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Periodicity in the May's host parasitoid equation

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Abstract.

In this paper we consider the May's host parasitoid equation,

$$x_{n+1} = \frac{\alpha x_n^2}{(1+x_n)x_{n-1}}, \alpha > 1. \tag{1}$$

We show that with initial conditions $x_{-1} = x_0 = 1$ there are values of α giving periodic solutions of prime period n for all integers $n \geq 7$. There are no non-equilibrium periodic solutions of periods 2, 3, 4, 5 or 6.

§1. Introduction

May's host parasitoid equation (1) shows some rhythmic behavior, and some results about its long term behavior are given in [2]. In this paper we indicate that in the case $x_{-1} = x_0 = 1$, for different values of α there are periodic solutions of prime period n for all natural numbers $n \geq 7$. A key observation is that for appropriate values of α there are in this situation arbitrarily long decreasing sequences

$x_0 > x_1 > \ldots > x_n$. The techniques used are based on those of [3] and [4].

§2. Getting periodic solutions

With equation (1) and initial conditions $x_{-1} = x_0 = 1$, the value of x_n will be a rational function of α. We will often write $x_n(\alpha)$ to emphasize this dependence. Direct computation (with simplification of fractions) gives

$x_1(\alpha) = \frac{\alpha}{2}$,
$x_2(\alpha) = \frac{\alpha^3}{4+2\alpha}$,

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$x_3(\alpha) = \frac{\alpha^6}{(2+\alpha)(4+2\alpha+\alpha^3)}$,
$x_4(\alpha) = \frac{2\alpha^{10}}{(4+2\alpha+\alpha^3)(8+8\alpha+2\alpha^2+2\alpha^3+\alpha^4+\alpha^6)}$,
etc..

We will later need the observation that the functions $x_i(\alpha)$ are rational functions in α with non-negative coefficients in both numerator and denominator, hence are continuous for $\alpha > 1$. This can be formally established by induction on i.
The existence of periodic solutions of given period will eventually follow from the following theorem.

Theorem 1. *With May's host parasitoid equation with $x_{-1} = x_0 = 1$, and for $k \geq 0$,*

(i) *α gives a periodic solution of period $2k+2$ if and only if $x_k(\alpha) = x_{k+1}(\alpha)$;*

(ii) *α gives a periodic solution of period $2k+3$ if and only if $x_k(\alpha) = x_{k+2}(\alpha)$.*

Proof. We prove (i); the proof of (ii) is similar and is omitted. Equation (1) can be rewritten as

$$x_{n-1} = \frac{\alpha x_n^2}{(1+x_n)x_{n+1}}. \tag{2}$$

If: Assume that $x_k(\alpha) = x_{k+1}(\alpha)$, $k \geq 0$. We can generate the sequence in both directions starting with $x_k = x_{k+1}$. The similarity of equations (1) and (2) gives us equations $x_{k-1} = x_{k+2}$, $x_{k-2} = x_{k+3}$, and in general $x_{k-i} = x_{k+i+1}$. This means we get $x_0 = x_{2k+1} = 1$ and $x_{-1} = x_{2k+2} = 1$. So $x_{-1} = x_{2k+1}$, $x_0 = x_{2k+2}$. Since (1) is a second order difference equation it follows that the seuence $\{x_n\}$ is periodic of period $2k+2$.

Only if: Assume the sequence is periodic of period $2k+2$. Then $x_{-1} = x_{2k+1} = 1$, and $x_0 = x_{2k+2} = 1$. Using (1) and (2) and working both ways, we get $x_1 = x_{2k}, x_2 = x_{2k-1}, \ldots, x_k = x_{k+1}$. Q.E.D.

We can use theorem 1 to show the existence of some periodic solutions algebraically. For example, setting the expressions for $x_3(\alpha)$ and $x_4(\alpha)$ that we have above equal and simplifying we get the equation
$f(\alpha) = 8 + 8\alpha + 2\alpha^2 + 2\alpha^3 - 3\alpha^4 - 2\alpha^5 + \alpha^6 = 0$.

Descartes' Rule of Signs says there are zero or two positive solutions. We know (and can check) that $\alpha = 2$ is a solution (giving the equilibrium $x_n = 1$ for all n). Since $f'(2) = -24$ (in particular, $f'(2) \neq 0$), we know that $\alpha = 2$ is not a repeated root. In fact, since $f(\alpha)$ has a positive leading coefficient we know the other root is greater than 2. (In fact, the value of α giving a period 8 solution is approximately 2.43). Similarly we could show that a periodic solution of period 7 comes when α is approximately 4.71, and a period 9 solution comes when α is approximately 1.89.

We can also show there is no non-equilibrium solution of period 6 in this manner. Setting $x_2(\alpha) = x_3(\alpha)$ and simplifying gives the equation $\alpha^3 - 2\alpha - 4 = 0$. Descartes' Rule of Signs says there is one positive root. It is $\alpha = 2$, giving the equilibrium solution.

Computationally this algebraic approach to proving the existence of periodic solutions becomes unmanageable as k increases, and another, non-algebraic approach (still using theorem 1) is needed.

§3. Long decreasing sequences

Another observation which plays a key role in establishing periodicity is contained in the following theorem.

Theorem 2. *With the May's host parasitoid equation with initial conditions $x_{-1} = x_0 = 1$, for any positive integer N there is a value of α greater than 1, $\alpha = c_N$, with*

$$x_0(c_N) > x_1(c_N) > \ldots > x_N(c_N).$$

Proof. We first show that if $1 \geq x_k \geq \frac{1}{2^{2^k-1}}$ for $k = 0, 1, \ldots n-1$ and $\alpha < 2$, then $x_n \geq \frac{1}{2^{2^n-1}}$. This is done, of course, using induction on n: The fact is clearly true for $n = 1$, where $1 \geq x_1 = \frac{\alpha}{2} \geq \frac{1}{2} = \frac{1}{2^{2^1-1}}$. So assume it is true for $n-1$. Then

$$x_n = \frac{\alpha x_{n-1}^2}{(1+x_{n-1})x_{n-2}} > \frac{x_{n-1}^2}{(1+x_{n-1})x_{n-2}} \geq \frac{x_{n-1}^2}{(1+x_{n-1})} \geq \frac{x_{n-1}^2}{2} \geq \left(\frac{1}{2^{2^{n-1}-1}}\right)^2 \cdot \frac{1}{2} = \frac{1}{2^{(2^{n-1}-1)2}} \cdot \frac{1}{2} = \frac{1}{2^{2^n-2+1}} = \frac{1}{2^{2^n-1}},$$

as desired.

Next we show that if $\alpha < 2$ and if $1 + x_k \geq \alpha$ for $0 \leq k \leq n$, then

$x_0 > x_1 > \ldots > x_n$. Again the proof uses induction. For $n = 1$, the conclusion says $x_0 = 1 > x_1 = \frac{\alpha}{2}$, which is true as $\alpha < 2$. Assume $n > 1$ and the claim is true for all $k < n$. Then $x_n = \frac{\alpha}{1+x_{n-1}} \cdot \frac{x_{n-1}}{x_{n-2}} \cdot x_{n-1}$. Since $1 + x_{n-1} \geq a$ and, by induction, $x_{n-2} > x_{n-1}$, we get $x_{n-1} > x_n$, as desired.

Now, take $\alpha = c_N = \frac{1}{2^{2^N-1}}$. By the first observation we get $1+x_k \geq \alpha$ for all $k \leq N$ (assuming $x_k \leq 1$ for $k \leq N$), and so by the second observation

$x_0 > x_1 > \ldots > x_N$ (and indeed $x_k \leq 1$ for $k \leq N$). Q.E.D.

With values of α decreasing towards 1 we get arbitrarily long decreasing sequences $x_{-1} = x_0 = 1 > x_1 > x_2 > \ldots > x_N$. It is shown in [2] that none of these sequences have limit 0, in contrast to the situation where $\alpha \leq 1$ (treated in [2], and in the case $\alpha = 1$ also covered in [1]).

§4. The existence of periodic solutions

We are now in a position to prove the existence of periodic solutions to the May's host parasitoid equation.

Theorem 3. *For every positive integer $n \geq 7$, there exists a value of $\alpha > 1$ so that the May's host parasitoid equation (1) has a periodic solution of prime period n and with initial conditions $x_{-1} = x_0 = 1$.*

Proof. The cases of $n = 7$, 8, and 9 were dealt with previously.

Continuing the notation introduced in theorem 2, let $c_N \in (1, 2)$ satisfy the inequalities $x_0(c_N) > x_1(c_N) > \ldots > x_N(c_N)$. Following Theorem 1 we want to find an α so that (for odd periodicity) $x_0(\alpha) > x_1(\alpha) > \ldots > x_{N+1}(\alpha)$ and $x_N(\alpha) = x_{N+2}(\alpha)$, $(N \geq 4)$. Define a set A_N by

$A_N = \{\alpha | \alpha \geq c_{N+2}$ and $x_0(c) > x_1(c) > \ldots > x_{N+1}(c)$ and $x_N(c) \geq x_{N+2}(c)$ for all c in $[c_{N+2}, \alpha]\}$.

The set A_N is non-empty, as $c_{N+2} \in A_N$. Also, A_N is bounded above by the value b of α giving a period 9 solution, since $x_0(b) > x_1(b) > x_2(b) > x_3(b) > x_4(b)$ but $x_3(b) = x_5(b)$, so $b \notin A_N$ $(N \geq 4)$. Denote by a_N the least upper bound of A_N.

We claim that

$x_0(a_N) > x_1(a_N) > \ldots > x_{N+1}(a_N)$ and that $x_N(a_N) = x_{N+2}(a_N)$.

Because of the continuity of the functions $x_k(\alpha)$, we get

$(*)\qquad x_0(a_N) \geq x_1(a_N) \geq \ldots \geq x_{N+1}(a_N)$ and $x_N(a_N) \geq x_{N+2}(a_N)$.

If $x_0(a_N) = x_1(a_N)$, then $x_1(a_N) = 1$ and we have the equilibrium solution, This means $a_N = 2$, but this can't happen since A_N has an upper bound $b < 2$ and a_N is supposed to be the least upper bound of A_N. If $x_k(a_N) = x_{k+1}(a_N)$ for any $k = 1, 2, \ldots, N-1$, take k to be minimal with this property. The as before, using equations (1) and (2), we get $x_{k+2} = x_{k-1}$. This contradicts $x_{k-1}(a_N) > x_k(a_N) \geq x_{k+1}(a_N) \geq x_{k+2}(a_N)$, so we can't have $x_k(a_N) = x_{k+1}(a_N)$. Thus inequalities are strict through $x_{N-1} > x_N$. If $x_N = x_{N+1}$, then (again using equations (1) and (2)) we get $x_{N+2} = x_{N-1} > x_N$, contradicting $(*)$. Finally, if $x_N > x_{N+2}$, again using continuity we can find a $\delta > 0$ so that $a_N + \delta \in A_N$, contradicting the fact that a_N is an upper bound for A_N. Then by theorem 1, a_N gives a periodic solution of period $2N + 3$.

Similar arguments using the sets

$B_N = \{\alpha | \alpha \geq c_{N+2}$ and $x_0(c) > x_1(c) > \ldots > x_{N-1}(c) \geq x_N(c)$ for all c in $[c_{N+2}, a]\}$

give even period periodic solutions of period greater than or equal to 10.

The constructions involving the sets A_N and B_N guarantee that the indicated period of the solution is its prime period, and in fact the period is made up of just one cycle of the sequence. Q.E.D.

Note in the proof of theorem 3 that it was essential to have an upper bound b of the sets A_N and B_N which was less than 2 to rule out the possibility that $a_N = 2$ and we got the equilibrium solution. The rest of the argument would work for A_1, except that in this case a_1 is 2 and our period 5 solution is the equilibrium solution.

Problems:

1. Is there only one value of α for which equation (1) has a one cycle period n solution $(n \geq 7)$?
2. Show multicycle period n solutions exist (for some values of n).
3. Give periodicity results for arbitrary x_{-1} and x_0.
4. Show that in general solutions are bounded.
5. Give an invariant for equation (1).

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