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Kähler Ricci solitons

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§1. Introduction

We give a survey of Ricci solitons in a Kähler background. The emphasis is on joint work with Christina Tønnesen–Friedman and Galliano Valent [11].

Let (M, J) be a complex manifold. Consider pairs (g, V) consisting of a Kähler metric g and a real holomorphic vector field V on M, such that JV is an isometry of g and

$$\rho - \lambda\Omega = L_V\Omega, \tag{1}$$

where ρ is the Ricci form, Ω is the Kähler form and λ is a constant. Such structures are called *quasi-Einstein Kähler metrics* or *Kähler Ricci solitons* [4, 5, 7, 8, 12].

Remark 1. Quasi-Einstein metrics are solitons for the Hamilton flow [8]

$$\frac{d}{dt}g_t = -r_t + \frac{\overline{s_t}}{n}g_t, \tag{2}$$

where r_t is the Ricci curvature tensor and $\overline{s_t}$ is the average scalar curvature of g_t. Indeed, if g_0 is quasi-Einstein then $(\Phi_{-t})^*g_0$ solves (2), where $\Phi_t = \exp(tV)$. Thus if g_0 is quasi-Einstein but not Einstein, then g_t does not converge to an Einstein metric – it flows along V as a soliton.

Remark 2. Friedan [6] studied quasi-Einstein metrics in connection with bosonic σ-models. He showed that the one-loop renormalizability

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of the model is ensured if and only if the metric of the target space is quasi-Einstein.

Remark 3. If M is a compact manifold, then the equation

$$\rho - \lambda\Omega = L_V\Omega,$$

implies that $[\rho] = \lambda[\Omega] \in H^2(M,\mathbb{R})$. Thus a necessary condition for M to admit a quasi-Einstein Kähler metric is that $c_1(M) = [\frac{\rho}{2\pi}]$ has a sign. If $c_1 \leq 0$, then Calabi, Yau and Aubin showed that there exist Kähler-Einstein metrics. However, for $c_1 > 0$ we do not always have Kähler-Einstein metrics and the quasi-Einstein Kähler metrics serve as suitable generalizations.

Remark 4. Let M be compact and assume $c_1(M)$ is positive. Assume M is a Kähler-Ricci soliton with non-trivial V. Recall [2], that for any compact Kähler manifold, using Hodge theory and Kähler identities, we have

$$\rho - \rho_H = \sqrt{-1}\partial\overline{\partial}\varphi_\Omega = L_{\frac{1}{2}\nabla\varphi_\Omega}\Omega,$$

where ρ_H is the harmonic part of ρ and φ_Ω is called the Ricci potential. Indeed, we have $\varphi_\Omega = -Gs$, where G is the Green's operator of the Laplacian and s is the scalar curvature. Furthermore, the Futaki invariant of the Kähler class [2] associates to each holomorphic vector field X the integral

$$(m!)^{-1}\int_{M^{2m}} X(\varphi_\Omega)\Omega^m.$$

It follows easily that the Futaki invariant of the Kähler class on the vector field V is given as the L^2-norm of V. This observation tells us that the existence of quasi-Einstein Kähler metrics with non-trivial vector fields is an obstruction to the existence of Kähler-Einstein metrics.

Remark 5. In the compact case, Tian and Zhu [12] have proved uniqueness (modulo automorphisms) for Kähler-Ricci solitons with a fixed vector field.

The pair (g, V) is said to be *generalized quasi-Einstein* if

$$\rho - \rho_H = L_V\Omega, \tag{3}$$

where ρ_H is the harmonic part of the Ricci form ρ.

Remark 6. If (g, V) is quasi-Einstein Kähler then certainly

$$\rho_H = \lambda\Omega,$$

so we are indeed talking about a generalization.

Remark 7. The notion of generalized quasi-Einstein Kähler metrics is a generalization of constant scalar curvature Kähler metrics. Indeed, Guan [7] proved that a generalized quasi-Einstein Kähler metric has constant scalar curvature if and only if the Futaki invariant of the Kähler class vanishes. Thus, these metrics behave very much like extremal Kähler metrics [2].

§2. Constructions

In order to construct these solitons we look for Kähler metrics with at least one symmetry X. Therefore, the equations can be formulated in terms of complex coordinates on the Kähler quotient and a momentum map z. Recall for example the LeBrun Ansatz [9] (in real dimension four) for scalar-flat Kähler metrics

$$g = e^u w(dx^2 + dy^2) + w dz^2 + w^{-1}(dt + A)^2$$

$$w_{xx} + w_{yy} + (we^u)_{zz} = 0$$

$$u_{xx} + u_{yy} + (e^u)_{zz} = 0,$$

or the ansatz [10] for Kähler-Einstein metrics with g as above, where

$$w = \frac{u_z}{-2\lambda(z+B)}$$

$$u_{xx} + u_{yy} + z^2\left(z^{-2}(e^u)_z\right)_z = 0.$$

Here (x,y) are isothermal coordinates on the Kähler quotient, (u,w) are functions of (x,y,z), $X = \frac{\partial}{\partial t}$, A is a 1-form in (x,y,z) and (λ, B) are constants.

In higher dimensions a Kähler metric on M^{2m} with a symmetry X is given as

$$g = h + w dz^2 + w^{-1}\omega^2$$

where h is a Kähler metric on the Kähler quotient $B^{2(m-1)}$, ω is a connection 1-form and w is a function of the moment map coordinate z and of the complex coordinates ξ^μ on $B^{2(m-1)}$. The complex structure and the Kähler form are given by

$$J\omega = -w dz \quad \text{and} \quad \Omega = dz \wedge \omega + \Omega_h,$$

where Ω_h is the Kähler form on $B^{2(m-1)}$. Then the Kähler condition gives

$$\frac{\partial^2 h_{\mu\bar{\nu}}}{\partial z^2} + 4\frac{\partial^2 w}{\partial \xi^\mu \partial \overline{\xi^\nu}} = 0. \tag{4}$$

To obtain the equations $\rho - \lambda\Omega = L_V\Omega$ for a soliton we furthermore assume that $V = -\frac{1}{2}cJX$ (for a constant c), and we have: let

$$u = \log(\frac{\det h}{w}) \quad \text{and} \quad w = \frac{c + u_z}{-2\lambda(z+B)};$$

then

$$4\frac{\partial^2 u}{\partial\xi^\mu \partial\overline{\xi^\nu}} = -2\lambda\left(h_{\mu\nu} - (z+B)\frac{\partial h_{\mu\nu}}{\partial z}\right). \tag{5}$$

In four dimensions this last equation is

$$u_{xx} + u_{yy} + z^2\left(z^{-2}(e^u)_z\right)_z + c\, z^2\left(z^{-2}e^u\right)_z = 0.$$

To solve these equations we proceed as follows. Let (B, g_B) be an $(m-1)$-dimensional compact Kähler manifold with scalar curvature s_B. Assume that the Kähler form Ω_B is such that the deRham class $[\frac{\Omega_B}{2\pi}]$ is contained in the image of $H^2(B,\mathbb{Z}) \to H^2(B,\mathbb{R})$. Let L be a holomorphic line bundle such that $c_1(L) = [\frac{-\Omega_B}{2\pi}]$. On the total space M of $(L-0) \xrightarrow{\pi} B$ we can form an S^1-symmetric Kähler metric

$$g = zg_B + wdz^2 + w^{-1}\omega^2$$

where z, being the coordinate of $(a,b) \subset (0,\infty]$, becomes the moment map of g with the obvious S^1 action on L, w is a positive function depending only on z, and ω is the connection one-form of the connection induced by g on the S^1-bundle

$$(L-0) \xrightarrow{(\pi,z)} B \times (a,b).$$

That is $d\omega = \Omega_B$. Clearly, condition (4) is satisfied. From equation (5) we see that the base g_B must be *Kähler-Einstein* and

$$\left(\frac{z^{m-1}}{w}\right)_z + c\frac{z^{m-1}}{w} = -2\lambda z^m + \frac{s_B}{(m-1)}z^{m-1}.$$

§3. Global Metrics

If w^{-1} satisfies $w^{-1}(a) = 0$ and $(w^{-1})'(a) = 2$, then we can add a copy of B at $z = a$ and extend the Kähler metric g over the zero-section of the bundle $L \to B$. If moreover $b < \infty$, $w^{-1}(b) = 0$, and $(w^{-1})'(b) = -2$ then we can add another copy of B at $z = b$ and extend g to a Kähler metric on the total space of the $\mathbb{CP}_1$-bundle $\mathbb{P}(\mathcal{O} \oplus L)$ [11].

Theorem 1. *Let (B, g_B) be a non-positive compact Kähler-Einstein manifold of dimension $(m-1)$. Assume that $[\frac{\Omega_B}{2\pi}]$ is an integer cohomology class. Let L be a holomorphic line bundle on B such that $c_1(L) = [\frac{-\Omega_B}{2\pi}]$. Let X denote the Hamiltonian vector field generating the natural S^1 action on L and let J denote the complex structure on the total space of $L \to B$. Then, for a given $a > 0$ and a given $c > 0$, there exists a complete Kähler metric g on the total space of the bundle $L \to B$ such that the pair $(g, -\frac{1}{2}cJX)$ is quasi-Einstein, satisfying the equation*

$$\rho - \lambda\Omega = L_{-\frac{1}{2}cJX}\Omega,$$

where

$$\lambda = \frac{s_B - 2(m-1)}{2a(m-1)}.$$

Remark 8. Notice that there is no hope of producing quasi-Einstein metrics on the compact manifold $\mathbb{P}(\mathcal{O} \oplus L) \to B$. This follows from the fact that c_1 has no sign for $s_B \leq 0$: take any compact metric on M of the type described above. Since $\int_C \rho > 0$ and $\int_{E_0} \rho \wedge \Omega_B^{m-2} < 0$ when $s_B \leq 0$, we conclude that $c_1 = [\frac{\rho}{2\pi}]$ does not have a sign.

Remark 9. Koiso and Guan [8, 7] considered only *positive* s_B and obtained solutions on

$$M = \mathbb{P}(\mathcal{O} \oplus K^{\frac{p}{m}}) \to \mathbb{C}P_{m-1}; \qquad p = 1, \ldots, m-1,$$

where K is the canonical line bundle. For $p = m+1, m+2, \ldots$ they found complete non-compact quasi-Einstein Kähler metrics on the total space of $K^{\frac{p}{m}} \to B$. Note that $s_B = \frac{2m(m-1)}{p}$.

§4. Generalized Quasi-Einstein Manifolds

We use the same approach as above, but we stay in real dimension four. Therefore, B is a compact Riemann surface, and g_B is Kähler-Einstein ($\rho_B = \frac{s_B}{2}\Omega_B$). Finally, we assume that g can be extended to a smooth Kähler metric on the compact manifold $M = \mathbb{P}(\mathcal{O} \oplus L) \to B$. Now, taking traces, the equation $\rho - \rho_H = L_V\Omega$ implies

$$s - \overline{s} = -c\Delta z. \tag{6}$$

Conversely, if (6) is satisfied, then the Ricci potential is given as

$$\varphi_\Omega = -Gs = cz + \kappa,$$

where κ is some constant. Then $\nabla\varphi_\Omega = c\nabla z = 2V$, and since any compact Kähler metric satisfies

$$\rho - \rho_H = L_{\frac{1}{2}\nabla\varphi_\Omega}\Omega,$$

we conclude that (g, V) is generalized quasi-Einstein. Now, inserting

$$s = \frac{s_B}{z} - \frac{(\frac{z}{w})_{zz}}{z}$$

into (6), we see that there are in fact solutions w of the resulting equation satisfying the boundary conditions for compact metrics [11]:

Theorem 2. *Let $M = \mathbb{P}(\mathcal{O} \oplus L) \to B$, where L is a non-trivial holomorphic line bundle on a compact Riemann surface B. Then any Kähler class on M admits a generalized quasi-Einstein Kähler metric (g, V), where $V = \frac{1}{2}\nabla\varphi_\Omega$ and $W := V - iJV$ is a multiple of the holomorphic vector field generating the natural $\mathbb{C}^*$ action on L.*

Remark 10. If the genus of B is less than 2, then the metrics in the above theorem were constructed by Guan [7]. In particular, if the genus of B is equal to 0, $L = K^{\frac{1}{2}}$ and the Kähler class is a multiple of $c_1(M)$, we have the quasi-Einstein Kähler metric on $\mathbb{C}P_2 \# \overline{\mathbb{C}P_2}$ constructed by Koiso [8].

Remark 11. If the genus of B is at least 2, we notice that, in contrast with the family of extremal Kähler metrics constructed by Tønnesen-Friedman [13], the generalized quasi-Einstein Kähler metrics above do exhaust the Kähler cone.

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