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Stabilities with respect to a weight function in Volterra difference equations

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Abstract.

We establish some results on ρ-stabilities of the zero solution of a system of linear Volterra difference equations. Furthermore, for the equations with nonnegative coefficients we give an easy criterion for the ρ-UAS of the zero solution.

§1. Introduction

In this paper we are concerned with a system of linear Volterra difference equations

(1)
$$x(n+1) = \sum_{k=0}^{n} Q(n-k)x(k) \ (= (Q*x)(n)), \qquad n \in \mathbf{Z}^+ := \{0, 1, 2, \cdots\},$$

where $x = \{x(n)\}$ is a sequence of r column vectors, and $Q := \{Q(n)\}$ is a sequence of $r \times r$ matrices which is ρ-summable, that is, $\sum_{n=0}^{\infty} \rho(n)$ $\|Q(n)\| < \infty$, for a weight function $\rho : \mathbf{Z}^+ \to (0, \infty)$. Here and hereafter, we call ρ a weight function if ρ satisfies

$$\rho(0) = 1, \qquad \rho(n+m) \le \rho(n)\rho(m) \quad \text{for } \forall n, \ m \in \mathbf{Z}^+.$$

The functions $\rho(n) \equiv L^n$, $\rho(n) \equiv L^n(1+n)^p$, $\rho(n) = L^n(1+\log(1+n))^p$ with positive costants L and p are typical ones of weight functions. In this paper, we will introduce the concept of stabilities with respect to ρ (ρ-stabilities, in short), which relates to the decay rate of solutions of Eq. (1) and is exactly same as the usual concept of stabilities in case of $\rho(n) \equiv 1$. As one of our main results in this paper, we will give a result which is a characterization of the ρ-uniform asymptotic stability property of the zero solution of Eq. (1) in connection with the ρ-summability

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of the resolvent matrix of Eq. (1), as well as the invertibility of the characteristic matrix $zE - \sum_{n=0}^{\infty} Q(n)z^{-n}$ for z belonging to the domain $|z| \geq 1/l$, where E denotes the $r \times r$ unit matrix, and l is a number defined by $l := \lim_{n\to\infty} \sqrt[n]{\rho(n)}$ $(= \inf_n \sqrt[n]{\rho(n)})$ and it is assumed to be positive. In fact, in case of $\rho(n) \equiv 1$, the result is identical with the one obtained by Elaydi and Murakami [2]. Furthermore, under the restriction that the matrices $Q(n)$ are all nonnegative, we will give an easy criterion for the ρ-UAS of the zero solution of Eq. (1).

The following result is a typical one which is obtainable as an immediate consequence of our main theorems in Sections 2 and 3:

Theorem 1. *Assume that $Q(n) \geq 0, \forall n \geq 0$ and $\sum_{n=0}^{\infty} \|Q(n)\| < \infty$. Then the zero solution of Eq. (1) is UAS, if and only if*

$$\det\left(zE - \sum_{j=0}^{\infty} Q(j)\right) \neq 0 \quad (\forall |z| \geq 1).$$

On the other hand, the $(1+n)^{p+2}$-UAS can **never** *be realized for the zero solution of Eq. (1), when $\sum_{n=0}^{\infty}(n^p\|Q(n)\|) = \infty$ for some positive integer p.*

Indeed, the former part of the theorem is a direct consequence of Theorem 4 (in Section 3) with $\rho(n) \equiv 1$. Also, the latter part of the theorem follows from Theorem 3 (in Section 2), because the $(1+n)^{p+2}$-UAS of the zero solution of Eq. (1) implies that the resolvent $R = \{R(n)\}_{n\in\mathbf{Z}^+}$ of Eq. (1) satisfies $\sup_{n\geq 0}\{(1+n)^{p+2}\|R(n)\|\} < \infty$, and hence $\sum_{n=0}^{\infty}(n^p\|R(n)\|) < \infty$.

§2. Some results on ρ-stabilities

Given $\forall \tau \in \mathbf{Z}^+$ and $\phi : [0, \tau] := \{0, 1, \ldots, \tau\} \to \mathbf{C}^r$. A function $x : \mathbf{Z}^+ \to \mathbf{C}^r$ is called *a solution* of Eq. (1) through (τ, ϕ) if

$$x(n+1) = (Q * x)(n) \quad (\forall n \geq \tau), \quad x(s) = \phi(s) \ (\forall s = 0, 1, \ldots, \tau),$$

and is denoted by the symbol $x(\cdot; \tau, \phi)$.

The zero solution of Eq. (1) is said to be;

(1) *uniformly stable with respect to ρ* (ρ-US) if $\forall \epsilon > 0, \exists \delta > 0 :$

$$\|\phi\|_l := \sup_{-\tau\leq\theta\leq 0} (|\phi(\tau+\theta)|l^{\theta}) < \delta \Longrightarrow \rho(n)|x(n+\tau; \tau, \phi)| < \epsilon \ (\forall n \geq 0),$$

(2) *uniformly asymptotically stable with respect to* ρ (ρ-UAS) if the zero solution is ρ-US and if $\exists\delta_0 > 0$, $\forall\epsilon > 0, \exists N(\epsilon) > 0$:

$$\|\phi\|_l < \delta_0 \Longrightarrow \rho(n)|x(n+\tau;\tau,\phi)| < \epsilon \quad (\forall n \geq N(\epsilon)).$$

In the above, l is the (nonnegative) constant given by

$$l := \lim_{n\to\infty} \sqrt[n]{\rho(n)} \ (= \inf_n \sqrt[n]{\rho(n)}).$$

We remark that if $\rho(n) \equiv 1$, then $l = 1$, and hence the concept of the above ρ-stabilities is exactly same as the concept of the usual stabilities.

In the remainder, we always assume $l > 0$ and the following condition on the weight function ρ:

$$(2) \quad \exists(\text{constant})\ C > 0: \quad \rho(n)l^s \leq C\rho(n+s-1) \quad (\forall n \geq 0,\ s \geq 1).$$

We note that Condition (2) is satisfied (with $C = l$) for the concrete functions $\rho(n) \equiv l^n$, $\rho(n) \equiv l^n(1+n)^p$, and so on.

Notice that the *Z-transform* $\tilde{Q}(z) := \sum_{n=0}^{\infty} Q(n)z^{-n}$ is convergent for any $|z| \geq 1/l$, because $\rho(n) \geq l^n$ and $\sum_{n=0}^{\infty} \|Q(n)\|\rho(n) < \infty$.

A family of $r \times r$ matrices $R = \{R(n)\}_{n\in\mathbf{Z}^+}$ is called the *resolvent* of Eq. (1), if $R(0) = E$ and $R(n+1) = (Q * R)(n)$ for any $n \geq 0$. It is easy to see that the Z-transform $\tilde{R}(z)$ of the resolvent R exists for z with sufficiently large $|z|$. Considering the Z-transform of the both sides of $R(n+1) \equiv (Q * R)(n)$, one can get the relation $z\tilde{R}(z) - zR(0) = \tilde{Q}(z)\tilde{R}(z)$, or $(zE - \tilde{Q}(z))\tilde{R}(z) = zE$.

One of the crucial results in this section is the following:

Theorem 2. *Assume that $Q = \{Q(n)\}$ is ρ-summable for a weight function ρ satisfying Condition (2). Then the following statements are equivalent:*

(1) $\det(zE - \tilde{Q}(z)) \neq 0 \quad (\forall |z| \geq 1/l)$.

(2) *R is ρ-summable; that is,* $\displaystyle\sum_{n=0}^{\infty}(\|R(n)\|\rho(n)) < \infty$.

(3) *The zero solution of Eq. (1) is ρ-UAS.*

The equivalence (i) $\Longleftrightarrow$ (ii) can be established by applying the theory of Banach algebras [6, Chapter 11] to the space

$$L_\rho(\mathbf{Z}^+;\mathbf{C}) := \{a = \{a(n)\}_{n\in\mathbf{Z}^+} \subset \mathbf{C} \mid \|a\| := \sum_{n=0}^{\infty} \rho(n)|a(n)| < \infty\}.$$

Indeed, the space $L_\rho(\mathbf{Z}^+;\mathbf{C})$ with the convolution product is a commutative Banach algebra, and any nontrivial character χ of $L_\rho(\mathbf{Z}^+;\mathbf{C})$ is given by $\chi = \phi_w$, that is, $\chi(a) \equiv \sum_{n=0}^{\infty} a(n)w^n$, for some $w \in \mathbf{C}$ with $|w| \le l$. By this fact and the relation $(zE - \tilde{Q}(z))\tilde{R}(z) = zE$, one can verify the equivalence; cf. [3, Proof of Corollary 1 and pp.1213–1214]. Also, the implications (ii)$\Longrightarrow$(iii) and (iii)$\Longrightarrow$(i) can be verified by applying the representation formula of solutions of Eq. (1)

$$x(n+\tau;\tau,\phi) = R(n)\phi(\tau) + \sum_{j=0}^{n-1} R(n-j-1) \sum_{s=1}^{\tau} Q(j+s)\phi(\tau - s),$$

using Condition (2); cf. [2, Proof of Theorem 2].

Applying Theorem 2 with $l = 1$ and $\rho(n) \equiv (1+n)^p$ (p is a nonnegative integer), we get the following corollary:

Corollary 1. Assume that $\sum_{n=0}^{\infty}(\|Q(n)\|n^p) < \infty$. Then the following statements are equivalent:

(1) $\det(zE - \tilde{Q}(z)) \neq 0 \quad (\forall |z| \ge 1)$.

(2) $\sum_{n=0}^{\infty}(\|R(n)\|n^p) < \infty$.

(3) The zero solution of Eq. (1) is $(1+n)^p$-UAS.

In case of $p = 0$, Corollary1 is exactly tha same as [2, Theorem 2]. As shown in the following result, we cannot drop the n^p-summability condition on $Q(n)$ in Corollary1 under *the sign condition* for $Q(n)$.

Theorem 3. *Assume that* $\sum_{n=0}^{\infty} \|Q(n)\| < \infty$, *and that for sufficiently large* n, *each component* $q_{ij}(n)$ *of* $Q(n)$ *has the same sign; that is,*

$$\exists n_0 \ge 0 : \quad q_{ij}(n) \ge 0 \quad (\forall n \ge n_0, \ i,j = 1,\dots,r)$$

or

$$q_{ij}(n) \le 0 \quad (\forall n \ge n_0, \ i,j = 1,\dots,r).$$

If $\sum_{n=0}^{\infty}(\|R(n)\|n^p) < \infty$ *for some positive integer* p, *then* $\sum_{n=0}^{\infty}(\|Q(n)\|n^p) < \infty$.

Proof. The theorem can be proved by a method similar to the one in [5]; so, we will give only a sketch of the proof. In the following, we may consider the case $p \ge 1$ and assume that the norm of $r \times r$ matrix $A = (a_{ij})$ is given by the l^1 norm, that is, $\|A\| = \sum_{i,j=1}^{r} |a_{ij}|$. Observe that

$\tilde{R}(z) = \sum_{n=0}^{\infty} R(n)z^{-n}$ is convergent on $|z| \geq 1$, and $(zE - \tilde{Q}(z))\tilde{R}(z) = zE$; hence $\tilde{R}(z)$ is invertible and

$$\tilde{Q}(x) \equiv xE - x(\tilde{R}(x))^{-1} \ (=: F(x)) \quad (\forall x \geq 1).$$

It follows that $F \in C^p([1, \infty))$, moreover if $M > n_0$ and $h > 0$, then

$$\begin{aligned} \left\| \frac{F(1+h) - F(1)}{h} \right\| &= \left\| \sum_{n=0}^{\infty} Q(n) \frac{(1+h)^{-n} - 1}{h} \right\| \\ &\geq \left(\sum_{n=n_0}^{M} - \sum_{n=0}^{n_0 - 1} \right) \|Q(n)\| \frac{1 - (1+h)^{-n}}{h} \end{aligned}$$

by the sign condition on Q. Noting that

$$\sup_{0 \leq n \leq M} \left| \frac{1 - (1+h)^{-n}}{h} - n \right| \leq 1$$

for small $h > 0$, we can establish that

$$\sum_{n=0}^{\infty} n\|Q(n)\| < \infty, \quad F'(x) = \frac{d}{dx}\tilde{Q}(x) = \sum_{n=0}^{\infty} (-n)Q(n)x^{-n-1} \quad (\forall x \geq 1)$$

by a contradiction; cf. [5, Proof of Theorem 2]. Repeating the above argument, we can derive that

$$\sum_{n=0}^{\infty} n^k \|Q(n)\| < \infty,$$

$$F^{(k)}(x) = \sum_{n=0}^{\infty} (-1)^k n(n+1) \cdots (n+k-1) Q(n) x^{-n-k} \quad (\forall x \geq 1)$$

for $k = 1, 2, \ldots, p$. Letting $k = p$, we get the required one. Q.E.D.

§3. Stabilities in equations with nonnegative matrices

For any two $l \times q$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, the inequality $A \geq B$ means $a_{ij} \geq b_{ij}$ for all $i = 1, \cdots, l$, $j = 1, \cdots, q$. If $A \geq 0$, then A is called a nonnegative matrix. Also, a (column) vector b is said to be strictly positive, if all components of b are positive. We use the symbol $b \gg 0$ to denote that b is strictly positive.

In this section, we treat Eq. (1) with nonnegative matrices $Q(n)$, $\forall n \in \mathbf{Z}^+$, and establish the following result on the ρ-stabilities.

Theorem 4. *Assume that $Q(n) \geq 0, \forall n \geq 0$ and that $Q = \{Q(n)\}$ is ρ-summable for a weight function ρ satisfying Condition (2). Then the following statements are equivalent:*

(1) $\det(zE - \sum_{n=0}^{\infty} Q(n)l^n) \neq 0 \quad (\forall |z| \geq 1/l)$.
(2) *The zero solution of Eq. (1) is ρ-UAS.*
(3) *For any $b \gg 0$ and any solution y of $y(n+1) = (Q*y)(n)+b/l^n$, $l^n y(n)$ tends to a strictly positive vector as $n \to \infty$.*
(4) *For some $b \gg 0$ and some solution y of $y(n+1) = (Q*y)(n)+b/l^n$, $l^n y(n)$ tends to a strictly positive vector as $n \to \infty$.*

Applying Theorem 4 with $l = 1$ and $\rho(n) \equiv (1+n)^p$ (p is a nonnegative integer), we get the following corollary:

Corollary 2. Assume that $Q(n) \geq 0, \forall n \geq 0$ and that $\sum_{n=0}^{\infty}(\|Q(n)\| n^p) < \infty$. Then the following statements are equivalent:

(1) $\det(zE - \sum_{n=0}^{\infty} Q(n)) \neq 0 \quad (\forall |z| \geq 1)$.
(2) The zero solution of Eq. (1) is $(1+n)^p$-UAS.
(3) For any $b \gg 0$, any solution of $y(n+1) = (Q*y)(n) + b$ tends to a strictly positive vector as $n \to \infty$.
(4) For some $b \gg 0$, some solution of $y(n+1) = (Q*y)(n) + b$ tends to a strictly positive vector as $n \to \infty$.

In Corollary2, (or Theorem 4), we cannot always drop the nonnegativity condition on $Q(n)$ as the following example shows.

Example 1. Let us consider the scalar equation (1) with

$$Q(0) = 0, \qquad Q(n) = (-1)^{n-1}\frac{\alpha}{n^2}(n = 1,\ 2, \ldots),$$

where $\alpha = 6/\pi^2$. Then $\sum_{n=0}^{\infty} Q(n) = \alpha \times (\pi^2/12) = 1/2$; thus, Condition (i) in Corollary2 is satisfied. But the zero solution of Eq. (1) is not UAS, because the equation $\det(zE - \tilde{Q}(z)) = 0$ has a root $z = -1$.

Without assuming the nonnegative condition on $Q(n)$, one can obtain the following result (like a comparison theorem in ODE) on stabilities in Eq. (1) by checking Condition (i) in Theorem 2. Here and hereafter, for any matrix $A = (a_{ij})$ we employ the notation $|A| = (|a_{ij}|)$.

Theorem 5. *Assume that $Q = \{Q(n)\}$ is ρ-summable for a weight function ρ satisfying Condition (2), and that*

$$\det\left(zE - \sum_{n=0}^{\infty}(|Q(n)|l^n)\right) \neq 0 \quad (\forall |z| \geq 1/l).$$

Then the zero solution of Eq. (1) is ρ-UAS.

Example 2. Let $|\alpha| < 1/2$, $|\beta| < 1$, $|\gamma| < 1$, and consider Eq. (1) with

$$Q(n) \equiv \begin{pmatrix} \alpha^{n+1} & 0 \\ \beta^n & \gamma/(n+1)(n+2) \end{pmatrix}.$$

Then

$$\sum_{n=0}^{\infty} |Q(n)| = \begin{pmatrix} |\alpha|/(1-|\alpha|) & 0 \\ 1/(1-|\beta|) & |\gamma| \end{pmatrix}.$$

Observe that $|\alpha|/(1-|\alpha|)$ and $|\gamma|$ are the eigenvalues for the nonnegative matrix $\sum_{n=0}^{\infty} |Q(n)|$, and that they are smaller than 1 by the requirements on α and γ. Therefore, the zero solution of Eq. (1) is UAS by Theorem 5 with $l = 1$.

Before concluding the paper, we will give a sketch of the proof of Theorem 4. To do this, we first recall the following result which is well known as Perron-Frobenius's Theorem [1, 4]

Lemma 1. Let P be a nonnegative square matrix. Then

(1) The spectral radius $r := r_\sigma(P)$ is an eigenvalue of P and there exists a nonnegative eigenvector x, $x \neq 0$ such that $Px = rx$.
(2) $r_\sigma(P) = \sup\{\nu \in \mathbf{R} : \ Px \geq \nu x \quad \text{for some nonzero } x \geq 0\}$.
(3) If $\lambda > r_\sigma(P)$, then $(\lambda E - P)^{-1} \geq 0$.

The following result is an immediate consequence of Perron-Frobenius's Theorem (ii).

Proposition 1. Let P and Q be $r \times r$ matrices such that $|P| \leq Q$. Then $r_\sigma(P) \leq r_\sigma(Q)$.

In the proof of the theorem, the following proposition plays a crucial role.

Proposition 2. Assume that $Q(n) \geq 0, \forall n \geq 0$ and $\displaystyle\sum_{n=0}^{\infty}(\|Q(n)\|\rho(n)) < \infty$. Then the following statements are equivalent:

(1) $\det(zE - \tilde{Q}(z)) \neq 0 \quad (\forall |z| \geq 1/l)$.
(2) $r_\sigma(\sum_{n=0}^{\infty}(Q(n)l^n)) < 1/l$; in other words, $\det(zE - \sum_{n=0}^{\infty} Q(n)l^n) \neq 0 \quad (\forall |z| \geq 1/l)$.
(3) For some $b \gg 0$, there is a $y^* \gg 0$ such that $\displaystyle\left((1/l)E - \sum_{k=0}^{\infty} Q(k)l^k\right)y^* = b$.

Proof. (ii)$\Longrightarrow$(i). This is a direct consequence of Proposition 1, because of the inequality $|\tilde{Q}(z)| \leq \sum_{n=0}^{\infty} Q(n)l^n$ for $|z| \geq 1/l$.

(i)$\Longrightarrow$(ii). Let us consider the continuous function $f(t) = t - r_\sigma(\tilde{Q}(t))$ defined on $[1/l, \infty)$. Assume $f(1/l) \leq 0$. Since $f(\infty) = \infty$, there is some $\lambda_1 \geq 1/l$ such that $f(\lambda_1) = 0$; in other words, $\lambda_1 = r_\sigma(\sum_{n=0}^{\infty} Q(n)\lambda_1^{-n})$. Then λ_1 is an eigenvalue of the matrix $\sum_{n=0}^{\infty} Q(n)\lambda_1^{-n}$ by Perron-Frobenius's Theorem (i), which contradicts (i). Thus we must get $f(1/l) > 0$, which is equivalent to (ii).

(ii)$\Longrightarrow$(iii). This is a direct consequence of Perron-Frobenius's Theorem (iii).

(iii)$\Longrightarrow$(ii). Since $r := r_\sigma(\sum_{n=0}^{\infty} Q(n)l^n)$ equals the spectral radius of the transposed matrix of $\sum_{n=0}^{\infty} Q(n)l^n$, Perron-Frobenius's Theorem (i) implies that there is a nonzero row vector $c \geq 0$ such that $c(\sum_{n=0}^{\infty} Q(n)l^n) = rc$. Then $rcy^* = c(\sum_{n=0}^{\infty} Q(n)l^n)y^* = c(y^*/l - b)$, or $cb = (1/l - r)cy^*$. Since $cy^* > 0$ and $cb > 0$, we must get $r < 1/l$, which shows (ii). Q.E.D.

Sketch of Proof of Theorem 4. The equivalence relation (i) $\Longleftrightarrow$ (ii) is an immediate consequence of Theorem 2 and Proposition 2. Also, the implication (iii) $\Longrightarrow$ (iv) is obvious.

(i) $\Longrightarrow$ (iii). Given any $b \gg 0$ and a solution y of $y(n+1) = (Q * y)(n) + b/l^n$ on $[\sigma, \infty)$ such that $y \equiv \phi$ on $[0, \sigma]$. By Proposition 2, there is a $y^* \gg 0$ such that $\left((1/l)E - \sum_{k=0}^{\infty} Q(k)l^k\right)y^* = b$. We will certify that $\lim_{n\to\infty}\{l^n y(n)\} = y^*$. Observe that the solution y is written by the representation formula

$$y(n) = R(n-\sigma)\phi(\sigma) + \sum_{s=\sigma}^{n-1} R(n-s-1)\left(\sum_{j=-\sigma}^{-1} Q(s-\sigma-j)\phi(\sigma+j) + \frac{b}{l^s}\right).$$

Since $\sum_{n=0}^{\infty} \|R(n)\|\rho(n) < \infty$ by Theorem 2, one can easily deduce that $l^n\|R(n-\sigma)\phi(\sigma)\| \to 0$ and $l^n\|\sum_{s=\sigma}^{n-1} R(n-s-1)\{\sum_{j=-\sigma}^{-1} Q(s-\sigma-j)\phi(\sigma+j)\}\| \to 0$ as $n \to \infty$. Then

$$\begin{aligned}
\lim_{n\to\infty}\{l^n y(n)\} &= \lim_{n\to\infty} \sum_{s=\sigma}^{n-1} \{R(n-s-1)l^{n-s}b\} \\
&= \lim_{n\to\infty} \sum_{w=0}^{n-\sigma-1} \{R(w)l^{w+1}b\} \\
&= l\tilde{R}(1/l)b \\
&= l\tilde{R}(1/l)\left(\frac{1}{l}E - \sum_{k=0}^{\infty} Q(k)l^k\right)y^* = y^*,
\end{aligned}$$

because of $z(zE - \tilde{Q}(z))^{-1} = \tilde{R}(z)$ for $|z| \geq 1/l$.

(iv) $\Longrightarrow$ (i). Let y be a solution of $y(n+1) = (Q * y)(n) + b/l^n$ satisfying $\lim_{n\to\infty}(l^n y(n)) = y^* \gg 0$, where $b \gg 0$ is some vector. Since $\sum_{k=0}^{\infty} \|Q(k)\|\rho(k) < \infty$, one can easily check that $\lim_{n\to\infty}(l^n \sum_{s=0}^{\infty} Q(n-s)y(s)) = \{\sum_{k=0}^{\infty} l^k Q(k)\}y^*$. Hence

$$\begin{aligned} y^* = \lim_{n\to\infty}(l^{n+1}y(n+1)) &= \lim_{n\to\infty}\left(l^{n+1}\sum_{s=0}^{n} Q(n-s)y(s) + lb\right) \\ &= l\{\sum_{k=0}^{\infty} l^k Q(k)\}y^* + lb \end{aligned}$$

or

$$\left(\frac{1}{l}E - \sum_{k=0}^{\infty} Q(k)l^k\right)y^* = b;$$

thus Condition (iii) of Proposition 2 is satisfied. Therefore, (i) follows from Proposition 2. Q.E.D.

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