

3-dimensional i.i.d. binary random vectors governed by Jacobian elliptic space curve dynamics

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Abstract.

Sufficient conditions have been recently given for a class of ergodic maps of an interval onto itself: $I = [0, 1] \subset \mathbb{R} \rightarrow I$ and its associated binary function to generate a sequence of independent and identically distributed (i.i.d.) binary random variables. Jacobian elliptic Chebyshev map, its derivative and second derivative induce Jacobian elliptic space curve. A mapping of the space curve with its coordinates, e.g., X, Y and Z , onto itself is introduced which defines 3 projective onto mappings, represented in the form of rational functions of $\{x_n, y_n, z_n\}_{n=0}^{\infty}$. Such mappings with their absolutely continuous invariant measures as functions of elliptic integrals and their associated binary function can generate a 3-dimensional sequence of i.i.d. binary random vectors.

§1. Introduction

Bernoulli shift and its associated binary function can produce a sequence of independent and identically distributed (i.i.d.) binary random variables (BRVs) [1], [2]. Tent map [3], closely related to the Bernoulli map, and its associated binary function can also generate a sequence of i.i.d. BRVs. Ulam and von Neumann[4] showed that the logistic map is topologically conjugate to the tent map via the homeomorphism $h^{-1}(\omega) = \frac{2}{\pi} \sin^{-1} \sqrt{\omega}$. They also pointed out that the logistic map is a strong candidate for pseudo-random number generation (PRNG) even though it has a non-uniform absolutely continuous invariant (ACI) measure. A number of analog chaos techniques, which use a chaotic real-valued trajectory itself, have also been proposed [5],[6]. Binary sequences

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using chaos, however, play an important role in several applications such as spreading spectrum codes [7], [8], [9], [10], pseudo-random number generators [11] and cryptosystems [12], [13].

Motivated by this situation, we have shown that a class of ergodic maps with its unique ACI measure satisfying equi-distributivity property (EDP) can generate a sequence of i.i.d. binary random variables if its associated binary function satisfies constant summation property (CSP) [14]. Fortunately, many well-known 1-dimensional maps, which are topologically conjugate to the tent map via homeomorphism [3], satisfy EDP. The Bernoulli map, logistic map and Chebyshev polynomial are good examples [15]. These maps are governed by duplication formulae. In other words, *a duplication formula gives chaotic dynamics*. It is well known that elliptic functions satisfy an addition theorem [16]. We introduced a Jacobian elliptic Chebyshev rational map as a rational function version of Chebyshev polynomial [17]. This map as well as the other well known maps mentioned above are mappings from an interval onto itself.

Modern cryptosystems, however, need more and more pseudo-random numbers. In fact, to break DES of 64 bits, it takes 2^{43} steps and the success rate is 85% if 2^{43} pairs (plaintext, ciphertext) are known [18]. Furthermore, the size of new block ciphers such as AES becomes large, e.g., 512 bits.

This situation motivated us to discuss a closed smooth space curve defined by an algebraic relation between the Jacobian elliptic function, its derivative and second derivative. These duplication formulae give a real-valued sequence $\{x_n, y_n, z_n\}_{n=0}^{\infty}$ generated by 3-dimensional dynamics with Cartesian coordinates, e.g., X, Y and Z . Such 3-dimensional dynamics forces us to define a mapping from such a space curve onto itself and three projection mappings from an interval onto itself associated with coordinates, i.e., $x_{n+1} = \tau_x(x_n)$, $y_{n+1} = \tau_y(y_n)$ and $z_{n+1} = \tau_z(z_n)$, respectively. The former $\tau_x(\cdot)$, being single-valued, is the same as the Jacobian elliptic Chebyshev rational map. On the contrary, the latter two $\tau_y(\cdot)$ and $\tau_z(\cdot)$, being multi-valued, consist of single-valued mappings, each of which is a rational function of x_n, y_n, z_n and have their ACI measures with EDP. Hence, every bit of binary expansion of real-valued vector (x_n, y_n, z_n) satisfies CSP. This implies that the mapping from the space curve onto itself governed by duplication formulae gives a sequence of i.i.d. 3-dimensional binary random vectors.

§2. Related theories

We will begin by describing some of the related theories which play an important role in evaluating statistical properties of a sequence of binary random variables generated by a real-valued sequence.

2.1. EDP and CSP

Perhaps the simplest mathematical object that can display chaotic behavior is a class of one-dimensional maps $\omega_{n+1} = \tau(\omega_n)$, where $\omega_n = \tau^n(\omega_0) \in I = [d, e]$, $n = 0, 1, 2, \dots$ and $\tau(\cdot) : I \rightarrow I$.

Consider a piecewise monotonic (PM) onto ergodic map $\tau(\cdot)$ that satisfies the following properties:

- i): there is a (trivial) partition $d = d_0 < \dots < d_{N_\tau} = e$ of I such that for each integer $i = 1, \dots, N_\tau$, ($N_\tau > 2$) the restriction of $\tau(\cdot)$ to the interval $I_i = [d_{i-1}, d_i]$, denoted by $\tau_i(\omega)$, is a C^2 function; as well as
- ii): $\tau(I_i) = (d, e)$, that is, τ has N_τ monotonic onto maps τ_i ;
- iii): τ has a unique ACI measure, denoted by $f^*(\omega)d\omega$.

The following four definitions are important to evaluate statistical properties of $\{\omega_n\}_{n=0}^\infty$.

Definition 1 (Perron-Frobenius operator [19]). The Perron-Frobenius operator P_τ acting on function of bounded variation $F(\omega) \in L^\infty$ for $\tau(\omega)$ is defined as

$$P_\tau F(\omega) = \frac{d}{d\omega} \int_{\tau^{-1}([d, \omega])} F(y) dy = \sum_{i=0}^{N_\tau-1} |g_i'(\omega)| F(g_i(\omega)),$$

where $g_i(\omega)$ is the i -th preimage of ω and N_τ denotes the number of preimages.

The ACI measure $f^*(\omega)d\omega$ satisfies

$$(1) \quad P_\tau f^*(\omega) = f^*(\omega).$$

Birchoff Individual Ergodic Theorem [19] tells us that for a stationary real-valued sequence $\{F(\omega_n)\}_{n=0}^\infty$, the time average of $\{F(\omega_n)\}_{n=0}^\infty$, defined by

$$(2) \quad \langle F \rangle = \lim_{T \rightarrow \infty} (1/T) \sum_{n=0}^{T-1} F(\omega_n)$$

is equal almost everywhere to the expectation of $F(\omega)$, defined by

$$(3) \quad \mathbf{E}_\omega[F(\tau^n)] = \int_I F(\tau^n(\omega))f^*(\omega)d\omega.$$

From the stationarity of process, we denote $\mathbf{E}_\omega[F(\tau^n)]$ by $\mathbf{E}_\omega[F]$. Consider two sequences $\{G(\tau^n(\omega))\}_{n=0}^\infty$ and $\{H(\tau^n(\omega))\}_{n=0}^\infty$, where $G(\omega)$, $H(\omega) \in L^\infty$. The second-order cross-covariance function between these sequences from a seed $\omega = \omega_0$ is defined by

$$(4) \quad \rho(\ell, G, H) = \int_I (G(\omega) - \mathbf{E}_\omega[G]) \cdot (H(\tau^\ell(\omega)) - \mathbf{E}_\omega[H])f^*(\omega)d\omega,$$

where $\ell = 0, 1, 2, \dots$. The operator P_τ is useful in evaluating correlation functions because it has the following important property:

$$(5) \quad \int_I G(\omega)P_\tau\{H(\omega)\}d\omega = \int_I G(\tau(\omega))H(\omega)d\omega.$$

Using this property, we get

$$(6) \quad \rho(\ell, G, H) = \int_I P_\tau^\ell\{(G(\omega) - \mathbf{E}_\omega[G])f^*(\omega)\}(H(\omega) - \mathbf{E}_\omega[H])d\omega.$$

Bernoulli map with its uniform ACI measure $f^*(\omega)d\omega = d\omega$ is defined as

$$(7) \quad \tau_B(\omega) = 2\omega(\bmod 1) = \begin{cases} 2\omega, & 0 < \omega < \frac{1}{2}, \\ 2\omega - 1, & \frac{1}{2} \leq \omega < 1. \end{cases}$$

If ω is represented by its binary expansion as $\omega = 0.d_1(\omega)d_2(\omega)\dots$, then the binary expansion of $\tau_B(\omega)$ is given by $\tau_B(\omega) = 0.d_2(\omega)d_3(\omega)\dots$. This implies that $\tau_B(\cdot)$ shifts the digits one place to the left. The functions $d_k(\cdot)$, called Rademacher functions, furnish us with a model of independent tosses of a fair coin [2]. A sequence $\{d_k(\omega)\}_{k=0}^\infty$ can be regarded as a sequence of i.i.d. BRVs in the sense that for almost every ω , $d_k(\omega)$ can imitate coin tossing.

Another map and its associated binary function are as follows. Consider piecewise linear map of p branches with $f^*(\omega)d\omega = d\omega$, given by [3] ($N_\tau = p$),

$$(8) \quad N_p(\omega) = (-1)^{\lfloor p\omega \rfloor} p\omega(\bmod p), \quad \omega \in [0, 1].$$

In particular, $N_2(\omega)$ is referred to as the tent map. Introduce its associated BRV defined as

$$(9) \quad a_k = \begin{cases} 0, & \text{for } N_2^k(\omega) \leq \frac{1}{2}, \\ 1, & \text{for } N_2^k(\omega) > \frac{1}{2}. \end{cases}$$

Then for $\omega = 0.d_1(\omega)d_2(\omega)\cdots$, we get $a_0(\omega) = d_1(\omega)$, $a_k(\omega) = d_k(\omega) \oplus d_{k+1}(\omega)$, $k \geq 1$, where \oplus denotes a modulo 2 addition (or exclusive-or) operation. Hence $N_2(\omega)$ and its associated binary functions $a_k(\cdot)$ can generate a sequence of i.i.d. BRVs.

Naturally, the important question arises, that can any other map and its associated binary function generate a sequence of i.i.d. BRVs? We have got an affirmative answer to this question [14], [15], which is firstly, the map should satisfy EDP and secondly, the binary function should satisfy CSP.

Definition 2 (EDP [14]). *If a piecewise-monotonic onto map $\tau(\omega)$ satisfies*

$$(10) \quad |g'_i(\omega)|f^*(g_i(\omega)) = \frac{1}{N_\tau}f^*(\omega), \quad 0 \leq i \leq N_\tau - 1,$$

then the map is said to satisfy equi-distributivity property (EDP).

Definition 3 (CSP [14],[15]). *For a class of maps with EDP, if its associated function $G(\cdot)$ satisfies*

$$(11) \quad \frac{1}{N_\tau} \sum_{i=0}^{N_\tau-1} G(g_i(\omega)) = \mathbf{E}_\omega[G] \text{ or } P_\tau\{G(\omega)f^*(\omega)\} = \mathbf{E}_\omega[G]f^*(\omega)$$

then $G(\cdot)$ is said to satisfy constant summation property (CSP).

CSP guarantees no-correlation between two functions $G(\cdot)$ and $\forall H(\cdot)$, i.e., $\rho(\ell, G, H) = 0$, $\ell > 0$ [15]. Fortunately, EDP is satisfied by many well-known maps and is invariant under topological conjugation.

Definition 4 (topological conjugation [19]). *Two transformations $\bar{\tau} : \bar{I} \rightarrow \bar{I}$ and $\tau : I \rightarrow I$ on intervals \bar{I} and I are called topological conjugate if there is a homeomorphism $h : \bar{I} \xrightarrow{\text{onto}} I$ as $\tau(\omega) = h \circ \bar{\tau} \circ h^{-1}(\omega)$.*

Suppose $\tau(\cdot)$ and $\bar{\tau}(\cdot)$ have their ACI measures $f^*(\omega)d\omega$ and $\bar{f}^*(\bar{\omega})d\bar{\omega}$ respectively. Then, under the topological conjugation, these ACI measures have the relation

$$(12) \quad f^*(\omega) = \left| \frac{dh^{-1}(\omega)}{d\omega} \right| \bar{f}^*(h^{-1}(\omega)).$$

The relation between $\tau(\cdot)$ and $\bar{\tau}(\cdot)$ via h is represented diagrammatically as follows :

$$(13) \quad \begin{array}{ccc} I & \xrightarrow{\tau} & I \\ h^{-1} \downarrow & & \uparrow h \\ \bar{I} & \xrightarrow{\bar{\tau}} & \bar{I} \end{array}$$

Remark 1. If we take $N_2(\bar{\omega})$ as $\bar{\tau}(\bar{\omega})$, then $f^*(\omega)$ is simply represented by the derivative of $h^{-1}(\omega)$. Hence, if $h(\bar{\omega})$ can be given in an inverse function form, then its integrand gives an ACI measure within normalization factor. Most famous example of inverse functions is sin function, *i.e.*, $\omega = \int_0^{\sin \omega} \frac{du}{\sqrt{1-u^2}}$.

This remark provides a starting point for discussion. In fact, Ulam and von Neumann [4] gave the logistic map

$$(14) \quad L_2(\omega) = 4\omega(1-\omega), \quad \omega \in [0, 1]$$

with $f^*(\omega)d\omega = \frac{d\omega}{\pi\sqrt{\omega(1-\omega)}}$ which is topologically conjugate to $N_2(\bar{\omega})$ using $h^{-1}(\omega) = \frac{2}{\pi} \sin^{-1} \sqrt{\omega}$.

2.2. Binary function

In our previous study [14], we proposed methods to obtain binary sequences from chaotic real-valued sequences $\{\tau^n(\omega)\}_{n=0}^{\infty}$. We define a (non-trivial) partition $d = t_0 < t_1 < \dots < t_{2M} = e$ of $[d, e]$ and T denotes the set of thresholds $\{t_r\}_{r=0}^{2M}$. Then the following binary function is obtained

$$(15) \quad C_T(\omega) = \sum_{r=0}^{2M} (-1)^r \Theta_{t_r}(\omega),$$

where $\Theta_t(\omega)$ is the threshold function such that

$$(16) \quad \Theta_t(\omega) = \begin{cases} 0, & \text{for } \omega < t \\ 1, & \text{for } \omega \geq t. \end{cases}$$

§3. Duplication formula gives chaos

The example mentioned above shows that duplication formula gives chaos. To observe it, several examples are listed as follows.

(1) logistic map: Transformation $x = \sin^2 \theta$ gives $\left(\frac{dx}{d\theta}\right)^2 = 4x(1-x)$. Let $x_n = \sin^2 \theta_n$, $\theta_{n+1} = 2\theta_n$. Then we get 2-dimensional sequences $\{(x_n, y_n)\}_{n=0}^{\infty}$, given by

$$(17) \quad \begin{aligned} x_{n+1} &= L_2(x_n) = 4x_n(1-x_n), \\ y_{n+1} &= \left(\frac{1}{2} \cdot \frac{dL_2(x_n)}{d\theta_n}\right)^2 = 4L_2(x_n)(1-L_2(x_n)). \end{aligned}$$

(2) Chebyshev map of degree 2: Grossmann and Thomae [3] observed that Chebyshev polynomial maps of degree p ($p = 2, 3, \dots$) [20] with its

ACI measure $f^*(\omega)d\omega = \frac{d\omega}{\pi\sqrt{1-\omega^2}}$, defined by

$$(18) \quad T_p(\omega) = \cos(p \cos^{-1} \omega), \omega \in [-1, 1]$$

is topologically conjugate to $N_p(\omega)$ via $h(\bar{\omega}) = \cos \pi \bar{\omega}$. Transformation $x = \cos \theta$ gives $(\frac{dx}{d\theta})^2 = 1 - x^2$. Let $x_n = \cos \theta_n$, $\theta_{n+1} = 2\theta_n$. Then we get 2-dimensional sequences $\{(x_n, y_n)\}_{n=0}^\infty$, given by

$$(19) \quad x_{n+1} = T_2(x_n) = 2x_n^2 - 1, \quad y_{n+1}^2 = \left(\frac{1}{2} \cdot \frac{dT_2(x_n)}{dx_n}\right)^2 = 1 - (T_2(x_n))^2.$$

(3) Schröder and Böttcher map: ¹ Schröder [22] and Böttcher [23] gave a rational function version of $L_2(\cdot)$ with parameter k , defined as

$$(20) \quad R_2^{\text{sn}^2}(\omega, k) = \frac{4\omega(1-\omega)(1-k^2\omega)}{(1-k^2\omega^2)^2}, \omega \in [0, 1]$$

with its ACI measure

$$(21) \quad f^*(\omega, k)d\omega = \frac{d\omega}{2K(k)\sqrt{\omega(1-\omega)(1-k^2\omega)}}$$

via $h^{-1}(\omega) = \frac{1}{K(k)} \text{sn}^{-1}(\sqrt{\omega}, k)$, where $\text{sn}(\omega, k)$ is the inverse function of the elliptic integral with modulus k ($|k| < 1$) and $K(k)$ is the complete elliptic integral, each of which is given respectively as

$$(22) \quad u = \int_0^{\text{sn}(u, k)} \frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}}, \quad K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}}.$$

Transformation $x = \text{sn}^2 u$ gives $(\frac{dx}{du})^2 = 4x(1-x)(1-k^2x)$. Let $x_n = \text{sn}^2 u_n$, $u_{n+1} = 2u_n$. Then we get 2-dimensional sequences $\{(x_n, y_n)\}_{n=0}^\infty$, given by

$$(23) \quad x_{n+1} = R_2^{\text{sn}^2}(x_n, k) = \frac{4x_n(1-x_n)(1-k^2x_n)}{(1-k^2x_n^2)^2}$$

$$(24) \quad y_{n+1}^2 = \left(\frac{1}{2} \cdot \frac{dR_2^{\text{sn}^2}(x_n, k)}{dx_n}\right)^2 \\ = 4R_2^{\text{sn}^2}(x_n, k)(1 - R_2^{\text{sn}^2}(x_n, k))(1 - k^2R_2^{\text{sn}^2}(x_n, k)).$$

¹see [21] for a historical review of rational maps.

§4. **Jacobian elliptic space curve and 3-dimensional dynamics**

We know that the Jacobian elliptic function $\text{cn}(u, k)^2$ is an inverse function of an elliptic integral of the first kind in the Legendre-Jacobi normal form [16]

$$(25) \quad u = \int_{\text{cn}(u,k)}^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2+k^2t^2)}}.$$

Kohda and Fujisaki [17] introduced the Jacobian elliptic Chebyshev rational map with positive integer p

$$(26) \quad R_p^{\text{cn}}(\omega, k) = \text{cn}(p \text{cn}^{-1}(\omega, k), k), \quad \omega \in [-1, 1]$$

which is topologically conjugate to the tent map $N_p(u)$ via homeomorphism $h^{-1}(\omega, k) = \frac{\text{cn}^{-1}(\omega, k)}{2K(k)}$ and has its ACI measure

$$(27) \quad f^*(\omega, k)d\omega = \frac{d\omega}{2K(k)\sqrt{(1-\omega^2)(1-k^2+k^2\omega^2)}}.$$

This map is a rational function version of the Chebyshev polynomial

$$(28) \quad T_p(\omega) = \cos(p \cos^{-1} \omega), \quad \omega \in [-1, 1].$$

We know that $R_p^{\text{cn}}(\omega, k)$ satisfies the semi-group property

$$(29) \quad R_r^{\text{cn}}(R_s^{\text{cn}}(\omega, k), k) = R_{rs}^{\text{cn}}(\omega, k)$$

for integers r, s and when $p = 2$,

$$(30) \quad R_2^{\text{cn}}(\omega, k) = \frac{1 - 2(1 - \omega^2) + k^2(1 - \omega^2)^2}{1 - k^2(1 - \omega^2)^2}.$$

Let us concentrate on the Jacobian real elliptic function with $p = 2$ [16]. As shown in Fig. 1, the Jacobian elliptic function $X = \text{cn}(u, k)$, its derivative $Y = \frac{d}{du} \text{cn } u = -\text{sn } u \text{ dn } u$ and the second derivative $Z = \frac{d^2}{du^2} \text{cn } u$ give the Jacobian elliptic space curve, given by

$$(31) \quad Y^2 = (1 - X^2)(1 - k^2 + k^2 X^2), \quad Z = X(-1 + 2k^2(1 - X^2)).$$

² $\text{cn}(u, 0)$ simply reduces to $\cos u$.

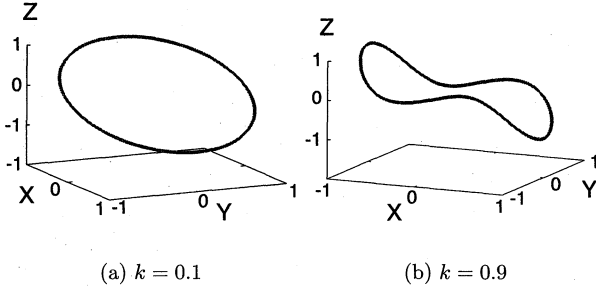


Fig. 1. Two Jacobian elliptic space curves (X, Y, Z) .

Let $u_{n+1} = 2u_n$, $x_n = \text{cn } u_n$, $y_n = \frac{dx_n}{du_n}$ and $z_n = \frac{d^2x_n}{du_n^2}$. Then we get a 3-dimensional dynamics, given by

$$(32) \quad \begin{cases} x_{n+1} &= R_2^{\text{cn}}(x_n, k) = \tau_x(x_n, k), \\ y_{n+1}^2 &= \left(\frac{1}{2} \frac{dx_{n+1}}{du_n}\right)^2 = (1 - x_{n+1}^2)(1 - k^2 + k^2 x_{n+1}^2) = \tau_y^2(y_n, k), \\ z_{n+1} &= \frac{1}{4} \frac{d^2x_{n+1}}{du_n^2} = \tau_z(z_n(x_n), k) = \tau_z(x_n, k) \\ &= \frac{k^2 - 1 + 2(1 - k^2)x_n^2 + k^2x_n^4}{1 - k^2(1 - x_n^2)^2} \{1 - 2\left(\frac{1 - k^2 + k^2x_n^4}{1 - k^2(1 - x_n^2)^2}\right)^2\}. \end{cases}$$

This gives a mapping from such a space curve onto itself which induces three projective onto mappings associated with coordinates, e.g., X, Y, Z , denoted by $\tau_x(\cdot), \tau_y(\cdot), \tau_z(\cdot)$. The first one is shown in Fig.2(a), which has a symmetric ACI measure, defined by

$$f_X^*(x, k)dx = \frac{dx}{2K(k)\sqrt{(1 - x^2)(1 - k^2 + k^2x^2)}}$$

in Fig.3(a).

In addition, it has been shown [24] that the projective onto map τ_y is symmetric and has a symmetric ACI measure as shown in Figs.2(b) and 3(b), respectively. (see Appendix A for theoretical expression of τ_y) Its associated symmetric binary function, e.g., binary expansion of real-valued orbit $\{x_n\}_{n=0}^\infty$ or $\{y_n\}_{n=0}^\infty$ can generate a sequence of i.i.d. binary random variables [24].

Here we consider the map τ_z and examine whether it has its symmetric ACI measure [25]. Squaring the second expression of Eq.(31) with $k \neq 0$ gives the relation

$$(33) \quad X^6 - \frac{1}{k^2}(-1 + 2k^2)X^4 + \frac{1}{4k^4}(-1 + 2k^2)^2X^2 - \frac{Z^2}{4k^4} = 0$$

which implies that for a given Z , X^2 has the following three real-valued solutions at most.

$$(34) \quad X^2(Z) = \begin{cases} \xi_1^2(Z), & \text{for } k \leq \sqrt{1/2} \text{ (} R(Z, k) > 0 \text{)} \\ \xi_1^2(Z), & \text{for } k > \sqrt{1/2} \text{ and } R(Z, k) > 0 \\ \xi_i^2(Z), \ 2 \leq i \leq 4, & \text{for } k > \sqrt{1/2} \text{ and } R(Z, k) < 0, \end{cases}$$

where $R(Z, k) = \frac{b^2(Z, k)}{4} + \frac{a^3(k)}{27}$, $a(k) = -\frac{1}{12k^4}(-1 + 2k^2)^2$, $b(Z, k) = \frac{1}{4 \cdot 27} \left\{ \frac{(-1+2k^2)^3}{k^6} - \frac{27}{k^4} Z^2 \right\}$.

On the space curve, 3-dimensional dynamics has a unique ACI measure with respect to each coordinate. Fig. 3(c) shows comparison between the marginal distribution taken from experiments and theoretical calculations, where the theoretical distributions of τ_z is given as follows

$$(35) \quad f_z^*(z, k) dz = \begin{cases} \frac{1}{2K(k)} f_Z(\xi_1(Z), k) dz, & \text{for } 0 < k \leq \sqrt{1/2} \\ \frac{1}{2K(k)} f_Z(\xi_1(Z), k) dz, & \text{for } k > \sqrt{1/2}, r(k) \leq |z| < 1 \\ \frac{1}{2K(k)} \sum_{\ell=2}^4 f_Z(\xi_\ell(Z), k) dz & \text{for } k > \sqrt{1/2}, |z| \leq r(k) \end{cases}$$

where

$$(36) \quad \begin{aligned} r(k) &= \sqrt{\frac{2}{27}(-1 + 2k^2)^3}, \\ f_Z(\xi_\ell(Z), k) dz &= \frac{dz}{\sqrt{(1 - \xi_\ell^2(Z))(1 - k^2 + k^2 \xi_\ell^2(Z)) | -6k^2 \xi_\ell^2(Z) + 2k^2 - 1|}}. \end{aligned}$$

Finally, we notice that theoretical distribution $f_X^* dx$ is also given by integrand of elliptic integral for inverse function $\text{cn}^{-1}(u, k)$ (see Eq.(25)). The same is true for $f_Y^* dy$. In fact, inverse function $\left(\frac{d \text{cn}(u, k)}{du} \right)^{-1} = (-\text{sn}(u, k) \text{dn}(u, k))^{-1}$ is defined by Eq.(45) and Eq.(46) (see Appendix B). Similarly $f_Z^* dz$ is expressed in the inverse function form, as given by Eq.(49) and Eq.(50) (see Appendix C).

§5. I.I.D. binary random vectors

We shall now look into the relation between (z_n, z_{n+1}) . Eqs.(33) and (34) tell us that the relation $z_{n+1} = \tau_z(\xi_1(z_n))$ is one-to-one when $k < \sqrt{1/2}$ but the graph of z_n versus z_{n+1} is *one-to-many* when $k >$

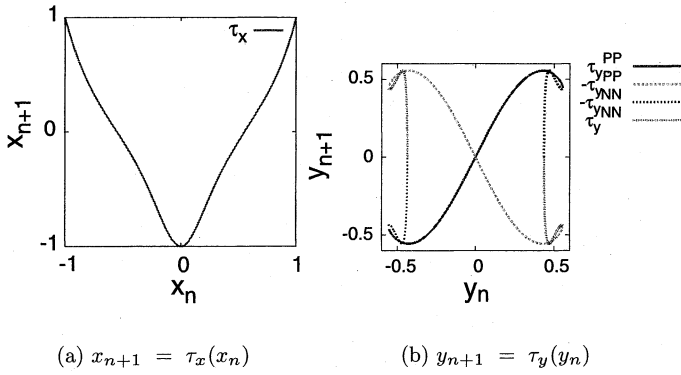


Fig. 2. Three projection mappings when $k = 0.9$.

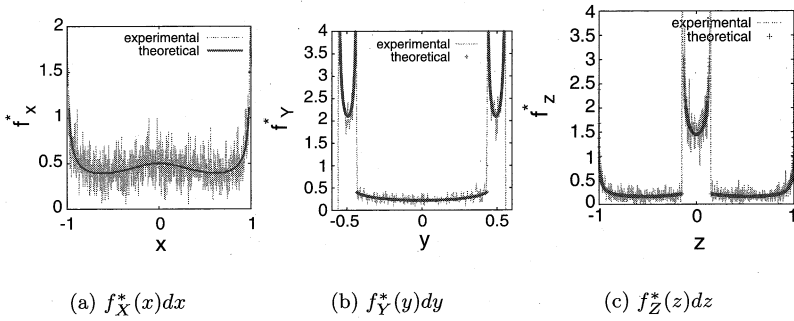


Fig. 3. Three marginal distributions when $k = 0.9$

$\sqrt{1/2}$. Namely, the latter case gives a closed curve as shown in Fig. 2(c). Suppose that $k > \sqrt{1/2}$ and that $X_1(x)$ is the first bit of normalized x in binary representation, such as

$$\frac{x+1}{2} = 0.X_1(x)X_2(x)\cdots X_i(x)\cdots, X_i(x) \in \{0, 1\}.$$

We denote $X_1(x)$ by X_1 and $1 - X_1(x)$ by $\overline{X_1}$. Similarly $Z_1(z)$ and $1 - Z_1(z)$ are denoted by Z_1 and $\overline{Z_1}$ respectively. In addition, $D(\frac{dz}{dx})$ and $1 - D(\frac{dz}{dx})$ are represented by D_z and $\overline{D_z}$ respectively, where $D(\frac{dz}{dx}) = 0$ (or 1) when $\frac{dz}{dx} < 0$ (or when $\frac{dz}{dx} \geq 0$).

Then, we can obtain a piecewise-monotonic onto map τ_z defined by

$$(37) \quad \tau_z = X\overline{Z}\overline{D}\tau_z^{1-} + \overline{X}Z\overline{D}\tau_z^{1+} + X\overline{Z}\overline{D}\tau_z^{2-} + \overline{X}Z\overline{D}\tau_z^{2+} \\ + \overline{X}\overline{Z}D\tau_z^{3-} + XZD\tau_z^{3+} + \overline{X}\overline{Z}\overline{D}\tau_z^{4-} + XZ\overline{D}\tau_z^{4+}$$

where $\tau_z^{i-} = \tau_z(-\xi_i(z))$, $\tau_z^{i+} = \tau_z(\xi_i(z))$, $1 \leq i \leq 4$ and where $\xi_i^2(z)$ is defined by Eq.(34).

It can be shown that for uniform ACI measure $f_U^*(u)du = du$,

$$(38) \quad \left. \begin{aligned} P_{\tau_x}\{C_{T_x}(x)f_X^*(x)\} &= \mathbf{E}_u[C_{T_x}]f_X^*(x), & x = \text{cn } u \\ P_{\tau_y}\{C_{T_y}(y)f_Y^*(y)\} &= \mathbf{E}_u[C_{T_y}]f_Y^*(y), & y = -\text{sn } u \text{ dn } u \\ P_{\tau_z}\{C_{T_z}(z)f_Z^*(z)\} &= \mathbf{E}_u[C_{T_z}]f_Z^*(z), & z = \frac{d(-\text{sn } u \text{ dn } u)}{du} \end{aligned} \right\}$$

holds, where $\{C_{T_x}(x_n)\}_{n=0}^\infty$, $\{C_{T_y}(y_n)\}_{n=0}^\infty$ and $\{C_{T_z}(z_n)\}_{n=0}^\infty$ are symmetric binary sequences with their sets of symmetric thresholds T_x, T_y and T_z associated with real-valued sequences $\{x_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty$ and $\{z_n\}_{n=0}^\infty$.

This implies that $\rho(\ell, C_{T_x}, C_{T_x}) = \rho(\ell, C_{T_y}, C_{T_y}) = \rho(\ell, C_{T_z}, C_{T_z}) = 0$, for $\ell \geq 0$. [14]

It should be noted that $C_{T_x}(x)$, $C_{T_y}(\tau_y^\ell(y))$, $C_{T_z}(\tau_z^m(z))$ are not always independent each other for $\ell = m = 0$, that is, e.g., $\mathbf{E}_u[C_{T_x}C_{T_y}] \neq \mathbf{E}_u[C_{T_x}]\mathbf{E}_u[C_{T_y}]$ even if each of them is a sequence of i.i.d. BRVs. This is inevitable as long as these sequences are generated from a single seed $u = u_0$. However, we can design appropriate sets of thresholds T_x, T_y, T_z satisfying $\mathbf{E}_u[C_{T_x}C_{T_y}] = \mathbf{E}_u[C_{T_x}]\mathbf{E}_u[C_{T_y}]$ (see [14] for details).

§6. Conclusion

We discussed a real-valued dynamics on the Jacobian elliptic space curve between Jacobian elliptic function, its derivative and second derivative, governed by their duplication formulae. Furthermore, we showed that a mapping of the space curve onto itself: $R^3 \rightarrow R^3$ which defines 3

projective onto mappings with their ACI measures satisfying EDP and can generate sequences of 3-dimensional i.i.d. binary random vectors when using their associated symmetric binary functions, e.g., bits of binary expansions of these real-valued x_n, y_n, z_n as shown in Fig. 4.

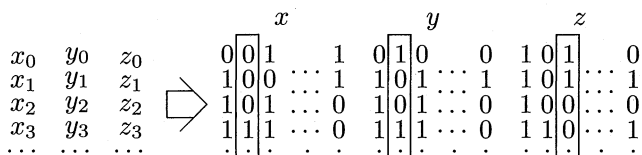


Fig. 4. Method of generating multidimensional i.i.d. binary vectors

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§**Appendix A.** Derivation of the theoretical expression of τ_y

The first expression in Eq.(31) gives

$$(39) \quad y_n^2 = (1 - x_n^2)(1 - k^2 + k^2 x_n^2).$$

Solving Eq.(39), we get for $k \neq 0$

$$(40) \quad x_n^2 = \frac{2k^2 - 1 \pm \sqrt{1 - 4k^2 y_n^2}}{2k^2}.$$

Eq.(30) and Eq.(32) give

$$(41) \quad R_2^{\text{cn}}(x_n, k) = \frac{1 - 2(1 - x_n^2) + k^2(1 - x_n^2)^2}{1 - k^2(1 - x_n^2)^2}$$

and

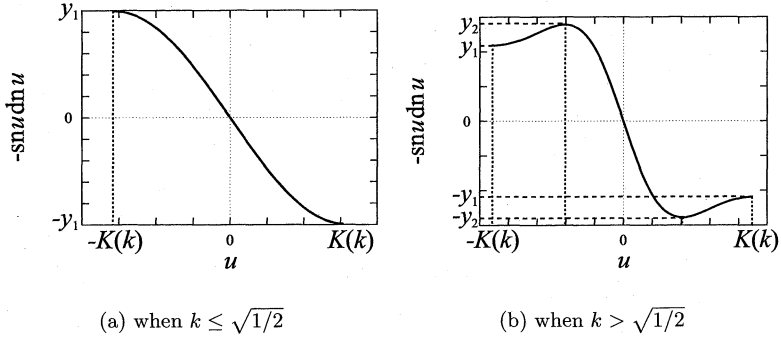
$$(42) \quad y_{n+1} = \sqrt{(1 - (R_2^{\text{cn}}(x_n, k))^2)(1 - k^2 + k^2(R_2^{\text{cn}}(x_n, k))^2)}.$$

Substituting Eq.(40) and Eq.(41) into Eq.(42), we have

$$(43) \quad y_{n+1} = \frac{2\sqrt{2}ky\sqrt{2k^2 - 1 \pm \sqrt{1 - 4k^2 y^2}}}{(2k^2 - 1 + 2k^2 y^2 \pm \sqrt{1 - 4k^2 y^2})^2} \times \left\{ 1 - 2k^2 y^2 \pm (2k^2 - 1)\sqrt{1 - 4k^2 y^2} \right\}.$$

where three \pm signs on R.H.S. are either + or -. Denote two maps by $\tau_y^{PP}(y)$ and $\tau_y^{NN}(y)$ when + and - are chosen on the R.H.S. of Eq.(43), respectively. Then

$$(44) \quad \begin{aligned} \tau_y(y) &= X_1(D \oplus Y_1)\tau_y^{PP}(y) + \overline{X_1(D \oplus Y_1)}(-\tau_y^{PP}(y)) \\ &+ X_1\overline{(D \oplus Y_1)}\tau_y^{NN}(y) + \overline{X_1(D \oplus Y_1)}(-\tau_y^{NN}(y)) \end{aligned}$$

§Appendix B. Inverse function Y [24]Fig. 5. $y = -\operatorname{sn} u \operatorname{dn} u$ ($y_1 = \sqrt{1-k^2}$ and $y_2 = 1/2k$, $k \neq 0$).When $0 < k \leq \sqrt{1/2}$,

$$(45) \quad u = \int_{-\operatorname{sn} u \operatorname{dn} u}^0 f_Y^+(y) dy.$$

When $k > \sqrt{1/2}$,

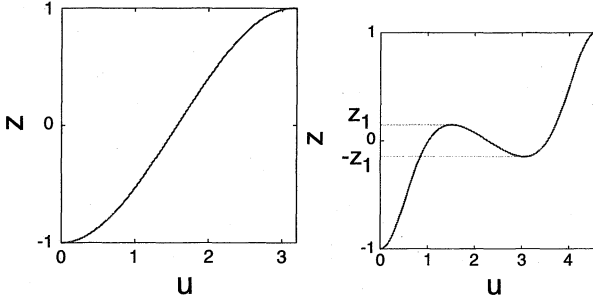
$$(46) \quad u = \begin{cases} \int_{-\operatorname{sn} u \operatorname{dn} u}^0 f_Y^+(y) dy, & \text{for } |u| \leq \operatorname{cn}^{-1} \sqrt{\frac{2k^2-1}{2k^2}} \\ \int_{\frac{1}{2k}}^0 f_Y^+(y) dy - \int_{-\operatorname{sn} u \operatorname{dn} u}^{\frac{1}{2k}} f_Y^-(y) dy, & \text{for } -K(k) \leq u < -\operatorname{cn}^{-1} \sqrt{\frac{2k^2-1}{2k^2}} \\ \int_{-\frac{1}{2k}}^0 f_Y^+(y) dy - \int_{-\operatorname{sn} u \operatorname{dn} u}^{-\frac{1}{2k}} f_Y^-(y) dy, & \text{for } \operatorname{cn}^{-1} \sqrt{\frac{2k^2-1}{2k^2}} < u \leq K(k) \end{cases}$$

where [24]

$$f_Y^\pm(y) dy = \frac{\sqrt{2k}}{\sqrt{(2k^2-1 \pm \sqrt{1-4k^2y^2})(1-4k^2y^2)}} dy$$

where the \pm sign on R.H.S is either $+$ or $-$ and is to be decided on the basis whether there is f_Y^+ or f_Y^- on the L.H.S.

§Appendix C. Inverse function Z [25]



(a) $k \leq \sqrt{1/2}$ (b) $k > \sqrt{1/2}$
 Fig. 6. $z = \text{cn } u(-1 + 2k^2 - 2k^2 \text{cn}^2 u)$, ($z_1 = r(k)$)

When $k \leq \sqrt{1/2}$ (see Fig.6(a)), simple differential calculation gives

$$(47) \quad \frac{d(\text{cn } u(-1 + 2k^2 - 2k^2 \text{cn}^2 u))}{du} = \sqrt{(1 - \text{cn}^2 u)(1 - k^2 + k^2 \text{cn}^2 u)} \times \{6k^2 \text{cn}^2 u - 2k^2 + 1\}.$$

Integrating each side of Eq.(47) over u , we have

$$(48) \quad u = \int_{-1}^{\text{cn } u(-1+2k^2-2k^2 \text{cn}^2 u)} \frac{dZ}{\sqrt{(1-X^2(Z))(1-k^2+k^2 X^2(Z))\{6k^2 X^2(Z)-2k^2+1\}}},$$

where $X^2(Z)$ is given by Eq.(34). ACI measure of the map τ_z is defined in the form of inverse of elliptic functions, i.e., elliptic integral.

$$(49) \quad u(z) = \int_{-1}^z f_z(\xi_1(Z))dZ, \quad \text{for } -1 \leq z \leq 1, k \leq \sqrt{1/2}.$$

The same discussion applies to $k > \sqrt{1/2}$ case with care to constants of integration (see Fig.6(b)).

(50)

$$\left\{ \begin{array}{l} u_1(z) = \int_{-1}^z f_Z(\xi_1(Z))dZ, \quad \text{for } -1 \leq z < -r(k) \\ u_2(z) = u_1(-r(k)) + \int_{-r(k)}^z f_Z(\xi_2(Z))dZ, \quad \text{for } -r(k) \leq z < 0 \\ u_3(z) = u_2(0) + \int_0^z f_Z(\xi_4(Z))dZ, \quad \text{for } 0 \leq z < r(k) \\ u_4(z) = u_3(r(k)) - \int_{r(k)}^z f_Z(\xi_3(Z))dZ, \quad \text{for } r(k) \geq z > -r(k) \\ u_5(z) = u_4(-r(k)) + \int_{-r(k)}^z f_Z(\xi_4(Z))dZ, \quad \text{for } -r(k) \leq z < 0 \\ u_6(z) = u_5(0) + \int_0^z f_Z(\xi_2(Z))dZ, \quad \text{for } 0 \leq z < r(k) \\ u_7(z) = u_6(r(k)) + \int_{r(k)}^z f_Z(\xi_1(Z))dZ, \quad r(k) \leq z \leq 1 \end{array} \right.$$

where $f_Z(X_i(Z))$ is given by Eq.(35) and

$$r(k) = \sqrt{\frac{2}{27}(-1 + 2k^2)^3}.$$

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