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Characterization of equilibrium paths in the two-sector model with sector specific externality

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Abstract.

We study the two-sector economy with sector-specific externality following Benhabib, Nishimura and Venditti [2002]. We focus on the external effect of capital-labor ratio and provide the characterization of equilibrium paths in the case Benhabib, Nishimura and Venditti did not discuss explicitly.

§1. Introduction

The aim of this paper is to characterize the local behavior of the equilibrium paths around the steady state in the two-sector model with sector specific externality. It is well known that externalities may cause the indeterminacy of equilibrium paths in an infinite horizon model. Benhabib and Farmer [1994] has showed that indeterminacy could occur in the one-sector growth model with both externality and increasing returns. In their model, the production function is constant return to scale from the private perspective, while it is increasing return to scale from the social perspective. Since then, there have been many papers about the existence of indeterminate equilibria in dynamic general equilibrium models. However, most of the literature dealt with models in which the production function is increasing return to scale from the social perspective, until the publication of Benhabib and Nishimura [1998, 1999]. They proved that indeterminacy may arise in an economy in which the production function from the social perspective is constant return to scale in the continuous time framework. Benhabib, Nishimura

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and Venditti [2002] studied the two-sector model with sector specific external effects in discrete time. They assumed that each sector has Cobb-Douglas technology with positive sector specific externalities and there is an infinitely-lived representative agent with linear utility function. Under these assumptions, they provided conditions in which indeterminacy may occur even if the production function is decreasing return to scale from the social perspective.

In this paper, we study the same model as in Benhabib, Nishimura and Venditti [2002][1], focus on the external effect of capital-labor ratio in the pure capital goods sector and provide the characterization of equilibrium paths in the case allowing negative externality, as was not explicitly discussed. We will show how the degree of externality affects the local behavior of the equilibrium path around the steady state.

In Section 2 we describe the model. We discuss the existence of a steady state and give the local characterization of equilibrium path around the steady state in Section 3. Section 4 is the Appendix.

§2. The model

We consider the two-sector model introduced by Benhabib, Nishimura and Venditti [2002]. There exists an infinitely-lived representative agent with single period utility function given by

$$u(C_t) = C_t.$$

There are goods : consumption goods, C, and capital goods, K. Following Benhabib, Nishimura and Venditti, we assume that each good is produced with a Cobb-Douglas technology.

$$C_t = K_{1,t}^{\alpha_1} L_{1,t}^{\alpha_2}, \qquad \alpha_1 + \alpha_2 = 1 \tag{1}$$

$$Y_t = A_t K_{2,t}^{\beta_1} L_{2,t}^{\beta_2}, \qquad \beta_1 + \beta_2 = 1 \tag{2}$$

where A_t represents the externality and varies each period. We formulate the externality as follows:

$$A_t = \left(\frac{\bar{K}_{2,t}}{\bar{L}_{2,t}}\right)^b = \bar{K}_{2,t}^b \bar{L}_{2,t}^{-b}. \tag{3}$$

[2]

[1]See also Nishimura and Venditti [2002] for derivation of the results.

[2]We focus on the external effect of capital-labor ratio in the capital goods sector.

A bar over a variable denotes the economy-wide average. We assume that the representative firms take as given these economy -wide average.

Definition 2.1. *We call* $Y_t = A_t K_{2,t}^{\beta_1} L_{2,t}^{\beta_2}$ *the production function from the private perspective, and* $Y_t = A_t K_{2,t}^{\beta_1+b} L_{2,t}^{\beta_2-b}$ *the production function from the social perspective.*

The aggregate capital is divided between sectors,

$$K_t = K_{1,t} + K_{2,t},$$

and the labor endowment is normalized to one and divided between sectors,

$$L_{1,t} + L_{2,t} = 1.$$

The capital accumulation equation is

$$K_{t+1} = Y_t,$$

that is, the capital depreciates completely in one period.

2.1. Profit maximization

We denote by p_2, w_1, and w_2 respectively the price of capital goods, the rental price of the capital goods and the wage rate of labor[3]. Then, each representative firm maximizes its profits:

$$\pi_1 = K_1^{\alpha_1} L_1^{\alpha_2} - w_1 K_1 - w_2 L_1, \tag{4}$$

$$\pi_2 = p_2 A K_2^{\beta_1} L_2^{\beta_2} - w_1 K_2 - w_2 L_2. \tag{5}$$

From the first order condition with respect to K_i, L_i $(i = 1, 2)$, we have the following conditions:

$$\frac{w_1}{w_2} = \frac{\alpha_1 L_1}{\alpha_2 K_1}, \tag{6}$$

$$\frac{w_1}{w_2} = \frac{\beta_1 L_2}{\beta_2 K_2}. \tag{7}$$

From equations (6) and (7), we derive the next equation.

[3]We normalize the price of consumption goods to one.

$$\text{(8)} \qquad \frac{\frac{\alpha_1}{\alpha_2}}{\frac{\beta_1}{\beta_2}} = \frac{\frac{K_1}{L_1}}{\frac{K_2}{L_2}}.$$

Factor intensities may be determined by the coefficients of the Cobb-Douglas functions. If $\frac{\alpha_1}{\alpha_2} > (<) \frac{\beta_1}{\beta_2}$, the consumption (capital) goods sector is capital intensive from the private perspective.

Using equation (8), $L_1 + L_2 = 1$, and $K = K_1 + K_2$. L_2 is expressed the follows :

$$\text{(9)} \qquad L_2 = \frac{\alpha_1 \beta_2 K_2}{\alpha_2 \beta_1 K + (\alpha_1 \beta_2 - \alpha_2 \beta_1) K_2}.$$

Substituting equation (9) into equation (2), and solving for K_2, K_2 is exhibited as a function of K, Y, and A.

$$\text{(10)} \qquad K_2 = \hat{K}_2 (K,\ Y;\ A).$$

Then, by equation (9), $L_1 + L_2 = 1$, and $K = K_1 + K_2$, K_1, L_1 and L_2 also are exhibited as functions of K, Y, and A, respectively. Therefore,

$$\text{(11)} \quad K_1 = \hat{K}_1 (K,\ Y;\ A),\quad L_1 = \hat{L}_1 (K,\ Y;\ A),\quad L_2 = \hat{L}_2 (K,\ Y;\ A).$$

Denote by Δ^* the denominator of equation (9),

$$\text{(12)} \qquad \Delta^* (K,\ y;\ A) \equiv \alpha_2 \beta_1 K + (\alpha_1 \beta_2 - \alpha_2 \beta_1) \hat{K}_2 (K,\ Y;\ A).$$

We define the social production function as below:

(13)

$$\begin{aligned} T^* (Y,\ K;\ A) &= A \hat{K}_1 (K,\ Y;\ A)^{\alpha_1} \hat{L}_1 (K,\ Y;\ A)^{\alpha_2} \\ &= A \left(\frac{\alpha_2 \beta_1}{\Delta^* (K,\ Y;\ A)} \right)^{\alpha_2} \left(K - \hat{K}_2 (K,\ Y;\ A) \right)^{\alpha_1 + \alpha_2}. \end{aligned}$$

2.2. Utility maximization

The consumer optimization problem will be given by

$$\text{(14)} \qquad \max \sum_{t=0}^{\infty} \rho^t T^* (K_{t+1},\ K_t;\ A_t), \qquad \textit{subject to} \quad k_0,\ \{A_t\}_{t=0}^{\infty} \ \textit{given}$$

where $\rho \in (0,\ 1)$ is the discount factor.

Then, the Euler equation in this model[4] is

$$T_1^*\left(K_{t+1},\ K_t;\ A_t\right)+\rho T_2^*\left(K_{t+2},\ K_{t+1};\ A_t\right)=0. \tag{15}$$

The solution of equation (14) satisfies the following transversality condition

$$\lim_{t\to+\infty}\rho^t K_{t+1}T_1^*\left(K_{t+1},\ K_t;\ A_t\right)=0. \tag{16}$$

We express the solution of this problem in $\{K_t\}_{t=0}^{\infty}$. This path depends on the choice of sequence $\{A_t\}_{t=0}^{\infty}$. If the sequence $\{A_t\}_{t=0}^{\infty}$ satisfies

$$A_t=\left[\hat{K}_2\left(K_{t+1},\ K_t;\ A_t\right)\right]^b\left[\hat{L}_2\left(K_{t+1},\ K_t;\ A_t\right)\right]^{-b}, \tag{17}$$

then the sequence $\{K_t\}_{t=0}^{\infty}$ is called an equilibrium path[5]. Solving equation (17) for A_t, A_t is given as a function of $(K_{t+1},\ K_t)$, namely $A_t=\hat{A}_t\left(K_{t+1},\ K_t\right)$. Substituting this expression into T^*,

$$T\left(K_{t+1},\ K_t\right)\equiv T^*\left(K_{t+1},\ K_t;\ \hat{A}_t\left(K_{t+1},\ K_t\right)\right). \tag{18}$$

This is the same as the function obtained by solving the first order conditions with respect to $K_1,\ K_2,\ L_1$, and L_2 of the Lagrangian below:

$$\begin{aligned}\mathcal{L}=&K_{1,t}^{\alpha_1}L_{1,t}^{\alpha_2}+p_{2t}\left(A_tK_{2,t}^{\beta_1}L_{2,t}^{\beta_2}-K_{t+1}\right)\\&+w_{1t}\left(K_t-K_{1,t}-K_{2,t}\right)+w_{2t}\left(1-L_{1,t}-L_{2,t}\right).\end{aligned} \tag{19}$$

Using the envelope theorem we derive the equilibrium prices[6].

$$T_1\left(K_{t+1},\ K_t\right)=-p_{2t}, \tag{20}$$

$$T_2\left(K_{t+2},\ K_{t+1}\right)=w_{1t+1}. \tag{21}$$

Then the Euler equation becomes the following:

$$T_1\left(K_{t+1},\ K_t\right)+\rho T_2\left(K_{t+2},\ K_{t+1}\right)=0. \tag{22}$$

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[4]Where $T_1^*\left(K_{t+1},\ K_t;\ A_t\right)=\partial T^*\left(K_{t+1},\ K_t;\ A_t\right)/\partial K_{t+1}$ and $T_2^*\left(K_{t+1},\ K_t;\ A_t\right)=\partial T^*\left(K_{t+1},\ K_t;\ A_t\right)/\partial K_t$.

[5]On an equilibrium path, the representative firm's expectations correspond with the realized value.

[6]Using the envelope theorem, that is, $\frac{\partial T}{\partial K_{t+1}}=\frac{\partial \mathcal{L}}{\partial K_{t+1}}$ and $\frac{\partial T}{\partial K_t}=\frac{\partial \mathcal{L}}{\partial K_t}$, we get equations (20) and (21).

§3. Steady state

Definition 3.1. *A steady state is defined* $K_t = K_{t+1} = Y_t = K^*$ *and is given by the solution of* $T_1(K_{t+1},\ K_t) + \rho T_2(K_{t+2},\ K_{t+1}) = 0$.

The following lemmas are direct consequences of the results in Baierl, Nishimura and Yano [1998], Benhabib, Nishimura and Venditti [5] and Nishimura and Venditti [6].

Lemma 1. *In this model, there exists a unique stationary capital stock* K^* *such that:*

$$K^* = \frac{\alpha_1\beta_2(\rho\beta_1)^{\frac{1}{\beta_2 - b}}}{\beta_1[\alpha_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)]}. \tag{23}$$

To study local behavior of equilibrium path around a steady state K^*, we linearize the Euler equation (22) at the steady state K^* and obtain the following characteristic equation

$$\rho T_{21}\lambda^2 + [\rho T_{22} + T_{11}]\lambda + T_{12} = 0. \tag{24}$$

Lemma 2. *The characteristic equation (24) is equivalent to the expression*

$$\begin{aligned}\lambda^2 \;&+\; \left[\frac{\alpha_2}{\rho(\alpha_2\beta_1 - \alpha_1\beta_2)} - \frac{\alpha_2 - (\beta_2 - b)}{\alpha_2}\right] \\ &- \frac{\alpha_2}{\rho(\alpha_2\beta_1 - \alpha_1\beta_2)} \cdot \frac{\alpha_2 - (\beta_2 - b)}{\alpha_2} = 0.\end{aligned} \tag{25}$$

Lemma 3. *The characteristic roots of equation (26) are*

$$\lambda_1 = \frac{\alpha_2}{\rho(\alpha_2\beta_1 - \alpha_1\beta_2)},\ \lambda_2 = \frac{\alpha_2 - (\beta_2 - b)}{\alpha_2}. \tag{26}$$

The roots of the characteristic equation determine the local behavior of the equilibrium paths. The sign of λ_1 is determined by factor intensity differences from the private perspective[7], while the sign of λ_2 is determined by factor intensity differences from the social perspective.

In what follows, we provide the characterization of equilibrium paths in this model. In particular we can show that the local behavior of equilibrium path around the steady state changes with the degree of external effect in the capital goods sector.

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[7] If $\alpha_2\beta_1 - \alpha_1\beta_2 > (<)\, 0$, the capital gooss sector (the consumption goods) is capital intensive from the private perspective.

We define local indeterminacy following Benhabib, Nishimura and Venditti [2002].

Definition 3.2. *A steady state k^* is called locally indeterminate if there exists ε such that for any $k_0 \in (k^* - \varepsilon,\ k^* + \varepsilon)$, there are infinitely many equilibrium paths converging to the steady state.*

Proposition 1. *Suppose that the capital goods sector is capital intensive from the private perspective, that is $\alpha_2\beta_1 - \alpha_1\beta_2 > 0$. Then, the steady state is a saddle for $b < \beta_2$, and it is totally unstable for $b > \beta_2$.*

Proof. Note that $\alpha_1 = 1-\alpha_2$ and $\beta_1 = 1-\beta_2$. By substituting into equation (26), we obtain $\lambda_1 = \frac{\alpha_2}{\rho(\alpha_2-\beta_2)} > 0$. Denoting $\rho_1 \equiv \frac{\alpha_2}{(\alpha_2-\beta_2)} > 1$, then $\lambda_1 = \frac{\rho_1}{\rho}$ is always greater than 1 as $\rho_1 > \rho > 0$.

Since $\lambda_1 > 1$ and λ_2 depends on the degree of externality b, we have the following cases.

(i) Let $b < \beta_2$. $0 < \frac{(\beta_2-b)}{\alpha_2} < 1$. Hence $0 < \lambda_2 < 1$. Therefore, the steady state is a saddle point.

(ii) Let $b > \beta_2$. Then $\lambda_2 > 1$. Hence the steady state is totally unstable. Q.E.D.

Remark 1. The production function from the social perspective is represented as follow:

$$Y = \left(\frac{K_2}{L_2}\right)^{\beta_1+b} L_2. \tag{27}$$

Divide both sides by L_2, $b+\beta_1 > 1$ and denote $\left(\frac{K_2}{L_2}\right)^{\beta_1+b}$ by $h\left(\frac{K_2}{L_2}\right)$. When b is larger than β_2, the function $h\left(\frac{K_2}{L_2}\right)$ exhibits increasing return.

$$h\left(\frac{K_2}{L_2}\right) = \left(\frac{K_2}{L_2}\right)^{\beta_1+b} > \frac{K_2}{L_2}.$$

Next we state results under the assumption that consumption goods are capital intensive from the private perspective with $\alpha_2\beta_1 - \alpha_1\beta_2 < 0$.

Proposition 2. *Let $\rho_2 \equiv \frac{\alpha_2}{(\beta_2-\alpha_2)}$. Suppose that $\rho_2 < \rho < 1$ and $2\alpha_2 < \beta_2$. Then, the steady state is a saddle for $b < \beta_2 - 2\alpha_2$, it is locally indeterminate for $0 < \beta_2 - 2\alpha_2 < b < \beta_2$, and it is a saddle for $b > \beta_2$.*

Proof. In this case, $\lambda_1 = \frac{\alpha_2}{\rho(\alpha_2-\beta_2)} < 0$. From $2\alpha_2 < \beta_2$, $\rho_2 = \frac{\alpha_2}{(\beta_2-\alpha_2)} < 1$. Hence λ_1 can be rewritten as $\lambda_1 = \frac{-\rho_2}{\rho}$. Then, $-1 < \lambda_1 < 0$. The size of $\lambda_2 = 1 - \frac{(\beta_2-b)}{\alpha_2}$ is determined in the following way.

(i) For $b < \beta_2 - 2\alpha_2$, $2 < \frac{(\beta_2-b)}{\alpha_2}$. Hence $\lambda_2 < -1$. Then the steady state is a saddle point.

(ii) For $0 < \beta_2 - 2\alpha_2 < b < \beta_2$, $0 < \frac{(\beta_2-b)}{\alpha_2} < 2$. Hence $-1 < \lambda_2 < 1$.Therefore the steady state is locally indeterminate.

(iii) For $b > \beta_2$, $\frac{(\beta_2-b)}{\alpha_2} < 0$. Hence $1 < \lambda_2$. Therefore the steady state is a saddle point. Q.E.D.

Remark 2. If $0 < \rho < \rho_2$, the steady state is unstable for $b < \beta_2 - 2\alpha_2$, it is a saddle for $0 < \beta_2 - 2\alpha_2 < b < \beta_2$, and it is totally unstable for $b > \beta_2$. Therefore $b > \beta_2$ or increasing returns of $h\left(\frac{K_2}{L_2}\right)$ implies the total instability of the steady state.

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