

Chiral Potts model and the discrete Sine-Gordon model at roots of unity

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Abstract.

The discrete quantum Sine-Gordon model at roots of unity remarkably combines a classical integrable system with an integrable quantum spin system, whose parameters obey classical equations of motion. We show that the fundamental R -matrix of the model (which satisfies a difference property Yang–Baxter equation) naturally splits into a product of a singular “classical” part and a finite dimensional quantum part. The classical part of the R -matrix itself satisfies the quantum Yang–Baxter equation, and therefore can be factored out producing, however, a certain “twist” of the quantum part. We show that the resulting equation exactly coincides with the star-triangle relation of the N -state chiral Potts model. The associated spin model on the whole lattice is, in fact, more general than the chiral Potts and reduces to the latter only for the simplest (constant) classical background. In a general case the model is inhomogeneous: its Boltzmann weights are determined by non-trivial background solutions of the equations of motion of the classical discrete sine-Gordon model.

§1. Introduction

The fundamental works of Professor Akihiro Tsuchiya [40, 41] made an outstanding contribution to the theory of integrable quantum systems. Here we present some new developments in this field. The discovery of the chiral Potts model originated in [32, 31, 2] and finalized in [8] brought a lot of new interesting problems. First the Boltzmann weights of the model require high genus algebraic functions for their uniformization. The second (related) problem is that there is no difference property in the model. The uniformization problem has been solved in [4] using classical results on the rotation of a rigid body [37]. However, even with this uniformization we still do not know how to apply various methods which worked perfectly for all other two-dimensional solvable

models, basically because there is no a difference property. In particular, it took 16 years until the intriguingly simple conjecture of [1] for the local order parameter

$$(1.1) \quad \langle \omega^{\alpha j} \rangle = (k^2)^{\Delta_j}, \quad \Delta_j = \frac{j(N-j)}{2N^2}, \quad j = 1, \dots, N-1$$

has finally been proven [6] by a generalization of the method of [33]. Here N , $N \geq 2$, is the number of local spin states and k is a modulus of the algebraic curve appearing in the model, it also plays the role of the temperature like variable: the model is critical for $k = 0$, it reduces than to the Fateev–Zamolodchikov model [30]. For $N = 2$ the chiral Potts model coincides with the ordinary Ising model and formula (1.1) reduces to the famous Onsager–Yang result [39, 43] for the spontaneous magnetization of the Ising model.

In view of the above discussion the chiral Potts model may seem standing very far away from the “conventional” solvable model with the difference property. However, this is not so. It was shown [13] that the chiral Potts model is remarkably connected to the six-vertex model: its R-matrix satisfies the Yang–Baxter equations with two cyclic L-operators related to the R-matrix of the six-vertex model. Further, it turned out that the last connection can be extended by replacing the six-vertex model (with two states per edge) by an the n -state model [22] associated with the $U_q(\mathfrak{sl}(n))$ algebra. This resulted in the “ $\mathfrak{sl}(n)$ -generalized chiral Potts model” [15], which is a two-dimensional model with spins that each take N^{n-1} values.

Note, also that the chiral Potts model has very deep relations with the theory of cyclic representations of quantum groups at roots of unity [24, 23]. In particular, the R-matrix of the model can be interpreted [15, 23] as an intertwiner for two minimal cyclic representations of the $U_q(\widehat{\mathfrak{sl}(n)})$ algebra.

Next, in [10] it was shown that the $\mathfrak{sl}(n)$ -generalized chiral Potts model can be interpreted as a model on a three-dimensional simple cubic lattice consisting of n square-lattice layers. At each site there is an N -valued spin. And when $N = 2$ this three-dimensional model reduces (to within a minor modification of the boundary conditions) to the Zamolodchikov model [44, 45]. For $N > 2$ it gives a new solvable interaction-round-a-cube model in three dimensions [10]. The Boltzmann weights of this model satisfy [5, 34] the tetrahedron relations which ensure its integrability. However, it turns that the integrability of the model can be proved bypassing the tetrahedron relations. It is based on

a much simpler “restricted star-triangle relation” [11], which is a particular case of the star-triangle relation [8, 3] for the original chiral Potts model.

In this paper we discuss yet another remarkable connection of the chiral Potts model. This time we relate it with the quantum discrete sine-Gordon (QDSG) model [29, 19, 25]. In [14] it was shown that for rational values of the coupling constant the QDSG-model can be viewed as some special two-dimensional lattice model of statistical mechanics where (integer) spins take $N \geq 2$ values and where the Boltzmann weights are determined by solutions of the equation of motion of the *classical* discrete sine-Gordon model (the situation here is very similar to quantum field theory in a classical “background” field). Here we show that in the simplest case of the constant background the above lattice model exactly coincides with the chiral Potts model.

The organization of the paper is as follows. In Section 2 we review the formulation and basic properties of the classical and quantum discrete sine-Gordon models. Associated solutions of the Yang–Baxter equation are considered in Section 3. In Section 4 we review the basic definitions of the chiral Potts model and Section 5 we relate this model to the QDSG model.

§2. The quantum discrete Sine-Gordon model

2.1. Formulation of the model

The Sine-Gordon equation for a scalar function $\phi(x, t)$ in $1 + 1$ dimensions

$$(2.1) \quad -\partial_t^2 \phi + \partial_x^2 \phi = m^2 \sin \phi$$

is the most famous example of a system integrable on both the classical and quantum levels. Despite of an extensive literature devoted to this model (see [46],[28] for the references), it still continues to reveal its new features. In this paper we shall consider integrable generalizations [29, 19, 25] of this model (classical and quantum) for the case when both the space and time variables are discrete.

Following [25] let us give a brief description of the discrete quantum Sine-Gordon model. The classical model can then be obtained in an appropriate limit. Let q be a complex number, and $\mathcal{A}_L(q)$ be an algebra of power series¹ in variables w_n , $n = 0, \dots, 2L - 1$, $L \geq 2$, obeying the

¹We shall assume these series to be semi-infinite, i.e, the powers of w to be bounded from below, but not necessarily non-negative. The multiplication of such series is well defined.

following relations

$$(2.2) \quad \begin{aligned} w_n w_{n-1} &= q^2 w_{n-1} w_n, & \forall n, \\ w_n w_m &= w_m w_n, & |m - n| \geq 2, \end{aligned}$$

with the periodicity condition $w_{n+2L} = w_n$. The variables w constitute the set of the dynamical variables of the model at any fixed value of time. The evolution on one step of the discrete time acts as an automorphism τ of the algebra $\mathcal{A}_L(q)$

$$(2.3) \quad \tau : \mathcal{A}_L(q) \longrightarrow \mathcal{A}_L(q),$$

such that

$$(2.4) \quad \begin{aligned} \hat{w}_{2n} &= \tau(w_{2n}) = f(qw_{2n-1}) w_{2n} f(qw_{2n+1})^{-1}, \\ \hat{w}_{2n-1} &= \tau(w_{2n-1}) = f(q\hat{w}_{2n-2}) w_{2n-1} f(q\hat{w}_{2n})^{-1}. \end{aligned}$$

Here and below we use the notation $f(x)^{-1} = 1/f(x)$. The function f has the form

$$(2.5) \quad f(x) = f(\kappa^2, x), \quad f(\lambda, x) = \frac{1 + \lambda x}{\lambda + x}$$

where κ^2 is a fixed parameter of the model. One can easily show that the variables \hat{w}_n satisfy the same relations (2.2) as w_n . To get a geometric picture of the equations (2.4) consider the square lattice drawn diagonally as in Fig. 1, with $L \geq 2$ sites per row and impose periodic boundary conditions in the horizontal (spatial) direction. The time axis is assumed to be directed upwards. Now assign the variables w_n , $n = 0, \dots, 2L - 1$, to the sites belonging to some horizontal ‘‘saw’’ \mathcal{S} as shown in Fig 1. In the same way assign the variables \hat{w}_n to the saw $\hat{\mathcal{S}}$ shifted by one time unit from the initial saw \mathcal{S} .

With such arrangement both evolution equations (2.4) connect four w 's around a single square face of the lattice and can be written in a universal form

$$(2.6) \quad \begin{aligned} w_U &= f(qw_L) w_D f(qw_R)^{-1}, \\ w_L w_D &= q^{-2} w_D w_L, & w_D w_R &= q^{-2} w_R w_D, \end{aligned}$$

where the variables w are labeled as in Fig. 2.

The algebra (2.2) has two Casimir elements

$$(2.7) \quad C_1 = \prod_{n=0}^{L-1} w_{2n+1}, \quad C_2 = \prod_{n=0}^{L-1} w_{2n}.$$

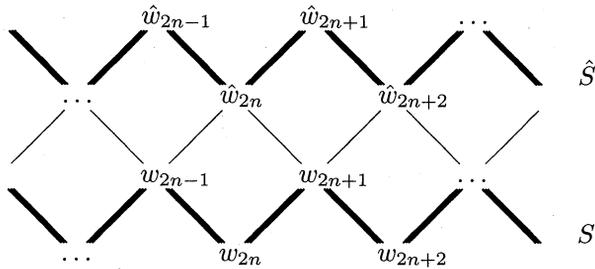


Fig. 1. Arrangement of the operators w_n and \hat{w}_n on the space-time lattice.

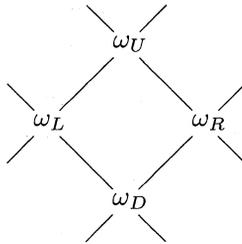


Fig. 2. Labeling of the four w 's around an elementary face. The indices stand for "up", "down", "left" and "right".

which commute with all w 's and are preserved by the evolution automorphism τ . Here we restrict ourselves to the case

$$(2.8) \quad C_1 = C_2 = 1.$$

This leaves $2L - 2$ independent generators, say, $w_0, w_1, \dots, w_{2L-3}$.

The quasiclassical limit of the model corresponds to the case $q \rightarrow 1$. The algebra (2.2) is then replaced by the Poisson algebra, where

$$(2.9) \quad \{w_n, w_{n-1}\} = 2w_n w_{n-1}, \quad \forall n,$$

but all other brackets $\{w_n, w_m\}$ vanish. The equation of motion (2.6) becomes

$$(2.10) \quad w_U = w_D f(w_L) / f(w_R).$$

One can show that in the appropriate continuous limit the equation (2.10) reduces to (2.1) when $\kappa^2 \rightarrow 0$. For the details see [25].

We will say that the evolution τ is Hamiltonian if there exists an invertible operator \mathbf{U} such that

$$(2.11) \quad \tau(a) = \mathbf{U}^{-1}a\mathbf{U}.$$

Let $r(\lambda, x)$, $\lambda \in \mathbb{C}$, be a solution of the following q -difference equation

$$(2.12) \quad r(\lambda, q^2x) = f(\lambda, qx)r(\lambda, x),$$

where the function f is given by (2.5). Define the operator \mathbf{U} as follows

$$(2.13) \quad \mathbf{U} = \prod_{k=0}^{L-1} r(\kappa^2, w_{2k}) \prod_{k=0}^{L-1} r(\kappa^2, w_{2k+1}).$$

Applying (2.11) for $a = w_n$ and using relation (2.12) one can easily see that (2.11) reduces then to the evolution equations (2.4). The calculations are rather trivial due to the locality of the commutation relations among w 's.

To proceed further we need to specify an actual realization of the Hilbert space of the model, supporting the commutation relations (2.2), where the equation (2.12) is well defined and has a unique solution. At the moment there is no classification of representations of this algebra suitable for this purpose. Only two cases are known so far. The first case is connected with non-compact modular representations of (2.2) and the Faddeev–Volkov model [29, 25] (see also [27, 35, 42, 21]). Recently [16, 17] this model was solved exactly and shown to be related to the discrete analog of the Riemann mapping theorem in two dimensions. For further details we refer the reader to the already mentioned papers [16, 17] and to [9].

In the present paper we consider the second known case when the equation (2.12) is well defined. It is when the parameter q is a root of unity. Originally this case was studied in [14].

2.2. The model at roots of unity

Here we are continuing the study of the discrete quantum Sine-Gordon model in the limit when q^2 approaches a root of unity. As it was shown in [14] the model in this case can be viewed as a (finite dimensional) integrable quantum system on top of an integrable classical one.

Let $(-q_0)$ be a primitive N -th root of -1 , then q_0^2 will always be a primitive N -th root of 1 ,

$$(2.14) \quad (-q_0)^N = -1, \quad q_0^{2N} = 1, \quad N \geq 1.$$

When $q = q_0$ the algebra $\mathcal{A}_L(q_0)$ has a (commutative) center $\mathcal{Z}(\mathcal{A}_L(q_0))$ generated by the elements w_n^N . It is easy to obtain the evolution equations for these commuting variables. Taking the N -th power of both sides of (2.6) for $q = q_0$ one obtains

$$(2.15) \quad w_U^N = w_D^N f(\kappa^{2N}, w_L^N) / f(\kappa^{2N}, w_R^N),$$

by using (2.14) and the simple identity

$$(2.16) \quad f(\kappa^{2N}, x^N) = \prod_{j=0}^{N-1} f(\kappa^2, q_0^{2j+1}),$$

where $f(\lambda, x)$ is defined in (2.5). Apart from the trivial replacement of κ^2 by κ^{2N} the N -th powers of w obey the same evolution equation as the one of the *classical* discrete Sine-Gordon model (2.10). In particular, when $N = 1$, $q = 1$, formula (2.15) exactly reduces to (2.10) as it, of course, should.

The quantum evolution of the model takes place in a finite dimensional factor of the algebra $\mathcal{A}_L(q_0)$ over its center $\mathcal{Z}(\mathcal{A}_L(q_0))$. In fact, let $\alpha = \{\alpha_0, \dots, \alpha_{2L-3}\}$ be a set of nonzero complex numbers and I_α be an ideal generated by $(w_n^N - \alpha_n)$, $n = 0, \dots, 2L-3$. Then the factor $\mathcal{A}_L(q_0)/I_\alpha$ for any α is isomorphic to a finite dimensional algebra generated by elements X_i, Z_i , $i = 1, \dots, L-1$ obeying the following relations

$$(2.17) \quad \begin{aligned} X_i Z_i &= q_0^2 Z_i X_i, & X_i^N &= Z_i^N = 1, \\ X_i Z_j &= Z_j X_i, & i &\neq j. \end{aligned}$$

The explicit isomorphism is achieved by the formulae

$$(2.18) \quad \begin{aligned} w_{2n} &= e^{P_{n+1}} X_{n+1}, & n &= 0, \dots, L-2, \\ w_{2n+1} &= e^{Q_{n+1} - Q_{n+2}} Z_{n+1}^{-1} Z_{n+2}, & n &= 0, \dots, L-3, \\ w_{2L-3} &= e^{P_{2L-3}} Z_{L-1}^{-1}, \end{aligned}$$

where for later convenience we have parametrized the α 's through a set of new variables P_i, Q_i , $i = 1, \dots, L-1$ as

$$(2.19) \quad \begin{aligned} \alpha_{2n} &= e^{P_{n+1}}, & n &= 0, \dots, L-2, \\ \alpha_{2n+1} &= e^{Q_{n+1} - Q_{n+2}}, & n &= 0, \dots, L-3, \\ \alpha_{2L-3} &= e^{P_{2L-3}}. \end{aligned}$$

The remaining two dependent generators w_{2L-2} and w_{2L-1} can be expressed from (2.18), (2.7) and (2.8).

One can introduce a Poisson algebra structure on $\mathcal{Z}(\mathcal{A}_L(q_0))$ with its Poisson action by derivations on the whole $\mathcal{A}_L(q_0)$. In fact, this structure naturally emerges from (2.2) in the limit $q \rightarrow q_0$. First, recall that the Poisson algebra P is by definition a commutative algebra with the structure of a Lie algebra, defined by a bracket $\{ , \}$ and the rule

$$(2.20) \quad \{a, bc\} = \{a, b\}c + b\{a, c\},$$

for any $a, b, c \in P$, which determines a distributive action of the bracket on the (commutative) product. Next, we can identify the algebras $\mathcal{A}_L(q)$ for all q as vector spaces of formal power series in generators by choosing some definite way of their ordering. For convenience set

$$(2.21) \quad q = e^h q_0$$

where h is a formal variable. Denoting by $a \cdot b$ the product of two elements in $\mathcal{A}_L(q)$ we can define the bracket

$$(2.22) \quad \{a, b\} = \lim_{h \rightarrow 0} \frac{a \cdot b - b \cdot a}{h}$$

when at least one of the elements a and b belongs to $\mathcal{Z}(\mathcal{A}_L(q_0))$. Using (2.2) it is easy to compute the Poisson brackets (2.22) between the generators w_n^N of $\mathcal{Z}(\mathcal{A}_L(q_0))$ and their action on w_n :

$$(2.23) \quad \begin{aligned} \{w_m^N, w_n^N\} &= 4N^2(\delta_{m+1,n} - \delta_{m,n+1})w_m^N w_n^N \\ \{w_m^N, w_n\} &= 2N(\delta_{m+1,n} - \delta_{m,n+1})w_m^N w_n \end{aligned}$$

where

$$(2.24) \quad \delta_{m,n} = \begin{cases} 1, & m = n \pmod{2L}, \\ 0, & \text{otherwise.} \end{cases}$$

The Poisson action of $\mathcal{Z}(\mathcal{A}_L(q_0))$ on the whole $\mathcal{A}_L(q_0)$ is extended from (2.23) by the same rule (2.20) where now $a \in \mathcal{Z}(\mathcal{A}_L(q_0))$ and $b, c \in \mathcal{A}_L(q_0)$ (of course, the multiplication there should now be assumed non-commutative).

For later use introduce the following notation

$$(2.25) \quad \exp(a) \circ b = \sum_{n \geq 0} \frac{1}{n!} \underbrace{\{a \cdots \{a, b\} \cdots\}}_n$$

for the Poisson action of $\exp(a)$ on b , where $a \in \mathcal{Z}(\mathcal{A}_L(\mathfrak{q}_0))$ and $b \in \mathcal{A}_L(\mathfrak{q}_0)$. It follows from (2.22) that

$$(2.26) \quad \exp\left(\frac{a}{h}\right) \cdot b \cdot \exp\left(-\frac{a}{h}\right) = \exp(a) \circ b + O(h).$$

If we use the parametrization (2.18) then (2.23) reduces to the only non-trivial brackets

$$(2.27) \quad \{Q_i, P_j\} = \delta_{ij}.$$

The standard quantization of the Poisson structure described above gives a realization of the algebra $\mathcal{A}_L(\mathfrak{q})$, $\mathfrak{q} = e^h \mathfrak{q}_0$, within a wider algebra of formal power series

$$(2.28) \quad \mathcal{X}(P, Q, X, Z, h) = (\text{series in } P_i, Q_i, X_i, Z_i, h)$$

in non-negative powers of $P_i, Q_i, X_i, Z_i, i = 0, \dots, L-1$ and h , where X_i, Z_i are defined in (2.17) and P_i, Q_i satisfy the relations

$$(2.29) \quad \begin{aligned} [Q_i, P_j] &= 2h\delta_{ij}, & [P_i, h] &= [Q_i, h] = 0, \\ [P_i, X_j] &= [P_i, Z_j] = [Q_i, X_j] = [Q_i, Z_j] = 0. \end{aligned}$$

The generators w_n are given by (2.18) where the elements P_i, Q_i now satisfy (2.29). This realization of $\mathcal{A}_L(\mathfrak{q})$ is most suitable for the study of the limit $\mathfrak{q} \rightarrow \mathfrak{q}_0$, $\mathfrak{q}_0^{2N} = 1$.

Note also that when $L = 2$ the algebra $\mathcal{A}_L(\mathfrak{q})$ has only two independent generators w_0 and w_1 which we denote as v and u respectively. From (2.2) we have

$$(2.30) \quad uv = \mathfrak{q}^2 vu$$

while the realization (2.18) contains only one pair of the P, Q and one pair of the X, Z operators

$$(2.31) \quad u = e^Q Z^{-1} \quad v = e^P X$$

where

$$(2.32) \quad XZ = \mathfrak{q}_0^2 ZX, \quad X^N = Z^N = 1,$$

$$(2.33) \quad [Q, P] = 2h \quad [P, h] = [Q, h] = 0,$$

where h and \mathfrak{q} are related by (2.21).

2.3. The integrability of the model

In this section we briefly address the matters related to the integrability of the discrete quantum Sine-Gordon model which so far have not been discussed. The details of the calculations can be found in [14].

Let W_q be the Weyl algebra generated by the invertible elements U and V ,

$$W_q : \quad UV = qVU ,$$

where $q \in \mathbb{C}$. Further, let $R(\lambda)$ be the R -matrix of the 6-vertex model acting in $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$(2.34) \quad \begin{aligned} R(\lambda) &= (\lambda q - \lambda^{-1} q^{-1})(e_{11} \otimes e_{11} + e_{22} \otimes e_{22}) \\ &+ (\lambda - \lambda^{-1})(e_{11} \otimes e_{22} + e_{22} \otimes e_{11}) \\ &+ (q - q^{-1})(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}) , \end{aligned}$$

where e_{ij} is the matrix unit and $L(\lambda)$ be an L -operator acting in $\mathbb{C}^2 \otimes W_q$

$$(2.35) \quad L(\lambda) = \begin{pmatrix} U & -\lambda V \\ \lambda V^{-1} & U^{-1} \end{pmatrix} .$$

These operators satisfy two Yang-Baxter equations

$$(2.36) \quad R_{12}(\lambda)R_{13}(\lambda\mu)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda\mu)R_{12}(\lambda)$$

and

$$(2.37) \quad R_{12}(\lambda)L_1(\lambda\mu)L_2(\mu) = L_2(\mu)L_1(\lambda\mu)R_{12}(\lambda)$$

which are written in the standard notations. As usual, define the transfer matrix

$$(2.38) \quad t(\lambda) = \text{trace}(L_0(\lambda\kappa)L_1(\lambda/\kappa)L_2(\lambda\kappa) \cdots L_{2L-1}(\lambda/\kappa)),$$

for a periodic chain of the length $2L$ with alternating rapidity variables $\lambda\kappa, \lambda/\kappa, \lambda\kappa$, etc., where κ is a constant. This transfer matrix acts in a direct product $W_q^{\otimes 2L}$ of $2L$ copies of the algebra W_q .

As a consequence of (2.37) the transfer matrices (2.5) with different λ form a commuting family. From the definitions (2.35) and (2.38) it is trivial to see that $t(\lambda)$ is a polynomial of the degree $2L$ in λ^2 . What is less trivial, the operators $U_i, V_i, i = 0, \dots, 2L - 1$, associated with different sites of the chain, enter coefficients of this polynomial only

trough $2L$ elements of the form (remind that we work with periodic boundary conditions)

$$(2.39) \quad w_n = U_n V_n^{-1} U_{n+1} V_{n+1},$$

which obey the commutation relations of the algebra (2.2)! The proof of this statement can be found in [14]. The next remarkable fact is that the transfer matrix (2.38) commutes with the evolution operator (2.11) and, therefore, it generates commuting integrals of motion of the discrete quantum Sine-Gordon model.

To see this let us search for another R -matrix acting in $W_q \otimes W_q$ which intertwines two L operators in the “quantum” space. In fact, this R -matrix has already appeared in Sect. 1. Assume that there exists a well defined element $r(\lambda, u) \in W_q \otimes W_q$, where

$$(2.40) \quad u = UV^{-1} \otimes UV,$$

and $r(\lambda, x)$ is the solution of the q -difference equation (2.12).

$$(2.41) \quad r(\lambda, q^2 x) = \frac{1 + qx}{\lambda + qx} r(\lambda, x).$$

Then, it is easy to show that

$$(2.42) \quad r(\lambda, u)(L(\lambda\mu) \otimes L(\mu)) = (L(\mu) \otimes L(\lambda\mu))r(\lambda, u).$$

With an account of (2.11), (2.39) and (2.40) this fact immediately implies the required commutativity of the evolution operator of (2.11) and the transfer matrix (2.38)

$$(2.43) \quad [t(\lambda), \mathbf{U}] = 0.$$

At this point we would like to note that so far we used only the defining relation (2.41) for the R -matrix $r(\lambda, x)$ rather than its explicit form. This concerns the Yang–Baxter equation (2.42) and the equivalence of the two forms (2.4) and (2.11) of the equations of motion in the model. An explicit form of $r(\lambda, x)$ for the case when q^2 is approaching a root of unity will be given in the next Section.

§3. The Yang–Baxter equation for $r(\lambda, x)$

3.1. General analysis

The reader may have noticed that the sequence of Yang–Baxter equations (2.36), (2.37), (2.42) in the previous Section which starts from the six-vertex model is very much resembling the one that appeared in

the analysis of the chiral Potts model in ref. [13]. In fact, our considerations here are quite similar but more general than in [13] and the relations with the Chiral Potts model will soon be very explicit.

The final point in the descending sequence of the Yang–Baxter equations mentioned above is the equation for R -matrix $r(\lambda, x)$ itself. The form of (2.42) requires that it should be written as the “braid relation”

$$(3.1) \quad r(\lambda, u)r(\lambda\mu, v)r(\mu, u) = r(\mu, v)r(\lambda\mu, u)r(\lambda, v),$$

where v and u denote respectively any two successive generators w_n and w_{n+1} from (2.40) which form a Weyl pair

$$(3.2) \quad uv = q^2vu.$$

Here $\lambda, \mu \in \mathbb{C}$ are the (multiplicative) rapidity variables. It might be tempting to assume that the two relations (2.41) and (3.2) are sufficient to imply the Yang–Baxter equation (3.1). As we shall see below this statement is not in general true. However, it would be quite interesting to understand how far one could advance in proving (3.1) using those two relation only.

Combining (2.41) and (3.2) one obtains

$$(3.3) \quad r(\lambda, u)v = v \frac{1 + q\lambda u}{\lambda + qu} r(\lambda, u), \quad r(\lambda, v)u = u \frac{q\lambda + v}{q + \lambda v} r(\lambda, v).$$

Denote the left and right hand sides of the Yang–Baxter equation (3.1) as Φ_L and Φ_R respectively. Then using the relations (3.2), (3.3) only one can show that for both $\Phi = \Phi_L$ or $\Phi = \Phi_R$

$$(3.4) \quad \Phi u = ug(u, v, q)\Phi, \quad \Phi v = v\Phi(g(v, u, q^{-1}))^{-1},$$

where

$$(3.5) \quad g(u, v, q) = (q\lambda^2\mu + q^2\lambda\mu u + v + q\lambda v u)(q\lambda + q^2u + \lambda\mu v + q^2\lambda^2\mu v u)^{-1}.$$

The calculations are elementary but a bit tedious. We present them in Appendix A. Fortunately, we have to derive only one of the equations in (3.4) since they follow from one another under the action of the automorphism

$$(3.6) \quad r(\lambda, x) \longrightarrow (r(\lambda, x))^{-1}, \quad u \longleftrightarrow v, \quad q \longrightarrow q^{-1},$$

which leaves (3.2), (3.3) unchanged. There are also additional “exchange” relations which follow from (3.4) and the fact that $r(\lambda^{-1}, x)^{-1}$ satisfies

the same q -difference equation (2.41) as $r(\lambda, x)$

$$(3.7) \quad \Phi u = u \Phi(\bar{g}(u, v, q))^{-1}, \quad \Phi v = v \bar{g}(v, u, q^{-1})\Phi.$$

As above, these relations are valid for both $\Phi = \Phi_L$ or $\Phi = \Phi_R$. The function $\bar{g}(u, v, q)$ is given by the RHS of (3.5) with λ and μ replaced by μ^{-1} and λ^{-1} respectively. The relations (3.4) and (3.7) immediately imply that

$$(3.8) \quad [\Psi, u] = [\Psi, v] = 0,$$

where

$$(3.9) \quad \Psi = \Phi_L \Phi_R^{-1}.$$

Apparently, this is the most general consequence for the Yang–Baxter equation (3.1) which one can obtain from (3.2),(3.3) without further requirements on the properties of the $r(\lambda, x)$. Therefore, if the representation of (2.2) were such that the commutativity with the elements u and v in (3.8) implies that the quantity Ψ is proportional to the unit operator, then Yang–Baxter equation(3.1) would be satisfied and, at the same time, equation (2.12) would have a unique solution.

Let $q = e^h q_0$, where h is a formal variable and q_0 is defined in (2.14). The most appropriate way to study (2.12) when q^2 approaches a root of unity, $h \rightarrow 0$, is to consider $r(\lambda, x)$ and, hence, the left and right hand sides of equation (3.1) as asymptotic expansions in h . Define the Euler dilogarithm function as

$$(3.10) \quad Li_2(x) = - \int_0^x \frac{\log(1-t)}{t} dt.$$

The Yang–Baxter equation (3.1) involves only one Weyl pair (3.2) which we parametrize as in (2.31). Then it is convenient to rewrite (2.41) as

$$(3.11) \quad r(\lambda, e^{T+h} q_0 z) = \frac{1 + \lambda x}{\lambda + x} r(\lambda, e^{T-h} z/q_0).$$

Here we set $x = e^T z$, where z is a formal variable such that $z^N = 1$. Next, impose the following normalization condition on $r(\lambda, x)$

$$(3.12) \quad r(1, x) = 1.$$

Then equation (2.41) uniquely determine the asymptotics expansion of $r(\lambda, x)$ in the form

$$(3.13) \quad r(\lambda, e^T z) = \exp \{ c_{-1}(T, z)h^{-1} + c_0(T, z) + c_1(T, z)h + \dots \},$$

where the coefficients $c_k(T, z)$ are series in T and z . In principle, all these coefficients can be calculated explicitly from (3.11). In particular, retaining only the first two terms in (3.13) one obtains

$$(3.14) \quad r(\lambda, x) = \exp\left(\frac{H(\lambda^N, x^N)}{N^2 h}\right) \bar{r}(\lambda, x)(1 + O(h))$$

where $x = e^T z$,

$$(3.15) \quad H(a, b) = -\frac{1}{2} \{Li_2(-ab) + Li_2(-a/b) + \log^2 b + \pi^2/6\}$$

and

$$(3.16) \quad \bar{r}(\lambda, x) = \left(\frac{\lambda^N + x^N}{1 + \lambda^N x^N}\right)^{(N-1)/2N} \prod_{j=0}^{N-1} \left(\frac{1 + \lambda x q_0^{2j+1}}{\lambda + x q_0^{2j+1}}\right)^{j/N}$$

The branches of the function $Li_2(x)$ are assumed to be chosen such that $H(1, x) = 0$, so that the normalization condition (3.12) is satisfied. The expansion (3.14) can be easily deduced from formula (3.11) of ref. [12].

3.2. The case $q \rightarrow 1$

First, consider the case $q \rightarrow 1$. This corresponds to $N = 1, q_0 = 1, q = e^h$ in our previous notations. Eq.(2.31) now reads

$$(3.17) \quad u = e^Q, \quad v = e^P.$$

It follows from (3.13) that both elements $r(\lambda, u)$ and $r(\lambda, v)$ have the form of exponentials

$$(3.18) \quad \exp(\mathcal{X}/h),$$

where \mathcal{X} is a series

$$(3.19) \quad \mathcal{X} = \sum_{k, \ell, m=0}^{\infty} c_{k\ell m} h^k Q^\ell P^m,$$

in non-negative powers of P, Q and h . Of course, for $r(\lambda, u)$ and $r(\lambda, v)$ the above series do not contain mixed terms (they involve only pure powers of P or Q). However, below we will consider more general series (3.19). We will always assume them to be “normally ordered” such that all P 's appear to the right from Q 's. The product of such two exponentials is again an exponential of the same form. Indeed, from the Campbell–Hausdorff formula we have

$$(3.20) \quad e^{\mathcal{X}/h} e^{\mathcal{Y}/h} = e^{\mathcal{Z}/h},$$

$$(3.21) \quad \mathcal{Z} = \mathcal{X} + \mathcal{Y} + \frac{1}{2h}[\mathcal{X}, \mathcal{Y}] + \frac{1}{12h^2} \{[\mathcal{X}, [\mathcal{X}, \mathcal{Y}]] + [\mathcal{Y}, [\mathcal{Y}, \mathcal{X}]]\} + \dots$$

where for each term of the last series the number of commutators coincides with the power of the parameter h in the denominator. If both \mathcal{X} and \mathcal{Y} are of the form (3.19), then due to (2.33) each commutator produces one extra power of h in the numerator so that all negative powers of h in (3.21) exactly cancel out. Therefore \mathcal{Z} is a series of the form (3.19) as well. It should be noted that each coefficient $c_{k\ell m}$ therein will now be an infinite numerical series in the coefficients of \mathcal{X} and \mathcal{Y} . For the moment let us assume that all these numerical series converge and, hence, \mathcal{Z} is well defined. Repeatedly applying (3.20) one concludes that the ratio Ψ of the left and right hand sides of the Yang–Baxter equation in (3.9) is also an exponential of the form (3.18). Then it is easy to show that the most general form of Ψ satisfying (3.8), (3.17) is

$$(3.22) \quad \Psi = \text{const} \exp \{ i \pi (mP + nQ) / h \},$$

where m and n are arbitrary integer constants. Finally setting the spectral parameters $\lambda = \mu = 1$ in (3.1) and using (3.12) one concludes that $\Psi \equiv 1$. Since we worked with exponentials of asymptotic series in h this implies that the Yang–Baxter equation (3.1) is satisfied as an equality of such exponentials in any given order in h .

It should be noted that the series in h in the exponents of (3.13) and (3.20) can be consistently truncated at any level. In particular, setting $N = 1$ and taking into account only the two leading terms in (3.13) one obtains the following Yang–Baxter equation

$$(3.23) \quad r(\lambda, u)r(\lambda\mu, v)r(\mu, u) = r(\mu, v)r(\lambda\mu, u)r(\lambda, v)(1 + O(h)),$$

where

$$(3.24) \quad r(\lambda, x) = e^{H(\lambda, x)/h}(1 + O(h)).$$

Writing (3.23) in full and substituting (3.17) one obtains

$$(3.25) \quad \begin{aligned} & \exp(H(\lambda, e^Q)/h) \exp(H(\lambda\mu, e^P)/h) \exp(H(\mu, e^Q)/h) \\ & = \exp(H(\mu, e^P)/h) \exp(H(\lambda\mu, e^Q)/h) \exp(H(\lambda, e^P)/h)(1 + O(h)). \end{aligned}$$

Introducing a special notation for the leading term of the Campbell–Hausdorff composition in (3.20)

$$(3.26) \quad \exp\left(\frac{\mathcal{X}}{h}\right) \exp\left(\frac{\mathcal{Y}}{h}\right) = O(1) \exp\left(\frac{\mathcal{X} * \mathcal{Y}}{h}\right), \quad h \rightarrow 0,$$

we can rewrite the equality of the singular-in- \hbar exponents in (3.23) as

$$(3.27) \quad H(\lambda, u) * H(\lambda\mu, v) * H(\mu, v) = H(\mu, v) * H(\lambda\mu, u) * H(\lambda, u).$$

Below we show that this formula is equivalent to some identity for the dilogarithm function (3.10).

To complete the proof of the Yang–Baxter equation we have to ensure that the abovementioned numerical series involved in the product of the R -matrices converge. Apparently this can be done “order by order” in perturbations in \hbar using a special structure of the Campbell–Hausdorff series (3.21) and the fact that the numerical coefficients there rapidly decrease (see, e.g., [20], chapter 2, §6.4). We will not analyze this problem in its general setting here, but restrict our considerations to the truncated equation (3.23). In this case we are able not only to show that products of the R -matrices in (3.23) exist but also explicitly calculate normal symbols of the two sides of (3.23) up to the order of $O(\hbar^0)$. The details of the calculations are presented below.

3.3. The 12-term dilogarithm identity

Let A be an operator given as a power series in \hat{P} and \hat{Q} satisfying²

$$(3.28) \quad A = \sum_{k,l=0}^{\infty} a_{kl} \hat{Q}^k \hat{P}^l$$

ordered such that all \hat{P} 's are placed on the right from \hat{Q} 's. We will call the series ordered in this way as normal form of the operator A . As usual the normal symbol $: A(Q, P) :$ of the operator A is defined by the same series (3.28) as its normal form but with \hat{P} and \hat{Q} replaced by commuting variables P and Q respectively.

To calculate the normal symbol of the product of two different operators one has to use a simple identity for the normal symbol of an elementary (unordered) monomial

$$(3.29) \quad A_{kl} = \hat{P}^k \hat{Q}^l$$

$$(3.30) \quad : A_{kl}(Q, P) := (-2\hbar)^{k+l} \exp(PQ/2\hbar) \frac{\partial^k}{\partial Q^k} \frac{\partial^l}{\partial P^l} \exp(-PQ/2\hbar)$$

²In this subsection we will denote the operators P and Q from (2.33) as \hat{P} and \hat{Q} and use the letters P and Q for commuting arguments of the normal symbols. We hope this will not cause any confusion.

which can be easily proved from (2.33) by induction. Following [18] we can write the RHS of (3.30) in an integral form

$$(3.31) \quad (3.30) = \int \frac{dP' dQ'}{2\pi h} \exp((Q - Q')(P - P')/2h) P'^k Q'^l$$

assuming h to be negative real and considering P' and Q' as complex conjugated variables $P' = (Q')^* \in \mathbb{C}$. The integral is assumed to be over the whole complex plane. Then the normal symbol of the product of two operators A and B can be written in a compact way

$$(3.32) \quad : AB(Q, P) : \\ = \int \frac{dQ' dP'}{2\pi h} \exp((Q - Q')(P - P')/2h) : A(Q, P') :: B(Q', P) : .$$

Applying this for the left hand side of (3.23) one obtains

$$(3.33) \quad : \Phi_L(Q, P) : \\ = r(\lambda, e^Q) \int e^{(Q-Q')(P-P')/2h} r(\lambda\mu, e^{P'}) r(\mu, e^{Q'}) \frac{dQ' dP'}{2\pi h} .$$

This integral can be explicitly calculated to within terms of the order of $O(h^0)$ using the saddle point approximation. A similar treatment of the quantum five-term identity can be found in [26]. Substituting (3.24) into (3.33) one obtains

$$(3.34) \quad : \Phi_L(Q, P) := (\det \mathbf{M}(x', y'))^{-1/2} \exp(E_L/2h)(1 + O(h)) , \\ E_L = H(\lambda, x) + H(\lambda\mu, y') + H(\mu, x') + 2 \log(x/x') \log(y/y')$$

where

$$(3.35) \quad x = e^Q, \quad y = e^P, \quad x' = e^{Q'}, \quad y' = e^{P'},$$

and the coordinates P' , Q' of the saddle point are determined by the equations

$$(3.36) \quad x = \frac{x'(1 + \lambda\mu y')}{\lambda\mu + y'}, \quad y = \frac{y'(1 + \mu x')}{\mu + x'} .$$

The two by two matrix \mathbf{M} coincides (up two a factor of $2h$) with the matrix of the quadratic form in the exponent of the integrand near the saddle point

$$(3.37) \quad \mathbf{M}(x', y') = \begin{pmatrix} \frac{x'(\mu^2 - 1)}{(x' + \mu)(1 + \mu x')} & 1 \\ 1 & \frac{y'(\lambda^2 \mu^2 - 1)}{(1 + \lambda\mu y')(y' + \lambda\mu)} \end{pmatrix} .$$

The equations (3.36) determine two distinct saddle points. Instead of solving these equations with respect to x' and y' , it is more convenient to regard x' and y' as independent variable and consider (3.36) as a definition of x and y . In the same way one can calculate the normal symbol of the RHS of (3.23). Then for any x' and y' the corresponding saddle point for $:\Phi_R(Q, P):$ is given by

$$(3.38) \quad y'' = \frac{x'(\lambda\mu x'y' + x' + \lambda y' + \mu)}{\mu x'y' + \lambda x' + y' + \lambda\mu}, \quad x'' = \frac{y'(\lambda\mu x'y' + x' + \lambda y' + \lambda^2\mu)}{\lambda^2\mu x'y' + \lambda x' + y' + \lambda\mu},$$

where

$$(3.39) \quad x'' = e^{P''}, \quad y'' = e^{Q''},$$

and P'' and Q'' are integration variables in the corresponding integral for $:\Phi_R(Q, P):$. The expression for $:\Phi_R(Q, P):$ is obtained from (3.34) merely by replacing x' and y' therein with x'' and y'' respectively. Equating the singular-in- \hbar exponents of the normal symbols we thus obtain the following identity.

The twelve-term dilogarithm identity. Let λ, μ, x' and y' be arbitrary complex numbers and x, y, x'', y'' are given by (3.36), (3.38). Then

$$(3.40) \quad \begin{aligned} & H(\lambda, x) + H(\lambda\mu, y') + H(\mu, x') + 2\log(x/x')\log(y/y') \\ &= H(\mu, x'') + H(\lambda\mu, y'') + H(\lambda, y) + 2\log(x/y'')\log(y/y''), \end{aligned}$$

where the function H is defined by (3.15). The branches of the dilogarithms and logarithms are to be chosen as analytic continuation from the case when λ, μ, x' and y' are all real and positive.

Since we have shown (by explicit calculations) that products of the R -matrices in the Yang–Baxter equation (3.23) do exist and are well defined (to within the required order in \hbar) the identity (3.40) is a corollary of our proof of the Yang–Baxter equation in the previous subsection. However, is very easy to check directly the dilogarithm identity of this type once it is written. First, computing partial derivatives of the first order in the variables λ, μ, x' or y' one can check that the difference between the LHS and RHS of (3.40) considered as a function of these four independent variables is a constant. To check that this constant is actually equal to zero one has to set $x' = y' = 1$. The four variable dilogarithm identity (3.40) seems to be new, but, of course, can be obtained by a repeated application of the classical five term identity for

the dilogarithm function (3.10) (for the most comprehensive survey on this subject see [36]).

Note that (3.36), (3.38) imply

$$(3.41) \quad x = \frac{y''(1 + \mu x'')}{\mu + x''}, \quad y = \frac{x''(1 + \lambda \mu y'')}{\lambda \mu + y''}.$$

Regarding λ and μ as constants and x'' , y'' as independent variables we can write the RHS of (3.40) as some function $\mathcal{F}_0(x'', y'')$. Then the identity (3.40) can be interpreted as an invariance of this function under the (bi)rational substitution (3.38) of two variables

$$(3.42) \quad \mathcal{F}_0(x'', y'') = \mathcal{F}_0(x', y').$$

Note that the substitution (3.38) is an involution, i.e, it reduces to identity substitution on the second iteration.

Similarly, the order of $O(h^0)$ in (3.23) (which is equivalent to the equality of the determinants in (3.34) and in the corresponding expression for $:\Phi_R(Q, P):$) gives a polynomial invariant

$$(3.43) \quad \mathcal{F}_1(x'', y'') = (1 + \mu x'')(1 + \lambda \mu y'')$$

of the substitution (3.38)

$$(3.44) \quad \mathcal{F}_1(x'', y'') = \mathcal{F}_1(x', y').$$

Of course, the last formula can be readily checked by direct calculations.

3.4. The case of a generic root of unity

Now let q_0 be a primitive N -th root of -1 as defined in (2.14) and

$$(3.45) \quad q = e^h q_0.$$

The generators u and v are now given by the general formulae (2.31–2.33) and the leading asymptotics of the R -matrix by (3.14). Our previous proof of the truncated Yang–Baxter equation (3.23) can be easily modified for this case. One has to generalize the series in (3.19) to series in P , Q , X , Z and h (of course the powers of X and Z will not exceed $N - 1$). The subsequent arguments are very similar to those of the case $q \rightarrow 1$ and we will not present them here.

Let us rather consider consequences of the Yang–Baxter equation (3.23) in this case. The R -matrix (3.14) factorizes into a “classical” and a “quantum” part. The former is the singular-in- h exponential belonging to the center of the algebra $\mathcal{Z}(\mathcal{A}_2(q_0))$ while the latter belongs to the finite dimensional factor of this algebra over its center, which is

generated only by the elements X and Z obeying (2.32). It turns out that the Yang–Baxter equation (3.23) remarkably splits into two separate (quantum) Yang–Baxter equations for the classical and the quantum parts of the R -matrix. The trick is similar to that used in [12] for the asymptotics of the quantum five-term identity of ref. [26]. The equation for the classical parts is equivalent to (3.25) while the one for the quantum parts appears to be equivalent (as we shall see in the following sections) to the star-triangle relation of the N -state chiral Potts model.

Substitute (3.14) into (3.23) and move all singular-in- \hbar exponentials to the right. To do this one has to use formula (2.26), since these exponentials in general do not commute with the elements \bar{r} . For instance, for the LHS of (3.23) one obtains

(3.46)

$$\begin{aligned}
 & e^{\tilde{H}(\lambda, e^Q)/\hbar} \bar{r}(\lambda, e^Q Z^{-1}) e^{\tilde{H}(\lambda\mu, e^P)/\hbar} \bar{r}(\lambda\mu, e^P X) \\
 & \qquad \qquad \qquad \times e^{\tilde{H}(\mu, e^Q)/\hbar} \bar{r}(\mu, e^Q Z^{-1}) \\
 & = \bar{r}(\lambda, e^Q Z^{-1}) \left(e^{\tilde{H}(\lambda, e^Q)} \circ \bar{r}(\lambda\mu, e^P X) \right) \\
 & \quad \times \left(e^{\tilde{H}(\lambda\mu, e^P)} \circ e^{\tilde{H}(\mu, e^Q)} \circ \bar{r}(\mu, e^Q Z^{-1}) \right) \\
 & \qquad \qquad \times e^{\tilde{H}(\lambda, e^Q)/\hbar} e^{\tilde{H}(\lambda\mu, e^P)/\hbar} e^{\tilde{H}(\mu, e^Q)/\hbar} (1 + O(\hbar)),
 \end{aligned}$$

where

$$(3.47) \qquad \tilde{H}(a, b) = N^{-2} H(a^N, b^N)$$

with the function H given by (3.15). Perform similar transformations for the RHS of (3.23). After that all singular exponentials in the LHS cancel with those in the RHS side due to (3.25) (more precisely, one has to use (3.25) with the function H replaced by a function \tilde{H} , defined in (3.47)). The remaining equation is that for the quantities of the order of $O(\hbar^0)$ so that one can now set $\hbar = 0$. The operators P and Q then become commuting variables. More precisely, they become elements of the Poisson algebra with the bracket

$$(3.48) \qquad \{Q, P\} = 2h$$

as it was explained in Section 2.2. As a result we obtain the following “twisted” Yang–Baxter equation for the elements (3.16)

$$(3.49) \quad \begin{aligned} & \bar{r}(\lambda, e^Q Z^{-1}) \bar{r}(\lambda \mu, e^{P'} X) \bar{r}(\mu, e^{Q''} Z^{-1}) \\ & = \bar{r}(\mu, e^P X) \bar{r}(\lambda \mu, e^{Q'} Z^{-1}) \bar{r}(\lambda, e^{P''} X), \end{aligned}$$

where we have used the notation (2.25). Using (3.48) one can show that

$$(3.50) \quad \begin{aligned} e^{NP'} &= \frac{1 + \lambda^N e^{NQ}}{\lambda^N + e^{NQ}} e^{NP}, & e^{NP''} &= \frac{1 + \lambda^N \mu^N e^{NQ'}}{\lambda^N \mu^N + e^{NQ'}} e^{NP}, \\ e^{NQ'} &= \frac{\mu^N + e^{NP}}{1 + \mu^N e^{NP}} e^{NQ}, & e^{NQ''} &= \frac{\lambda^N \mu^N + e^{NP'}}{1 + \lambda^N \mu^N e^{NP'}} e^{NQ}. \end{aligned}$$

The Yang–Baxter equation (3.49) contains four (complex) parameters: two rapidities λ and μ and two arbitrary parameters P and Q . The connection of this equation with the star-triangle relation will be considered in Section 5.1. Note here two important properties of the R -matrix (3.16). Introducing

$$(3.51) \quad \omega = 1/q_0^2, \quad \omega^{1/2} = -1/q_0,$$

where q_0 is defined in (2.14) one can show that

$$(3.52) \quad \frac{\bar{r}(\lambda, \omega^n x)}{\bar{r}_l(x, z)} = \left(\frac{1 + \lambda^N x^N}{1 + \lambda^{-N} x^N} \right)^{\frac{n}{N}} \prod_{j=1}^n \frac{1 - \omega^{-1/2} \lambda^{-1} x \omega^j}{1 - \omega^{-1/2} \lambda x \omega^j}$$

and

$$(3.53) \quad \prod_{j=0}^{N-1} \bar{r}(\lambda, \omega^j x) = 1.$$

§4. Chiral Potts model

We define the chiral Potts model in the usual way [8], [7]. Consider the square lattice \mathcal{L} , drawn diagonally as in Fig. 3, with L sites per row. At each site i there is a spin σ_i , which takes values $0, \dots, N - 1$. There is an associated lattice \mathcal{L}' denoted by dotted lines, such that each edge of \mathcal{L} passes through a vertex of \mathcal{L}' .

To each vertical (horizontal) line on the “dotted” lattice \mathcal{L}' assign a rapidity variable p (q). In general they may be different for different lines. In fact a convenient level of generality that we shall use here is to

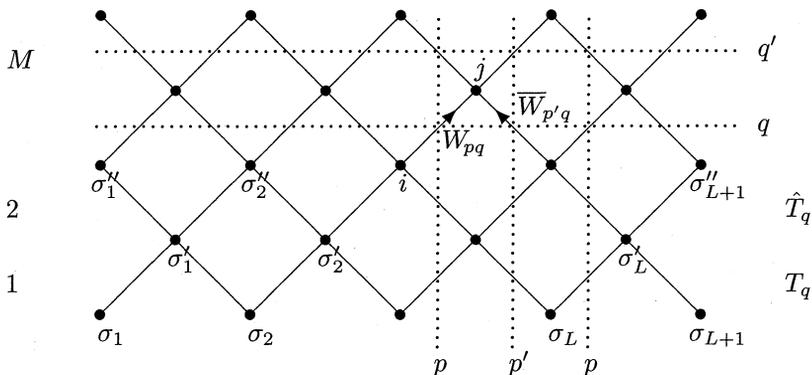


Fig. 3. The square lattice \mathcal{L} of M rows with L sites per row. T_q is the transfer matrix of an odd row, \hat{T}_q of an even row. Three vertical and two horizontal dotted rapidity lines are shown.

allow the vertical rapidities to be alternating p, p', p, p', \dots , as indicated³. Each edge of \mathcal{L} is assigned a Boltzmann weight depending on the two spins adjacent to the edge and on the two rapidities passing through the edge. For example, consider a typical SW \rightarrow NE edge (i, j) of \mathcal{L} (with j above i). The spins σ_i, σ_j interact with Boltzmann weight $W_{pq}(\sigma_i - \sigma_j)$ (or $W_{p'q}(\sigma_i - \sigma_j)$). Similarly, on all SE \rightarrow NW edges the spins interact with Boltzmann weight $\overline{W}_{pq}(\sigma_i - \sigma_j)$ (or $\overline{W}_{p'q}(\sigma_i - \sigma_j)$), where again j is above i . The explicit form of the Boltzmann weights will be given below.

Each rapidity variable p (or q) is represented by a four-vector $p = (a_p, b_p, c_p, d_p)$ which specify a point on the algebraic curve \mathcal{C}_k defined by any two of the following four equations (the complimentary pair of equations follows from the other two)

$$(4.1) \quad \begin{aligned} a_p^N + k'b_p^N &= kd_p^N, & k'a_p^N + b_p^N &= kc_p^N, \\ ka_p^N + k'c_p^N &= d_p^N, & kb_p^N + k'd_p^N &= c_p^N, \end{aligned}$$

³Here and below the letter q denotes a rapidity variable; it is different from the parameter q in (2.2).

where $k^2 + k'^2 = 1$. The modulus of the curve, k , is considered as a fixed parameter of the model. It is convenient to also use another set of the “ p -variables”,

$$(4.2) \quad x_p = a_p/d_p, \quad y_p = b_p/c_p, \quad s_p = d_p/c_p$$

and

$$(4.3) \quad t_p = x_p y_p = a_p b_p / c_p d_p.$$

In these variables the curve (4.1) reads

$$(4.4) \quad x_p^N + y_p^N = k(1 + x_p^N y_p^N), \quad kx_p^N = 1 - k's_p^{-N}, \quad ky_p^N = 1 - k's_p^N.$$

With these definitions the Boltzmann weights have the form

$$(4.5) \quad \frac{\overline{W}_{pq}(n)}{\overline{W}_{pq}(0)} = \prod_{j=1}^n \frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j} = (s_p s_q)^n \prod_{j=1}^n \frac{\omega x_p - \omega^j x_q}{y_q - \omega^j y_p},$$

$$\frac{W_{pq}(n)}{W_{pq}(0)} = \prod_{j=1}^n \frac{d_p b_q - a_p c_q \omega^j}{b_p d_q - c_p a_q \omega^j} = (s_p / s_q)^n \prod_{j=1}^n \frac{y_q - \omega^j x_p}{y_p - \omega^j x_q}$$

where $n \in \mathbb{Z}$ and ω is a primitive root of unity of degree N . The weights satisfy the periodicity conditions $W_{pq}(n + N) = W_{pq}(n)$, $\overline{W}_{pq}(n + N) = \overline{W}_{pq}(n)$. They also satisfy the star-triangle relation [8], [3], [38]:

$$(4.6) \quad \sum_{d=0}^{N-1} \overline{W}_{qr}(b-d) W_{pr}(a-d) \overline{W}_{pq}(d-c) = R_{pqr} W_{pq}(a-b) \overline{W}_{pr}(b-c) W_{qr}(a-c),$$

for all rapidities p, q, r and all integers (spins) a, b, c . Here R_{pqr} is a spin-independent function, defined by

$$(4.7) \quad R_{pqr} = f_{pq} f_{qr} / f_{pr},$$

where

$$(4.8) \quad f_{pq}^N = \prod_{j=0}^{N-1} \left(\overline{W}_{pq}^{(f)}(j) / W_{pq}(j) \right)$$

and

$$(4.9) \quad \overline{W}_{pq}^{(f)}(n) = \sum_{a=0}^{l-1} \overline{W}_{pq}(a) \omega^{na}.$$

It follows from (4.5) that

$$(4.10) \quad \frac{\overline{W}_{pq}^{(f)}(n)}{\overline{W}_{pq}^{(f)}(0)} = \prod_{j=1}^n \frac{c_p b_q - a_p d_q \omega^j}{b_p c_q - d_p a_q \omega^j} = \prod_{j=1}^n \frac{y_q - \omega^j x_q s_p s_q}{y_p - \omega^j x_q s_p s_q}.$$

Note also, that the normalization factor $\overline{W}_{pq}^{(f)}(0)$ can also be written in a product form by using the identity (2.44) of [7], namely

$$(4.11) \quad \prod_{j=0}^{N-1} \overline{W}_{pq}^{(f)}(j) = \overline{W}_{pq}(0)^N N^{N/2} e^{i\pi(N-1)(N-2)/12} \prod_{j=1}^{N-1} \frac{(t_p - \omega^j t_q)^j}{(x_p - \omega^j x_q)^j (y_p - \omega^j y_q)^j}.$$

We define row-to-row transfer matrices T and \hat{T} as in [7]. Let $\sigma = \sigma_1, \dots, \sigma_L$ be the spins in the lower row of Figure 3. Similarly, let $\sigma' = \sigma'_1, \dots, \sigma'_L$ be the spins in the next row, and $\sigma'' = \sigma''_1, \dots, \sigma''_L$ those in the row above that. Let T be the N^L by N^L matrix with elements

$$(4.12) \quad T_{\sigma\sigma'} = \prod_{J=1}^L W_{pq}(\sigma_J - \sigma'_J) \overline{W}_{p'q}(\sigma_{J+1} - \sigma'_J),$$

similarly, let \hat{T} be the N^L by N^L matrix with elements

$$(4.13) \quad \hat{T}_{\sigma'\sigma''} = \prod_{J=1}^L \overline{W}_{pq}(\sigma'_J - \sigma''_J) W_{p'q}(\sigma'_J - \sigma''_{J+1}).$$

Let Y be a formal variable such that $Y^N = 1$. Define

$$(4.14) \quad \begin{aligned} F(p, q; Y) &= N^{-1} \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} \omega^{-ab} W_{pq}(a) Y^b = \sum_{a=0}^{N-1} W_{pq}^{(f)}(-a) Y^a \\ \overline{F}(p, q; Y) &= N^{-1} \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} \omega^{ab} \overline{W}_{pq}^{(f)}(a) Y^b = \sum_{a=0}^{N-1} \overline{W}_{pq}(a) Y^a \end{aligned}$$

where $W_{pq}^{(f)}(n)$ is defined similarly to (4.9). It is easy to see that

$$(4.15) \quad F(p, q; \omega^n) = W_{pq}(n), \quad \overline{F}(p, q; \omega^n) = \overline{W}_{pq}^{(f)}(n)$$

for any $n \in \mathbb{Z}$ and that the functions F and \bar{F} are uniquely determined by the following recurrence relations

$$(4.16) \quad \frac{F(p, q; Y)}{F(p, q; \omega^{-1}Y)} = \frac{s_p(y_q - x_p Y)}{s_q(y_p - x_q Y)}, \quad F(p, q; 1) = W_{pq}(0)$$

$$(4.17) \quad \frac{\bar{F}(p, q; Y)}{\bar{F}(p, q; \omega^{-1}Y)} = \frac{y_q - x_q s_p s_q Y}{y_p - x_p s_p s_q Y}, \quad \bar{F}(p, q; 1) = \bar{W}_{pq}^{(f)}(0).$$

Now choose the following matrix realization of the algebra (2.32) with $q_0^2 = \omega^{-1}$, $\omega^N = 1$,

$$(4.18) \quad (X)_{a,b} = \delta_{a,b+1}, \quad (Z)_{a,b} = \omega^a \delta_{a,b},$$

where $a, b \in \mathbb{Z}_N$ and

$$(4.19) \quad \delta_{a,b} = \begin{cases} 1, & a = b \pmod{N}, \\ 0, & \text{otherwise.} \end{cases}$$

From (4.15) and (4.14) it follows that

$$(4.20) \quad (F(p, q; Z))_{a,b} = \delta_{a,b} W_{pq}(a), \quad (\bar{F}(p, q; X))_{a,b} = \sum_{k=0}^{N-1} \bar{W}_{pq}(k) \delta_{a,b+k}.$$

Using these one can equivalently rewrite [38] the star-triangle relation (4.6) as the following relation between F and \bar{F}

$$(4.21) \quad F(p, q; Z^{-1}) \bar{F}(p, r; X) F(q, r; Z^{-1}) = R_{pqr}^{-1} \bar{F}(q, r; X) F(p, r; Z^{-1}) \bar{F}(p, q; X).$$

§5. The connection of the discrete quantum Sine-Gordon and chiral Potts models

5.1. The star-triangle relation

We now want to identify equation (3.49) with the star-triangle relation in the form (4.21). Equation (3.49) contains four continuous parameters λ, μ, P, Q , while (4.21) involves the modulus of the curve (4.1) and three rapidities p, q, r , representing points on that curve. We have to establish a relationship between these two sets of parameters. Denote

$$(5.1) \quad \begin{aligned} A &= 1 + (e^Q/\lambda)^N, & B &= 1 + (e^Q \lambda)^N \\ C &= 1 + (e^P/\mu)^N, & D &= 1 + (e^P \mu)^N \end{aligned}$$

and consider the curve (4.1) with the modulus

$$(5.2) \quad k^2 = \frac{[AC(1-B) - BD(1-C)][BC(1-A)(1-D) - AB]}{CD[A - B(1-D)][B - A - BC(1-A)]}.$$

Now choose three rapidities variables p , q and r such that

$$(5.3) \quad \begin{aligned} \omega^{-1/2} \lambda^{-1} e^Q &= x_p / y_q, & \omega^{-1/2} \lambda e^Q &= x_q / y_p, \\ \omega^{-1/2} \mu^{-1} e^P &= x_q s_q s_r / y_r, & \omega^{-1/2} \mu e^P &= x_r s_r s_q / y_q, \end{aligned}$$

where ω and q_0 are related by (3.51). One can check that the last four equations are consistent with (5.2). In fact, the expression (5.2) is a corollary of (5.3). This can be verified by direct substitution of (5.3) into (5.1) and (5.2) with an account of the relations (4.4). Comparing now (4.10) and the second equation in (4.5) with equation (3.52) we see that the relations (5.3) allow us to identify the first factors in the left and right sides of (4.21) with the corresponding factors in (3.49)

$$(5.4) \quad F(p, q, Z^{-1}) = \bar{r}(\lambda, e^Q Z^{-1}), \quad \bar{F}(q, r, X) = \bar{r}(\mu, e^P X),$$

provided the normalization factors $W_{pq}(0)$ and $\bar{W}_{qr}(0)$ (which so far were at our disposal) are appropriately chosen to satisfy (3.53). To identify the remaining factors in (3.49) and (4.21) we need four more pairs of relations

$$(5.5) \quad \begin{aligned} \omega^{-1/2} (\lambda\mu)^{-1} e^{P'} &= x_p s_p s_r / y_r, & \omega^{-1/2} \lambda\mu e^{P'} &= x_r s_r s_p / y_p, \\ \omega^{-1/2} (\lambda\mu)^{-1} e^{Q'} &= x_p / y_r, & \omega^{-1/2} \lambda\mu e^{Q'} &= x_r / y_p, \\ \omega^{-1/2} \lambda^{-1} e^{P''} &= x_p s_p s_q / y_q, & \omega^{-1/2} \lambda e^{P''} &= x_r s_r s_p / y_p, \\ \omega^{-1/2} \mu^{-1} e^{Q''} &= x_q / y_r, & \omega^{-1/2} \mu e^{Q''} &= x_r / y_q. \end{aligned}$$

Mysteriously enough, all these relations are corollaries of (5.3), (3.50) and (4.4). The calculations are simple but too long to be presented here. We quote only a few formulae important for the derivation of (5.5). It follows from (5.3) that

$$(5.6) \quad \lambda^2 = t_q / t_p, \quad \mu^2 = t_r / t_q,$$

where the t -variables are defined in (4.3) and that

$$(5.7) \quad e^{2Q} = \omega x_p x_q / y_p y_q, \quad e^{2P} = \omega x_q x_r s_q^2 s_r^2 / y_q y_r.$$

Then one can rewrite (3.50) as

$$(5.8) \quad \begin{aligned} e^{2Q'} &= \omega x_p x_r / y_p y_r, & e^{2P'} &= \omega x_p x_r s_p^2 s_r^2 / y_p y_r, \\ e^{2Q''} &= \omega x_p x_r / y_p y_r, & e^{2P''} &= \omega x_p x_q s_p^2 s_q^2 / y_p y_q. \end{aligned}$$

Similarly to (5.4) we have from (5.5)

$$(5.9) \quad \begin{aligned} \bar{F}(p, r, X) &= \bar{r}(\lambda\mu, e^{P'} X), & F(p, r, Z^{-1}) &= \bar{r}(\lambda\mu, e^{Q'} Z^{-1}), \\ F(q, r, Z^{-1}) &= \bar{r}(\mu, e^{Q''} Z^{-1}), & \bar{F}(p, q, X) &= \bar{r}(\lambda, e^{P''} X). \end{aligned}$$

Thus we have proved the equivalence of the twisted Yang–Baxter equation (3.49) with the star-triangle relation of the chiral Potts model (4.21). Note that the normalization of the weights of the chiral Potts model assumed by the relations (5.4), (5.9) is such that the factor R_{pqr} , (4.7) in (4.21) is equal to 1, as it follows from (3.53).

Let us summarize the results of this section. The four relations (5.3) allow us to express the parameters k, p, q, r entering the star-triangle relation (4.6) or (4.21) through λ, μ, P and Q from the twisted Yang–Baxter equation (3.49) and vice versa. Remarkably, these four relations imply (5.5) enabling us to prove the equivalence of the two Yang–Baxter equations (3.49) and (4.6). This equivalence allows us to look at the chiral Potts model from a new angle.

5.2. The evolution operator

Consider now the evolution operator (2.13). Substituting there the expression for the R -matrix (3.14) we can factorize \mathbf{U} into classical and quantum parts [14]

$$(5.10) \quad \mathbf{U} = \mathbf{U}_{\text{cl}} \mathbf{U}_{\text{quant}}$$

where

$$(5.11) \quad \mathbf{U}_{\text{cl}} = \exp\left(-\mathcal{H}_{\text{cl}}^{(0)}/h\right) \exp\left(-\mathcal{H}_{\text{cl}}^{(1)}/h\right) (1 + O(h)),$$

$$(5.12) \quad \mathcal{H}_{\text{cl}}^{(k)} = -\sum_{n=0}^{L-1} \tilde{H}(\kappa^2, w_{2n+k}), \quad k = 0, 1,$$

with the function \tilde{H} defined in (3.47) and (3.15). Further,

$$(5.13) \quad \mathbf{U}_{\text{quant}} = \prod_{n=0}^{L-1} \bar{r}(\kappa^2, \tau_{\text{cl}}(w_{2n})) \prod_{n=0}^{L-1} \bar{r}(\kappa^2, w_{2n+1}),$$

$$(5.14) \quad \tau_{\text{cl}}(a) = \left(\exp(\mathcal{H}_{\text{cl}}^{(1)}) \circ \exp(\mathcal{H}_{\text{cl}}^{(0)}) \circ a \right) = \lim_{h \rightarrow 0} \mathbf{U}_{\text{cl}}^{-1} a \mathbf{U}_{\text{cl}},$$

where we used the notation (2.26). Let us now remove the constraints (2.8) and consider a special case of the realization (2.18) when $q \rightarrow q_0$, $q_0^{2N} = 1$,

$$(5.15) \quad w_{2n} = \alpha_{2n} X_{n+1}, \quad w_{2n+1} = \alpha_{2n+1} Z_{n+1}^{-1} Z_{n+2}, \quad n = 0, \dots, L-1,$$

such that

$$(5.16) \quad \alpha_{2n} = \alpha, \quad \alpha_{2n+1} = \beta, \quad \forall n,$$

where α and β are fixed constants. It follows then from (2.10) that

$$(5.17) \quad \tau(\alpha_{2n}) = \alpha, \quad \tau(\alpha_{2n+1}) = \beta, \quad \forall n,$$

i.e., the configuration (5.16) is stationary with respect to the classical evolution.

Consider the algebraic curve (4.1) with the modulus

$$(5.18) \quad k^2 = \frac{(1 - \alpha^N \beta^N)(1 - \alpha^N \beta^{-N})}{(1 - \alpha^N \kappa^{2N})(1 - \alpha^N \kappa^{-2N})},$$

and choose two rapidities p and q such that

$$(5.19) \quad \begin{aligned} \omega^{-1/2} \beta / \kappa^2 &= x_p / y_q, & \omega^{-1/2} \beta \kappa^2 &= x_q / y_p, \\ \omega^{-1/2} \alpha / \kappa^2 &= x_p s_p s_q / y_q, & \omega^{-1/2} \alpha \kappa^2 &= x_q s_p s_q / y_p. \end{aligned}$$

Note that the last four equations are consistent with the relation (5.18) and imply the latter as their corollary. The expression (5.13) can now be rewritten as

$$(5.20) \quad \mathbf{U}_{\text{quant}} = \prod_{n=1}^L \bar{F}(p, q, X_n) \prod_{n=1}^L F(p, q, Z_n^{-1} Z_{n+1}),$$

where we have used (4.14). Taking into account (4.20) the matrix elements of this operator are

$$(5.21) \quad \left[\mathbf{U}_{\text{quant}} \right]_{a_1, a_2, \dots, a_L}^{b_1, b_2, \dots, b_L} = \prod_{n=1}^L \bar{W}_{pq}(a_n - b_n) \prod_{n=1}^L W_{pq}(b_{n+1} - b_n),$$

where $W_{pq}(a-b)$ and $\bar{W}_{pq}(a-b)$ are the Boltzmann weights of the chiral Potts model defined in (4.5).

Let $M \geq 1$ be an integer. Obviously, the trace

$$(5.22) \quad Z_{\text{chiral Potts}} = \text{Tr} \left[\mathbf{U}_{\text{quant}} \right]^M$$

is the partition function of the chiral Potts model for a (non-diagonal) square lattice of size L by M with periodic boundary conditions in both directions (to avoid confusion, note that usually the periodic boundary conditions in the chiral Potts model are imposed for the diagonal (i.e., 45° -rotated) square lattice, corresponding to the transfer matrix (4.12)).

§6. Concluding remarks

We have presented a new interpretation of the chiral Potts model where it is arising as a particular case of a more general Ising-type model on a square lattice with local spins taking $N \geq 2$ values at each site. The Boltzmann weights of this model are determined by solutions of the classical discrete sine-Gordon model (which is an integrable model of classical field theory). In this setting the chiral Potts model corresponds to the simplest (constant) solution of the above classical model. It would be interesting to consider more general spin models, corresponding to non-trivial solutions of the “background” classical field theory. Our construction also sheds some light on the origin of the “non-difference” property in the chiral Potts model. Further discussion of this point will be given in [9].

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Appendix A. Derivation of Eqs.(3.4)–(3.8).

The derivation of the first equation in (3.8) for $\Phi = \Phi_L$ goes through the following sequence of transformations by using (3.3) and (3.2)

(A.1)

$$\begin{aligned}
 \Phi_L u &= r(\lambda, u)r(\lambda\mu, v)r(\mu, u) u \\
 &= u r(\lambda, u)(q\lambda\mu + v)(q + \lambda\mu v)^{-1}r(\lambda\mu, v)r(\mu, u) \\
 &= u \left(q\lambda\mu + v \frac{1 + q\lambda u}{\lambda + qu} \right) \left(q + \lambda\mu v \frac{q\lambda + v}{q + \lambda v} \right)^{-1} r(\lambda, u)r(\lambda\mu, v)r(\mu, u) \\
 &= ug(u, v, q)\Phi_L
 \end{aligned}$$

where the function g is the same as in (3.4).

For $\Phi = \Phi_R$ the sequence of transformation is longer

$$\begin{aligned}
 \Phi_R u &= r(\mu, v)r(\lambda\mu, u)r(\lambda, v) u \\
 &= r(\mu, v) u r(\lambda\mu, u) \frac{q\lambda + v}{q + \lambda v} r(\lambda, v) \\
 &= r(\mu, v) u (q\lambda(\lambda\mu + qu) + v(1 + q\lambda\mu u)) \times \\
 &\quad \times \left[(q(\lambda\mu + qu) + \lambda v(1 + q\lambda\mu u))^{-1} r(\lambda\mu, u)r(\lambda, v) \right] \\
 &= r(\mu, v)(q\lambda(\lambda\mu + qu) + q^2 v(1 + q\lambda\mu u)) u \left[\dots \right] \\
 &= (q\lambda^2\mu + q^2 v + q\lambda u(q\mu + v))r(\mu, v) u \left[\dots \right] \\
 &= \left\{ u (q\lambda^2\mu + q^2\lambda\mu u + v + q\lambda v u) \frac{q\mu + v}{q + \mu v} \right\} r(\mu, v) \left[\dots \right] \\
 &= \left\{ \dots \right\} \left(\lambda(q\mu + v) + u \frac{q\mu + v}{q + \mu v} (q^2 + \lambda^2\mu q^{-1}v) \right)^{-1} \Phi_R \\
 &= \left\{ \dots \right\} \frac{q + \mu v}{q\mu + v} (q\lambda + q^2 u + \lambda\mu v + q^2\lambda^2\mu v u)^{-1} \Phi_R \\
 &= ug(u, v, q)\Phi_R.
 \end{aligned}$$

(A.2)

The second equation in (3.8) is obtained from (A.1) and (A.2) with the help of the automorphisms (3.6).

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