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On a problem of arrangements related to the hypergeometric integrals of confluent type

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Abstract.

The general hypergeometric integrals on the complex matrix space are considered. We study the twisted algebraic de Rham cohomology group associated with the integrand of the general hypergeometric integrals. After reviewing some fundamental facts, we discuss the exterior product structure of the cohomology group.

§1. Introduction

Let $f_0, f_1, \ldots, f_n$ be polynomial functions of degree 1 on $\mathbb{C}^r$and let $H_j = \ker f_j$ be the hyperplane in $\mathbb{C}^r$defined as zeros of $f_j = 0$. Then $\mathcal{A} = \{H_0, \ldots, H_n\}$ defines an arrangement in $\mathbb{C}^r$. Many authors are interested in the topology of the complement of these hyperplanes. One of the important topics related to this topological problem is the theory of hypergeometric functions on the Grassmannian manifold initiated by K. Aomoto [1] and I.M. Gelfand [3]. It is given by the integral

$$F = \int_C f_0^{\alpha_0} \cdots f_n^{\alpha_n} du_1 \wedge \cdots \wedge du_r,$$

where α_j are complex constants and C is some r-dimensional chain. The integral gives a function on the space of coefficients of the polynomials f_j.

The typical example is the Gauss hypergeometric function

$${}_2F_1(a,b,c;x) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m m!} x^m = A \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b} du,$$

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where the polynnomials are $f_0 = 1, f_1 = u, f_2 = 1 - u, f_3 = 1 - xu$ and the chain C is an open segment $(0,1) \subset \mathbb{R} \subset \mathbb{C}$ and A is a constant.

Let $\mathcal{L}$ be the local system on $X = \mathbb{C}^r \setminus \cup_{j=0}^n H_j$ defined by the function $f_0^{-\alpha_0} \cdots f_n^{-\alpha_n}$. Then to investigate the hypergeometric function it is important to study and compute the cohomology group $H^*(X, \mathcal{L})$ and the homology group $H_*(X, \mathcal{L}^\vee)$. Note that elements of the cohomology group $H^*(X, \mathcal{L})$ are represented by rational differential forms by virtue of algebraic de Rham theory due to Deligne and Grothendieck.

This paper concerns an analogous problem of arragements of hyperplanes in the complex affine space $\mathbb{C}^r$ related to the hypergeometric integrals of confluent type. We focus on the computation of the twisted algebraic de Rham cohomology group.

§2. General hypergeometric integrals

2.1. Regular elements of $\mathrm{GL}_N(\mathbb{C})$

To define the general hypergeometric integrals, we use a class of maximal abelian subgroups of $\mathrm{GL}_N(\mathbb{C})$ which are obtained as centralizers of regular elements. We start with explaining about regular elements.

Definition 1. *An element $a \in \mathrm{GL}_N(\mathbb{C})$ is a regular element when the orbit $O(a) = \{gag^{-1} \mid g \in \mathrm{GL}_N(\mathbb{C})\}$ under the adjoint action is of maximum dimension.*

One can see that $a \in \mathrm{GL}_N(\mathbb{C})$ is regular if any two Jordan cells of the Jordan normal form of a have different eigenvalues and in this case $\dim_{\mathbb{C}} O(a) = \dim_{\mathbb{C}} \mathrm{GL}_N(\mathbb{C}) - N$. Then the centralizer $Z(a)$ of a is a maximal abelian subgroup of $\mathrm{GL}_N(\mathbb{C})$ of dimension N. The size of Jordan cells gives a partition of N.

Example 2. *If $a \in \mathrm{GL}_4(\mathbb{C})$ is a regular element, then a is similar to the following Jordan normal form. Partitions of 4 mean possible*

structure of Jordan normal forms.

$$\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & a_4 \end{pmatrix} \longleftrightarrow (1,1,1,1)$$

$$\begin{pmatrix} a_1 & 1 & & \\ & a_1 & & \\ & & a_3 & \\ & & & a_4 \end{pmatrix} \longleftrightarrow (2,1,1)$$

$$\begin{pmatrix} a_1 & 1 & & \\ & a_1 & & \\ & & a_3 & 1 \\ & & & a_3 \end{pmatrix} \longleftrightarrow (2,2)$$

$$\begin{pmatrix} a_1 & 1 & & \\ & a_1 & 1 & \\ & & a_1 & \\ & & & a_4 \end{pmatrix} \longleftrightarrow (3,1)$$

$$\begin{pmatrix} a_1 & 1 & & \\ & a_1 & 1 & \\ & & a_1 & 1 \\ & & & a_1 \end{pmatrix} \longleftrightarrow (4)$$

where $a_i \neq a_j$ $(i \neq j)$.

2.2. Maximal abelian groups

We shall describe the centralizer more explicitly. Assume that a regular element a is of Jordan normal form and that the Jordan cells of a have the size $n_1, \ldots, n_\ell$ and arrayed in non increasing order: $n_1 \geq n_2 \geq \cdots \geq n_\ell > 0$. Hence the partition $\lambda = (n_1, \ldots, n_\ell)$ of N is associated with a. The centralizer $Z(a)$ depends on the partition λ but not on the Jordan normal form. So we denote the centralizer by H_λ.

Lemma 3. *The centralizer* H_λ *is written as*

$$H_\lambda = J(n_1) \times \cdots \times J(n_\ell) \subset \mathrm{GL}_N(\mathbb{C}),$$

where

$$J(n) = \left\{ h = \begin{pmatrix} h_0 & h_1 & \ldots & h_{n-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & h_1 \\ & & & h_0 \end{pmatrix} ; h_0 \neq 0 \right\}.$$

An element $h \in H_\lambda$ is denoted as

$$h = (h^{(1)}, \dots, h^{(\ell)}), \quad h^{(k)} = h_0^{(k)} I + h_1^{(k)} \Lambda + \cdots + h_{n-1}^{(k)} \Lambda^{n-1} \in J(n_k),$$

where $\Lambda = (\delta_{i+1,j}) \in \mathrm{Mat}_n(\mathbb{C})$ is the shift matrix.

Remark 4. *We have the isomorphism* $J(n) \to \left(\mathbb{C}[T]/(T^n)\right)^\times$ *by the obvious correspondence*

$$h_0 I + h_1 \Lambda + \cdots + h_{n-1} \Lambda^{n-1} \mapsto h_0 + h_1 T + \cdots + h_{n-1} T^{n-1} \quad (mod\, T^n).$$

This identification motivates the definition of generalized Veronese map in Subsection 4.2.

Lemma 5. *There is an isomorphism* $J(n) \simeq \mathbb{C}^\times \times \mathbb{C}^{n-1}$ *given by*

$$h \mapsto (h_0, \theta_1(h), \dots, \theta_{n-1}(h)),$$

where $\theta_m(h)$ $(m = 0, 1, \dots)$ *is defined by*

$$\log h = \log(h_0 I + h_1 \Lambda + \cdots + h_{n-1} \Lambda^{n-1}) = \sum_{m=0}^{n-1} \theta_m(h) \Lambda^m, \quad \theta_0(h) = \log h_0.$$

It is easily seen that $\theta_m(h), m \geq 1$, are rational functions of $h_0, \dots, h_{n-1}$, with a pole of order m along $h_0 = 0$, given explicitly by

$$\theta_m(h) = \sum (-1)^{k_1 + \cdots + k_m - 1} \frac{(k_1 + \cdots + k_m - 1)!}{k_1! \cdots k_m!} \left(\frac{h_1}{h_0}\right)^{k_1} \cdots \left(\frac{h_m}{h_0}\right)^{k_m},$$

where the sum is taken over the indices $(k_1, \dots, k_m)$ such that $k_1 + 2k_2 + \cdots + mk_m = m$. For example, $\theta_1(h), \theta_2(h), \theta_3(h)$ are written as

$$\begin{aligned} \theta_1(h) &= \frac{h_1}{h_0}, \\ \theta_2(h) &= \frac{h_2}{h_0} - \frac{1}{2}\left(\frac{h_1}{h_0}\right)^2, \\ \theta_3(h) &= \frac{h_3}{h_0} - \left(\frac{h_1}{h_0}\right)\left(\frac{h_2}{h_0}\right) + \frac{1}{3}\left(\frac{h_1}{h_0}\right)^3. \end{aligned} \tag{1}$$

2.3. Definition of the integrals

By virtue of Lemma 5, the characters of the universal covering group $\tilde{H}_\lambda$ of H_λ are described as follows. First, the character $\chi_n : \tilde{J}(n) \to \mathbb{C}^\times$ is given by

$$\chi_n(h; \alpha) = \exp\left(\alpha_0 \theta_0(h) + \cdots + \alpha_{n-1} \theta_{n-1}(h)\right),$$

$$\alpha = (\alpha_0, \dots, \alpha_{n-1}) \in \mathbb{C}^n.$$

Since H_λ is a product of $J(n_k)$'s, the character $\chi : \tilde{H}_\lambda \to \mathbb{C}^\times$ is written as

$$\chi(h;\alpha) = \prod_{k=1}^{\ell} \chi_{n_k}(h^{(k)};\alpha^{(k)}) = \prod_{k=1}^{\ell} \exp\left(\sum_{m=0}^{n_k-1} \alpha_m^{(k)} \theta_m(h^{(k)})\right),$$

$$h = (h^{(1)}, \dots, h^{(\ell)})$$

with $\alpha = (\alpha^{(1)}, \dots, \alpha^{(\ell)}), \alpha^{(k)} = (\alpha_0^{(k)}, \dots, \alpha_{n_k-1}^{(k)}) \in \mathbb{C}^{n_k}$.

We assume that the character χ satisfies the condition

$$\sum_{k=1}^{\ell} \alpha_0^{(k)} = -r-1, \quad \alpha_{n_k-1}^{(k)} \neq 0 \ (n_k \geq 2), \ \notin \mathbb{Z} \ (n_k = 1) \tag{2}$$

and let us consider its Radon transform. Let $u = (u_1, \dots, u_r)$ be the coordinates of $\mathbb{C}^r$, the variables of integration. Consider N polynomials $f_j^{(k)} = z_{0j}^{(k)} + u_1 z_{1j}^{(k)} + \cdots + u_r z_{rj}^{(k)}$ of degree 1 in u which will be substituted into the character χ. The coefficients of $f_j^{(k)}$ give a column vector $z_j^{(k)} = (z_{0j}^{(k)}, \dots, z_{rj}^{(k)})^t \in \mathbb{C}^{r+1}$. Hence the polynomials $f_j^{(k)}$ are given by specifying a matrix $z = (z^{(1)}, \dots, z^{(\ell)}) \in \mathrm{Mat}_{r+1,N}(\mathbb{C})$ with $z^{(k)} = (z_0^{(k)}, \dots, z_{n_k-1}^{(k)}) \in \mathrm{Mat}_{r+1,n_k}(\mathbb{C})$. Note that the set of polynomials $f_j^{(k)}$ is written as

$$(f_0^{(1)}, \dots, f_{n_1-1}^{(1)}, \dots, f_0^{(\ell)}, \dots, f_{n_\ell-1}^{(\ell)}) = \vec{u}z, \quad \vec{u} = (1, u_1, \dots, u_r).$$

Let $Z_{r,\lambda} \subset \mathrm{Mat}_{r+1,N}(\mathbb{C})$ be the Zariski open subset consisting of $z = (z^{(1)}, \dots, z^{(\ell)})$ satisfying the following condition.

Condition(*): For any $(m_1, \dots, m_\ell) \in \mathbb{Z}_{\geq 0}^\ell$ such that $0 \leq m_k \leq n_k \quad (k = 1, \dots, \ell)$, $m_1 + \cdots + m_\ell = r+1$, we have

$$\det(z_0^{(1)}, \dots, z_{m_1-1}^{(1)}, \dots, z_0^{(\ell)}, \dots, z_{m_\ell-1}^{(\ell)}) \neq 0.$$

Definition 6. *For a character of $\chi(\cdot\,;\alpha)$ of $\tilde{H}_\lambda$ satisfying (2) and $z \in Z_{r,\lambda}$, the general hypergeometric integral (GHGI) is*

$$I(z,\alpha,c) = \int_c \chi(\vec{u}z;\alpha)du, \quad du = du_1 \wedge \cdots \wedge du_r, \tag{3}$$

where c is a cycle of some homology group defined by using $\chi(\vec{u}z;\alpha)$.

Remark 7. *From the explicit form of θ_m, the integrand $\chi(\vec{u}z;\alpha)$ of the integral $I(z,\alpha;c)$ is a multivalued holomorphic function of $u \in \mathbb{C}^r$ with the branch locus on the union of hyperplanes $\cup_{k=1}^{\ell} H^{(k)}$, where $H^{(k)} = \{u \in \mathbb{C}^r \mid \vec{u} \cdot z_0^{(k)} = 0\}$.*

Remark 8. *For an appropriate chosen cycle c, GHGI satisfies a holonomic system of differential equations on $Z_{r,\lambda}$.*

2.4. Classical HGF

We explain how the Gauss hypergeometric function and the associated functions of confluent type are related to the general hypergeometric integral.

Gauss hypergeometric function is a holomorphic solution of

$$x(1-x)y'' + \{c-(a+b+1)x\}y' - aby = 0 \tag{4}$$

at $x = 0$ with the condition $y(0) = 1$. Confluent type functions of ${}_2F_1$ we know are solutions respectively of the following 2nd order linear differential equations on $\mathbb{P}^1$:

$$\begin{aligned}
&\text{(Kummer)} && xy'' + (c-x)y' - ay = 0,\\
&\text{(Bessel)} && x^2y'' + xy' + (x^2-a^2)y = 0,\\
&\text{(Hermite)} && y'' - xy' + ay = 0,\\
&\text{(Airy)} && y'' - xy = 0.
\end{aligned}$$

The solutions of these equations have the integral representations:

$$\begin{aligned}
&\text{(Kummer)} && {}_1F_1(a,c;x) = A\int_0^1 e^{xu}u^{a-1}(1-u)^{c-a-1}du,\\
&\text{(Bessel)} && J_a(x) = A\int_C e^{x(u-1/u)}u^{-a-1}du,\\
&\text{(Hermite)} && H_a(x) = A\int_C e^{xu-\frac{1}{2}u^2}u^{-a-1}du,\\
&\text{(Airy)} && \mathrm{Ai}(x) = A\int_C e^{xu-\frac{1}{3}u^3}du,
\end{aligned}$$

where C in each integral is an appropriate chain (see [6]). We would like to explain these functions are all understood as GHGI for various partitions λ of 4. The correspondence between the functions (or differential equations) and the partitions of 4 are

Gauss	Kummer	Bessel	Hermite	Airy
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
$(1,1,1,1)$	$(2,1,1)$	$(2,2)$	$(3,1)$	(4)

We discuss the correspondence in the cases of Gauss and Airy.

2.4.1. *Gauss case* This is one of the simplest cases of general hypergeometric function considered by K. Aomoto and I. M. Gelfand. For the partition $\lambda = (1,1,1,1)$ we associate the maximal abelian subgroup $H = H_{(1,1,1,1)}$ consisting of diagonal matrices $h = \mathrm{diag}(h_0, \dots, h_3)$. Then the character $\chi : \tilde{H} \to \mathbb{C}^\times$ has the form

$$\chi(h;\alpha) = h_0^{\alpha_0} h_1^{\alpha_1} h_2^{\alpha_2} h_3^{\alpha_3}. \tag{5}$$

Substitute into χ the following linear functions:

$$h_0(u) = 1, \quad h_1(u) = u, \quad h_2(u) = 1-u, \quad h_3(u) = 1 - xu. \tag{6}$$

Define the parameters α in χ as $\alpha = (b-c, a-1, c-a-1, -b)$, then we have

$${}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \chi(h(u);\alpha)du.$$

As is well known, the right-hand side gives an analytic continuation of ${}_2F_1$, which is defined in the unit disk $|x| < 1$, along any path in $\mathbb{C} \setminus \{0,1\}$. Note that $x = 0, 1, \infty$ is the set of singular points of the Gauss hypergeometric equation (4).

Let us explain the meaning of the choice of polynomials $h_j(u)$ given in (6) and how far the Gauss hypergeometric integral is generalized by considering GHGI. As explained in Subsection 2.3, the polynomials (6) are determined by the matrix of the form

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -x \end{pmatrix}. \tag{7}$$

Note that the point $x = 0, 1$ are excluded as values of x, which are singular points of (4). On the other hand, for GHGI (3), the polynomials f_j to be substituted into the character χ are given by a 2×4 matrix $z \in Z (= Z_{1,(1,1,1,1)})$. Let X be the set of matrices of the form (7) with the condition $x \neq 0, 1$. Then the meaning of the subset $X \subset Z$ is as follows.

Define the left and right action of $\mathrm{GL}_2(\mathbb{C})$ and H on Z by

$$\mathrm{GL}_2(\mathbb{C}) \times Z \times H \ni (g, z, h) \mapsto gzh \in Z. \tag{8}$$

Proposition 9. *For any $z \in Z$, we can find $g \in \mathrm{GL}_2(\mathbb{C})$ and $h \in H$ such that*

$$gzh = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -x \end{pmatrix}, \tag{9}$$

and $x \in \mathbb{C} \setminus \{0, 1\}$ is uniquely determined by z:

$$x = \frac{[1,2][0,3]}{[0,2][1,3]}, \qquad [i,j] := \det(z_i, z_j). \tag{10}$$

This propsition implies that each orbit $O(z)$ in Z by the above action has the unique representative of the form (7). Thus we see that the quotient space $\mathrm{GL}_2(\mathbb{C})\backslash Z/H$ has a realization by the global slice $X \subset Z$. On the level of integrals we can show that GHGI in this setting with appropriately chosen cycle c is related to the Euler integral representation of Gauss hypergeometric function by

$$I(z, \alpha, c) = (\det g)\chi(h; \alpha)^{-1} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b} du,$$

where z and x are related by (9) and (10). Thus, not only the Gauss hypergeometric function is obtained by restricting the general hypergeometric integral on X, but also it is essentially the same as GHGI on $Z_{1,(1,1,1,1)}$.

2.4.2. *Airy case* We shall explain that Airy's integral $\mathrm{Ai}(x) = \int_c e^{xu-u^3/3} du$ is essentially the same as GHGI for the partition $\lambda = (4)$. Recall that the maximal abelian subgroup of $\mathrm{GL}_4(\mathbb{C})$ in this case is

$$H_{(4)} = \left\{ h = \begin{pmatrix} h_0 & h_1 & h_2 & h_3 \\ & h_0 & h_1 & h_2 \\ & & h_0 & h_1 \\ & & & h_0 \end{pmatrix} \right\} \subset \mathrm{GL}_4(\mathbb{C}).$$

Let $\chi : \tilde{H}_{(4)} \to \mathbb{C}^\times$ be a character of the universal covering group of $H_{(4)}$. It is written as

$$\chi(h; \alpha) = h_0^{\alpha_0} \exp(\alpha_1\theta_1(h) + \alpha_2\theta_2(h) + \alpha_3\theta_3(h))$$

for some $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^4$, where $\theta_1, \theta_2, \theta_3$ are those given in (1). To realize $\mathrm{Ai}(x)$ as GHGI, take the character χ with the parameter

$$\alpha = (-2, 0, 0, -1)$$

and substitute into χ the polynomial of u:

$$h_0(u) = 1,\ h_1(u) = u,\ h_2(u) = 0,\ h_3(u) = -xu. \tag{11}$$

And then consider the integral

$$\int \chi(h(u);\alpha)du.$$

That this integral coincides with $\mathrm{Ai}(x)$ is easily seen using the expression of $\theta_3(h)$ given in (1):

$$\chi(h(u);\alpha) = h_0(u)^{-2}\exp(-\theta_3(h(t))) = \exp\left(xu - \frac{1}{3}u^3\right).$$

As in Subsection 2.3, the polynomials $h_j(u)$ in (11) are determined by the matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -x \end{pmatrix}. \tag{12}$$

When we consider the Airy integral, there is no restriction on x and can take arbitrary complex number. This is related with the fact that Airy's differential equation has no singular point in $\mathbb{C}$. So in the matrix (12), x is arbitrary complex number. In order to understand why the particular matrix (12) is chosen to obtain the Airy integral from GHGI of type $\lambda = (4)$, we proceed in a similar way as in Gauss hypergeometric case. Let $Z_{(4)} = \{z = (z_0, \ldots, z_3) \in \mathrm{Mat}_{2,4}(\mathbb{C}); \det(z_0.z_1) \neq 0\}$ and let $X_{(4)}$ be the subset of $Z_{(4)}$ consisting of the matrices of the form (12) with $x \in \mathbb{C}$. Define the left and right action of $\mathrm{GL}_2(\mathbb{C})$ and $H_{(4)}$ on $Z_{(4)}$by

$$\mathrm{GL}_2(\mathbb{C}) \times Z_{(4)} \times H_{(4)} \ni (g, z, h) \mapsto gzh \in Z_{(4)}. \tag{13}$$

Proposition 10. *For any $z \in Z_{(4)}$, we can find $g \in \mathrm{GL}_2(\mathbb{C})$ and $h \in H_{(4)}$ such that*

$$gzh = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -x \end{pmatrix}, \tag{14}$$

and $x \in \mathbb{C}$ is uniquely determined by z:

$$x = \frac{1}{[0,1]^2}\{[0,1][2,1] + [0,2]^2 - [0,1][0,3]\}, \qquad [i,j] := \det(z_i, z_j). \tag{15}$$

This proposition implies that the quotient space $\mathrm{GL}_2(\mathbb{C})\backslash Z_{(4)}/H_{(4)}$ has a realization as the global slice $X_{(4)} \subset Z_{(4)}$:

$$\mathrm{GL}_2(\mathbb{C})\backslash Z_{(4)}/H_{(4)} \simeq X_{(4)} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -x \end{pmatrix}; x \in \mathbb{C} \right\}.$$

On the level of integrals we can show that GHGI in this setting with appropriately chosen cycle c is related to the Airy integral by

$$I(z, \alpha, c) = (\det g)\chi(h; \alpha)^{-1} \int_C e^{xu - \frac{1}{3}u^3} du,$$

where z and x are related by (14) and (15). This explains that GHGI for the partition (4) on $Z_{(4)}$ is essentially the same as the classical Airy integral.

Remark 11. *The reason for choosing the parameter α in χ as*

$$\alpha = (-2, 0, 0, -1)$$

is explained by considering the group of symmetry for the generalized Airy function which is given by the analogue of Weyl group $N_{\mathrm{GL}_4(\mathbb{C})}(H_{(4)})/H_{(4)}$. See [9].

§3. Twisted de Rham cohomology

This section concerns the explicit computation of the twisted de Rham cohomology group associated with GHGI (3).

Let $\chi = \chi(\vec{u}z; \alpha)$ with $z \in Z_{r,\lambda}$ and $\alpha \in \mathbb{C}^N$ satisfying (2) and let $\mathcal{A} = \{H^{(1)}, \dots, H^{(\ell)}\}$ be the arrangement of hyperplanes in $\mathbb{C}^r$, where $H^{(k)} = \{u \in \mathbb{C}^r; \vec{u} \cdot z_0^{(k)} = 0\} \subset \mathbb{C}^r$. Put $N(\mathcal{A}) = \bigcup_{k=1}^{\ell} H^{(k)}$. Then χ is a multivalued holomorphic function on $\mathbb{C}^r \setminus N(\mathcal{A})$ whose logarithmic exterior derivative $d \log \chi$ has poles on $H^{(1)}, \dots, H^{(\ell)}$ of order $n_1, \dots, n_\ell$, respectivley.

Let $\Omega^p(*\mathcal{A})$ be the set of rational p-forms having poles at most on $N(\mathcal{A})$. Define the twisted differentiation $\nabla : \Omega^p(*\mathcal{A}) \to \Omega^{p+1}(*\mathcal{A})$ by

$$\nabla(\eta) = (\chi^{-1} \cdot d \cdot \chi)(\eta) = d\eta + \Big(d \log \chi(\vec{u}z; \alpha)\Big) \wedge \eta.$$

Since $d\chi$ has poles only on $N(\mathcal{A})$, ∇ really sends $\Omega^p(*\mathcal{A})$ to $\Omega^{p+1}(*\mathcal{A})$. Note also $\nabla \circ \nabla = \chi^{-1} \cdot d^2 \cdot \chi = 0$. It follows that we have the twisted algebraic de Rham complex

$$C_{z,\alpha}(*\mathcal{A}) : \ \Omega^0(*\mathcal{A}) \xrightarrow{\nabla} \Omega^1(*\mathcal{A}) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^r(*\mathcal{A}) \to 0.$$

We sometimes write $C_{z,\alpha}$ instead of $C_{z,\alpha}(*\mathcal{A})$.

Definition 12. *The twisted algebraic de Rham cohomology is*

$$H^p(C_{z,\alpha}(*\mathcal{A})) := \frac{\operatorname{Ker}\{\nabla : \Omega^p(*\mathcal{A}) \to \Omega^{p+1}(*\mathcal{A})\}}{\operatorname{Im}\{\nabla : \Omega^{p-1}(*\mathcal{A}) \to \Omega^p(*\mathcal{A})\}} \quad (p \geq 0).$$

We know the following cases about the computation of the cohomology groups which will be described explicitly below.

(1) $r = 1$, i.e. the case where GHGI is defined by 1-dimensional integral.
(2) r is general and $\lambda = (1, \ldots, 1)$. This is the case of Aomoto–Gelfand hypergeometric integral.
(3) r is general and $\lambda = (N)$. The case of generalized Airy integral.
(4) r is general and $\lambda = (q+1, 1, \ldots, 1)$.

3.1. 1-dimensional case

In the case the general hypergeometric integral is given by 1-dimensional integral, we have the following result.

Proposition 13. *Let $\lambda = (n_1, \ldots, n_\ell)$ be a partition of N. For any $z \in Z_{1,\lambda}$, we have*

(1) $H^p(C_{z,\alpha}) = 0$ *for* $p \neq 1$,
(2) $\dim_{\mathbb{C}} H^1(C_{z,\alpha}) = N - 2$,

As a basis of $H^1(C_{z,\alpha})$ we can take the following 1-forms. In the case $n_1 = 1$,

$$d\theta_0(\vec{u}z^{(k)}) - d\theta_0(\vec{u}z^{(k+1)}), \quad (2 \leq k \leq N-1);$$

in the case $n_1 \geq 2$,

(16)
$$d\theta_1(\vec{u}z^{(1)}), \ldots, d\theta_{n_1-2}(\vec{u}z^{(1)}), d\theta_0(\vec{u}z^{(k)}), \ldots, d\theta_{n_k-1}(\vec{u}z^{(k)}), \quad (2 \leq k \leq \ell).$$

3.2. Generalized Airy case

The generalized Airy integral is the GHGI for the partition $\lambda = (N)$ of N. The generalized Airy integral is introduced by Gelfand, Retahk and Serganova in [4]. In this case the space of coefficients of polynomial of degree 1 is

$$Z_{r,(N)} = \{z = (z_0, \ldots, z_{N-1}) \in \operatorname{Mat}_{r+1,N}(\mathbb{C}) \mid \det(z_0, \ldots, z_r) \neq 0\}.$$

We assume that

$$z_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \alpha_{N-1} \neq 0. \tag{17}$$

Then the integrand of the generalized Airy integral has the form

$$\chi(\vec{u}z;\alpha) = e^{f(u)}, \quad f(u) = \sum_{m=1}^{N-1} \alpha_m \theta_m(\vec{u}z). \tag{18}$$

Note that by the assumption (17), f is a polynomial of $u = (u_1, \ldots, u_r)$ of degree $N-1$.

Lemma 14. *For any $z \in Z_{r,(N)}$ satisfying (17), f is a polynomial having only isolated critical points with the Milnor number*

$$\mu(f) = \binom{N-2}{r}.$$

This fact can be seen as follows. Put $z = (z', z'')$ with $z' = (z_0, \ldots, z_r)$. Since $\det z' \neq 0$ by the definition of $Z_{r,(N)}$, we can make the change of variables $\vec{u} = (1, v_1, \ldots, v_r)z'$. If we define the weights of v_k as $wt(v_k) = k$, then the highest weight part of f is $\alpha_{N-1}\theta_{N-1}(\vec{v}(I_{r+1}, 0))$ which is weighted homogeneous of weight $N-1$ having isolated critical point at 0. Then Lemma 14 follows from this fact.

Proposition 15. [7] *For the r-dimensional generalized Airy integral, we assume the condition (17) . Then for any $z \in Z_{r,(N)}$, we have*

(1) $H^p(C_{z,\alpha}) = 0$ *for* $p \neq r$,

(2) $\dim_{\mathbb{C}} H^r(C_{z,\alpha}) = \binom{N-2}{r}$.

To state the result on the choice of a basis of top cohomology group $H^r(C_{z,\alpha})$, we prepare some notations. Let $\mathcal{Y}(r,l)$ be the set of Young diagrams contained in $r \times l$ box, namely $Y \in \mathcal{Y}(r,l)$ if the length is $\ell(Y) \leq r$ and any "parts of Y" is not greater than l. The number of boxes in the diagram Y is denoted by $|Y|$. For any $Y \in \mathcal{Y}(r,l)$, let $s_Y(v)$ be the Schur polynomial of $v_1, \ldots, v_r$ associated with Y which is a symmetric polynomial of degree $|Y|$. Since $s_Y(v)$ is a symmetric polynomial, we can write it as a polynomial of the elementary symmetric functions $e_1(v), \ldots, e_r(v)$ of v. It is denoted as $S_Y(u)$:

$$s_Y(v) = S_Y(e(v)).$$

Proposition 16. [7] *For the r-dimensional generalized Airy integral, the following r-form gives a basis of $H^r(C_{z,\alpha})$:*

$$S_Y(u)du_1 \wedge \cdots \wedge du_r, \quad Y \in \mathcal{Y}(r, N-r-2). \tag{19}$$

Remark 17. (1) *In the case $r = 1$, namely the integral is one dimensional, the basis (19) above is written as*

$$du, udu, \ldots, u^{N-3}du.$$

(2) *Another choice of a basis is that given in Proposition 13:*

$$d(\theta_1(\vec{u}z)), \ldots, d(\theta_{N-2}(\vec{u}z)).$$

It is an analogue of flat basis of the Jacobi ring of singularity of A_{N-2} type [10].

3.3. $\lambda = (q+1, 1, \ldots, 1)$ **case**

In the case where the partition of N has the form $\lambda = (q+1, 1, \ldots, 1)$, we can give an explicit form of a basis of the cohomology group. Recall that the space of coefficients of polynomials of degree 1 in the definition of GHGI is

$$Z_{r,\lambda} = \{z = (z_0, \ldots, z_{N-1}) \in \mathrm{Mat}_{r+1,N}(\mathbb{C}) \mid \text{the condition (*)}\},$$

where the condition (*) is that, for any $0 \le k \le q+1$ and $q < j_k < j_{k+1} < \cdots < j_r \le N-1$, there holds

$$\det(z_0, \ldots, z_{k-1}, z_{j_k}, \ldots, z_{j_r}) \neq 0.$$

In this case the integrand of the GHGI has the form $\chi(\vec{u}z; \alpha) = e^{g(u,z)} \prod_{j=q+1}^{N-1} f_j^{\alpha_j}$, where

$$g(u,z) = \sum_{k=0}^{q} \alpha_k \theta_k(f_0, f_1, \ldots, f_q), \quad f_j = \vec{u}z_j, \quad (0 \le j < N).$$

We assume that

$$z_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{20}$$

and

$$\alpha_0 + \sum_{j>q} \alpha_j = -r-1, \ \alpha_q \neq 0, \ \alpha_j \notin \mathbb{Z}, \ (j > q). \tag{21}$$

As a consequence of the assumptions (20) and $\alpha_q \neq 0$, we see that $g(u,z)$ is a polynomial of u of degree q. Put $H_j := \ker f_j$ $(j = q+1, \dots, N-1)$ and $\mathcal{A} = \{H_{q+1}, \dots, H_{N-1}\}$. Moreover from the assumption (21), the integrand $\chi(\vec{u}z;\alpha)$ of GHGI is a multivalued holomorphic function on $\mathbb{C}^r$ with the branch locus $N(\mathcal{A}) = \cup_{j=q+1}^{N-1} H_j$.

Proposition 18. [8] *In the case $\lambda = (q+1, 1, \dots, 1)$, we assume for the GHGI the conditions (20) and (21). Then we have*

(1) $H^p(C_{z,\alpha}) = 0$ *for* $p \neq r$,

(2) $\dim_{\mathbb{C}} H^r(C_{z,\alpha}) = \binom{N-2}{r}$.

Let us give a basis for the top cohomology group $H^r(C_{z,\alpha})$. To this end, we use the same notation as in the generalized Airy case for $\mathcal{Y}(r,l)$, $s_Y(v)$, $S_Y(u)$. For $0 \leq k \leq r$, take a Yang diagram $Y \in \mathcal{Y}(k, q-1-k)$ and $J = (j_{k+1}, \dots, j_r)$ satisfying

$$q+1 \leq j_{k+1} < \cdots < j_r < N$$

and define the logarithmic r-form

$$\omega_{Y,J} := S_Y(f_1, \dots, f_k) df_1 \wedge \cdots \wedge df_k \wedge \frac{df_{j_{k+1}}}{f_{j_{k+1}}} \wedge \cdots \wedge \frac{df_{j_r}}{f_{j_r}} \in \Omega^r(\mathcal{A}).$$

Proposition 19. [8] *Under the assumption (20) and (21), we can take as a basis of $H^r(C_{z,\alpha})$ the following set of r-forms:*

$$\bigcup_{k=0}^{r} \left\{ \omega_{Y,J} \;\middle|\; \begin{array}{l} Y \in \mathcal{Y}(k, q-1-k), \\ J = (j_1, \dots, j_{r-k}) \text{ such that } q+1 \leq j_1 < \cdots < j_{r-k} \leq N \end{array} \right\}.$$

§4. Exterior power structure

In the case the partition λ of N is general, we compute the cohomology group for particular points z of the parameter space $Z_{r,\lambda}$which will be called the Veronese points. Combining this fact with the holonomicity of the general hypergeometric system of type λ, we can show the purity of the cohomology groups and give the rank of the top cohomology group at any point $z \in Z_{r,\lambda}$. Recall that, in the case of generalized Airy integral, we gave in Proposition 16 a basis of the r-th cohomology group expressed in terms of Schur functions:

$$S_Y(u) du_1 \wedge \cdots \wedge du_r, \qquad Y \in \mathcal{Y}(r, N-r-2).$$

This basis is related to the exterior product structure of the cohomology group at the Veronese points.

4.1. Veronese map for $\lambda = (1, \dots, 1)$

We first recall the Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^r$ in order to motivate the construction of generalized Veronese map given in Subsection 4.2. Consider the map

$$\psi : \mathbb{C}^2 \ni \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \mapsto \begin{pmatrix} v_0^r \\ v_0^{r-1} v_1 \\ \vdots \\ v_1^r \end{pmatrix} \in \mathbb{C}^{r+1}. \tag{22}$$

It induces the Veronese embedding $\bar{\psi} : \mathbb{P}^1 \ni [v_0, v_1] \mapsto [v_0^r, v_0^{r-1} v_1, \dots, v_1^r] \in \mathbb{P}^r$.

Using the map ψ, we define, in the case $\lambda = (1, \dots, 1)$, the map $\Psi_{(1,\dots,1)} : \mathrm{Mat}_{2,N}(\mathbb{C}) \to \mathrm{Mat}_{r+1,N}(\mathbb{C})$ by $(z_0, z_1, \dots, z_{N-1}) \mapsto (\psi(z_0), \psi(z_1), \dots, \psi(z_{N-1}))$ or by

$$\begin{pmatrix} z_{00} & \dots & z_{0,N-1} \\ z_{10} & \dots & z_{1,N-1} \end{pmatrix} \mapsto \begin{pmatrix} (z_{00})^r & \dots & (z_{0,N-1})^r \\ (z_{00})^{r-1} z_{10} & \dots & (z_{0,N-1})^{r-1} z_{1,N-1} \\ \vdots & & \vdots \\ (z_{10})^r & \dots & (z_{1,N-1})^r \end{pmatrix}.$$

We see that it induces the map $\Psi_{(1,\dots,1)} : Z_{1,(1,\dots,1)} \to Z_{r,(1,\dots,1)}$ which we call the Veronese map. This fact is seen as follows. Note that $z = (z_0, \dots, z_{N-1}) \in Z_{1,(1,\dots,1)}$ if and only if $\det(z_i, z_j) \neq 0$ for any $i \neq j$. Take any indices $0 \leq i_0 < i_1 < \dots < i_r \leq N-1$. Then we see that

$$\det(\psi(z_{i_0}), \psi(z_{i_1}), \dots, \psi(z_{i_r})) = \prod_{k<l} \det(z_{i_k}, z_{i_l}) \neq 0.$$

We will define in the next subsection the map $\Psi_\lambda : Z_{1,\lambda} \to Z_{r,\lambda}$ for any partition λ of N which coincides with $\Psi_{(1,\dots,1)}$ in the case $\lambda = (1, \dots, 1)$. To this end we restate the map ψ as follows. Let V be a vector space of $\dim_{\mathbb{C}} V = 2$ and let $S^r V$ be the r-th symmetric tensor product. Then we have $\dim_{\mathbb{C}} S^r V = r + 1$. Define the map

$$\psi : V \ni v \mapsto \overbrace{v \otimes \dots \otimes v}^{r \text{ times}} \in S^r V.$$

Let e_0, e_1 be a basis of V, and let $\mathbf{e}_0, \dots, \mathbf{e}_r$ be a basis of $S^r V$ defined by

$$\mathbf{e}_k = \sum_{i_1 + i_2 + \dots + i_r = k} e_{i_1} \otimes \dots \otimes e_{i_r}.$$

Then from

$$(v_0e_0 + v_1e_1)^{\otimes r} = \sum_{k=0}^{r} v_0^{r-k} v_1^k \mathbf{e}_k,$$

we see that the map ψ, expressed in terms of the basis e_0, e_1 of V and $\mathbf{e}_0, \ldots, \mathbf{e}_r$ of S^rV, is the same as the map (22).

4.2. Veronese map for $\lambda = (n_1, \ldots, n_\ell)$

Let us generalize the Veronese map $\Psi_{(1,\ldots,1)}$ to the case where the partition λ of N is general. To do this we at first generalize the map ψ given by (22). Let V be a vector space of $\dim_{\mathbb{C}} V = 2$ and let $R_n = \mathbb{C}[T]/(T^n)$ be the quotient of the polynomial ring $\mathbb{C}[T]$ by the ideal (T^n) generated by the n-th power T^n. Consider $V_n := V \otimes R_n$, which is an R_n-module, and its r-th symmetric tensor product S^rV_n as an R_n-module. Define the map

$$\psi_n : V_n \ni v \mapsto \overbrace{v \otimes \cdots \otimes v}^{r} \in S^rV_n. \tag{23}$$

Let us express ψ_n using the basis of V_n as a vector space over $\mathbb{C}$:

$$e_i \otimes T^j \qquad (i = 0, 1,\ 0 \le j < n)$$

and that of S^rV_n:

$$\mathbf{e}_i \otimes T^j \qquad (0 \le i \le r,\ 0 \le j < n).$$

We identify an element $v = \sum_{i,j} v_{ij} e_i \otimes T^j \in V_n$ with the matrix

$$\begin{pmatrix} v_{00} & v_{01} & \cdots & v_{0,n-1} \\ v_{10} & v_{11} & \cdots & v_{1,n-1} \end{pmatrix} \in \mathrm{Mat}_{2,n}(\mathbb{C})$$

and an element $w = \sum_{i,j} w_{ij} \mathbf{e}_i \otimes T^j \in S^rV_n$ with

$$\begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0,n-1} \\ \vdots & \vdots & & \vdots \\ w_{r0} & w_{r1} & \cdots & w_{r,n-1} \end{pmatrix} \in \mathrm{Mat}_{r+1,n}(\mathbb{C}).$$

The map $\psi_n : V_n \to S^rV_n$ defined by (23) induces a map $\psi_n : \mathrm{Mat}_{2,n}(\mathbb{C}) \to \mathrm{Mat}_{r+1,n}(\mathbb{C})$.

Example 20. *Let $r = 2$. Then the map $\psi_3 : \mathrm{Mat}_{2,3}(\mathbb{C}) \to \mathrm{Mat}_{3,3}(\mathbb{C})$ is given by*

$$v \mapsto \begin{pmatrix} v_{00}^2 & 2v_{00}v_{01} & 2v_{00}v_{02} + v_{01}^2 \\ v_{00}v_{01} & v_{00}v_{11} + v_{01}v_{10} & v_{00}v_{12} + v_{02}v_{10} + v_{01}v_{11} \\ v_{01}^2 & 2v_{01}v_{11} & 2v_{10}v_{12} + v_{11}^2 \end{pmatrix}.$$

Definition 21. *For a partition $\lambda = (n_1, \dots, n_\ell)$ of N, define $\Psi_\lambda : \mathrm{Mat}_{2,N}(\mathbb{C}) \to \mathrm{Mat}_{r+1,N}(\mathbb{C})$ by*

$$z = (z^{(1)}, \dots, z^{(\ell)}) \mapsto (\psi_{n_1}(z^{(1)}), \dots, \psi_{n_\ell}(z^{(\ell)})).$$

We call this map the **Veronese map of type** λ.

Lemma 22. $\Psi_\lambda(Z_{1,\lambda}) \subset Z_{r,\lambda}$.

This fact can be shown as follows. For $z \in Z_{1,\lambda}$, put $w = \Psi_\lambda(z)$ and write it as $w = (w^{(1)}, \dots, w^{(\ell)}), w^{(k)} = (w_0^{(k)}, \dots, w_{n_k-1}^{(k)}) \in \mathrm{Mat}_{r+1,n_k}(\mathbb{C})$. Then for any $\mu = (m_1, \dots, m_\ell)$ such that $0 \le m_k \le n_k$ and $m_1 + \cdots + m_\ell = r+1$, we have

$$\begin{aligned}\det(w_0^{(1)}, \dots, w_{m_1-1}^{(1)}, \dots, w_0^{(\ell)}, \dots, w_{m_\ell-1}^{(\ell)}) \\ = C_\mu \prod_k \det(z_0^{(k)}, z_1^{(k)})^{m_k(m_k-1)/2} \prod_{k<l} \det(z_0^{(k)}, z_0^{(l)})^{m_k m_l}.\end{aligned}$$

with some nonzero constant C_μ. The map $\Psi_\lambda : Z_{1,\lambda} \to Z_{r,\lambda}$ is also called the Veronese map of type λ. The set $\Psi_\lambda(Z_{1,\lambda})$ and its point are called respectively the **Veronese image** and a **Veronese point**.

4.3. Exterior power structure of the cohomology group

The following theorem asserts that the cohomology groups for the r-dimensional GHGI of type λ at Veronese points can be constructed from that for 1 dimensional GHGI of type λ.

Theorem 23. *Let $z \in Z_{1,\lambda}$ be such that $z_0^{(1)} = (1,0)^t$ and let*

$$\tilde{z} = \Psi(z) \in Z_{r,N}, \quad \tilde{\alpha} = \alpha + (-r+1, 0, \dots, 0). \tag{24}$$

Then we have

(1) $H^p(C_{\tilde{z},\tilde{\alpha}}) = 0, \quad (p \ne r)$,

(2) *the top cohomology group has the exterior product structure:*

$$H^r(C_{\tilde{z},\tilde{\alpha}}) \simeq \bigwedge^r H^1(C_{z,\alpha}), \tag{25}$$

(3) $\dim_{\mathbb{C}} H^r(C_{\tilde{z},\tilde{\alpha}}) = \binom{N-2}{r}$.

Example 24. *[Generalized Airy case $\lambda = (N)$]*
Let $z \in Z_{1,(N)}$ and $\tilde{z} = \Psi_{(N)}(z)$. Theorem 23 says the following.

- *If we take a basis of* $H^1(C_{z,\alpha})$ *as*

$$\varphi_i = u^i du, \quad (0 \le i \le N-3),$$

then the isomorphism (25) gives the correspondence

$$\varphi_{i_1} \square \cdots \square \varphi_{i_r} \mapsto S_Y(v) dv.$$

Here $\square$ *denotes the exterior product in* $\bigwedge^r H^1(C_{z,\alpha})$,

$$i_1 > i_2 > \cdots > i_r \ge 0, \quad Y = (i_1 - r + 1, i_2 - r + 2, \ldots, i_r) \in \mathcal{Y}(r, N-r-2)$$

and

$$v = (v_1, \ldots, v_r), \quad dv = dv_1 \wedge \cdots \wedge dv_r.$$

- *If one take a basis of* $H^1(C_{z,\alpha})$ *as*

$$\varphi_i = d(\theta_i(\vec{u}z)), \quad (1 \le i \le N-2),$$

then the isomorphism (25) gives the correspondence

$$\varphi_{i_1} \square \cdots \square \varphi_{i_r} \mapsto d(\theta_{i_1}(\vec{v}\tilde{z})) \wedge \cdots \wedge d(\theta_{i_r}(\vec{v}\tilde{z})).$$

Example 25. *[*$\lambda = (n_1, \ldots, n_\ell)$ *case]*
Let $z \in Z_{1,\lambda}$. *Then as a basis of* $H^1(C_{z,\alpha})$ *we can take, for* $n_1 \ge 2$ *for example,*

$$d(\theta_1(\vec{u}z^{(1)})), \ldots, d(\theta_{n_1-2}(\vec{u}z^{(1)}))$$
$$d(\theta_0(\vec{u}z^{(k)})), d(\theta_1(\vec{u}z^{(k)})), \ldots, d(\theta_{n_k-1}(\vec{u}z^{(k)})), \quad (2 \le k \le \ell).$$

Put $\tilde{z} = \Psi_\lambda(z)$. *Then as a basis of* $H^r(C_{\tilde{z},\tilde{\alpha}})$ *we can take the* r*-forms obtained by choosing* r *forms* $d(\theta_j(\vec{v}\tilde{z}^{(k)}))$ *and taking the exterior product of them.*

Combining Theorem 23 and the holonomicity of the general hypergeometric system, we can know the purity and the dimension of the cohomology group for any $z \in Z_{r,\lambda}$.

Corollary 26. *Let* $\lambda = (n_1, \ldots, n_\ell)$ *be a partition of* N *and let* $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(\ell)})$, $\alpha^{(k)} = (\alpha_0^{(k)}, \ldots, \alpha_{n_k-1}^{(k)}) \in \mathbb{C}^{n_k}$ *satisfy the condition (2). Then for any point* z *of* $Z_{r,\lambda}$ *satisfying* $z_0^{(1)} = (1, 0, \ldots, 0)^t$, *we have*

(1) $H^p(C_{z,\alpha}) = 0$ *for* $p \ne r$,
(2) $\dim_{\mathbb{C}} H^r(C_{z,\alpha}) = \binom{N-2}{r}$.

Question: In the case $\lambda = (n_1, \ldots, n_\ell)$, we consider the r-forms constructed in Example 25 at any point $z \in Z_{r,\lambda}$. Do these r-forms give a basis for $H^r(C_{z,\alpha})$?

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