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Golden optimal path in discrete-time dynamic optimization processes

Seiichi Iwamoto and Masami Yasuda

Abstract.

We are concerned with dynamic optimization processes from a viewpoint of Golden optimality. A path is called Golden if any state moves to the next state repeating the same Golden section in each transition. A policy is called Golden if it, together with a relevant dynamics, yields a Golden path. The probelm is whether an optimal path/policy is Golden or not. This paper minimizes a quadratic criterion and maximizes a square-root criterion over an infinite horizon. We show that a Golden path is optimal in both optimizations. The Golden optimal path is obtained by solving a corresponding Bellman equation for dynamic programming. This in turn admits a Golden optimal policy.

§1. Introduction

Recently the Golden optimal solution, its duality, and its equivalence have been discussed in static optimization problems [4, 5, 6]. In this paper we consider the Golden optimal solution in dynamic optimization problems.

We consider two typical types of criterion — quadratic and square-root — in a deterministic optimization. We minimize quadratic criteria

$$I(x) = \sum_{n=0}^{\infty} \left[x_n^2 + (x_n - x_{n+1})^2\right], \quad J(x) = \sum_{n=0}^{\infty} \left[(x_n - x_{n+1})^2 + x_{n+1}^2\right]$$

and maximize square-root criteria

$$K(x) = \sum_{n=0}^{\infty} \beta^n \left(\sqrt{x_n} + \sqrt{x_n - x_{n+1}}\right),$$

$$L(x) = \sum_{n=0}^{\infty} \beta^n \left(\sqrt{x_n - x_{n+1}} + \sqrt{x_{n+1}}\right),$$

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respectively. Here β is $0 < \beta < 1$. The differences between I and J and between K and L are

$$\begin{aligned} J(x) &= I(x) - x_0^2 \\ L(x) &= K(x) + \sum_{n=0}^{\infty} \left(\beta^{n-1} - \beta^n\right) \sqrt{x_n} \quad (\beta^{-1} = 0). \end{aligned}$$

We show that a Golden path is optimal in these four optimization problems. The Golden optimal path is obtained by solving Bellman equation for dynamic programming [3, 7].

§2. Golden paths

A real number

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

is called *Golden number* [1, 2, 8]. It is the larger of the two solutions to quadratic equation

$$x^2 - x - 1 = 0. \tag{1}$$

Sometimes (1) is called *Fibonacci quadratic equation.* The Fibonacci quadratic equation has two real solutions: ϕ and its *conjugate* $\overline{\phi} := 1 - \phi$. We note that

$$\phi + \overline{\phi} = 1, \ \phi \cdot \overline{\phi} = -1.$$

Further we have

$$\begin{aligned} \phi^2 &= 1 + \phi, \ \overline{\phi}^2 = 2 - \phi \\ \phi^2 + \overline{\phi}^2 &= 3, \ \phi^2 \cdot \overline{\phi}^2 = 1. \end{aligned}$$

A point $(2-\phi)x$ splits an interval $[0, x]$ into two intervals $[0, (2-\phi)x]$ and $[(2-\phi)x, x]$. A point $(\phi - 1)x$ splits the interval into $[0, (\phi - 1)x]$ and $[(\phi - 1)x, x]$. In either case, the length constitutes the Golden ratio $(2 - \phi) : (\phi - 1) = 1 : \phi$. Thus both divisions are the *Golden section.*

Definition 2.1. *A sequence $x : \{0, 1, \ldots\} \to R^1$ is called Golden if and only if either*

$$\frac{x_{t+1}}{x_t} = \phi - 1 \ \textit{or} \ \frac{x_{t+1}}{x_t} = 2 - \phi.$$

Lemma 2.1. *A Golden sequence x is either*

$$x_t = x_0(\phi - 1)^t \ \textit{or} \ x_t = x_0(2 - \phi)^t.$$

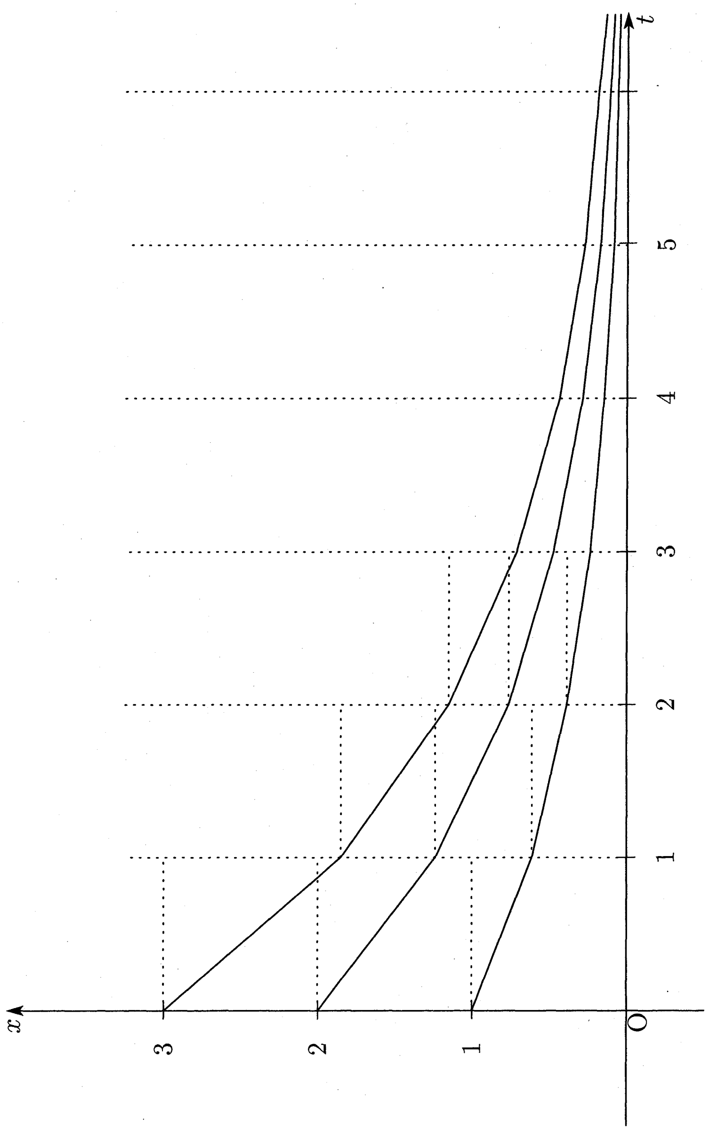

Fig. 1 Golden paths $x = c(\phi - 1)^t \; c = 1, 2, 3$

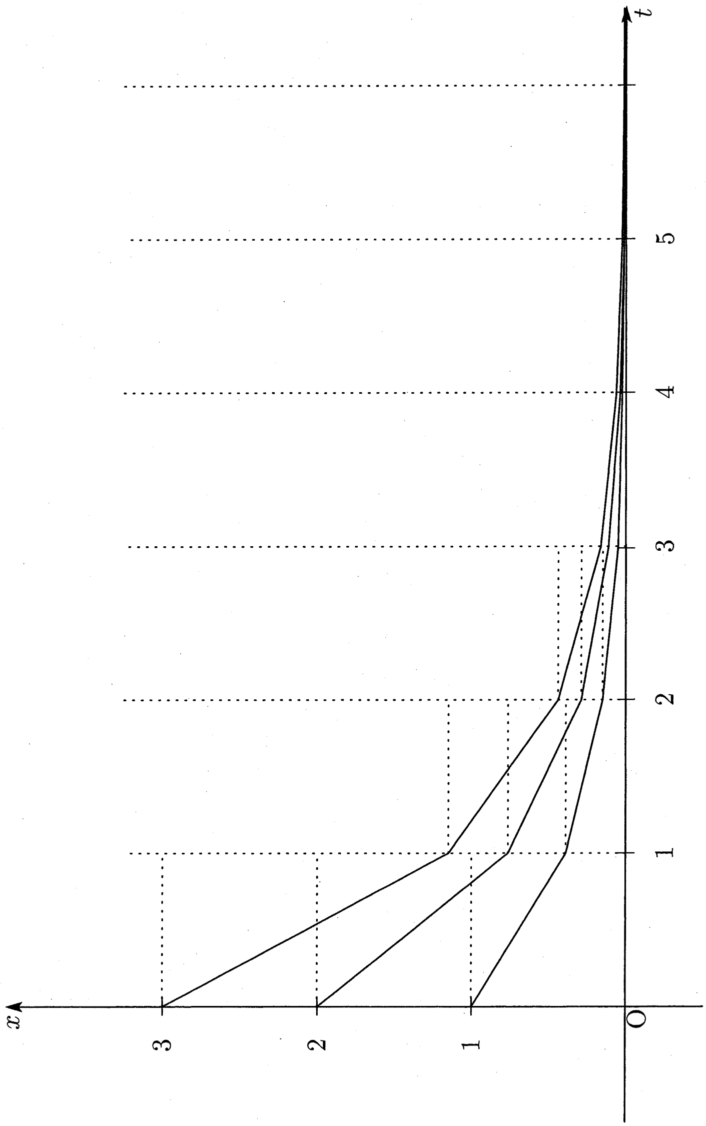

Fig. 2 Golden paths (c) $x = c(2-\phi)^t$ $c = 1, 2, 3$

We remark that

$$(\phi - 1)^t = \phi^{-t}, \ (2 - \phi)^t = (1 + \phi)^{-t}$$

where

$$\phi - 1 = \phi^{-1} \approx 0.618, \ 2 - \phi = (1 + \phi)^{-1} \approx 0.382$$

Let us introduce a controlled linear dynamics with real parameter b as follows.

$$x_{t+1} = bx_t + u_t \ t = 0, 1, \ldots \tag{2}$$

where $u : \{0, 1, \ldots\} \to R^1$ is called *control.* If $u_t = px_t$, the control u is called *proportional*, where p is a real constant, *proportional rate.* A sequence x satisfying (2) is called *path.*

Definition 2.2. *A proportional control u on dynamics (2) is called Golden if and only if it generates a Golden path x.*

Lemma 2.2. *A proportional control $u_t = px_t$ on (2) is Golden if and only if*

$$p = -b + \phi - 1 \ \textit{or} \ p = -b + 2 - \phi. \tag{3}$$

§3. Control processes

This section minimizes two quadratic cost functions

$$\sum_{n=0}^{\infty} \left[x_n^2 + (x_n - x_{n+1})^2\right] \text{ and } \sum_{n=0}^{\infty} \left[(x_n - x_{n+1})^2 + x_{n+1}^2\right].$$

Both problems are solved as a control process with criterion

$$\sum_{n=0}^{\infty} \left(x_n^2 + u_n^2\right) \text{ and } \sum_{n=0}^{\infty} \left(u_n^2 + x_{n+1}^2\right)$$

under a common additive dynamics with a given initial state

$$x_{n+1} = bx_n + u_n, \ x_0 = c$$

where $c \in R^1$.

3.1. Quadratic in current state

Let us now consider a control process with an additive transition $T(x, u) = bx + u$. Here b is a constant, which represents a characteristics of the process :

$$\mathrm{C}(c) \qquad \begin{array}{lll} \text{minimize} & \displaystyle\sum_{n=0}^{\infty} (x_n^2 + u_n^2) & \\ \text{subject to} & \text{(i)} \quad x_{n+1} = bx_n + u_n & \\ & \text{(ii)} \quad -\infty < u_n < \infty & n \geq 0 \\ & \text{(iii)} \quad x_0 = c. & \end{array}$$

Let $v(c)$ be the minimum value of $\mathrm{C}(c)$. Then the value function v satisfies Bellman equation [3]:

$$v(x) = \min_{-\infty<u<\infty} \left[x^2 + u^2 + v(bx + u)\right]. \tag{4}$$

Eq. (4) has a quadratic form $v(x) = vx^2$, where $v \in R^1$.

Lemma 3.1. *The control process* $\mathrm{C}(c)$ *with characteristic value* b $(\in R^1)$ *has a proportional optimal policy* $f^\infty, f(x) = px$, *and a quadratic minimum value function* $v(x) = vx^2$, *where*

$$v = \frac{b^2 + \sqrt{b^4 + 4}}{2}, \quad p = -\frac{v}{1+v}b.$$

The proportional optimal policy f^∞ splits at any time an interval $[0, x]$ into $[0, (b+p)x] = \left[0, \dfrac{bx}{1+v}\right]$ and $\left[\dfrac{bx}{1+v}, x\right]$. In particular, when $b = 1$, the quadratic coefficient v is reduced to the *Golden number*

$$\phi = \frac{1 + \sqrt{5}}{2} \approx 1.618$$

Further the division of $[0, x]$ into $\left[0, \dfrac{x}{1+\phi}\right]$ and $\left[\dfrac{x}{1+\phi}, x\right]$ is Golden. A quadratic function $w(x) = ax^2$ is called *Golden* if $a = \phi$.

Theorem 3.1. *The control process* $\mathrm{C}(c)$ *with characteristic value* $b = 1$ *has a Golden optimal policy* $f^\infty, f(x) = (1-\phi)x$, *and the Golden quadratic minimum value function* $v(x) = \phi x^2$.

3.2. Qquadratic in next state

Here we consider the cost function $r : X{\times}U \to R^1$ which is quadratic in current control and next state :

$$r(x,u) \;=\; u^2 + (bx+u)^2.$$

Then a control process is represented by the following sequential minimization problem :

$$\mathrm{C}'(c) \qquad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{n=0}^{\infty} \left(u_n^2 + x_{n+1}^2\right) \\ \text{subject to} & \text{(i)} \quad x_{n+1} = bx_n + u_n \\ & \text{(ii)} \quad -\infty < u_n < \infty \\ & \text{(iii)} \quad x_0 = c. \end{array} \qquad n \geq 0$$

The value function v satisfies Bellman equation [3]:

$$v(x) \;=\; \min_{-\infty<u<\infty} \left[u^2 + (bx+u)^2 + v(bx+u)\right]. \tag{5}$$

Eq. (5) has a quadratic solution $v(x) = vx^2$, where $v \in R^1$.

Lemma 3.2. *The control process* C$'(c)$ *with characteristic value* b *has a proportional optimal policy* $f^\infty, f(x) = px$, *and a quadratic minimum value function* $v(x) = vx^2$, *where*

$$v \;=\; \frac{b^2 - 2 + \sqrt{b^4+4}}{2}, \quad p \;=\; -\frac{1+v}{2+v}b.$$

The policy f^∞ splits an interval $[0,\ x]$ into $\left[0,\ \dfrac{bx}{2+v}\right]$ and $\left[\dfrac{bx}{2+v},\ x\right]$. When $b = 1$, the coefficient v is reduced to the *inverse Golden number*

$$\phi^{-1} = \phi - 1 = \frac{-1+\sqrt{5}}{2} \approx 0.618$$

Further the division of $[0,\ x]$ into $[0,\ (2-\phi)x]$ and $[(2-\phi)x,\ x]$ is Golden. A quadratic function $w(x) = ax^2$ is called *inverse Golden* if $a = \phi^{-1}$.

Theorem 3.2. *The control process* C$'(c)$ *with characteristic value* $b = 1$ *has a Golden optimal policy* $f^\infty, f(x) = (1-\phi)x$, *and the inverse Golden quadratic minimum value function* $v(x) = (\phi - 1)x^2$.

§4. Allocation processes

This section maximizes two discounted square-root reward functions

$$\sum_{n=0}^{\infty} \beta^n \left(\sqrt{x_n} + \sqrt{x_n - x_{n+1}}\right) \text{ and } \sum_{n=0}^{\infty} \beta^n \left(\sqrt{x_n - x_{n+1}} + \sqrt{x_{n+1}}\right).$$

Both problems are solved as an allocation process with criterion

$$\sum_{n=0}^{\infty} \beta^n \left(\sqrt{x_n} + \sqrt{u_n}\right) \text{ and } \sum_{n=0}^{\infty} \beta^n \left(\sqrt{u_n} + \sqrt{x_{n+1}}\right)$$

under a common subtractive dynamics with a given initial state

$$x_{n+1} = x_n - u_n, \ x_0 = c$$

where $c \geq 0$.

4.1. Square-root in current state

Let us now consider an allocation process with a subtractive transition $T(x, u) = x - u$:

$$\mathrm{A}(c) \qquad \begin{array}{lll} \text{Maximize} & \displaystyle\sum_{n=0}^{\infty} \beta^n \left(\sqrt{x_n} + \sqrt{u_n}\right) & \\ \text{subject to} & \text{(i)} \quad x_{n+1} = x_n - u_n & \\ & \text{(ii)} \quad 0 \leq u_n \leq x_n & n \geq 0 \\ & \text{(iii)} \quad x_0 = c. & \end{array}$$

Let $v(c)$ be the maximum value of $\mathrm{A}(c)$. Then the maximum value function v satisfies the following Bellman equation:

$$v(x) = \max_{0 \leq u \leq x} \left[\sqrt{x} + \sqrt{u} + \beta v(x - u)\right]. \tag{6}$$

Eq. (6) has a square-root form $v(x) = v\sqrt{x}$, where $v \in R^1$.

Let us adopt a proportional policy f^∞ ($f(x) = px$) with proportional rate p ($0 < p < 1$). Then state x under the control $u = px$ goes deterministically to the next state $T(x, u) = x - u = x - px = (1 - p)x$. Thus we have $x = (1 - p)x + px$. The state transition of control process $\mathrm{A}(c)$ governed by the proportional policy f^∞ means that the current control $u = px$ splits the state interval $[0, x]$ into two intervals $[0, (1 - p)x]$ and $[(1 - p)x, x]$. When the split yields a *Golden section*, the proportional policy f^∞ ($f(x) = px$) is called *Golden.*

Lemma 4.1. *The allocation process* A(c) *has a proportional optimal policy* $f^\infty, f(x) = px$, *and a square-root maximum value function* $v(x) = v\sqrt{x}$, *where*

$$v = \frac{2}{1-\beta^2}, \; p = \frac{(1-\beta^2)^2}{(1+\beta^2)^2}.$$

We remark that the coefficient v is the solution to

$$v = 1 + \sqrt{1 + (\beta v)^2}, \; v \geq 2.$$

Let us solve $1 - p = \phi - 1$ or $2 - \phi$. Then we have the following result.

Theorem 4.1. *When* $\beta = \phi(1 - \sqrt{\phi - 1}) \approx 0.346$ *or* $\beta = \sqrt{\phi} - \sqrt{\phi - 1} \approx 0.486$, *the proportional policy* $f^\infty, f(x) = px$, *is Golden optimal.*

4.2. Square-root in next state

Now we consider an allocation process with transition $T(x, u) = x - u$:

$$\mathrm{A}'(c) \qquad \begin{array}{lll} \text{Maximize} & \displaystyle\sum_{n=0}^{\infty} \beta^n \left(\sqrt{u_n} + \sqrt{x_{n+1}}\right) & \\ \text{subject to} & \text{(i)} \quad x_{n+1} = x_n - u_n & \\ & \text{(ii)} \quad 0 \leq u_n \leq x_n & n \geq 0 \\ & \text{(iii)} \quad x_0 = c. & \end{array}$$

Let $v(c)$ be the maximum value of A$'(c)$. Then the maximum value function v satisfies an optimality equation:

$$v(x) = \operatorname*{Max}_{0 \leq u \leq x} \left[\sqrt{u} + \sqrt{x-u} + \beta v(x-u)\right]. \tag{7}$$

Eq. (7) has a square-root solution $v(x) = v\sqrt{x}$, where $v \in R^1$.

Let us adopt a proportional policy f^∞ ($f(x) = px$) with p ($0 < p < 1$). Then the current control $u = px$ splits the interval $[0, x]$ into $[0, (1-p)x]$ and $[(1-p)x, x]$.

Lemma 4.2. *The allocation process* A$'(c)$ *has a proportional optimal policy* $f^\infty, f(x) = px$, *and a square-root maximum value function* $v(x) = v\sqrt{x}$, *where*

$$v = \frac{\beta + \sqrt{2-\beta^2}}{1-\beta^2}, \; p = \frac{1 - \beta\sqrt{2-\beta^2}}{2}.$$

Note that the coefficient v is the positive solution to

$$v = \sqrt{1 + (1 + \beta v)^2}.$$

By solving $1 - p = \phi - 1$, we have the following result.

Theorem 4.2. *When* $\beta = \sqrt{1 - 2\sqrt{2\phi - 3}} \approx 0.168$, *the proportional policy* $f^\infty, f(x) = px$, *is Golden optimal.*

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Seiichi Iwamoto
Department of Economic Engineering
Graduate School of Economics
Kyushu University
Fukuoka 812-8581, Japan

Masami Yasuda
Department of Mathematical Science
Faculty of Science, Chiba University
Chiba 263-8522, Japan

E-mail address: `iwamoto@en.kyushu-u.ac.jp`
`yasuda@math.s.chiba-u.ac.jp`