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Asymptotic properties of solutions of the discrete analogue of the Emden-Fowler equation

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Abstract.

This contribution investigates the discrete analogue of the Emden-Fowler differential equation. The main attention is paid to the asymptotic behavior of solution of this equation. This work takes up again the research started in paper [4], where the authors have already presented some asymptotic estimates of the solution of the considered equation. The aim of this paper is to improve these estimates.

§1. Introduction

The well-known Emden-Fowler differential equation has been investigated from many points of view, e.g. in the book [1, Chapter VII]. The special case of it is the equation

$$u''(t) - t^{-2}u^n(t) = 0$$

with $n > 1$.

Substitution

$$t = \exp(s), \quad u(s) = Cz^\alpha(s),$$

where

$$\alpha = -1/(n-1) \quad \text{and} \quad C^{n-1} = -\alpha,$$

leads (under the supposition $z \neq 0$) to the equation

$$z'' + \frac{(\alpha-1)(z')^2}{z} - z' + 1 = 0. \tag{1}$$

We will investigate the discrete analogue of equation (1), i.e. the second-order difference equation

$$\Delta^2 v(k) + \frac{(\alpha-1)(\Delta v(k))^2}{v(k)} - \Delta v(k) + 1 = 0, \tag{2}$$

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where $\alpha \in \mathbb{R}$, $\alpha < 0$, $k \in N(a) := \{a, a+1, \dots\}$, $a \in \mathbb{N}$, and $\Delta v(k) = v(k+1) - v(k)$.
Equation (2) can be rewritten as a system of two first-order difference equations (for the details, see [4])

$$\begin{aligned} \Delta u_1(k) &= u_1(k) - \frac{\alpha - 1}{k(1 + u_2(k))} \cdot (1 + u_1(k))^2, \\ \Delta u_2(k) &= \frac{1}{k+1}(-u_2(k) + u_1(k)), \end{aligned} \tag{3}$$

where u_1, u_2, v and Δv satisfy

$$v(k) = k(1 + u_2(k)), \tag{4}$$

$$\Delta v(k) = 1 + u_1(k). \tag{5}$$

§2. Summary of the previous results

System (3) can be seen as a special case of the general system of two difference equations

$$\begin{aligned} \Delta u_1(k) &= f_1(k, u_1(k), u_2(k)), \\ \Delta u_2(k) &= f_2(k, u_1(k), u_2(k)) \end{aligned} \tag{6}$$

with $f_1, f_2 : N(a) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.
In paper [4] one can find sufficient conditions guaranteeing the existence of at least one solution $u(k) = (u_1^*(k), u_2^*(k))$, $k \in N(a)$, of system (6) satisfying

$$\begin{aligned} b_1(k) &< u_1^*(k) < c_1(k), \\ b_2(k) &< u_2^*(k) < c_2(k) \end{aligned}$$

where $b_i, c_i : N(a) \to \mathbb{R}$, $i = 1, 2$, are auxiliary functions such that $b_i(k) < c_i(k)$ for every $k \in N(a)$.
The main result of [4] can be summarized in the following theorem.

Theorem 1. *Let $b_i(k), c_i(k), b_i(k) < c_i(k), i = 1, 2,$ be real functions defined on $N(a)$ and let $f_i : N(a) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, i = 1, 2$ be functions that are continuous with respect to their last two arguments.*
Suppose that for all points (u_1, u_2), such that $b_1(k) \le u_1 \le c_1(k)$ and $b_2(k) \le u_2 \le c_2(k)$ for some $k \in N(a)$, the following four conditions

hold:

(7)
$$u_1 = b_1(k) \quad \Rightarrow \qquad f_1(k,u_1,u_2) < b_1(k+1) - b_1(k),$$
(8)
$$u_1 = c_1(k) \quad \Rightarrow \qquad f_1(k,u_1,u_2) > c_1(k+1) - c_1(k),$$
(9)
$$u_2 = b_2(k) \quad \Rightarrow \quad b_2(k+1) - b_2(k) < f_2(k,u_1,u_2) < c_2(k+1) - b_2(k),$$
(10)
$$u_2 = c_2(k) \quad \Rightarrow \quad b_2(k+1) - c_2(k) < f_2(k,u_1,u_2) < c_2(k+1) - c_2(k).$$

Let, moreover, function $F(w) = w + f_2(k,u_1,w)$ be monotone for every fixed $(k,u_1) \in \{(k,u_1) : k \in N(a), b_1(k) \le u_1 \le c_1(k)\}$ on the interval $b_2(k) \le w \le c_2(k)$.

Then there exists a solution $u = (u_1^*(k), u_2^*(k))$ of system (6) satisfying the inequalities

$$\begin{aligned} b_1(k) &< u_1^*(k) < c_1(k), \\ b_2(k) &< u_2^*(k) < c_2(k) \end{aligned}$$

for every $k \in N(a)$.

Applying this general result to equation (3), in the same paper it was shown that there exists a solution of system (3) satisfying for k sufficiently large the conditions

$$-\left(\frac{1}{k}\right)^{\nu_i} < u_i(k) < \left(\frac{1}{k}\right)^{\nu_i}$$

for $i = 1,2$, where $0 < \nu_2 < \nu_1 < 1$. Rewritten in the terms of the second order equation (2) , it gives

$$|v(k) - k| < k \cdot \left(\frac{1}{k}\right)^{\nu_2} \tag{11}$$

and

$$|\Delta v(k) - 1| < \left(\frac{1}{k}\right)^{\nu_1} . \tag{12}$$

When this result was presented, the question came from the audience, whether this estimate of the solution could not be improved. Our present contribution gives a partial answer to this question (the investigation of this problem continues).

§3. New asymptotic estimate of the solution

We show that inequalities (11), (12) can be changed with the couple of inequalities (22), (23) below, where (23) substantially improves (12).

Theorem 2. *Let numbers ν_1, ν_2, $1 < \nu_1 < 2$, $0 < \nu_2 < 1$, $1 + \nu_2 > \nu_1$, be given. Then the system of difference equations (3) has for sufficiently large $a \in \mathbb{N}$ a solution $u(k) = (u_1(k), u_2(k))$ such that*

$$\frac{\alpha-1}{k} - \left(\frac{1}{k}\right)^{\nu_1} < u_1(k) < \frac{\alpha-1}{k} + \left(\frac{1}{k}\right)^{\nu_1}, \tag{13}$$

$$-\left(\frac{1}{k}\right)^{\nu_2} < u_2(k) < \left(\frac{1}{k}\right)^{\nu_2} \tag{14}$$

for $k \in N(a)$.

Proof. Verify the conditions of Theorem 1 with

$$\begin{aligned} f_1(k, u_1, u_2) &= u_1 - \frac{\alpha-1}{k(1+u_2)}(1+u_1)^2, \\ f_2(k, u_1, u_2) &= \frac{1}{k+1}(-u_2+u_1), \end{aligned}$$

$$b_1(k) = \frac{\alpha-1}{k} - \left(\frac{1}{k}\right)^{\nu_1}, \qquad c_1(k) = \frac{\alpha-1}{k} + \left(\frac{1}{k}\right)^{\nu_1}$$

and

$$b_2(k) = -\left(\frac{1}{k}\right)^{\nu_2}, \quad c_2(k) = \left(\frac{1}{k}\right)^{\nu_2}.$$

With respect to condition (7) we have to verify that

$$f_1(k, u_1, u_2) < b_1(k+1) - b_1(k)$$

if

$$u_1 = \frac{\alpha-1}{k} - \left(\frac{1}{k}\right)^{\nu_1} \text{ and } -\left(\frac{1}{k}\right)^{\nu_2} \le u_2 \le \left(\frac{1}{k}\right)^{\nu_2},$$

which gives

$$\begin{aligned} \frac{\alpha-1}{k} - \left(\frac{1}{k}\right)^{\nu_1} - \frac{\alpha-1}{k(1+u_2)}\left(1 + \frac{\alpha-1}{k} - \left(\frac{1}{k}\right)^{\nu_1}\right)^2 & \\ < \frac{\alpha-1}{k+1} - \left(\frac{1}{k+1}\right)^{\nu_1} - \frac{\alpha-1}{k} + \left(\frac{1}{k}\right)^{\nu_1}&. \end{aligned} \tag{15}$$

Denote $L_{(15)}$ and $R_{(15)}$ the left-hand and the right-hand side of inequality (15), respectively.

In the following estimate of $L_{(15)}$ we will use the fact that $\alpha < 0$, i.e. also $\alpha - 1 < 0$, and the fact that $1/(1-x) < 1 + 2x$ for $x > 0$ sufficiently close to zero.

$$
\begin{aligned}
L_{(15)} &\leq \frac{\alpha-1}{k} - \left(\frac{1}{k}\right)^{\nu_1} - \frac{\alpha-1}{k(1+u_2)}\left(1+\frac{\alpha-1}{k}\right)^2 \\
&\leq \frac{\alpha-1}{k} - \left(\frac{1}{k}\right)^{\nu_1} - \frac{\alpha-1}{k(1-(\frac{1}{k})^{\nu_2})}\left(1+\frac{\alpha-1}{k}\right)^2 \\
&\leq \frac{\alpha-1}{k} - \left(\frac{1}{k}\right)^{\nu_1} \\
&\quad - \frac{\alpha-1}{k}\left(1+2\left(\frac{1}{k}\right)^{\nu_2}\right)\left(1+2\,\frac{\alpha-1}{k}+\frac{(\alpha-1)^2}{k^2}\right) \\
&= \frac{\alpha-1}{k} - \frac{1}{k^{\nu_1}} \\
&\quad - \frac{\alpha-1}{k}\left(1+2\,\frac{\alpha-1}{k}+\frac{(\alpha-1)^2}{k^2}+2\,\frac{1}{k^{\nu_2}}\right. \\
&\qquad\qquad\qquad \left. +4\,\frac{\alpha-1}{k^{1+\nu_2}}+2\,\frac{(\alpha-1)^2}{k^{2+\nu_2}}\right) \\
&= -\frac{1}{k^{\nu_1}} - 2\,\frac{(\alpha-1)^2}{k^2} - \frac{(\alpha-1)^3}{k^3} \\
&\quad -2\,\frac{\alpha-1}{k^{1+\nu_2}} - 4\,\frac{(\alpha-1)^2}{k^{2+\nu_2}} - 2\,\frac{(\alpha-1)^3}{k^{3+\nu_2}}\,.
\end{aligned}
$$

As, due to the assumptions of the Theorem, $1 < \nu_1 < 2$, $0 < \nu_2 < 1$ and $1 + \nu_2 > \nu_1$, one can state that $L_{(15)}$ is negative for k sufficiently large.

As for $R_{(15)}$, it can be simplified to

$$
R_{(15)} = -\frac{\alpha-1}{k(k+1)} + \left(\frac{1}{k}\right)^{\nu_1} - \left(\frac{1}{k+1}\right)^{\nu_1} > 0.
$$

Thus, inequality (15) holds.

Now let us prove inequality (9), i.e.

$$
f_1(k, u_1, u_2) > c_1(k+1) - c_1(k)
$$

if

$$
u_1 = \frac{\alpha-1}{k} + \left(\frac{1}{k}\right)^{\nu_1} \quad \text{and} \quad -\left(\frac{1}{k}\right)^{\nu_2} \leq u_2 \leq \left(\frac{1}{k}\right)^{\nu_2},
$$

which gives

$$
\begin{aligned}
(16)\qquad & \frac{\alpha-1}{k}+\left(\frac{1}{k}\right)^{\nu_1}-\frac{\alpha-1}{k(1+u_2)}\left(1+\frac{\alpha-1}{k}+\left(\frac{1}{k}\right)^{\nu_1}\right)^2\\
&\qquad > \frac{\alpha-1}{k+1}+\left(\frac{1}{k+1}\right)^{\nu_1}-\frac{\alpha-1}{k}-\left(\frac{1}{k}\right)^{\nu_1}.
\end{aligned}
$$

Again, denote $L_{(16)}$ and $R_{(16)}$ the left-hand and the right-hand side of (16), respectively.

This time we will use the fact that $1/(1+x) > 1-2x$ for $x>0$.

$$
\begin{aligned}
L_{(16)} \;\geq\; & \frac{\alpha-1}{k}+\left(\frac{1}{k}\right)^{\nu_1}-\frac{\alpha-1}{k(1+u_2)}\left(1+\frac{\alpha-1}{k}\right)^2\\
\geq\; & \frac{\alpha-1}{k}+\left(\frac{1}{k}\right)^{\nu_1}-\frac{\alpha-1}{k(1+(\frac{1}{k})^{\nu_2})}\left(1+\frac{\alpha-1}{k}\right)^2\\
\geq\; & \frac{\alpha-1}{k}+\left(\frac{1}{k}\right)^{\nu_1}\\
& -\frac{\alpha-1}{k}\left(1-2\left(\frac{1}{k}\right)^{\nu_2}\right)\left(1+2\,\frac{\alpha-1}{k}+\frac{(\alpha-1)^2}{k^2}\right)\\
=\; & \frac{\alpha-1}{k}+\frac{1}{k^{\nu_1}}\\
& -\frac{\alpha-1}{k}\left(1+2\,\frac{\alpha-1}{k}+\frac{(\alpha-1)^2}{k^2}-2\,\frac{1}{k^{\nu_2}}\right.\\
& \qquad\qquad\left.-4\,\frac{\alpha-1}{k^{1+\nu_2}}-2\,\frac{(\alpha-1)^2}{k^{2+\nu_2}}\right)\\
=\; & \frac{1}{k^{\nu_1}}-2\,\frac{(\alpha-1)^2}{k^2}-\frac{(\alpha-1)^3}{k^3}\\
& +2\,\frac{\alpha-1}{k^{1+\nu_2}}+4\,\frac{(\alpha-1)^2}{k^{2+\nu_2}}+2\,\frac{(\alpha-1)^3}{k^{3+\nu_2}}\\
=\; & O\left(\frac{1}{k^{\nu_1}}\right).
\end{aligned}
$$

For the estimate of $R_{(16)}$, let us use the fact (gained with the help of the Mean Value Theorem) that

$$
(17)\qquad \left(\frac{1}{k+1}\right)^{\nu_1}-\left(\frac{1}{k}\right)^{\nu_1}=-\nu_1\left(\frac{1}{\xi}\right)^{\nu_1+1},
$$

where $k \leq \xi \leq k+1$. That gives

$$R_{(16)} \leq -\frac{\alpha-1}{k(k+1)} - \frac{\nu_1}{(k+1)^{1+\nu_1}} = O\left(\frac{1}{k^2}\right)$$

and inequality (16) is verified.

The proof of inequalities (10) and (11) is easier.

First let us take the left part of inequality (10):

$$b_2(k+1) - b_2(k) < f_2(k, u_1, u_2)$$

if

$$u_2 = -\left(\frac{1}{k}\right)^{\nu_2} \quad \text{and} \quad \frac{\alpha-1}{k} - \left(\frac{1}{k}\right)^{\nu_1} \leq u_1 \leq \frac{\alpha-1}{k} + \left(\frac{1}{k}\right)^{\nu_1},$$

which gives

$$-\left(\frac{1}{k+1}\right)^{\nu_2} + \left(\frac{1}{k}\right)^{\nu_2} < \frac{1}{k+1}\left(\left(\frac{1}{k}\right)^{\nu_2} + u_1\right). \tag{18}$$

Using a relation similar to (17), the left-hand side of (18) can be estimated as follows

$$-\left(\frac{1}{k+1}\right)^{\nu_2} + \left(\frac{1}{k}\right)^{\nu_2} \leq \frac{\nu_2}{k^{1+\nu_2}}.$$

The right-hand side of (18) can be estimated as

$$\begin{aligned} \frac{1}{k+1}\left(\left(\frac{1}{k}\right)^{\nu_2} + u_1\right) &\geq \frac{1}{k+1}\left(\left(\frac{1}{k}\right)^{\nu_2} + \frac{\alpha-1}{k} - \left(\frac{1}{k}\right)^{\nu_1}\right) \\ &= \frac{1}{(k+1)k^{\nu_2}} + O\left(\frac{1}{k^2}\right). \end{aligned}$$

That proves inequality (18) because $\nu_2 < 1$ and thus

$$\frac{\nu_2}{k^{1+\nu_2}} < \frac{1}{(k+1)k^{\nu_2}} + O\left(\frac{1}{k^2}\right)$$

for k sufficiently large.

The right part of inequality (10), i.e.

$$f_2(k, u_1, u_2) < c_2(k+1) - b_2(k)$$

if

$$u_2 = -\left(\frac{1}{k}\right)^{\nu_2} \quad \text{and} \quad \frac{\alpha-1}{k} - \left(\frac{1}{k}\right)^{\nu_1} \leq u_1 \leq \frac{\alpha-1}{k} + \left(\frac{1}{k}\right)^{\nu_1},$$

in our case gives

$$\frac{1}{k+1}\left(\left(\frac{1}{k}\right)^{\nu_2}+u_1\right)<\left(\frac{1}{k+1}\right)^{\nu_2}+\left(\frac{1}{k}\right)^{\nu_2}. \tag{19}$$

The maximum possible value of the left-hand side of (19) is

$$\frac{1}{k+1}\left(\left(\frac{1}{k}\right)^{\nu_2}+\frac{\alpha-1}{k}+\left(\frac{1}{k}\right)^{\nu_1}\right)=O\left(\frac{1}{k^{1+\nu_2}}\right),$$

meanwhile the right-hand side of (19) is $O\left(k^{-\nu_2}\right)$ and hence inequality (19) holds.

Condition (11) can be proved in a very similar way. First its left part:

$$b_2(k+1)-c_2(k)<f_2(k,u_1,u_2)$$

if

$$u_2=\left(\frac{1}{k}\right)^{\nu_2} \quad \text{and} \quad \frac{\alpha-1}{k}-\left(\frac{1}{k}\right)^{\nu_1}\leq u_1\leq\frac{\alpha-1}{k}+\left(\frac{1}{k}\right)^{\nu_1}.$$

That gives

$$-\left(\frac{1}{k+1}\right)^{\nu_2}-\left(\frac{1}{k}\right)^{\nu_2}<\frac{1}{k+1}\left(-\left(\frac{1}{k}\right)^{\nu_2}+u_1\right).$$

Multiplying it by -1, we get

$$\left(\frac{1}{k+1}\right)^{\nu_2}+\left(\frac{1}{k}\right)^{\nu_2}>\frac{1}{k+1}\left(\left(\frac{1}{k}\right)^{\nu_2}-u_1\right). \tag{20}$$

As the left-hand side of (20) is $O\left(k^{-\nu_2}\right)$ and the right-hand side is $O\left(k^{-(1+\nu_2)}\right)$, inequality (20) has be fulfilled.

Now prove the right part of condition (11):

$$f_2(k,u_1,u_2)<c_2(k+1)-c_2(k)$$

if

$$u_2=\left(\frac{1}{k}\right)^{\nu_2} \quad \text{and} \quad \frac{\alpha-1}{k}-\left(\frac{1}{k}\right)^{\nu_1}\leq u_1\leq\frac{\alpha-1}{k}+\left(\frac{1}{k}\right)^{\nu_1},$$

i.e.

$$\frac{1}{k+1}\left(-\left(\frac{1}{k}\right)^{\nu_2}+u_1\right)<\left(\frac{1}{k+1}\right)^{\nu_2}-\left(\frac{1}{k}\right)^{\nu_2}.$$

Again we will change the signes:

$$\frac{1}{k+1}\left(\left(\frac{1}{k}\right)^{\nu_2}-u_1\right)>\left(\frac{1}{k}\right)^{\nu_2}-\left(\frac{1}{k+1}\right)^{\nu_2}. \tag{21}$$

The left-hand side of (21) can be expressed as

$$\frac{1}{(k+1)k^{\nu_2}} + O\left(\frac{1}{k^2}\right),$$

the maximum possible value of the right hand side is $\nu_2/k^{1+\nu_2}$ and thus inequality (21) holds.

Finally, the function

$$F(w) = w + f_2(k, u_1, w) = w + \frac{1}{k+1}(-w + u_1)$$

is monotone for every fixed (k, u_1) such that $k \in N(a), b_1(k) \le u_1 \le c_1(k)$ on the interval $b_2(k) \le w \le c_2(k)$ since its derivative

$$F'(w) = 1 - \frac{1}{k+1} = \frac{k}{k+1}$$

is positive for $k \in N(a)$. Then, in accordance with the statement of Theorem 1, there exists a solution of system (3) satisfying inequalities (13) and (14). Q.E.D.

Now we are able to formulate the final result.

Theorem 3. *Let numbers ν_1, ν_2, $1 < \nu_1 < 2$, $0 < \nu_2 < 1$, $1 + \nu_2 > \nu_1$, be given. Then there exists a solution $v(k)$ of equation (2), such that*

$$|v(k) - k| < k \cdot \left(\frac{1}{k}\right)^{\nu_2} \tag{22}$$

and

$$\left|\Delta v(k) - 1 - \frac{\alpha - 1}{k}\right| < \left(\frac{1}{k}\right)^{\nu_1} \tag{23}$$

for k sufficiently large.

Proof. The statement is a simple consequence of Theorem 3 since u_1, u_2, v and Δv are connected by formulae (4), (5). Q.E.D.

Remark 1. The proof of Theorem 1 (see [4]) connects the so called retract technique with Lyapunov approach. For further results we refer e.g. to [2, 3, 5, 6].

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