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Quantizing the Bäcklund transformations of Painlevé equations and the quantum discrete Painlevé VI equation

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Dedicated to Professor Akihiro Tsuchiya on the occasion of his retirement of Nagoya University

Abstract.

Based on the works by Kajiwara, Noumi and Yamada, we propose a canonically quantized version of the rational Weyl group representation which originally arose as symmetries or the Bäcklund transformations in Painlevé equations. We thereby propose a quantization of discrete Painlevé VI equation as a discrete Hamiltonian flow commuting with the action of $W(D_4^{(1)})$.

§1. Introduction

Let $A = [a_{ij}]_{i,j=0}^{l}$ be a generalized Cartan matrix of affine type and $W(A)$ be the corresponding Weyl group. We denote the generators by s_i $(i = 0, \cdots, l)$. Let $\boldsymbol{F}_{cl} := \mathbf{C}(a_0, \cdots, a_l, f_0, \cdots, f_l)$ be the field of rational functions generated by commuting variables $a_0, \cdots, a_l, f_0, \cdots, f_l$. Let u_{ij} be integers that satisfy

(i) $u_{ij} = 0$ if $i = j$ or $a_{ij} = 0$,

(ii) $u_{ij} : u_{ji} = -a_{ij} : a_{ji}$ otherwise.

Theorem 1 (KNY, case $A_l^{(1)}$). *For $i, j = 0, \cdots, l$ put*

$$s_i(a_j) := a_j a_i^{-a_{ij}}, \quad s_i(f_j) := f_j \left(\frac{a_i + f_i}{1 + a_i f_i} \right)^{u_{ij}}. \tag{1}$$

Then these formulas define a group homomorphism $W(A_l^{(1)}) \to \mathrm{Aut}(\boldsymbol{F}_{cl})$.

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This is the typical formula of the affine Weyl group symmetry or the Bäcklund transforation for difference Painlevé equation in its symmetric form: the case $A_2^{(1)}$ gives the symmetry of the difference Painlevé IV equation[KNY]. Moreover, this action is a Poisson map with respect to the bracket

$$\{f_i, f_j\} = u_{ij} f_i f_j, \quad \{a_i, a_j\} = \{a_i, f_j\} = 0. \tag{2}$$

A naive expect is that there exist a quantization of this representation realized as adjoint actions of some suitable operators (quantum Hamiltonian action). One of the aim of this note is to answer this problem. In the type A case, we introduce the letters $F_0, \cdots, F_l$ subject to the quantized relation of (2),

$$F_i F_{i+1} = q^{-1} F_{i+1} F_i, \quad F_i F_j - F_j F_i = 0 \ (j \not\equiv i \pm 1)$$

as well as central letters $a_0, \cdots, a_l$. Let $\boldsymbol{F}$ be the skew field defined by these relations. We will construct the affine Weyl group action on $\boldsymbol{F}$ in the form

$$s_i(\phi) = S_i \phi S_i^{-1}$$

for any $\phi \in \boldsymbol{F}$. The "Hamiltonian" S_i is actually given by some infinite product which is rather familiar in q- analysis, despite that it involves non-commutative letters (Section 2, Theorem 2). It is also shown that the construction works for other affine Weyl groups as well (Section 3, Theorems 3 and 4).

Rescent studies of Painlevé systems enabled us to understand their discrete symmetries (Bäcklund transformations) and the (discrete) time evolution tranformation from the one, namely the affine Weyl group actions of the above type [NY1][S]. Based on this knowledge together with our quantization of the affine Weyl group action, we can quantize discrete (multiplicative) Painlevé type equations. In principle, if we choose some lattice direction in the affine Weyl group as the generator of a discrete time evolution, then this discrete dynamics commutes with the simple reflections corresponding to the roots that are perpendicular to the evolution direction in the lattice. We apply this idea to quantize the q- difference Painlevé III equation studied by Kajiwara and Kimura [KK] and also to quantize Jimbo–Sakai's q- difference Painlevé VI system[JS]: See (11), (12) in Section 2 and Theorems 5, 6 for results.

§2. Quantizing the Weyl group action: Type A case

Let $Q = \boldsymbol{Z}\alpha_0 + \cdots + \boldsymbol{Z}\alpha_l$ be the root lattice of type $A_l^{(1)}$ with simple roots $\alpha_0, \cdots, \alpha_l$ and $\boldsymbol{C}[Q] = \boldsymbol{C}[e^{\alpha_0}, \cdots, e^{\alpha_l}]$ be its group algebra. The

Weyl group action $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$ gives rise to the action on $\boldsymbol{C}[Q]$, and we can identify the previously used letter a_j as e^{α_j}:

$$s_i(a_j) = a_i^{-a_{ij}} a_j.$$

Let $\boldsymbol{K}$ be the quotient field of the group algebra $\boldsymbol{C}[Q]$, namely $\boldsymbol{K} = \boldsymbol{C}(a_0, \cdots, a_l)$.

Remark Let $\partial_0, \cdots, \partial_l$ be the "dual" letters such that

$$[\partial_j, \alpha_k] := \partial_j \alpha_k - \alpha_h \partial_j = a_{jk},$$

then we have

$$e^{\pi\sqrt{-1}\alpha_i\partial_i} \cdot \alpha_j \cdot e^{-\pi\sqrt{-1}\alpha_i\partial_i} = s_i(\alpha_j).$$

That is, the Weyl group action on $\boldsymbol{K}$ can be realized as adjoint actions. This remark applies to the latter cases as well.

For type $A_l^{(1)}$ case ($l > 1$), we introduce the cannonically quantized letters $F_0, \cdots, F_l$ corresponding to (2):

$$F_i F_{i+1 \bmod l+1} = q^{-1} F_{i+1 \bmod l+1} F_i, \; F_i F_j = F_j F_i \quad (i - j \not\equiv \pm 1). \tag{3}$$

(Here and in what follows we regard the subscripts as elements in $\boldsymbol{Z}/(l+1)\boldsymbol{Z}$.)

Let $\boldsymbol{K}\langle F_0, \cdots, F_l\rangle$ be the $\boldsymbol{K}$- algebra generated by the above letters (3). It can be shown in a standard way that this algebra is an Ore domain (cf. [B]). Let $\boldsymbol{F} := \boldsymbol{K}(F_0, \cdots, F_l)$ be the quotient skew field of $\boldsymbol{K}\langle F_0, \cdots, F_l\rangle$. The above relations (3) actually quantize the Poisson bracket (2). In fact, letting $q \to 1$ and think of the Poisson structure

$$\{\phi, \psi\} := \lim_{q \to 1} \frac{1}{q-1}[\phi, \psi]$$

on the commutative algebra $\boldsymbol{F} \bmod (q-1)$. Then we have

$$\{F_i, F_{i+1}\} \equiv \frac{1}{q-1}(F_i F_{i+1} - F_{i+1} F_i) = -F_i F_{i+1}$$

according to the defining relation (3).

Note that $a_i \in \boldsymbol{K}$ is central in $\boldsymbol{F}$. Let us introduce the following multiplication operator

$$\Psi(z, F_i) = \Psi_q(z, F_i) := \frac{(qF_i, q)_\infty (F_i^{-1}, q)_\infty}{(zqF_i, q)_\infty (zF_i^{-1}, q)_\infty} \tag{4}$$

where z and q are central letters and $(x,q)_\infty := \prod_{m=0}^{\infty}(1+xq^m)$. The right hand side of (4) should be understood in the q- adic completion $\boldsymbol{F}((q))$ of $\boldsymbol{F}$. We put

$$\rho_i := e^{\frac{1}{2}\pi\sqrt{-1}\alpha_i\partial_i}, \quad S_i := \Psi(z,F_i)\rho_i.$$

Note that ρ_i commutes with the variables F_j, $0 \le j \le l$.We are interested in the adjoint action of S_i, $Ad(S_i) : \phi \in \boldsymbol{F}((q)) \mapsto S_i\phi S_i^{-1} \in \boldsymbol{F}((q))$.

The statement of the following theorem essentially goes back to [FV].

Theorem 2. *We have*

$$Ad(S_i)^2 = id, \tag{5}$$

$$S_iS_j = S_jS_i \quad (j \not\equiv i \pm 1),$$

$$S_iS_{i+1}S_i = S_{i+1}S_iS_{i+1} \tag{6}$$

where the index should read modulo $l+1$. Hence $s_i \mapsto Ad(S_i)$ defines a group homomorphism

$$W(A_l^{(1)}) \to \mathrm{Aut}(\boldsymbol{F}((q))).$$

Let us calculate $Ad(S_i)F_j = S_iF_jS_i^{-1}$ first. We have

$$\begin{aligned} Ad(S_i)(F_{i-1}) &= \frac{1+a_iF_i}{a_i+F_i}F_{i-1}, \\ Ad(S_i)(F_{i+1}) &= F_{i+1}\frac{a_i+F_i}{1+a_iF_i}, \\ Ad(S_i)(F_j) &= F_j \quad (i-j \not\equiv \pm 1). \end{aligned} \tag{7}$$

In fact these are the defining recurrence relation for the multiplication operator (4), that is, one can recover the formula of $\Psi(a_i,F_i)$ from these modulo pseudo constants.

Now we can check $\mathrm{Ad}S_i^2 = id(5)$ from these formula. Equivalently, since

$$S_i^2 = \Psi(a_i,F_i)\Psi(a_i^{-1},F_i)\rho_i^2,$$

(5) follows from the fact that $\Psi(a_i,F_i)\Psi(a_i^{-1},F_i)$ is a pseudo constant:

$$\Psi(a_i,F_i)\Psi(a_i^{-1},F_i) = \Psi(a_i,qF_i)\Psi(a_i^{-1},qF_i). \tag{8}$$

As for (6), we can do the similar computation to check it as the adjoint action on $\boldsymbol{F}((q))$, namely, compare the result when adjointly applied to generaters F_i. However, (6) is satisfied as an identity of elements in $\boldsymbol{F}((q))$. Actually (6) follows from the dilogarithmic identity: suppose F and G satisfies $FG = qGF$, then we have

$$(G,q)_\infty(F,q)_\infty = (F,q)_\infty(GF,q)_\infty(G,q)_\infty.$$

From this we can show ([FV], [Ki])

$$\Psi(x,F_i)\Psi(xy,F_{i+1})\Psi(y,F_i) = \Psi(y,F_{i+1})\Psi(xy,F_i)\Psi(x,F_{i+1}) \tag{9}$$

where x, y are central letters, which is equivalent to (6).

Introduce the diagram automorphism by

$$\omega : a_i \mapsto a_{i+1}, \quad F_i \mapsto F_{i+1 \bmod l+1},$$

then ω and $s_i := Ad(S_i)$ generate the extended affine Weyl group $\tilde{W}(A_l^{(1)})$ acting on $\boldsymbol{F}((q))$. As is well known, we have the commuting elements

$$\left\{\begin{array}{l} T_1 := s_1 s_2 \cdots s_l \omega^{-1} \\ T_2 := s_2 \cdots s_l \omega^{-1} s_1 \\ \quad\vdots \\ T_l := s_l \omega^{-1} s_1 \cdots s_{l-1}. \end{array}\right. \tag{10}$$

They are mutually conjugate. If we take T_1 as a discrete time evolution operator, then the group $\langle s_0 s_1 s_0, s_2, s_3, \cdots, s_l\rangle \simeq W(A_{l-1}^{(1)})$ commutes with the T_1 action. This gives the quantization of the "q- difference" version of type A discrete system with Painlevé type symmetry which is extensively studied by Noumi and Yamada [NY1][NY2].

Example Let $l = 2$. Note that $a_0 a_1 a_2 =: p$ is invariant under $\tilde{W}(A_2^{(1)})$, and the same holds for $F_0 F_1 F_2 =: c$ since c commutes with everything. The action of $T_1 = s_1 s_2 \omega^{-1}$ is given by

$$T_1(a_0) = p^{-1} a_0,\ T_1(a_1) = p a_1,\ T_1(a_2) = a_2 \tag{11}$$

and

$$F_0 T_1(F_0) = c\frac{1 + a_1 F_1^{-1}}{1 + a_1 F_1}, \quad T_1^{-1}(F_1) F_1 = c\frac{1 + a_0^{-1} F_0}{1 + a_0^{-1} F_0^{-1}}. \tag{12}$$

This T_1 action commutes with $\langle s_0 s_1 s_0, s_2\rangle \simeq W(A_1^{(1)})$ and gives a quantization of the $q\mathrm{P}_{\mathrm{III}}$ system studied in [KK] (where p should be regarded as q).

In this $l=2$ case, the Hamiltonian for the diagram automorphism ω can be found as follows, so that the above T_1 flow is actually a discrete Hamiltonian flow. We put

$$\theta(X) := (X,q)_\infty (qX^{-1},q)_\infty$$

and

$$\Omega := \left(\theta(F_0^{-1}F_1)\theta(qF_1)^2\theta(F_2^{-1}F_0^{-1})\right)^{-1} \times p^{-\partial_1'}\rho_1\rho_2,$$

where

$$[\partial_1', \alpha_0] = -1,\ [\partial_1', \alpha_1] = 1,\ [\partial_1', \alpha_2] = 0.$$

(Note : If we realize the $A_2^{(1)}$ root system in $\boldsymbol{R}^3 \oplus \boldsymbol{R}\delta$ by $\alpha_0 = e_3 - e_1 + \delta, \alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3$, where e_1, e_2, e_3 are the standard orthogonal basis of $\boldsymbol{R}^3$ and δ the canonical null root so that $p = e^\delta$, then ∂_1' stands for the derivation corresponding to e_1.) Then we can easily check

$$\Omega a_i \Omega^{-1} = a_{i+1 \bmod 3},\ \Omega F_i \Omega^{-1} = F_{i+1 \bmod 3}$$

and therefore

$$T_1 = \mathrm{Ad}(S_1 S_2 \Omega^{-1}). \tag{13}$$

We have

$$\begin{aligned} & S_1 S_2 \Omega^{-1} \\ = \ & \Psi(a_1, F_1)\rho_1 \Psi(a_2, F_2)\rho_2 \cdot (p^{-\partial_1'}\rho_1\rho_2)^{-1}\theta(F_0^{-1}F_1)\theta(qF_1)^2\theta(F_2^{-1}F_0^{-1}) \\ = \ & \Psi(a_1, F_1)\Psi(a_1 a_2, F_2)\theta(F_0^{-1}F_1)\theta(qF_1)^2\theta(F_2^{-1}F_0^{-1})p^{\partial_1'}. \end{aligned}$$

§3. General case

If the Dynkin diagram for the genralized Cartan matrix is simply laced, the construction in the last section applies to obtain the corresponding Weyl group action. As for the non-simply laced case, the construction can be reduced to the rank two cases: B_2 type and G_2 type (cf. [NY3]).

As before let $\boldsymbol{K} := \boldsymbol{C}(a_1, a_2)$ be the quotient field of the group algebra $\boldsymbol{C}[Q]$, where Q stands for the rank two root lattice in problem and we identify the letter a_j with $e^{\alpha_j} \in \boldsymbol{C}[Q]$. We introduce $\partial_j (j = 1, 2)$ such that $[\partial_j, \alpha_k] = a_{jk}$ and put $\rho_j := e^{\frac{1}{2}\pi\sqrt{-1}\alpha_j \partial_j}$. Then the Weyl group action s_j on $\boldsymbol{K}$ is given by the adjoint action of ρ_j:

$$s_j(\alpha_k) := \alpha_k - a_{jk}\alpha_j = \rho_j \alpha_k \rho_j^{-1}, \quad s_j(a_k) = a_j^{-a_{jk}} a_k = \rho_j a_k \rho_j^{-1}.$$

<u>B_2 case.</u> Let $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$ and define the skew field $\boldsymbol{F} = \boldsymbol{K}(F_1, F_2)$, where

$$F_2F_1 = q^2F_1F_2.$$

<u>G_2 case.</u> Let $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ and define the skew field $\boldsymbol{F} = \boldsymbol{K}(F_1, F_2)$, where

$$F_2F_1 = q^3F_1F_2.$$

Using these, we have Hamiltonian Weyl group action on $\boldsymbol{F}$ in both cases:

Theorem 3. *For type B_2 case, put*

$$S_1 := \Psi_q(a_1, F_1)\rho_1, \quad S_2 := \Psi_{q^2}(a_2^2, F_2)\rho_2. \tag{14}$$

Then we have

$$Ad(S_j)^2 = id,$$

$$S_1S_2S_1S_2 = S_2S_1S_2S_1. \tag{15}$$

Theorem 4. *For type G_2 case, put*

$$S_1 := \Psi_q(a_1, F_1)\rho_1, \quad S_2 := \Psi_{q^3}(a_2^3, F_2)\rho_2. \tag{16}$$

We have

$$Ad(S_j)^2 = id,$$

$$S_1S_2S_1S_2S_1S_2 = S_2S_1S_2S_1S_2S_1. \tag{17}$$

For example, in the B_2 case we have

$$\begin{aligned}
S_1F_2S_1^{-1} &= \Psi_q(a_1, F_1)F_2\Psi_q(a_1, F_1)^{-1} \\
&= F_2\Psi_q(a_1, q^{-2}F_1)\Psi_q(a_1, F_1)^{-1} \\
&= F_2\frac{(F_1q^{-1}, q)_\infty(F_1^{-1}q^2, q)_\infty}{(a_1F_1q^{-1}, q)_\infty(a_1F_1^{-1}q^2, q)_\infty}\frac{(a_1F_1q, q)_\infty(a_1F_1^{-1}, q)_\infty}{(F_1q, q)_\infty(F_1^{-1}, q)_\infty} \\
&= F_2\frac{(1 + F_1q^{-1})(1 + F_1)}{(1 + a_1F_1q^{-1})(1 + a_1F_1)}\frac{(1 + a_1F_1^{-1})(1 + a_1F_1^{-1}q)}{(1 + F_1^{-1})(1 + F_1^{-1}q)} \\
&= F_2\frac{(a_1 + F_1)(a_1 + F_1q^{-1})}{(1 + a_1F_1)(1 + a_1F_1q^{-1})}.
\end{aligned}$$

The property $Ad(S_j)^2 = id$ can be checked by continuing this computation, or we can conclude it immediately from the pseudoconstant property (8).

In principle we can calculate the expressions for $Ad(S_1S_2S_1S_2)F_j$ and $Ad(S_2S_1S_2S_1)F_j$ $(j = 1, 2)$ also to check (15) at the adjoint level, though it is a quite lengthy way. In fact (15) holds as an identity in $\boldsymbol{F}((q))$. Let us introduce a square root of F_2, namely let $\sqrt{-F_2}$ be the letter satisfying

$$\sqrt{-F_2}^2 = -F_2, \quad F_1\sqrt{-F_2} = q^{-1}\sqrt{-F_2}F_1.$$

Then we have

$$\Psi_{q^2}(a_2^2, F_2) = \Psi_q(a_2, \sqrt{-F_2})\Psi_q(a_2, -\sqrt{-F_2}),$$

so that (15) can be reduced to the type A identity (9) . For short, we write $a_1 =: a, a_2 =: b$,

$$\Psi_q(a, F_1) =: \Psi_1^a,\ \Psi_{q^2}(b^2, F_2) =: \Psi_2^b,\ \Psi_q(b, \pm\sqrt{-F_2}) =: \Psi_\pm^b.$$

Then $S_1S_2S_1S_2 = S_2S_1S_2S_1$ is equivalent to

$$\Psi_1^a\Psi_2^{ab}\Psi_1^{ab^2}\Psi_2^b = \Psi_2^b\Psi_1^{ab^2}\Psi_2^{ab}\Psi_1^a,$$

or

$$\Psi_1^a\Psi_+^{ab}\Psi_-^{ab}\Psi_1^{ab^2}\Psi_+^b\Psi_-^b = \Psi_+^b\Psi_-^b\Psi_1^{ab^2}\Psi_+^{ab}\Psi_-^{ab}\Psi_1^a.$$

This can be verified as follows, which uses (9): $\Psi_\pm^x\Psi_1^{xy}\Psi_\pm^y = \Psi_1^y\Psi_\pm^{xy}\Psi_1^x$ as well as $\Psi_+^x\Psi_-^y = \Psi_-^y\Psi_+^x$ at the underlined places.

$$\begin{aligned}
\text{LHS} &= \Psi_1^a\Psi_+^{ab}\underline{\Psi_-^{ab}\Psi_1^{ab^2}\Psi_-^b}\Psi_+^b \\
&= \underline{\Psi_1^a\Psi_+^{ab}\Psi_1^b}\Psi_-^{ab^2}\Psi_1^{ab}\Psi_+^b \\
&= \Psi_+^b\Psi_1^{ab}\underline{\Psi_+^a\Psi_-^{ab^2}}\Psi_1^{ab}\Psi_+^b = \Psi_+^b\Psi_1^{ab}\Psi_-^{ab^2}\underline{\Psi_+^a\Psi_1^{ab}\Psi_+^b} \\
&= \Psi_+^b\underline{\Psi_1^{ab}\Psi_-^{ab^2}\Psi_1^b}\Psi_+^{ab}\Psi_1^a \\
&= \Psi_+^b\Psi_-^b\Psi_1^{ab^2}\Psi_-^{ab}\Psi_+^{ab}\Psi_1^a = \text{RHS}.
\end{aligned}$$

Proof for the G_2 case can be quite similary done as in the B_2 case: we use the cubic root $\zeta \neq 1$ of unity and $\sqrt[3]{F_2}$ of F_2 that satisfies $F_1\sqrt[3]{F_2} = q^{-1}\sqrt[3]{F_2}F_1$. We have

$$\Psi_{q^3}(a_2^3, F_2) = \Psi_q\left(a_2, \sqrt[3]{F_2}\right)\Psi_q\left(a_2, \zeta\sqrt[3]{F_2}\right)\Psi_q\left(a_2, \zeta^{-1}\sqrt[3]{F_2}\right).$$

As before, write $a_1 =: a, a_2 =: b$, $\Psi_q(a, F_1) =: \Psi^a_1, \Psi_{q^3}(a^3, F_2) =: \Psi^b_2$ and also

$$\Psi_q(b, \sqrt[3]{F_2}) =: \Psi^b_0, \quad \Psi_q(b, \zeta^{\pm 1}\sqrt[3]{F_2}) =: \Psi^b_\pm$$

for short. Then (17) is equivalent to

$$\Psi^a_1\Psi^{ab}_2\Psi^{a^2b^3}_1\Psi^{ab^2}_2\Psi^{ab^3}_1\Psi^b_2 = \Psi^b_2\Psi^{ab^3}_1\Psi^{ab^2}_2\Psi^{a^2b^3}_1\Psi^{ab}_2\Psi^a_1. \tag{18}$$

This time (9) means $\Psi^x_k\Psi^{xy}_1\Psi^y_k = \Psi^y_1\Psi^{xy}_k\Psi^x_1$ for $k = 0, \pm$ and $\Psi^x_0, \Psi^y_+, \Psi^z_-$ are commuting for any central x, y, z. We have

$$\begin{aligned}
\text{LHS of (18)} &= \Psi^a_1\Psi^{ab}_0\Psi^{ab}_+\underline{\Psi^{ab}_-\Psi^{a^2b^3}_1}\,\underbrace{\Psi^{ab^2}_0}\,\Psi^{ab^2}_+\,\underline{\Psi^{ab^2}_-}\,\underbrace{\Psi^{ab^3}_1\Psi^b_0}\,\Psi^b_+\Psi^b_- \\
&= \Psi^a_1\Psi^{ab}_0\Psi^{ab}_+\Psi^{ab^2}_1\Psi^{a^2b^3}_-\,\underline{\Psi^{ab}_1\Psi^{ab^2}_+\Psi^b_1}\,\Psi^{ab^3}_0\Psi^{ab^2}_1\Psi^b_+\Psi^b_- \\
&= \Psi^a_1\Psi^{ab}_0\underline{\Psi^{ab}_+\Psi^{ab^2}_1}\,\Psi^{a^2b^3}_-\,\underline{\Psi^b_+}\,\Psi^{ab^2}_1\,\underbrace{\Psi^{ab}_+}\,\Psi^{ab^3}_0\,\underbrace{\Psi^{ab^2}_1\Psi^b_+}\,\Psi^b_- \\
&= \underline{\Psi^a_1\Psi^{ab}_0\Psi^b_1}\,\Psi^{ab^2}_+\Psi^{ab}_1\Psi^{a^2b^3}_-\,\underbrace{\Psi^{ab^2}_1\Psi^{ab^3}_0\Psi^b_1}\,\Psi^{ab^2}_+\Psi^{ab}_1\Psi^b_- \\
&= \Psi^b_0\Psi^{ab}_1\underline{\Psi^a_0}\,\Psi^{ab^2}_+\,\underline{\Psi^{ab}_1}\,\Psi^{a^2b^3}_-\,\underline{\Psi^b_0}\,\Psi^{ab^3}_1\Psi^{ab^2}_0\Psi^{ab^2}_+\Psi^{ab}_1\Psi^b_- \\
&= \Psi^b_0\underline{\Psi^{ab}_1\Psi^{ab^2}_+\Psi^b_1}\,\Psi^{ab}_0\,\underbrace{\Psi^a_1\Psi^{a^2b^3}_-\Psi^{ab^3}_1}\,\Psi^{ab^2}_0\Psi^{ab^2}_+\Psi^{ab}_1\Psi^b_- \\
&= \Psi^b_0\Psi^b_+\Psi^{ab^2}_1\Psi^{ab}_+\underline{\Psi^{ab}_0}\,\Psi^{ab^3}_-\,\underline{\Psi^{a^2b^3}_1}\,\underbrace{\Psi^a_-}\,\underline{\Psi^{ab^2}_0}\,\Psi^{ab^2}_+\,\underbrace{\Psi^{ab}_1\Psi^b_-} \\
&= \Psi^b_0\Psi^b_+\Psi^{ab^2}_1\Psi^{ab}_+\Psi^{ab^3}_-\,\underline{\Psi^{ab}_0\Psi^{a^2b^3}_1\Psi^{ab^2}_0}\,\Psi^{ab^2}_+\,\underbrace{\Psi^a_-}\,\underbrace{\Psi^{ab}_1\Psi^b_-} \\
&= \Psi^b_0\Psi^b_+\Psi^{ab^2}_1\Psi^{ab}_+\Psi^{ab^3}_-\Psi^{ab^2}_1\Psi^{a^2b^3}_0\,\underline{\Psi^{ab}_1\Psi^{ab^2}_+\Psi^b_1}\,\Psi^{ab}_-\Psi^a_1 \\
&= \Psi^b_0\Psi^b_+\Psi^{ab^2}_1\,\underline{\Psi^{ab}_+}\,\Psi^{ab^3}_-\,\underline{\Psi^{ab^2}_1}\,\Psi^{a^2b^3}_0\,\underline{\Psi^b_+}\,\Psi^{ab^2}_1\Psi^{ab}_+\Psi^{ab}_-\Psi^a_1 \\
&= \Psi^b_0\Psi^b_+\underline{\Psi^{ab^2}_1\Psi^{ab^3}_-\Psi^b_1}\,\Psi^{ab^2}_+\,\underbrace{\Psi^{ab}_1\Psi^{a^2b^3}_0\Psi^{ab^2}_1}\,\Psi^{ab}_+\Psi^{ab}_-\Psi^a_1 \\
&= \Psi^b_0\Psi^b_+\Psi^b_-\Psi^{ab^3}_1\Psi^{ab^2}_-\Psi^{ab^2}_+\Psi^{ab^2}_0\Psi^{a^2b^3}_1\Psi^{ab}_0\Psi^{ab}_+\Psi^{ab}_-\Psi^a_1 \\
&= \Psi^b_2\Psi^{ab^3}_1\Psi^{ab^2}_2\Psi^{a^2b^3}_1\Psi^{ab}_2\Psi^a_1. = \text{RHS of (18)}
\end{aligned}$$

§4. Quantizing the discrete Painlevé equation

There is a discretization of the Painlevé VI equation proposed by Jimbo and Sakai [JS] and later it was reformulated under the affine Weyl group symmetry of type $D_5^{(1)}$: [S], [TM]. Using the ideas in the previous sections, here we propose its quantum (non-commutative) version in a quite straightforward way. Let us introduce the Dynkin diagram of type $D_5^{(1)}$ and its numbering:

$$\begin{array}{ccccc} 0 & & & & 5 \\ & \backslash & & / & \\ & 2 & - & 3 & \\ & / & & \backslash & \\ 1 & & & & 4 \end{array}$$

We denote the corresponding generalized Cartan matrix by $[a_{ij}]_{i,j=0}^5$ and the simple roots by $\{\alpha_i\}_0^5$. Let q be a formal central letter (or a complex parameter, $|q| < 1$). We introduce the field of rationals $\boldsymbol{K} = \boldsymbol{C}(a_0, \cdots, a_5)$, where $a_i = e^{\alpha_i}$ as in the previous sections.

The Weyl group $W = W(D_5^{(1)})$ acts on $\boldsymbol{K}$ by

$$s_i(a_j) = a_i^{-a_{ij}} a_j.$$

One checks that $a_0 a_1 a_2^2 a_3^2 a_4 a_5 =: p$ is invariant under the action of W. Moreover, this action can be extended by the diagram automorphisms $\sigma_{01}, \sigma_{45}, \tau$:

$$\begin{aligned} &\sigma_{01} : a_0 \leftrightarrow a_1^{-1}, a_j \mapsto a_j^{-1} \ (j \neq 0, 1), \\ &\sigma_{45} : a_4 \leftrightarrow a_5^{-1}, a_j \mapsto a_j^{-1} \ (j \neq 4, 5), \\ &\tau : a_j \leftrightarrow a_{5-j}^{-1} \ (j = 0, \cdots, 5). \end{aligned} \tag{19}$$

We denote the extended Weyl group by $\tilde{W} := \langle W, \sigma_{01}, \sigma_{45}, \tau \rangle$.

Let $\boldsymbol{F} = \boldsymbol{K}(F, G)$ be the skew field, where

$$FG = qGF. \tag{20}$$

The action of $\tilde{W}$ on $\boldsymbol{K}$ can be extended to $\boldsymbol{F} = \boldsymbol{K}(F, G)$ as follows.

Theorem 5. *We can extend the automorphisms s_j $(j = 0, \cdots, 5)$ of $\boldsymbol{K}$ as algebra automorphisms of $\boldsymbol{F}$ by putting*

$$s_2(F) := F \frac{a_0 a_1^{-1} G + a_2^2}{a_0 a_1^{-1} a_2^2 G + 1}, \quad s_j(F) := F \ (j \neq 2)$$

and

$$s_3(G) := \frac{a_3^2 a_4 a_5^{-1} F + 1}{a_4 a_5^{-1} F + a_3^2} G, \quad s_j(G) := G \ (j \neq 3).$$

They give rise to a homomorphism $W \to \mathrm{Aut}_{\text{skew field}}(\boldsymbol{F})$.

Moreover, the action of the diagram automorphisms $\sigma_{01}, \sigma_{45}, \tau$ on $\boldsymbol{K}$ can be extended as involutive antiautomorphisms on $\boldsymbol{F}$ by

$$\sigma_{01} : F \mapsto q^{-1}F^{-1},\ G \mapsto G$$

$$\sigma_{45} : F \mapsto F,\ G \mapsto q^{-1}G^{-1}$$

$$\tau : F \mapsto G,\ G \mapsto F$$

so that we have a homomorphism $\tilde{W} \to \mathrm{Aut}_{\boldsymbol{C}-lin}(\boldsymbol{F})$.

The proof is straightforward. Because of the noncommutativity, it seems inevitable to define $\sigma_{01}, \sigma_{45}, \tau$ actions on $\boldsymbol{F}$ as antiautomorphisms. For example, if we want to extend the action of τ on $\boldsymbol{K}$ to $\boldsymbol{F}$ by $\tau : F \leftrightarrow G$ (cf. [TM]), this cannot be compatible with the relation $FG = qGF$ if we insist τ to be an automorphism: $\tau(F)\tau(G) \neq q\tau(G)\tau(F)$.

Note that $\sigma := \sigma_{01}\sigma_{45}$ is an automorphism of $\boldsymbol{F}$ satisfying $\sigma s_j = s_{\sigma(j)}\tau$, where

$$(\sigma(0), \sigma(1), \sigma(2), \sigma(3), \sigma(4), \sigma(5)) = (1, 0, 2, 3, 5, 4).$$

We are interested in the action of

$$T_3 = s_2 s_1 s_0 s_2 \sigma_{01} s_3 s_4 s_5 s_3 \sigma_{45} = s_2 s_1 s_0 s_2 s_3 s_4 s_5 s_3 \sigma \in \tilde{W},$$

since in the commutative case this recovers the discrete Painlevé VI system [S]. Put $t := a_3^2 a_4 a_5$, then t is invariant under $\langle s_0, s_1, s_2 s_3 s_2 = s_3 s_2 s_3, s_4, s_5 \rangle = W^{T_3}$.

Theorem 6. *We have*

$$\begin{aligned}&(T_3(a_0), T_3(a_1), T_3(a_2), T_3(a_3), T_3(a_4), T_3(a_5))\\ &= (a_0, a_1, pa_2, p^{-1}a_3, a_4, a_5), \quad T_3(t) = p^{-1}t,\end{aligned}$$

$$T_3(F) = q^{-1}p^2t^{-2}\frac{G + tp^{-1}a_1^2}{G + t^{-1}pa_0^2}\,\frac{G + tp^{-1}a_1^{-2}}{G + t^{-1}pa_0^{-2}}F^{-1}, \tag{21}$$

$$T_3^{-1}(G) = q^{-1}t^{-2}G^{-1}\frac{F + ta_4^2}{F + t^{-1}a_5^2}\,\frac{F + ta_4^{-2}}{F + t^{-1}a_5^{-2}}. \tag{22}$$

This T_3 flow allows the symmetry of $W(D_4^{(1)})$, namely T_3 commutes with the subgroup $\langle s_0, s_1, s_2 s_3 s_2 = s_3 s_2 s_3, s_4, s_5, \sigma_{01}, \tau \rangle \simeq \langle W(D_4^{(1)}), \sigma_{01}, \tau \rangle$ of $\tilde{W}$.

Put $Z := tF, Y := \frac{t}{p}G, T := \left(\frac{t}{p}\right)^2$ and let us use the notation $T_3(X) =: \bar{X}$ for any X. We have $ZY = qYZ$, $\bar{T} = p^{-2}T$ and the formula (21), (22) can be rewritten as follows:

$$(23)\quad \bar{Z}Z = \frac{p}{q}\frac{Y+Ta_1^2}{Y+a_0^2}\frac{Y+Ta_1^{-2}}{Y+a_0^{-2}}, \qquad \bar{Y}Y = \frac{1}{pq}\frac{\bar{Z}+Ta_4^2}{\bar{Z}+a_5^2}\frac{\bar{Z}+Ta_4^{-2}}{\bar{Z}+a_5^{-2}}.$$

This system should be regarded as a *quantization of the discrete Painlevé VI equation.*

Likewise in the previous sections, we have the Hamiltonians for the $W(D_5^{(1)})$- action written in terms of infinite product Ψ. Put

$$S_2 := \Psi(a_2^2, a_0a_1^{-1}G)\rho_2,\ S_3 := \Psi(a_3^2, a_5a_4^{-1}F)\rho_3,\ S_j := \rho_j\ (j \neq 2, 3),$$

where $[\partial_j, a_k] = a_{jk}a_k$ and $\rho_j = e^{\frac{\pi}{2}\sqrt{-1}\alpha_j\partial_j}$. Then we have $s_j = \mathrm{Ad}(S_j)$ for $j = 0, \cdots, 5$.

We can also find a Hamiltonian for the diagram automorphism $\sigma = \sigma_{01}\sigma_{45}$. Let us introduce the letters ∂'_j by the relation $[\partial'_j, \alpha_k] = \delta_{jk}$ and put

$$\begin{aligned}\Sigma &:= \theta(qFG)\theta(G^{-1}F)\theta(qF)^4 \\ &\quad\times e^{\frac{\pi}{2}\sqrt{-1}(\alpha_0+\alpha_1)(\partial'_0+\partial'_1)}e^{\pi\sqrt{-1}\alpha_2\partial'_2}e^{\pi\sqrt{-1}\alpha_3\partial'_3}e^{\frac{\pi}{2}\sqrt{-1}(\alpha_4+\alpha_5)(\partial'_4+\partial'_5)}.\end{aligned}$$

Then we have $\sigma = Ad(\Sigma)$, that is, $\Sigma F\Sigma^{-1} = F^{-1}$, $\Sigma G\Sigma^{-1} = G^{-1}$ and $\Sigma a_j\Sigma^{-1} = a_{\sigma(j)}^{-1}$ hold. Thus we have

Theorem 7. *The quantum discrete Painlevé VI equation is a discrete Hamiltonian flow,*

$$T_3 = Ad(S_2S_1S_0S_2S_3S_4S_5S_3\Sigma).$$

Explicitly, we have

$$\begin{aligned}& S_2S_1S_0S_2S_3S_4S_5S_3\Sigma \\ =\ & \Psi(a_2^2, a_0a_1^{-1}G)\rho_2\rho_1\rho_0\Psi(a_2^2, a_0a_1^{-1}G)\rho_2 \\ & \times\Psi(a_3^2, a_4^{-1}a_5F)\rho_3\rho_4\rho_5\Psi(a_3^2, a_4^{-1}a_5F)\rho_3\Sigma \\ =\ & \Psi(a_2^2, a_0a_1^{-1}G)\Psi((a_0a_1a_2)^2, a_0^{-1}a_1G)\Psi(p^2(a_3a_4a_5)^{-2}, a_4a_5^{-1}F) \\ & \times\Psi(p^2a_2^2a_3^{-2}, a_4a_5^{-1}F)\theta(qFG)\theta(G^{-1}F)\theta(qF)^4p^{\partial},\end{aligned}$$

where $\partial = \frac{1}{2}(\partial_3' - \partial_2')$ so that $p^\partial t p^{-\partial} = tp^{-1}$. Note that we can realize the $D_5^{(1)}$ root lattice in $\boldsymbol{R}\delta \oplus \boldsymbol{R}^5 = \boldsymbol{R}\delta \oplus \boldsymbol{R}e_1 \oplus \cdots \oplus \boldsymbol{R}e_5$ by

$$\alpha_0 = \delta - e_1 - e_2,\ \alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3,$$

$$\alpha_3 = e_3 - e_4,\ \alpha_4 = e_4 - e_5,\ \alpha_5 = e_4 + e_5,$$

where e_j are regarded as the orthonormal basis. Then we have $t = e^{e_3}$, $\partial = -\partial/\partial e_3$. The root subsystem pependicular to e_3 is generated by $\alpha_0, \alpha_1, \alpha_2 + \alpha_3 = e_2 - e_4, \alpha_4, \alpha_5$ and isomorphic to $D_4^{(1)}$.

As for the classical (commutative) case, the discrete Painlevé system allows rather simple, so-called "seed" solutions, from which one can construct rich explicit solutions via Bäcklund transformations. In our quantized system, it seems however that such seed solutions are difficult to find because of the noncommutativity. For such issues as well as the consideration of the continuous limit, we hope to discuss elsewhere.

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