \notin \mathbb{Z}$ $(k \neq l)$ be a set of characteristic exponents at p_j for $j = 1, \dots, N$ and ∞. Then $(m_{j,1}, m_{j,2}, \dots, m_{j,n_j})$ is called the *spectral type* at p_j and

$$(m_{1,1}, m_{1,2}, \dots, m_{1,n_1}\,;\, m_{2,1}, m_{2,2}, \dots, m_{2,n_2}\,;\, \cdots \\ \cdots\,;\, m_{N,1}, m_{N,2}, \dots, m_{N,n_N}\,;\, m_{\infty,1}, m_{\infty,2}, \dots, m_{\infty,n_\infty})$$

the spectral type of the equation. For instance, the spectral type of the Gauss hypergeometric equation is, under some genericity condition, $(1,1\,;\,1,1\,;\,1,1)$ and that of the generalized hypergeometric equation ${}_{n+1}E_n$ is, under some genericity condition, $(\underbrace{1,1,\dots,1}_{n+1}\,;\,1,n\,;\,\underbrace{1,1,\dots,1}_{n+1})$, which is also denoted by $(1^{n+1}\,;\,1,n\,;\,1^{n+1})$ or $(1^{n+1}\,;\,1,n\,;\,1^{n+1})$. Actually, the generalized hypergeometric differential equation ${}_{n+1}E_n$ is

$$\left\{\theta_z\Big\{\prod_{1\le i\le n}(\theta_z+\beta_i-1)\Big\} - z\Big\{\prod_{1\le i\le n+1}(\theta_z+\alpha_i)\Big\}\right\}F = 0,$$

where $\theta_z = zd/dz$, and its characteristic exponents are

$$0,\ 1-\beta_1,\ 1-\beta_2,\ \dots,\ 1-\beta_n \quad \text{at} \quad z=0,$$

$$0,\ 1,\ \dots,\ n-1,\ \sum_{1\le i\le n}\beta_i - \sum_{1\le i\le n+1}\alpha_i \quad \text{at} \quad z=1,$$

$$\alpha_1,\ \alpha_2,\ \dots,\ \alpha_{n+1} \quad \text{at} \quad z=\infty.$$

Thus its spectral type is $(1^{n+1}\,;\,1,n\,;\,1^{n+1})$ under the genericity condition

$$\beta_i-\beta_j \notin \mathbb{Z}, \quad 1\le i<j\le n+1, \quad \sum_{i=1}^{n}\beta_i - \sum_{i=1}^{n+1}\alpha_i \notin \mathbb{Z},$$

$$\alpha_i-\alpha_j \notin \mathbb{Z}, \quad 1\le i<j\le n+1,$$

where $\beta_{n+1} = 1$.

Similarly, the spectral type of the Jordan–Pochhamer differential equation

$$
\begin{aligned}
&Q(z)F^{(n)} - \mu Q'(z)F^{(n-1)} + \frac{\mu(\mu+1)}{2}Q''(z)F^{(n-2)} - \cdots \\
&- R(z)F^{(n-1)} + (\mu+1)R'(z)F^{(n-2)} - \cdots = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
Q(z) &= (z-c_1)(z-c_2)\cdots(z-c_n), \\
R(z)/Q(z) &= \sum_{j=1}^{n} \alpha_j/(z-c_j)
\end{aligned}
$$

is $(1, n-1; 1, n-1; \cdots; 1, n-1)$ under the condition

$$\mu + \alpha_j \notin \mathbb{Z} \quad \text{for} \quad j = 1, \ldots, n, \quad \text{and} \quad \alpha_1 + \cdots + \alpha_n \notin \mathbb{Z},$$

since the characteristic exponents at $z = c_j, j = 1, \ldots, n$ are

$$0, 1, \ldots, n-2, \mu + n - 1 + \alpha_j,$$

and those at $z = \infty$ are

$$-(\mu+1), \ldots, -(\mu+n-1), -(\mu + \alpha_1 + \cdots + \alpha_n).$$

Yokoyama's classification of the Fuchsian differential equations accessory parameter free is asserted as follows [29]:

	rank	# of singularities on $\mathbb{P}^1$	spectral type
I (GHGF)	n	3	$1^n\,;1,n-1\,;1^n$
I* (Pochhammer)	n	$n-1$	$1,n-1;1,n-1;\ldots;1,n-1$
II	$2n$	3	$1^n,n;1^n,n;1,n-1,n$
II*	$2n$	4	$1^n,n;1^{n-1},n+1;1,2n-1;n,n$
III	$2n+1$	3	$1^{n+1},n;1^n,n+1;1,n,n$
III*	$2n+1$	4	$1^n,n+1;1^n,n+1;1,2n;n,n+1$
IV	6	3	$1^2,4;2^3;1^4,2$
IV*	6	4	$1^2,4;1^2,4;1^2,4;2,4$

We consider their solutions.

It is known that the function of the form

$$\int_C \prod_{i=1}^{n} t_i^{\alpha_{i+1}-\beta_i} \prod_{i=1}^{n+1} (t_i - t_{i-1})^{\beta_i-\alpha_i-1}\, dt_1 \cdots dt_n\,, \tag{1.1}$$

where $t_0 = 1,\ t_{n+1} = z$ and C a suitable cycle, satisfies the generalized hypergeometric equation ${}_{n+1}E_n$, which is the equation of type I. See, for instance, [17, 18].

It is also known that the function of the form

$$\int_C \prod_{j=1}^{n+1} (t-c_j)^{\alpha_j-1} dt, \tag{1.2}$$

where $t_{n+1} = z, \alpha_{n+1} = \mu + n$ and C a suitable cycle, satisfies the Jordan–Pochhammer equation, which is the equation of type I*. See, for instance, [28].

In the remaining sections, we give the solutions for all the equations of types II, III, IV, II*, III*, and IV* in terms of the integral of Euler type.

For our purpose, we apply the framework of the twisted homology theory developed by Aomoto in these decades [1, 2]. We refer the reader to [3, 14, 19, 20, 21, 22] for more knowledge about the twisted homology and its application to the integral representation of the solution to differential equations.

§2. The equation of type II

In this section, let $\mathcal{L}_z$ be the locally constant sheaf (the local system) determined by a function

$$u(t) = (t_n - t_0)^{\lambda_{0n}} \prod_{1\le i\le n} t_i^{\lambda_i} \prod_{0\le i\le n} (t_i - t_{i+1})^{\lambda_{i,i+1}} \tag{2.1}$$

on

$$T_z = \mathbb{C}^n \backslash \{ t_n - t_0 = 0 \} \cup \cup_{i=1}^n \{ t_i = 0 \} \cup \cup_{i=0}^n \{ t_i - t_{i+1} = 0 \},$$

where $t_0 = 1$ and $t_{n+1} = z$: the sheaf consisting of the local solutions of $dL = L\omega$ for $\omega = du(t)/u(t)$.

Let $H_n(T_z, \mathcal{L}_z)$ be the n-th homology group with coefficients in $\mathcal{L}_z$, $H_n^{\mathrm{lf}}(T_z, \mathcal{L}_z)$ the n-th locally finite homology group with coefficients in $\mathcal{L}_z$.

After fixing the variable z to be a real number satisfying $0 < z < 1$, the bounded chambers in the real locus $T_{\mathbb{R}}$ of $T = T_z$ are

$$\begin{pmatrix} 0 < t_s < t_{s+1} < \cdots < t_n < z, \\ t_s < t_{s-1} \cdots < t_1 < 1 \end{pmatrix} \tag{2.2}$$

for $1 \le s \le n$ and

$$\begin{pmatrix} z < t_n < 1 \\ 0 < t_s < t_{s+1} < \cdots < t_n, \\ t_s < t_{s-1} \cdots < t_1 < 1 \end{pmatrix} \tag{2.3}$$

for $1 \le s \le n$. Thus it follows from Theorem 5 of [3] that $\dim H_n^{\mathrm{lf}}(T_z, \mathcal{L}_z) = 2n$ under the condition

$$\lambda_{\infty=t_p=\cdots=t_q} = -\sum_{p\le k\le q} \lambda_k - \sum_{p\le k\le q+1} \lambda_{k-1,k} \notin \mathbb{Z}, \quad 1 \le p \le q \le n-1,$$

$$\lambda_{\infty=t_p=\cdots=t_n} = -\lambda_{0n} - \sum_{p\le k\le n} \lambda_k - \sum_{p\le k\le n+1} \lambda_{k-1,k} \notin \mathbb{Z}, \quad 1 \le p \le n.$$

Hereafter we denote the exponent of an irreducible component of the divisor $\widetilde{D} = \pi^{-1}(D)$ by λ_D, where $\pi : \widetilde{(\mathbb{P}^1(\mathbb{C}))^n} \to (\mathbb{P}^1(\mathbb{C}))^n$ is the minimal blow-up along the non-normally crossing loci of D. If λ_D is an integer, the irreducible component or the exponent itself is said to be *resonant*. For instance, $\lambda_{\infty=t_p=\cdots=t_q} \notin \mathbb{Z}$ means that the exponent of $\pi^{-1}\{\infty = t_p = \cdots = t_q\}$ is not resonant.

For each $i = 0, 1$, let γ_i be a simple loop which starts and ends at a base point z for $0 < z < 1$ and surrounds only the singular point i in counterclockwise direction. It corresponds to a generator of $\pi_1(\mathbb{C}\backslash\{0,1\})$, and induces an action of $\pi_1(\mathbb{C}\backslash\{0,1\})$ on the family of the homology group $H_n^{\mathrm{lf}}(T_z, \mathcal{L}_z)$ on $\mathbb{C}\backslash\{0,1\}$.

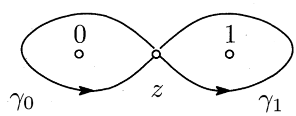

Fig. 1.

Considering the action of γ_0 on the chambers (2.3) shows that the multiplicity of holomorphic solutions around 0, which correspond to the eigenvalue 1, is n; considering the action of γ_0 on the chambers (2.2) shows that the eigenvalue

$$e\Big(\sum_{s\le k\le n} (\lambda_k + \lambda_{k,k+1})\Big) \tag{2.4}$$

for each $1 \le s \le n$ is multiplicity free. Moreover, it is seen that the chamber

$$\begin{pmatrix} 0 < t_s < t_{s+1} < \cdots < t_n < z, \\ 1 < t_1 < \cdots < t_{s-1} < t_s \end{pmatrix}$$

gives the eigenvector for the eigenvalue (2.4).

Similarly, considering the action of γ_1 on the chambers (2.2) shows that the multiplicity of the eigenvalue 1 is $n-1$, considering the action on the chambers (2.3) for $1 \le s \le n-1$ shows that the multiplicity of the eigenvalue

$$e(\lambda_{0n} + \lambda_{n,n+1})$$

is $n-1$, and the action on the chamber (2.3) for $s = n$ shows that the multiplicity of the eigenvalue

$$e(\lambda_{01} + \lambda_{0n} + \sum_{1\le k\le n} \lambda_{k,k+1})$$

is free.

Next, to know the eigenvalues of the action of $\gamma_\infty = \gamma_0^{-1}\gamma_1^{-1}$ and their multiplicities, we change the variables t_i into t_i^{-1} for $1 \le i \le n$. Then we have

$$u(t) = z^{\lambda_{n,n+1}} (t_0 - t_n)^{\lambda_{0n}} \prod_{1\le i\le n} t_i^{\lambda_i^\infty} \prod_{0\le i\le n} (t_{i+1} - t_i)^{\lambda_{i,i+1}},$$

where

$$\lambda_i^\infty = -\lambda_i - \lambda_{i-1,i} - \lambda_{i,i+1}, \qquad 1 \le i \le n-1,$$
$$\lambda_n^\infty = -\lambda_n - \lambda_{n-1,n} - \lambda_{n,n+1} - \lambda_{0n}$$

with $t_{n+1} = z^{-1}$ and $t_0 = 1$. After fixing the variable z^{-1} to be a real number satisfying $0 < z^{-1} < 1$, the bounded chambers are

$$\begin{pmatrix} 0 < t_s < t_{s+1} < \cdots < t_n < z^{-1}, \\ t_s < t_{s-1} \cdots < t_1 < 1 \end{pmatrix}$$

for $1 \le s \le n$ and

$$\begin{pmatrix} z^{-1} < t_n < 1 \\ 0 < t_s < t_{s+1} < \cdots < t_n, \\ t_s < t_{s-1} \cdots < t_1 < 1 \end{pmatrix}$$

for $1 \le s \le n$. Hence, the multiplicity of the eigenvalue

$$e(-\lambda_{n,n+1})$$

is free and that of

$$e(-\lambda_{0n} - \lambda_{s-1,s} - \sum_{s\le k\le n} (\lambda_k + \lambda_{k,k+1})), \quad 1 \le s \le n$$

is n.

Consequently, the spectral type turns out to be $(1^n, n; n, n-1, 1; 1^n, n)$ under the conditions

$$\sum_{s\le k\le l} (\lambda_k + \lambda_{k,k+1}) \notin \mathbb{Z}, \quad (1 \le s \le l \le n),$$

which is for the separation of the eigenvalues at 0,

$$\lambda_{01} + \lambda_{0n} + \sum_{1 \le k \le n} \lambda_{k,k+1} \notin \mathbb{Z},$$
$$\lambda_{0n} + \lambda_{n,n+1} \notin \mathbb{Z},$$
$$\lambda_{01} + \sum_{1 \le k \le n-1} \lambda_{k,k+1} \notin \mathbb{Z},$$

which is for the separation of the eigenvalues at 1, and

$$\lambda_{0n} + \sum_{s \le k \le n} (\lambda_k + \lambda_{k-1,k}) \notin \mathbb{Z}, \quad 1 \le s \le n,$$
$$\sum_{s \le k \le l-1} (\lambda_k + \lambda_{k-1,k}) \notin \mathbb{Z}, \quad 1 \le s < l \le n,$$

which is for the separation of the eigenvalues at ∞.

As a result, we have

Theorem 2.1. *The function of the form*

$$\int_C (t_n - t_0)^{\lambda_{0n}} \prod_{1 \le i \le n} t_i^{\lambda_i} \prod_{0 \le i \le n} (t_i - t_{i+1})^{\lambda_{i,i+1}} \, dt_1 \cdots dt_n,$$

where $t_0 = 1, t_{n+1} = z$ and C a suitable cycle, satisfies the equation of type II, whose rank is $2n$.

§3. The equation of type III

In this section, let $\mathcal{L}_z$ be the locally constant sheaf determined by a function

$$u(t) = (t_n - t_0)^{\lambda_{0n}} \prod_{1 \le i \le n-1} t_i^{\lambda_i} \prod_{0 \le i \le n} (t_i - t_{i+1})^{\lambda_{i,i+1}} \tag{3.1}$$

on

$$T_z = \mathbb{C}^n \backslash \cup \{ t_n - t_0 = 0 \} \cup \cup_{i=1}^{n-1} \{ t_i = 0 \} \cup \cup_{i=0}^{n} \{ t_i - t_{i+1} = 0 \},$$

where $t_0 = 1$ and $t_{n+1} = z$.

For a fixed variable z such that $0 < z < 1$, the bounded chambers in the real locus $T_{\mathbb{R}}$ of $T = T_z$ are

$$\begin{pmatrix} 0 < t_s < t_{s+1} < \cdots < t_n < z, \\ t_s < t_{s-1} \cdots < t_1 < 1 \end{pmatrix} \tag{3.2}$$

for $1 \le s \le n-1$ and

$$(3.3) \qquad \begin{pmatrix} z < t_n < 1 \\ 0 < t_s < t_{s+1} < \cdots < t_n, \\ t_s < t_{s-1} \cdots < t_1 < 1 \end{pmatrix}$$

for $1 \le s \le n$ (Remark that the latter one for $s = n$ means $(z < t_n < \cdots < t_1 < 1)$).

Thus $\dim H_n^{\mathrm{lf}}(T_z, \mathcal{L}_z) = 2n-1$ under the condition

$$\lambda_{\infty=t_p=\cdots=t_q} = -\sum_{p\le k\le q} \lambda_k - \sum_{p\le k\le q+1} \lambda_{k-1,k} \notin \mathbb{Z}, \quad 1 \le p \le q \le n-1,$$

$$\lambda_{\infty=t_p=\cdots=t_n} = -\lambda_{0n} - \sum_{p\le k\le n} \lambda_k - \sum_{p\le k\le n+1} \lambda_{k-1,k} \notin \mathbb{Z}, \quad 1 \le p \le n.$$

Considering the action of γ_0 on the cycles (3.2) for $1 \le s \le n-1$ and (3.3) for $1 \le s \le n$ as in Section 2, it is seen that the multiplicity of the eigenvalue

$$e(\sum_{s\le k\le n-1} (\lambda_k + \lambda_{k,k+1}) + \lambda_{n,n+1})$$

for each $1 \le s \le n-1$ is free and the multiplicity of the eigenvalue 1 is n.

Similarly, as for the action of γ_1, the multiplicity of the eigenvalue 1 is $n-1$, the multiplicity of the eigenvalue

$$e(\lambda_{0n} + \lambda_{n,n+1})$$

is $n-1$, and the multiplicity of the eigenvalue

$$e(\lambda_{01} + \lambda_{0n} + \sum_{1\le k\le n} \lambda_{k,k+1})$$

is free.

As for the multiplicities of the eigenvalues of the action of γ_∞, we consider

$$u(t) = z^{\lambda_{n,n+1}} (t_0 - t_n)^{\lambda_{0n}} \prod_{1\le i\le n} t_i^{\lambda_i^\infty} \prod_{0\le i\le n} (t_{i+1} - t_i)^{\lambda_{i,i+1}},$$

where

$$\lambda_i^\infty = -\lambda_i - \lambda_{i-1,i} - \lambda_{i,i+1}, \qquad 1 \le i \le n-1,$$

$$\lambda_n^\infty = -\lambda_{0n} - \lambda_{n-1,n} - \lambda_{n,n+1}$$

with $t_{n+1} = z^{-1}$ and $t_0 = 1$. This shows that the multiplicity of the eigenvalue

$$e(-\lambda_{0n} - \sum_{s\le k\le n-1} (\lambda_k + \lambda_{k-1,k}) - \lambda_{n-1,n} - \lambda_{n,n+1})$$

for each $1 \le s \le n$ is free, and the multiplicity of the eigenvalue

$$e(-\lambda_{n,n+1})$$

is $n-1$.

Therefore, the spectral type is $(1^{n-1}, n; 1, n-1, n-1; 1^n, n-1)$ under the condition

$$\sum_{s\le k\le n-1} (\lambda_k + \lambda_{k,k+1}) + \lambda_{n,n+1} \notin \mathbb{Z}, \quad 1 \le s \le n-1,$$

$$\sum_{s\le k\le l-1} (\lambda_k + \lambda_{k,k+1}) \notin \mathbb{Z}, \quad 1 \le s < l \le n-1,$$

which is for the separation of the eigenvalues at 0,

$$\lambda_{01} + \lambda_{0n} + \sum_{1\le k\le n} \lambda_{k,k+1} \notin \mathbb{Z},$$

$$\lambda_{0n} + \lambda_{n,n+1} \notin \mathbb{Z},$$

$$\lambda_{01} + \sum_{1\le k\le n-1} \lambda_{k,k+1} \notin \mathbb{Z},$$

which is for the separation of the eigenvalues at 1, and

$$\lambda_{0,n} + \sum_{s\le k\le n-1} (\lambda_k + \lambda_{k,k+1}) + \lambda_{s-1,s} \notin \mathbb{Z}, \quad 1 \le s \le n,$$

$$\sum_{s\le k\le l-1} (\lambda_k + \lambda_{k-1,k}) \notin \mathbb{Z}, \quad 1 \le s < l \le n,$$

which is for the separation of the eigenvalues at ∞.

Finally we reach the following.

Theorem 3.1. *The function of the form*

$$\int_C (t_n - t_0)^{\lambda_{0n}} \prod_{1\le i\le n-1} t_i^{\lambda_i} \prod_{0\le i\le n} (t_i - t_{i+1})^{\lambda_{i,i+1}}\, dt_1 \cdots dt_n,$$

where $t_0 = 1, t_{n+1} = z$ and C a suitable cycle, satisfies the equation of type III, whose rank is $2n-1$.

It is worthwhile to note that the function (3.1) can be obtained from (2.1) by the specialization $\lambda_n = 0$.

§4. The equation of type IV

In this section, let $\mathcal{L}_z$ be the locally constant sheaf determined by a function

$$u(t) = \prod_{i=1,2} t_i^{\lambda_i} \prod_{i=1,3,4} (t_i - t_0)^{\lambda_{0i}} \prod_{i=1,2,3,4} (t_i - t_{i+1})^{\lambda_{i,i+1}} \tag{4.1}$$

on

$$T_z = \mathbb{C}^4 \setminus \cup_{i=1}^{2} \{t_i = 0\} \cup \cup_{i=1,3,4}\{t_i - 1 = 0\} \cup_{i=1}^{4} \{t_i - t_{i+1} = 0\},$$

where $t_0 = 1$ and $t_5 = z$.

For a fixed z such that $0 < z < 1$, the bounded chambers in the real locus $T_{\mathbb{R}}$ of $T = T_z$ are

$$(0 < t_1 < t_2 < t_3 < t_4 < z), \quad \begin{pmatrix} 0 < t_2 < t_3 < t_4 < z, \\ t_2 < t_1 < 1 \end{pmatrix},$$

$$(z < t_4 < t_3 < t_2 < t_1 < 1), \quad \begin{pmatrix} z < t_4 < t_3 < 1, \\ 0 < t_1 < t_2 < t_3 \end{pmatrix},$$

$$\begin{pmatrix} z < t_4 < t_3 < 1, \\ 0 < t_2 < t_3, \\ t_2 < t_1 < 1 \end{pmatrix}, \quad \begin{pmatrix} z < t_4 < 1, \\ 0 < t_1 < t_2 < t_3 < t_4 \end{pmatrix},$$

and $$\begin{pmatrix} z < t_4 < 1, \\ 0 < t_2 < t_3 < t_4, \\ t_2 < t_1 < 1 \end{pmatrix}.$$

Thus $\dim H_n^{\mathrm{lf}}(T_z, \mathcal{L}_z) = 7$ under the condition

$$\lambda_{\infty=t_p=\cdots=t_q}=-\sum_{p\le k\le q}\lambda_k-\sum_{p\le k\le q+1}\lambda_{k-1,k}\notin\mathbb{Z},\quad 1\le p\le q\le 2,$$

$$\lambda_{\infty=t_p=\cdots=t_q}=-\sum_{p\le k\le q}\lambda_{0k}-\sum_{p\le k\le q+1}\lambda_{k-1,k}\notin\mathbb{Z},\quad 3\le p\le q\le 4,$$

$$\begin{aligned}\lambda_{\infty=t_1=\cdots=t_q}=&-\sum_{1\le k\le 2}\lambda_k-\sum_{3\le k\le q}\lambda_{0k}\\&-\sum_{1\le k\le q+1}\lambda_{k-1,k}\notin\mathbb{Z},\quad 3\le q\le 4,\end{aligned}$$

$$\lambda_{\infty=t_2=\cdots=t_q}=-\lambda_2-\sum_{3\le k\le q}\lambda_{0k}-\sum_{2\le k\le q+1}\lambda_{k-1,k}\notin\mathbb{Z},\quad 3\le q\le 4.$$

As for the action of γ_0, the multiplicities of the eigenvalues

$$e(\lambda_1+\lambda_2+\sum_{1\le k\le 4}\lambda_{k,k+1}),\quad e(\lambda_2+\sum_{2\le k\le 4}\lambda_{k,k+1})$$

are both multiplicity free, and the multiplicity of the eigenvalue 1 is 5; as for the action of γ_1, the multiplicities of the eigenvalues

$$e(\sum_{k=3,4}(\lambda_{0,k}+\lambda_{k,k+1})),\quad e(\lambda_{04}+\lambda_{45}),\quad 1$$

are all 2, and the multiplicity of the eigenvalue

$$e(\sum_{k=1,3,4}\lambda_{0,k}+\sum_{1\le k\le 4}\lambda_{k,k+1})$$

is free.

As for the action of γ_∞, we consider the function

$$u(t)=z^{\lambda_{45}}\prod_{1\le i\le 4}t_i^{\lambda_i^\infty}\prod_{i=1,3,4}(1-t_i)^{\lambda_{0i}}\prod_{i=1,2,3,4}(t_{i+1}-t_i)^{\lambda_{i,i+1}},$$

where

$$\lambda_i^\infty=\begin{cases}-\lambda_i-\lambda_{i-1,i}-\lambda_{i,i+1}, & i=1,2,\\ -\lambda_{0i}-\lambda_{i-1,i}-\lambda_{i,i+1}, & i=3,4,\end{cases}$$

with $t_5 = z^{-1}$ and $t_0 = 1$. It is seen that the multiplicity of each of the eigenvalues

$$e(-\lambda_{04} - \lambda_{34} - \lambda_{45}) \quad \text{and} \quad e(-\lambda_{45})$$

is 2, and the multiplicity of each of the eigenvalues

$$\begin{aligned} &e(-\lambda_1 - \lambda_2 - \sum_{k=1,3,4} \lambda_{0,k} - \sum_{1\le k\le 4} \lambda_{k,k+1}), \\ &e(-\lambda_2 - \sum_{k=3,4} \lambda_{0,k} - \sum_{1\le k\le 4} \lambda_{k,k+1}), \\ \text{and} \qquad &e(-\lambda_{03} - \lambda_{04} - \sum_{2\le k\le 4} \lambda_{k,k+1}) \end{aligned}$$

is free.

Thus the spectral type is (511; 2221; 22111) under some genericity condition.

At this stage, we impose a resonace condition

$$\text{(4.2)} \quad \lambda_{t_1=t_2=t_3=t_4=1} + 3 = \lambda_{01} + \lambda_{12} + \lambda_{23} + \lambda_{03} + \lambda_{34} + \lambda_{04} + 3 = 0.$$

As a result, a subspace which is invariant with respect to the action of the monodromy group emerges; it consists of the regularizable cycles, it has a spectral data (411; 222; 21111), and it gives the solution space of the equation of type IV. We proceed to that point.

First, it is important to note that the resonance (4.2) leads to the nontriviality of the kernel of the map $\iota : H_4(T, \mathcal{L}) \longrightarrow H_4^{\mathrm{lf}}(T, \mathcal{L})$. The dimension of the kernel is one, and thus the dimension of the image is 6. The image $\operatorname{Im}\iota$ is called the space of *regularizable cycles.* (See [19].)

Secondly, the chambers

$$(0 < t_1 < t_2 < t_3 < t_4 < z), \quad \begin{pmatrix} 0 < t_2 < t_3 < t_4 < z, \\ 1 < t_1 < \infty \end{pmatrix},$$

are regularizable even if the resonance condition (4.2) is imposed, and give the eigenvectors with the eigenvalues

$$e(\lambda_1 + \lambda_2 + \sum_{1\le k\le 4} \lambda_{k,k+1}), \quad e(\lambda_2 + \sum_{2\le k\le 4} \lambda_{k,k+1}),$$

with respect to the action of γ_0. Thus the spectral type at 0 of the space of regularizable cycles must be (114).

On the other hand, the chamber

$$(z < t_4 < t_3 < t_2 < t_1 < 1)$$

cannot be regularizable under the resonance condition (4.2). Thus (222) is the specral type at 1 of the space of regularizable cycles.

Moreover, if we consider

$$u(t) = z^{\lambda_{45}} \prod_{1\le i\le 4} t_i^{\lambda_i^\infty} \prod_{i=1,3,4} (1-t_i)^{\lambda_{0i}} \prod_{i=1,2,3,4} (t_{i+1}-t_i)^{\lambda_{i,i+1}}$$

for $0 < z^{-1} < 1$, it is seen that the chambers

$$(0 < t_1 < t_2 < t_3 < t_4 < z^{-1}), \quad \begin{pmatrix} 0 < t_2 < t_3 < t_4 < z^{-1}, \\ 1 < t_1 \end{pmatrix},$$

$$\text{and} \quad \begin{pmatrix} 0 < t_3 < t_4 < z^{-1}, \\ 1 < t_1 < t_2 \end{pmatrix},$$

are regularizable, even if the resonance condition (4.2) is imposed. Hence, either the eigenspace with the eigenvalue $e(-\lambda_{04} - \lambda_{34} - \lambda_{45})$ or that with $e(-\lambda_{45})$ becomes one dimensional space. It means that spectral type at ∞ is (11112).

Therefore, the spectral type of the space of regularizable cycles turns out to be (114; 222; 11112).

$$\begin{array}{llcclll} 1,1,5 & \text{at } 0 & & & 1,1,4 & \text{at } 0 \\ 1,2,2,2 & \text{at } 1 & \longrightarrow & (\text{IV}) & 2,2,2 & \text{at } 1 \\ 1,1,1,2,2 & \text{at } \infty & & & 1,1,1,1,2 & \text{at } \infty. \end{array}$$

Consequently, we have

Theorem 4.1. *The function of the form*

$$\int_C \prod_{i=1,2} t_i^{\lambda_i} \prod_{i=1,3,4} (t_i - t_0)^{\lambda_{0i}} \prod_{i=1,2,3,4} (t_i - t_{i+1})^{\lambda_{i,i+1}} \, dt_1 \cdots dt_4,$$

where

$$\lambda_{01} + \lambda_{12} + \lambda_{23} + \lambda_{03} + \lambda_{34} + \lambda_{04} + 3 = 0, \quad t_0 = 1, \quad t_5 = z,$$

and C a suitable cycle, satisfies the equation of type IV, whose rank is 6.

Remark. *The spectral type of the function ϕ which is obtained from the integral in the case III of rank* 5 *multiplied by* $(1-z)^{\lambda_{04}} = (1-t_4)^{\lambda_{04}}$ *is* $(311; 221; 2111)$, *and the characteristic exponent corresponding to the solution around* 1 *whose multiplicity is free is* $\lambda_{01} + \lambda_{12} + \lambda_{23} + \lambda_{03} + \lambda_{34} + \lambda_{04} + 3$. *Euler transfrom of ϕ under* (4.2) *is nothing but the integral in Theorem 4.1.*

§5. The equation of type IV*

In this section, let $\mathcal{L}_z$ be the locally constant sheaf determined by a function

$$u(t) = (t_2 - c)^{\lambda_{2c}} \prod_{i=1,2} \left\{ t_i^{\lambda_i} (t_i - t_0)^{\lambda_{0i}} (t_i - t_{i+1})^{\lambda_{i,i+1}} \right\} \tag{5.1}$$

on

$$T_z = \mathbb{C}^2 \backslash \{ t_2 - c = 0 \} \cup \\ \cup_{i=1}^{2} \left(\{ t_i = 0 \} \cup \{ t_i - t_0 = 0 \} \cup \{ t_i - t_{i+1} = 0 \} \right),$$

where $t_0 = 1$ and $t_3 = z$.

After fixing the variable z and parameter c to be real numbers satisfying $0 < z < c < 1$, the bounded chambers in the real locus $T_{\mathbb{R}}$ of $T = T_z$ are

$$\begin{array}{ll} (0 < t_1 < t_2 < z), & (0 < t_2 < z,\ t_2 < t_1 < 1), \\ (0 < t_1 < t_2,\ z < t_2 < c), & (z < t_2 < c,\ t_2 < t_1 < 1), \\ (0 < t_1 < t_2,\ c < t_2 < 1), & (c < t_2 < t_1 < 1). \end{array}$$

Thus $\dim H_2^{\mathrm{lf}}(T_z, \mathcal{L}_z) = 6$ under the condition

$$\lambda_{\infty = t_1 = t_2} = -\lambda_{2c} - \sum_{1 \le k \le 2} (\lambda_k + \lambda_{0k} + \lambda_{k,k+1}) \notin \mathbb{Z},$$

$$\lambda_{\infty = t_1} = -\lambda_1 - \lambda_{01} - \lambda_{12} \notin \mathbb{Z},$$

$$\lambda_{\infty = t_2} = -\lambda_2 - \lambda_{02} - \lambda_{2c} - \lambda_{12} - \lambda_{23} \notin \mathbb{Z}.$$

The eigenvalues of the action of γ_0 are

$$e(\lambda_1 + \lambda_2 + \lambda_{12} + \lambda_{23}), \quad e(\lambda_2 + \lambda_{23}),$$

each with multiplicity free, and 1 with multiplicity 4; the action of γ_c are

$$e(\lambda_{2c} + \lambda_{23})$$

with multiplicity 2, and 1 with multiplicity 4. To know the eigenvalues of the action of γ_1 and γ_∞, fix z and c to be $0 < c < 1 < z < \infty$. Then it turns out that the eigenvalues of the action of γ_1 are

$$e(\lambda_{02} + \lambda_{23}), \quad e(\lambda_{01} + \lambda_{02} + \lambda_{12} + \lambda_{23}),$$

each with multiplicity free, and 1 with multiplicity 4; those of the action of γ_∞ are

$$e(-\sum_{i=1,2} (\lambda_{0i} + \lambda_i + \lambda_{i,i+1}) - \lambda_{2c}), \quad e(-\lambda_{02} - \lambda_2 - \lambda_{23} - \lambda_{2c}),$$

each with multiplicity free, and $e(\lambda_{23})$ with multiplicity 4.

Therefore, the specral type of the space of regularizable cycle is $(411; 42; 411; 411)$ under the conditions:

$$\sum_{k=1,2} (\lambda_k + \lambda_{k,k+1}) \notin \mathbb{Z}, \qquad \lambda_k + \lambda_{k,k+1} \notin \mathbb{Z}, \qquad k = 1, 2,$$

which is for the separation of the eigenvalues at 0,

$$\lambda_{2c} + \lambda_{23} \notin \mathbb{Z},$$

which is for the separation of the eigenvalues at c,

$$\sum_{k=1,2} (\lambda_{0k} + \lambda_{k,k+1}) \notin \mathbb{Z}, \qquad \lambda_{0,k} + \lambda_{k,k+1} \notin \mathbb{Z}, \qquad k = 1, 2,$$

which is for the separation of the eigenvalues at 1, and

$$\lambda_{2c} + \sum_{k=1,2} (\lambda_k + \lambda_{0k} + \lambda_{k,k+1}) \notin \mathbb{Z}, \quad \lambda_{2c} + \lambda_{12} + \sum_{k=1,2} (\lambda_k + \lambda_{0k}) \notin \mathbb{Z},$$

$$\lambda_2 + \lambda_{02} + \lambda_{2c} + \lambda_{12} + \lambda_{23} \notin \mathbb{Z}, \quad \lambda_2 + \lambda_{02} + \lambda_{2c} + \lambda_{12} \notin \mathbb{Z},$$

$$\lambda_1 + \lambda_{01} \notin \mathbb{Z}, \quad \lambda_{23} \notin \mathbb{Z},$$

which is for the separation of the eigenvalues at ∞.

The combination of these facts leads to

Theorem 5.1. *The function of the form*

$$\int_C (t_2 - c)^{\lambda_{2c}} \prod_{i=1,2} \left\{ t_i^{\lambda_i} (t_i - t_0)^{\lambda_{0i}} (t_i - t_{i+1})^{\lambda_{i,i+1}} \right\} dt_1 dt_2,$$

where $t_0 = 1, t_3 = z$ and C a suitable cycle, satisfies the equation of type IV^*, whose rank is 6.

§6. The equation of type III*

In this section, let $\mathcal{L}_z$ be the locally constant sheaf determined by a function

$$(6.1) \qquad u(t) = (t_1 - c)^{\lambda_{1c}} (t_n - t_0)^{\lambda_{0n}} \prod_{1 \le i \le n} t_i^{\lambda_i} \prod_{1 \le i \le n} (t_i - t_{i+1})^{\lambda_{i,i+1}}$$

on

$$T_z = \mathbb{C}^n \backslash \{ t_1 - c = 0 \} \cup \{ t_n - t_0 = 0 \} \cup \cup_{i=1}^n \{ \{ t_i = 0 \} \cup \{ t_i - t_{i+1} = 0 \} \},$$

where $t_{n+1} = z$ and $t_0 = 1$. After fixing the variable z and parameter c to be real numbers satisfying $0 < z < c < 1$, the bounded chambers in the real locus $T_{\mathbb{R}}$ of $T = T_z$ are

$$\begin{pmatrix} 0 < t_s < t_{s+1} < \cdots < t_n < z, \\ t_s < t_{s-1} \cdots < t_1 < c \end{pmatrix}$$

for $1 \le s \le n$ and

$$\begin{pmatrix} z < t_n < 1 \\ 0 < t_s < t_{s+1} < \cdots < t_n, \\ t_s < t_{s-1} \cdots < t_1 < c \end{pmatrix}$$

for $1 \le s \le n$ and

$$(z < t_n < \cdots < t_1 < c).$$

Thus $\dim H_n^{\mathrm{lf}}(T_z, \mathcal{L}_z) = 2n+1$ under the condition

$$\lambda_{\infty = t_1 = \cdots = t_q} = -\lambda_{1c} - \sum_{1 \le k \le q} (\lambda_k + \lambda_{k,k+1}) \notin \mathbb{Z}, \quad 1 \le q \le n-1,$$

$$\lambda_{\infty = t_1 = \cdots = t_n} = -\lambda_{1c} - \lambda_{0n} - \sum_{1 \le k \le n} (\lambda_k + \lambda_{k,k+1}) \notin \mathbb{Z},$$

$$\lambda_{\infty = t_p = \cdots = t_q} = - \sum_{p \le k \le q} (\lambda_k + \lambda_{k-1,k}) - \lambda_{q,q+1} \notin \mathbb{Z}, \quad 2 \le p \le q \le n-1,$$

$$\lambda_{\infty = t_p = \cdots = t_n} = -\lambda_{0n} - \sum_{p \le k \le n} (\lambda_k + \lambda_{k-1,k}) - \lambda_{n,n+1} \notin \mathbb{Z}, \quad 2 \le p \le n.$$

The eigenvalues of the action of γ_0 are

$$e\Big(\sum_{s \le k \le n} (\lambda_k + \lambda_{k,k+1}) \Big), \quad 1 \le s \le n$$

with multiplicity free and 1 with multiplicity $n+1$; the action of γ_c are

$$e(\lambda_{1c} + \sum_{1\le k\le n} \lambda_{k,k+1})$$

with multiplicity free and 1 with multiplicity $2n$; the eigenvalues of the action of γ_1 are

$$e(\lambda_{0n} + \lambda_{n,n+1})$$

with multiplicity n and 1 with multiplicity $n+1$.

The eigenvalues of the action of γ_∞ is derived from

$$\begin{aligned} &u(t) \\ &= c^{\lambda_{1c}} z^{\lambda_{n,n+1}} (c^{-1} - t_1)^{\lambda_{1c}} (t_0 - t_n)^{\lambda_{0n}} \prod_{1\le i\le n} t_i^{\lambda_i^\infty} \prod_{0\le i\le n} (t_{i+1} - t_i)^{\lambda_{i,i+1}}, \end{aligned}$$

where

$$\lambda_i^\infty = \begin{cases} -\lambda_1 - \lambda_{1c} - \lambda_{12}, & i = 1, \\ -\lambda_i - \lambda_{i-1,i} - \lambda_{i,i+1}, & 2 \le i \le n-1, \\ -\lambda_{0n} - \lambda_n - \lambda_{n-1,n} - \lambda_{n,n+1}, & i = n, \end{cases}$$

with $t_{n+1} = z^{-1}$ and $t_0 = 1$. This shows the eigenvalues of the action of γ_∞ are

$$\begin{aligned} &e(-\lambda_{0n} - \lambda_{1c} - \sum_{1\le k\le n} (\lambda_k + \lambda_{k,k+1})), \\ &e(-\lambda_{0n} - \sum_{s\le k\le n} (\lambda_k + \lambda_{k-1,k}) - \lambda_{n,n+1}), \quad 2 \le s \le n, \end{aligned}$$

each with multiplicity free, and $e(-\lambda_{n,n+1})$ with multiplicity $n+1$.

Thus, the spectral type is $(1^n, n+1; n, n+1; 1, 2n; 1^n, n+1)$ under the conditions

$$\sum_{s\le k\le l-1} (\lambda_k + \lambda_{k,k+1}) \notin \mathbb{Z}, \quad 1 \le s \le l \le n,$$

which is for the separation of the eigenvalues at 0,

$$\lambda_{1c} + \sum_{1\le k\le n} \lambda_{k,k+1} \notin \mathbb{Z},$$

which is for the separation of the eigenvalues at c,

$$\lambda_{0n} + \lambda_{n,n+1} \notin \mathbb{Z},$$

which is for the separation of the eigenvalues at 1, and

$$\lambda_{1c} + \lambda_{0n} + \sum_{1\le k\le n-1} (\lambda_k + \lambda_{k,k+1}) + \lambda_n \notin \mathbb{Z},$$

$$\lambda_{0n} + \sum_{s\le k\le n} (\lambda_k + \lambda_{k-1,k}) \notin \mathbb{Z}, \qquad 2 \le s \le n,$$

$$\lambda_{1c} + \sum_{1\le k\le s-2} (\lambda_k + \lambda_{k,k+1}) + \lambda_{s-1}, \notin \mathbb{Z}, \qquad 2 \le s \le n,$$

which is for the separation of the eigenvalues at ∞.

It means

Theorem 6.1. *The function of the form*

$$\int_C (t_1 - c)^{\lambda_{1c}} (t_n - t_0)^{\lambda_{0n}} \prod_{1\le i\le n} t_i^{\lambda_i} \prod_{1\le i\le n} (t_i - t_{i+1})^{\lambda_{i,i+1}}\, dt_1 \cdots dt_n,$$

where $t_0 = 1, t_{n+1} = z$ and C a suitable cycle, satisfies the equation of type III, whose rank is $2n+1$.*

§7. The equation of type II*

In this section, let $\mathcal{L}_z$ be the locally constant sheaf determined by a function

$$u(t) = (t_1 - c)^{\lambda_{1c}} (t_n - t_0)^{\lambda_{0n}} \prod_{1\le i\le n-1} t_i^{\lambda_i} \prod_{1\le i\le n} (t_i - t_{i+1})^{\lambda_{i,i+1}} \tag{7.1}$$

on

$$T_z = \mathbb{C}^n \backslash \{ t_1 - c = 0 \} \cup \{ t_n - t_0 = 0 \} \cup_{i=1}^{n-1} \{ t_i = 0 \} \cup \cup_{i=1}^{n} \{ t_i - t_{i+1} = 0 \},$$

where $t_{n+1} = z$ and $t_0 = 1$.

After fixing the variable z and parameter c to be real numbers satisfying $0 < z < c < 1$, the bounded chambers in the real locus $T_{\mathbb{R}}$ of $T = T_z$ are

$$\begin{pmatrix} 0 < t_s < t_{s+1} < \cdots < t_n < z, \\ t_s < t_{s-1} \cdots < t_1 < c \end{pmatrix}$$

for $1 \leq s \leq n-1$ and

$$\begin{pmatrix} z < t_n < 1 \\ 0 < t_s < t_{s+1} < \cdots < t_n, \\ t_s < t_{s-1} \cdots < t_1 < c \end{pmatrix}$$

for $1 \leq s \leq n$ and

$$(z < t_n < \cdots < t_1 < c).$$

Thus $\dim H_n^{\mathrm{lf}}(T_z, \mathcal{L}_z) = 2n$ under the condition

$$\lambda_{\infty=t_1=\cdots=t_q} = -\lambda_{1c} - \sum_{1\leq k\leq q} (\lambda_k + \lambda_{k,k+1}) \notin \mathbb{Z}, \quad 1 \leq q \leq n-1,$$

$$\lambda_{\infty=t_1=\cdots=t_n} = -\lambda_{1c} - \lambda_{0n} - \sum_{1\leq k\leq n-1} (\lambda_k + \lambda_{k,k+1}) - \lambda_{n,n+1} \notin \mathbb{Z},$$

$$\lambda_{\infty=t_p=\cdots=t_q} = - \sum_{p\leq k\leq q} (\lambda_k + \lambda_{k-1,k}) - \lambda_{q,q+1} \notin \mathbb{Z}, \quad 2 \leq p \leq q \leq n-1,$$

$$\lambda_{\infty=t_p=\cdots=t_n} = -\lambda_{0n} - \sum_{p\leq k\leq n-1} (\lambda_k + \lambda_{k-1,k}) - \lambda_{n-1,n} - \lambda_{n,n+1} \notin \mathbb{Z}, \quad 2 \leq p \leq n.$$

The eigenvalues of the action of γ_0 are

$$e(\lambda_{n,n+1} + \sum_{s\leq k\leq n-1} (\lambda_k + \lambda_{k,k+1})), \quad 1 \leq s \leq n-1$$

with multiplicity free and 1 with multiplicity $n+1$; the action of γ_c are

$$e(\lambda_{1c} + \sum_{1\leq k\leq n} (\lambda_k + \lambda_{k,k+1}))$$

with multiplicity free and 1 with multiplicity $2n-1$; the eigenvalues of the action of γ_1 are

$$e(\lambda_{0n} + \lambda_{n,n+1})$$

with multiplicity n, and 1 with multiplicity n.

The eigenvalues of the action of γ_∞ is derived from

$$u(t) = c^{\lambda_{1c}} z^{\lambda_{n,n+1}} (c^{-1} - t_1)^{\lambda_{1c}} (t_0 - t_n)^{\lambda_{0n}} \prod_{1\leq i\leq n} t_i^{\lambda_i^\infty} \prod_{0\leq i\leq n} (t_{i+1} - t_i)^{\lambda_{i,i+1}},$$

where

$$\lambda_i^\infty = \begin{cases} -\lambda_1 - \lambda_{1c} - \lambda_{12}, & i = 1, \\ -\lambda_i - \lambda_{i-1,i} - \lambda_{i,i+1}, & 2 \leq i \leq n-1, \\ -\lambda_{0n} - \lambda_{n-1,n} - \lambda_{n,n+1}, & i = n, \end{cases}$$

with $t_{n+1} = z^{-1}$ and $t_0 = 1$. This shows the eigenvalues of the action of γ_∞ are

$$e(-\lambda_{0n} - \lambda_{1c} - \sum_{1 \leq k \leq n} (\lambda_k + \lambda_{k,k+1}) - \lambda_{n,n+1}),$$

$$e(-\lambda_{0n} - \sum_{s \leq k \leq n-1} (\lambda_k + \lambda_{k-1,k}) - \lambda_{n-1,n} - \lambda_{n,n+1}), \quad 2 \leq s \leq n,$$

each with multiplicity free, and $e(-\lambda_{n,n+1})$ with multiplicity n.

Thus, the spectral type is $(1^{n-1}, n+1; 1, 2n-1; n, n; 1^n, n)$ under the condition

$$\lambda_{n,n+1} + \sum_{s \leq k \leq n-1} (\lambda_k + \lambda_{k,k+1}) \notin \mathbb{Z}, \quad 1 \leq s \leq n-1,$$

$$\sum_{s \leq k \leq l-1} (\lambda_k + \lambda_{k,k+1}) \notin \mathbb{Z}, \quad 1 \leq s < l \leq n-1,$$

which is for the separation of the eigenvalues at 0,

$$\lambda_{1c} + \sum_{1 \leq k \leq n-1} \lambda_{k,k+1} \notin \mathbb{Z},$$

which is for the separation of the eigenvalues at c,

$$\lambda_{0n} + \lambda_{n,n+1} \notin \mathbb{Z},$$

which is for the separation of the eigenvalues at 1, and

$$\lambda_{1c} + \lambda_{0n} + \sum_{1\le k\le n-1} (\lambda_k + \lambda_{k,k+1}) \notin \mathbb{Z},$$

$$\lambda_{0n} + \sum_{s\le k\le n-1} (\lambda_k + \lambda_{k-1,k}) + \lambda_{n-1,n} \notin \mathbb{Z}, \qquad 2 \le s \le n,$$

$$\lambda_{1c} + \sum_{1\le k\le s-2} (\lambda_k + \lambda_{k,k+1}) + \lambda_{s-1} \notin \mathbb{Z}, \qquad 2 \le s \le n,$$

$$\sum_{s\le k\le l-1} (\lambda_k + \lambda_{k-1,k}) \notin \mathbb{Z}, \qquad 2 \le s < l \le n,$$

which is for the separation of the eigenvalues at ∞.

Finally, we obtain

Theorem 7.1. *The function of the form*

$$\int_C (t_1 - c)^{\lambda_{1c}} (t_n - t_0)^{\lambda_{0n}} \prod_{1\le i\le n-1} t_i^{\lambda_i} \prod_{1\le i\le n} (t_i - t_{i+1})^{\lambda_{i,i+1}}\, dt_1 \cdots dt_n,$$

where $t_0 = 1, t_{n+1} = z$ and C a suitable cycle, satisfies the equation of type II, whose rank is $2n$.*

§8. Supplements

Apart from the viewpoint of Okubo equations, Oshima recently studies the Fuchsian differential equations free from accessory parameters [24, 25, 26]. He demonstrates that there exist quite many examples of such equations; indeed, the cardinality of them is described in the following tables:

# of irreducible rigid Fuchsian differential systems with 3 singularities on $\mathbb{P}^1$

order	2	3	4	5	6	7	8	9	10	11	12	13	14
#	1	1	3	5	13	20	45	74	142	212	421	588	1004

# of irreducible rigid Fuchsian differential systems

order	2	3	4	5	6	7	8	9	10	11	12	13	14
#	1	2	6	11	28	44	96	157	306	441	857	1117	2032

On the other hand, in addition to Oshima's work, we refer the reader to the works by Katz [13], Simpson [27] Dettweiler and Reiter [5, 6], Kostov [15], Gleizer [7], and Crawley–Boavey and Shaw [4], who study intimately related topics from the viewpoint of rigid local systems (See also [10, 11, 12, 30]).

Finally, for our convenience, we illustrate the diagrams to express the integrands $u(t)$ treated in the present paper. Here $\underset{a}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\underset{b}{\circ}$ means $(a-b)^{\lambda_{ab}}$ up to a constant factor.

I (Generalized hypergeometric function) :

$$\begin{array}{ccccccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 & & 0 & & \\
 & & | & & | & & | & & & & | & & | & & \\
1 & \text{—} & t_1 & \text{—} & t_2 & \text{—} & t_3 & \text{—} & \cdots\cdots & \text{—} & t_{n-1} & \text{—} & t_n & \text{—} & z
\end{array}$$

1^{n+1}	at 0
$1, n$	at 1
1^{n+1}	at ∞

I* (Pochhammmer function):

$$\begin{tikzcd}
c_1 \arrow[dr, no head] & 0 \arrow[d, no head] & \\
c_2 \arrow[r, no head] & t \arrow[r, no head] & z \\
\vdots & & \\
c_n \arrow[uur, no head] & &
\end{tikzcd}$$

$1, n-1$	at 0
$1, n-1$	at c_1
$\cdots$	$\cdots$
$1, n-1$	at c_n
$1, n-1$	at ∞

II :

$$\begin{array}{cccccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 & & 0 & \\
 & & | & & | & & | & & & & | & & | & \diagup\ 1 \\
1 & \text{—} & t_1 & \text{—} & t_2 & \text{—} & t_3 & \text{—} & \cdots\cdots & \text{—} & t_{n-1} & \text{—} & t_n & \\
 & & & & & & & & & & & & & \diagdown\ z
\end{array}$$

$1^n, n$	at 0
$1, n-1, n$	at 1
$1^n, n$	at ∞

II* :

$$\begin{tikzcd}
 & 0 \arrow[d, dash] & 0 \arrow[d, dash] & 0 \arrow[d, dash] & & 0 \arrow[d, dash] & & \\
 & & & & & & & 1 \\
c \arrow[r, dash] & t_1 \arrow[r, dash] & t_2 \arrow[r, dash] & t_3 \arrow[r, dash] & \cdots\cdots \arrow[r, dash] & t_{n-1} \arrow[r, dash] & t_n \arrow[ur, dash] \arrow[dr, dash] & \\
 & & & & & & & z
\end{tikzcd}$$

$$\begin{array}{ll}
1^{n-1}, n+1 & \text{at } 0 \\
n, n & \text{at } 1 \\
1, 2n-1 & \text{at } c \\
1^n, n & \text{at } \infty
\end{array}$$

III :

$$\begin{tikzcd}
 & 0 \arrow[d, dash] & 0 \arrow[d, dash] & 0 \arrow[d, dash] & & 0 \arrow[d, dash] & & \\
 & & & & & & & 1 \\
1 \arrow[r, dash] & t_1 \arrow[r, dash] & t_2 \arrow[r, dash] & t_3 \arrow[r, dash] & \cdots\cdots \arrow[r, dash] & t_{n-1} \arrow[r, dash] & t_n \arrow[ur, dash] \arrow[dr, dash] & \\
 & & & & & & & z
\end{tikzcd}$$

$$\begin{array}{ll}
1^{n-1}, n & \text{at } 0 \\
1, n-1, n-1 & \text{at } 1 \\
1^n, n-1 & \text{at } \infty
\end{array}$$

III* :

$$\begin{tikzcd}
 & 0 \arrow[d, dash] & 0 \arrow[d, dash] & 0 \arrow[d, dash] & & 0 \arrow[d, dash] & 0 \arrow[d, dash] & \\
 & & & & & & & 1 \\
c \arrow[r, dash] & t_1 \arrow[r, dash] & t_2 \arrow[r, dash] & t_3 \arrow[r, dash] & \cdots\cdots \arrow[r, dash] & t_{n-1} \arrow[r, dash] & t_n \arrow[ur, dash] \arrow[dr, dash] & \\
 & & & & & & & z
\end{tikzcd}$$

$$\begin{array}{ll}
1^n, n+1 & \text{at } 0 \\
n, n+1 & \text{at } 1 \\
1, 2n & \text{at } c \\
1^n, n+1 & \text{at } \infty
\end{array}$$

IV :

$$\begin{tikzcd}
 & 0 \arrow[d, dash] & 0 \arrow[d, dash] & 1 \arrow[d, dash] & 1 \arrow[d, dash] & \\
1 \arrow[r, dash] & t_1 \arrow[r, dash] & t_2 \arrow[r, dash] & t_3 \arrow[r, dash] & t_4 \arrow[r, dash] & z
\end{tikzcd}
\qquad
\begin{array}{ll}
1^2, 4 & \text{at } 0 \\
2^3 & \text{at } 1 \\
1^4, 2 & \text{at } \infty
\end{array}$$

(with the resonance around $t_1 = t_2 = t_3 = t_4 = 1$)

IV* :

$$\begin{array}{ccccccc} & & 0 & & 0 & & 1 \\ & & | & & | & \diagup & \\ 1 & — & t_1 & — & t_2 & — & c \\ & & & & & \diagdown & \\ & & & & & & z \end{array}$$

$1, 1, 4$	at 0
$1, 1, 4$	at 1
$2, 4$	at c
$1, 1, 4$	at ∞

This diagram is very useful in several situations. In fact, when we find our integrands, it played a crucial role.

The integrands $u(t)$ in the case II of rank 4 and that of rank 6, written in subsections 5.5 and 5.10 of [9], are depicted as

$$\begin{array}{cccccc} & & 0 & & 0 & & 1 \\ & & | & & | & \diagup & \\ 1 & — & t_1 & — & t_2 & & \\ & & & & & \diagdown & \\ & & & & & & z \end{array} \quad \text{and} \quad \begin{array}{cccccccc} & & 0 & & 0 & & 0 & & 1 \\ & & | & & | & & | & \diagup & \\ 1 & — & t_1 & — & t_2 & — & t_3 & & \\ & & & & & & & \diagdown & \\ & & & & & & & & z \end{array} .$$

It is easy to guess that $u(t)$ for type II of rank $2n$ might be

$$\begin{array}{cccccccccccc} & & 0 & & 0 & & 0 & & & & 0 & & 0 & & 1 \\ & & | & & | & & | & & & & | & & | & \diagup & \\ 1 & — & t_1 & — & t_2 & — & t_3 & — & \cdots\cdots & — & t_{n-1} & — & t_n & & \\ & & & & & & & & & & & & & \diagdown & \\ & & & & & & & & & & & & & & z \end{array} .$$

The integrands $u(t)$ in the case II* of rank 4 and that of rank 6, written in subsections 5.2 and 5.7 of [9], are depicted as

$$\begin{array}{ccccccc} & & 0 & & & & 1 \\ & & | & & & \diagup & \\ c & — & t_1 & — & t_2 & & \\ & & & & & \diagdown & \\ & & & & & & z \end{array} \quad \text{and} \quad \begin{array}{ccccccccc} & & 0 & & 0 & & & & 1 \\ & & | & & | & & & \diagup & \\ c & — & t_1 & — & t_2 & — & t_3 & & \\ & & & & & & & \diagdown & \\ & & & & & & & & z \end{array} .$$

It is easy to guess that $u(t)$ for type II* of rank $2n$ might be

$$\begin{array}{ccccccccccccc} & & 0 & & 0 & & 0 & & & & 0 & & \\ & & | & & | & & | & & & & | & & \\ c & \text{---} & t_1 & \text{---} & t_2 & \text{---} & t_3 & \text{---} & \cdots\cdots & \text{---} & t_{n-1} & \text{---} & t_n \begin{array}{c}\diagup\\ \diagdown\end{array} \begin{array}{c}1\\ \\ z\end{array} \end{array} .$$

Similarly, the integrands $u(t)$ in the case III* of rank 5 and that of rank 7, written in subsections 5.4 and 5.9 of [9],

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & | & & | \\ c & \text{---} & t_1 & \text{---} & t_2 \begin{array}{c}\diagup\\ \diagdown\end{array} \begin{array}{c}1\\ \\ z\end{array} \end{array} \qquad \text{and} \qquad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & | & & | & & | \\ c & \text{---} & t_1 & \text{---} & t_2 & \text{---} & t_3 \begin{array}{c}\diagup\\ \diagdown\end{array} \begin{array}{c}1\\ \\ z\end{array} \end{array}$$

lead to the integrand

$$\begin{array}{ccccccccccccc} & & 0 & & 0 & & 0 & & & & 0 & & 0 \\ & & | & & | & & | & & & & | & & | \\ c & \text{---} & t_1 & \text{---} & t_2 & \text{---} & t_3 & \text{---} & \cdots\cdots & \text{---} & t_{n-1} & \text{---} & t_n \begin{array}{c}\diagup\\ \diagdown\end{array} \begin{array}{c}1\\ \\ z\end{array} \end{array}$$

for type III* of rank $2n+1$. Furthermore, considering the fact that the integrand in the case II is obtained from the integrand in the case II* by the specialization $c = 1$, the integrand in the case III of rank $2n+1$ might be

$$\begin{array}{ccccccccccccc} & & 0 & & 0 & & 0 & & & & 0 & & \\ & & | & & | & & | & & & & | & & \\ 1 & \text{---} & t_1 & \text{---} & t_2 & \text{---} & t_3 & \text{---} & \cdots\cdots & \text{---} & t_{n-1} & \text{---} & t_n \begin{array}{c}\diagup\\ \diagdown\end{array} \begin{array}{c}1\\ \\ z\end{array} \end{array} .$$

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