

## The probability of two $\mathbb{F}_q$ -polynomials to be coprime

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### Abstract.

By means of the adelic compactification  $\widehat{R}$  of the polynomial ring  $R := \mathbb{F}_q[x]$ ,  $q$  being a prime, we give a probabilistic proof to a density theorem:

$$\frac{\#\{(m, n) \in \{0, 1, \dots, N-1\}^2; \varphi_m \text{ and } \varphi_n \text{ are coprime}\}}{N^2} \rightarrow \frac{q-1}{q},$$

as  $N \rightarrow \infty$ , for a suitable enumeration  $\{\varphi_n\}_{n=0}^\infty$  of  $R$ . Then establishing a maximal ergodic inequality for the family of shifts  $\{\widehat{R} \ni f \mapsto f + \varphi_n \in \widehat{R}\}_{n=0}^\infty$ , we prove a strong law of large numbers as an extension of the density theorem.

### §1. Introduction

Dirichlet [2] discovered a density theorem that asserts the probability of two integers to be coprime be  $6/\pi^2$ , that is,

$$(1) \quad \lim_{N \rightarrow \infty} \frac{\#\{(m, n) \in \mathbb{N}^2; 1 \leq m, n \leq N, \gcd(m, n) = 1\}}{N^2} = \zeta(2)^{-1} = \frac{6}{\pi^2}.$$

The notion of density is something like a probability, but it is not exactly a probability. In order to give a rigorous probabilistic interpretation to this theorem, Kubota-Sugita [5] gave an adelic version of (1), that is, the probability of two adelic integers to be coprime is precisely  $6/\pi^2$ ,

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and they derived (1) from the adelic version. Soon after that, Sugita-Takanobu [11] established a strong law of large numbers (S.L.L.N. for short) in Kubota-Sugita [5]'s setting, and furthermore, discovered a new limit theorem which corresponds to the central limit theorem in usual cases.

In this paper, we discuss an analogy of these works for the polynomial ring  $\mathbb{F}_q[x] =: R$ ,  $q$  being a prime, using again the adelic compactification  $\widehat{R}$  of  $R$ . As a result, an S.L.L.N. holds in this case, too.

However, the proofs here are not a complete analogue of the previous ones. Indeed, in many points  $R$  and  $\widehat{R}$  resemble  $\mathbb{Z}$  and its adelic compactification  $\widehat{\mathbb{Z}}$  respectively, but in some points they are quite different. For example,  $\mathbb{Z}$  has a natural linear order, while  $R$  does not, so that we need to define an appropriate enumeration  $R = \{\varphi_n\}_{n=0}^\infty$ . And the family of shifts  $\{x \mapsto x + n\}_{n=0}^\infty$  in  $\widehat{\mathbb{Z}}$  forms a semigroup with respect to the addition of the parameter  $n$ , while the family of shifts  $\{f \mapsto f + \varphi_n\}_{n=0}^\infty$  in  $\widehat{R}$  does not, i.e., in general,  $\varphi_m + \varphi_n \neq \varphi_{m+n}$ . In particular, the latter is a strong obstacle in proving an S.L.L.N. (Theorem 2 below), which is finally overcome by adopting a modification of Stroock [10, § 5.3]'s method due to Miki [8].

## §2. Summary of theorems

We here present three theorems as well as definitions and a lemma to state them. The proof of the theorems will be given in the following sections.

**Definition 1.** Let  $q$  be a prime,  $\mathbb{F}_q := \mathbb{Z}/q\mathbb{Z} \cong \{0, 1, \dots, q-1\}$  be the finite field consisting of  $q$  elements, and  $R$  be the ring of all  $\mathbb{F}_q$ -polynomials, i.e.,  $R := \mathbb{F}_q[x]$ . We enumerate  $R$  as follows:

$$\varphi_n(x) := \sum_{i=1}^{\infty} b_i^{(q)}(n)x^{i-1}, \quad n = 0, 1, 2, \dots,$$

where  $b_i^{(q)}(n) \in \{0, 1, \dots, q-1\}$  denotes the  $i$ -th digit of  $n$  in its  $q$ -adic expansion, namely

$$n = \sum_{i=1}^{\infty} b_i^{(q)}(n)q^{i-1}, \quad n \in \mathbb{N} \cup \{0\}.$$

Both of infinite sums above are actually finite sums for each  $n$ .

The following density theorem is an analogue of (1).

**Theorem 1.** *The probability of two elements in  $R$  to be coprime is  $(q - 1)/q$ . More precisely<sup>1</sup>,*

$$(2) \quad \lim_{N \rightarrow \infty} \frac{\#\{(m, n) \in \{0, 1, \dots, N - 1\}^2; \gcd(\varphi_m, \varphi_n) = 1\}}{N^2} = \frac{q - 1}{q}.$$

More generally, for any  $f, g \in R$ , we have

$$(3) \quad \lim_{N \rightarrow \infty} \frac{\#\{(m, n) \in \{0, 1, \dots, N - 1\}^2; \gcd(f + \varphi_m, g + \varphi_n) = 1\}}{N^2} = \frac{q - 1}{q}.$$

The limit  $(q - 1)/q$  appearing in Theorem 1 is equal to  $\zeta_R(2)^{-1}$ , where

$$\zeta_R(s) := \left(1 - \frac{1}{q^{s-1}}\right)^{-1}$$

is the zeta function associated with  $R$ . See §4 below.

Let us introduce the adelic compactification  $\widehat{R}$  of  $R$ . We say  $p \in R$  is *irreducible*, if it is not a constant (or, an element of  $\mathbb{F}_q$ ) and if  $p$  cannot be divided by any  $f \in R$  with  $0 < \deg f < \deg p$ . Let  $\mathcal{P}$  denote the set of all *monic* irreducible polynomials.

**Definition 2.** For each  $p \in \mathcal{P}$ , we define a metric  $d_p$  on  $R$  by

$$d_p(f, g) = \inf\{q^{-n \deg p}; p^n | (f - g)\}, \quad f, g \in R.$$

Let  $R_p$  denote the completion of  $R$  by the metric  $d_p$ . It is a compact ring and has a unique Borel probability measure  $\lambda_p$  which is invariant under the shifts  $\{R_p \ni f \mapsto f + g\}_{g \in R_p}$  (Haar probability measure).

Now we define

$$\widehat{R} := \prod_{p \in \mathcal{P}} R_p, \quad \lambda := \prod_{p \in \mathcal{P}} \lambda_p.$$

The arithmetic operation ‘+’ and ‘×’ being defined coordinate-wise,  $\widehat{R}$  becomes a compact ring under the product topology. And  $\lambda$  becomes the unique Haar probability measure on  $\widehat{R}$ .

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<sup>1</sup>The function ‘gcd( $f, g$ )’ is assumed to return the greatest common divisor of  $f$  and  $g$  that is *monic*. In particular, if there is no common divisor other than constants (or, elements of  $\mathbb{F}_q$ ), we have  $\gcd(f, g) = 1$  and say ‘ $f$  and  $g$  are coprime’. When  $f = g = 0$ , any monic polynomial is their common divisor, so we do not define  $\gcd(0, 0)$ .

$\widehat{R}$  is metrizable with the following metric<sup>2</sup>:

$$d((f_1, f_2, \dots), (g_1, g_2, \dots)) := \sum_{i=1}^{\infty} 2^{-i} d_{p_i}(f_i, g_i),$$

$$f = (f_1, f_2, \dots), g = (g_1, g_2, \dots) \in \widehat{R}.$$

**Lemma 1.** *The diagonal set  $D := \{(f, f, \dots) \in \widehat{R}; f \in R\}$  is dense in  $\widehat{R}$ .*

*Proof.* According to the Chinese remainder theorem, for any  $k, m \in \mathbb{N}$  and any  $f_1, \dots, f_k \in R$ , there exists  $f \in R$  such that  $f = f_i \pmod{p_i^m}$ ,  $i = 1, \dots, k$ . This implies that  $D$  is dense in  $R \times R \times \dots$  with respect to the metric  $d$ .  $\square$

Identifying  $R$  with  $D$ , we can regard  $R$  as a dense subring of  $\widehat{R}$  by Lemma 1. Since  $R$  is countable, we have  $\lambda(R) = 0$ .

Now we can mention an S.L.L.N.

**Theorem 2.** *For each  $F \in L^1(\widehat{R}^l, \lambda^l)$ ,*

$$\lim_{N \rightarrow \infty} \frac{1}{N^l} \sum_{n_1, \dots, n_l=0}^{N-1} F(f_1 + \varphi_{n_1}, \dots, f_l + \varphi_{n_l})$$

$$= \int_{\widehat{R}^l} F(\hat{f}_1, \dots, \hat{f}_l) \lambda^l(d\hat{f}_1 \cdots d\hat{f}_l), \quad \lambda^l\text{-a.e.}(f_1, \dots, f_l).$$

As a special case of Theorem 2, we have an S.L.L.N.-version of Theorem 1.

**Definition 3.** For  $f, g \in \widehat{R}$ , we define

$$\rho_p(f) := \begin{cases} 1 & (f \in p\widehat{R}), \\ 0 & (f \notin p\widehat{R}), \end{cases}$$

$$X(f, g) := \prod_{p \in \mathcal{P}} (1 - \rho_p(f)\rho_p(g)).$$

Note that for  $f, g \in R$ ,  $X(f, g) = 1$  if and only if  $\gcd(f, g) = 1$ .

**Theorem 3.**

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{m, n=0}^{N-1} X(f + \varphi_m, g + \varphi_n) = \frac{q-1}{q}, \quad \lambda^2\text{-a.e.}(f, g).$$

<sup>2</sup>We enumerate  $\mathcal{P} = \{p_i\}_{i=1}^{\infty}$  in the order given by Definition 1.

§3.  $\widehat{R}$  — Preliminaries

3.1. Basic properties

Although all lemmas in this subsection can be proved essentially in the same way as in the case of  $\widehat{\mathbb{Z}}$ , we give them proofs to make this paper self-contained.

**Lemma 2.** *Let  $p, p' \in \mathcal{P}$ ,  $p \neq p'$ , and  $k \in \mathbb{N}$ .*

- (i)  $p^k R_p$  is a closed and open ball.
- (ii)  $p^k R_{p'} = R_{p'}$ .

*Proof.* (i) That

$$\begin{aligned} p^k R_p &= \{f \in R_p; d_p(f, 0) \leq q^{-k \deg p}\} \\ &= \{f \in R_p; d_p(f, 0) < q^{-(k-1) \deg p}\} \end{aligned}$$

shows  $p^k R_p$  is closed and open.

(ii) Since  $p^k R_{p'} \subset R_{p'}$  is clear, we show the converse inclusion. To this end, it is sufficient to show the existence of  $g \in R_{p'}$  for which  $p^k g = 1$ . For each  $m \in \mathbb{N}$ , there exists  $g_m \in R$  such that  $p^k g_m \equiv 1 \pmod{(p')^m}$ , i.e.,  $d_{p'}(p^k g_m, 1) \leq q^{-m \deg p'}$ . Then for  $n > m$ , we have  $p^k(g_n - g_m) \equiv 0 \pmod{(p')^m}$ , and hence

$$d_{p'}(p^k g_n, p^k g_m) = d_{p'}(g_n, g_m) \leq q^{-m \deg p'}$$

This implies  $\{g_m\}_{m=1}^\infty$  is a Cauchy sequence in  $R_{p'}$ . Then its limit  $g \in R_{p'}$  satisfies

$$d_{p'}(p^k g, 1) = \lim_{m \rightarrow \infty} d_{p'}(p^k g_m, 1) = 0,$$

in other words,  $p^k g = 1$ . □

**Lemma 3.** *Let  $f \in R$  and  $\deg f \geq 1$ .*

- (i) For<sup>3</sup>  $-\infty \leq \deg g \leq \deg f - 1$ , the set  $(f\widehat{R} + g)$  is closed and open.
- (ii)  $\widehat{R} = \cup_{g \in R; -\infty \leq \deg g \leq \deg f - 1} (f\widehat{R} + g)$ , which is a disjoint union.

*Proof.* (i) We may assume  $f$  to be monic. Let  $f = \prod_{p \in \mathcal{P}} p^{\alpha_p(f)}$  be the prime factor decomposition, where  $\alpha_p(f) = 0$  holds except for finite number of  $p \in \mathcal{P}$ . By Lemma 2,

$$(4) \quad f\widehat{R} = \prod_{p \in \mathcal{P}} fR_p = \prod_{p \in \mathcal{P}} p^{\alpha_p(f)} R_p,$$

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<sup>3</sup> $\deg 0 := -\infty$ .

where each  $p^{\alpha_p(f)}R_p$  is closed and open, and hence  $f\widehat{R}$  is closed and open, too. Since the shift  $\widehat{R} \ni f \mapsto (f + g) \in \widehat{R}$  is a homeomorphism,  $(f\widehat{R} + g)$  is closed and open, too.

(ii) Since  $R$  is dense in  $\widehat{R}$  and  $h \mapsto fh + g$  is a continuous and closed mapping, we have  $\overline{fR + g} = f\widehat{R} + g$ . On the other hand, since  $R = \cup_{g \in R; -\infty \leq \deg g \leq \deg f - 1} (fR + g)$ , we see

$$\widehat{R} = \bigcup_{\substack{g \in R; \\ -\infty \leq \deg g \leq \deg f - 1}} (f\widehat{R} + g).$$

Let us next show that the above union is disjoint. Let  $g, g'$  be distinct polynomials both of which are of lower degree than  $f$ . By (i),  $A := (f\widehat{R} + g) \cap (f\widehat{R} + g')$  is an open set. If  $A \neq \emptyset$ , then  $R \cap A \neq \emptyset$ , because  $R$  is dense in  $\widehat{R}$ . But then, for  $l \in R \cap A$ , we see that

$$d_p(l - g, 0) \leq p^{-\alpha_p(f)}, \quad d_p(l - g', 0) \leq p^{-\alpha_p(f)}, \quad p \in \mathcal{P},$$

which means that for any  $p \in \mathcal{P}$ ,  $p^{\alpha_p(f)}|(g - g')$ . Thus we see  $f|(g - g')$ , which is impossible. Consequently, we must have  $A = \emptyset$ .  $\square$

**Lemma 4.** For  $f \in R \setminus \{0\}$  and  $A \in \mathcal{B}(\widehat{R})$ , we have  $fA \in \mathcal{B}(\widehat{R})$  and that

$$(5) \quad \lambda(fA) = q^{-\deg f} \lambda(A).$$

*Proof.* Since  $\widehat{R}$  is a complete separable metric space and the multiplication  $\widehat{R} \ni g \mapsto fg \in \widehat{R}$  is injective and Borel measurable, it holds that  $fA \in \mathcal{B}(\widehat{R})$  (cf. [9, Chapter I Theorem 3.9]). Next, let  $\nu$  be a Borel probability measure on  $\widehat{R}$  defined by

$$\nu(A) = \frac{\lambda(fA)}{\lambda(f\widehat{R})}, \quad A \in \mathcal{B}(\widehat{R}).$$

Then  $\nu$  is clearly shift invariant, and hence  $\nu = \lambda$  by the uniqueness of the Haar measure. Thus we see  $\lambda(fA) = \lambda(f\widehat{R})\lambda(A)$ . Lemma 3 and the shift invariance of  $\lambda$  imply

$$1 = \lambda(\widehat{R}) = \sum_{\substack{g \in R; \\ -\infty \leq \deg g \leq \deg f - 1}} \lambda(f\widehat{R} + g) = q^{\deg f} \lambda(f\widehat{R}),$$

from which (5) immediately follows.  $\square$

**3.2. Zeta function associated with  $R$**

Let us define the zeta function associated with  $R$ :

$$(6) \quad \zeta_R(s) := \sum_{f \in R : \text{monic}} \frac{1}{N(f)^s}, \quad \text{Re } s > 1,$$

where

$$(7) \quad N(f) := \text{the number of residue classes } R/fR = q^{\deg f}.$$

Since the polynomial ring  $R$  is a unique factorization domain, and

$$N(fg) = N(f)N(g),$$

we have an Euler product representation of  $\zeta_R$ :

$$(8) \quad \zeta_R(s) = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{N(p)^s}\right)^{-1} = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{q^{s \deg p}}\right)^{-1}.$$

Surprisingly, the following extremely simple formula holds:

$$(9) \quad \zeta_R(s) = \left(1 - \frac{1}{q^{s-1}}\right)^{-1}.$$

Let us show (9). Let  $g(m) := \sum_{d|m} \mu\left(\frac{m}{d}\right)q^d$ , where  $\mu$  is the Möbius function. Then the Möbius inversion formula implies

$$q^n = \sum_{d|n} g(d), \quad n \in \mathbb{N}.$$

We must also recall that (See [7, 3.25. Theorem])

$$\#\{p \in \mathcal{P}; \deg p = m\} = \frac{1}{m}g(m).$$

Now noting that  $\log(1-t)^{-1} = \sum_{n=1}^{\infty} \frac{t^n}{n}$  ( $|t| < 1$ ),

$$\begin{aligned} \log \zeta_R(s) &= \sum_{p \in \mathcal{P}} \log \left(1 - \frac{1}{q^{s \deg p}}\right)^{-1} = \sum_{p \in \mathcal{P}} \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{q^{ns \deg p}} \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{q^{smn}} \#\{p \in \mathcal{P}; \deg p = m\} = \sum_{m,n=1}^{\infty} \frac{1}{mn} \frac{1}{q^{smn}} g(m) \\ &= \sum_{l=1}^{\infty} \frac{1}{l} \frac{1}{q^{sl}} \sum_{m|l} g(m) = \sum_{l=1}^{\infty} \frac{1}{l} \left(\frac{1}{q^{s-1}}\right)^l \end{aligned}$$

$$= \log \left( 1 - \frac{1}{q^s - 1} \right)^{-1}.$$

Thus we have (9).

Theorem 3 follows from the next lemma and Theorem 2.

**Lemma 5.**

$$\int_{\widehat{R}^2} X(f, g) \lambda^2(df dg) = \frac{q - 1}{q}.$$

*Proof.*

$$\begin{aligned} \int_{\widehat{R}^2} X(f, g) \lambda^2(df dg) &= \prod_{p \in \mathcal{P}} \int_{\widehat{R}^2} (1 - \rho_p(f) \rho_p(g)) \lambda^2(df dg) \\ &= \prod_{p \in \mathcal{P}} \left( 1 - \int_{\widehat{R}} \rho_p(f) \lambda(df) \int_{\widehat{R}} \rho_p(g) \lambda(dg) \right) \\ &= \prod_{p \in \mathcal{P}} (1 - q^{-\deg p} q^{-\deg p}) \\ &= \prod_{p \in \mathcal{P}} (1 - q^{-2 \deg p}). \end{aligned}$$

On the other hand, plugging  $s = 2$  into (8) and (9), we see that

$$\prod_{p \in \mathcal{P}} (1 - q^{-2 \deg p})^{-1} = \zeta_R(2) = \left( 1 - \frac{1}{q} \right)^{-1},$$

and hence

$$\int_{\widehat{R}^2} X(f, g) \lambda^2(df dg) = \frac{1}{\zeta_R(2)} = \frac{q - 1}{q}. \quad \square$$

### 3.3. Uniform distributivity of $\{\varphi_n\}_{n=0}^\infty$ in $\widehat{R}$

We begin with a characterization of continuous functions on  $\widehat{R}$ .

**Definition 4.** Let  $f \in \widehat{R}$  and  $h \in R \setminus \{0\}$ . When  $\deg h \geq 1$ , by Lemma 3(ii), there exists a unique  $g \in R$  such that  $-\infty \leq \deg g \leq \deg h - 1$  and  $f - g \in h\widehat{R}$ . This  $g$  is denoted by  $f \bmod h$ . When  $\deg h = 0$ , i.e.,  $h$  is non-zero constant, we always set  $f \bmod h := 0$ .

**Definition 5.** A function  $F : \widehat{R} \rightarrow \mathbb{R}$  is said to be *periodic*, if there exists  $h \in R$ ,  $\deg h \geq 1$ , such that

$$(10) \quad F(f) = F(f \bmod h) = \sum_{\substack{g \in R; \\ -\infty \leq \deg g \leq \deg h - 1}} F(g) \mathbf{1}_{h\widehat{R}+g}(f), \quad f \in \widehat{R}.$$



And  $F : \widehat{R} \rightarrow \mathbb{R}$  is said to be *almost periodic*, if there exists a sequence  $\{F_m\}_{m=1}^\infty$  of periodic functions that converges to  $F$  uniformly .

**Lemma 6.** *A function  $F : \widehat{R} \rightarrow \mathbb{R}$  is continuous, if and only if it is almost periodic.*

*Proof.* Lemma 3 implies that periodic functions on  $\widehat{R}$  are continuous, and hence their uniformly convergent limits, that is, almost periodic functions are continuous.

Conversely, let  $F$  be a continuous function on  $\widehat{R}$ . Since  $\widehat{R}$  is compact,  $F$  is uniformly continuous, in particular, for any  $\varepsilon > 0$ , there is  $\delta > 0$  such that for any  $h \in R$ ,  $d(0, h) < \delta$ , and any  $f \in \widehat{R}$ , it holds that  $|F(f) - F(f + h)| < \varepsilon$ . Now fix such an  $h \in R$ , and define a periodic function  $F'$  by

$$F'(f) := F(f \bmod h), \quad f \in \widehat{R}.$$

Then we have  $|F(f) - F'(f)| < \varepsilon$ ,  $f \in \widehat{R}$ . Thus  $F$  is almost periodic.  $\square$

We next introduce the following lemma, which shows an important property of our enumeration  $\{\varphi_n\}_{n=0}^\infty$ .

**Lemma 7.** *Let  $m \in \mathbb{N}$  and let  $h \in R$  be a monic polynomial of degree  $m$ . Then, for any  $j \in \mathbb{N}$ ,  $\{\varphi_n \bmod h; (j - 1)q^m \leq n < jq^m\}$  forms a complete residue system modulo  $h$ . Namely,*

$$\begin{aligned} \{\varphi_n \bmod h; (j - 1)q^m \leq n < jq^m\} &= \{g \in R; -\infty \leq \deg g < m\} \\ &= \{\varphi_n; 0 \leq n < q^m\}. \end{aligned}$$

*Proof.* This lemma is due to Hodges [4, p.71]. Since the enumeration  $\{\varphi_n\}_{n=0}^\infty$  is systematic, we can present a shorter proof here. Let  $j \in \mathbb{N}$  and let  $(j - 1)q^m \leq n < jq^m$ . According to the definition of  $\{\varphi_n\}_{n=0}^\infty$ , since

$$n = (n - (j - 1)q^m) + (j - 1)q^m, \quad 0 \leq n - (j - 1)q^m < q^m,$$

we have

$$\varphi_n = \varphi_{n-(j-1)q^m} + \varphi_{j-1} \varphi_{q^m},$$

where

$$\deg \varphi_{n-(j-1)q^m} < m, \quad \deg \varphi_{j-1} \varphi_{q^m} \begin{cases} \geq m & (j > 1), \\ = -\infty & (j = 1). \end{cases}$$

Noting that  $r := \varphi_{j-1} \varphi_{q^m} \bmod h$  is of degree  $< m$ , we see that

$$\{\varphi_n \bmod h; (j - 1)q^m \leq n < jq^m\}$$

$$\begin{aligned}
 &= \{(\varphi_{n-(j-1)q^m} + \varphi_{j-1}\varphi_{q^m}) \bmod h; (j-1)q^m \leq n < jq^m\} \\
 &= \{(\varphi_n + r) \bmod h; 0 \leq n < q^m\} \\
 &= \{\varphi_n; 0 \leq n < q^m\}. \quad \square
 \end{aligned}$$

Since  $\widehat{R}$  is compact and includes  $R$  densely, each continuous function  $F : \widehat{R} \rightarrow \mathbb{R}$  is determined by its values on  $R$ . In particular, the integral of  $F$  is determined by the sequence  $\{F(\varphi_n)\}_{n=0}^\infty$ . The following lemma indicates this fact explicitly.

**Lemma 8.** *The sequence  $\{\varphi_n\}_{n=0}^\infty$  is uniformly distributed in  $\widehat{R}$ , that is, for any continuous function  $F : \widehat{R} \rightarrow \mathbb{R}$ , it holds that*

$$(11) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} F(\varphi_n) = \int_{\widehat{R}} F(\hat{f}) \lambda(d\hat{f}).$$

*Proof.*

1° Let  $F$  be a periodic function, that is, let us assume  $F(f) = F(f \bmod h)$ ,  $f \in \widehat{R}$ , for some nonconstant monic  $h \in R$ . Then putting  $m := \deg h$  and  $j_0 := \lfloor \frac{N}{q^m} \rfloor$ , Lemma 7 implies that

$$\begin{aligned}
 &\frac{1}{N} \sum_{n=0}^{N-1} F(\varphi_n) \\
 &= \frac{1}{N} \sum_{n=j_0q^m}^{N-1} F(\varphi_n \bmod h) + \frac{1}{N} \sum_{j=1}^{j_0} \sum_{n=(j-1)q^m}^{jq^m-1} F(\varphi_n \bmod h) \\
 &= \frac{1}{N} \sum_{n=j_0q^m}^{N-1} F(\varphi_n \bmod h) + \frac{j_0}{N} \sum_{-\infty \leq \deg g < m} F(g).
 \end{aligned}$$

Letting  $\{t\}$  denote the fractional part of  $t > 0$ ,

$$\begin{aligned}
 &\left| \frac{1}{N} \sum_{n=0}^{N-1} F(\varphi_n) - \frac{1}{q^m} \sum_{-\infty \leq \deg g < m} F(g) \right| \\
 &= \left| \frac{1}{N} \sum_{n=j_0q^m}^{N-1} F(\varphi_n \bmod h) + \frac{1}{N} \left( \frac{N}{q^m} - \left\{ \frac{N}{q^m} \right\} \right) \sum_{-\infty \leq \deg g < m} F(g) \right. \\
 &\quad \left. - \frac{1}{q^m} \sum_{-\infty \leq \deg g < m} F(g) \right|
 \end{aligned}$$

$$\leq \frac{1}{N} \left\{ q^m \max_{-\infty \leq \deg g < m} |F(g)| + \left| \sum_{-\infty \leq \deg g < m} F(g) \right| \right\}$$

$\rightarrow 0 \quad \text{as } N \rightarrow \infty.$

Thus (11) holds for periodic functions.

2° Let  $F : \widehat{R} \rightarrow \mathbb{R}$  be a continuous function. By Lemma 6, for any  $\varepsilon > 0$ , there is a periodic function  $F_\varepsilon$  such that  $\|F - F_\varepsilon\|_\infty < \varepsilon$ . By 1°,

$$\begin{aligned} & \left| \frac{1}{N} \sum_{n=0}^{N-1} F(\varphi_n) - \int_{\widehat{R}} F(f) \lambda(df) \right| \\ &= \left| \frac{1}{N} \sum_{n=0}^{N-1} (F(\varphi_n) - F_\varepsilon(\varphi_n)) + \frac{1}{N} \sum_{n=0}^{N-1} F_\varepsilon(\varphi_n) - \int_{\widehat{R}} F_\varepsilon(f) \lambda(df) \right. \\ & \quad \left. + \int_{\widehat{R}} (F_\varepsilon(f) - F(f)) \lambda(df) \right| \\ &\leq 2\varepsilon + \left| \frac{1}{N} \sum_{n=0}^{N-1} F_\varepsilon(\varphi_n) - \int_{\widehat{R}} F_\varepsilon(f) \lambda(df) \right| \\ &\rightarrow 0 \quad (\text{first } N \rightarrow \infty, \text{ secondly } \varepsilon \rightarrow 0). \end{aligned}$$

Thus (11) holds for continuous functions. □

The following corollary follows from Lemma 8 and [9, Chapter III Lemma 1.1].

**Corollary 1.** For any continuous function  $F : \widehat{R}^2 \rightarrow \mathbb{R}$ , we have

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{m,n=0}^{N-1} F(\varphi_m, \varphi_n) = \int_{\widehat{R}^2} F(f, g) \lambda^2(df dg).$$

The assertion of Corollary 1 is referred to as *the weak convergence of the sequence of probability measures*<sup>4</sup>  $\{ \frac{1}{N^2} \sum_{m,n=0}^{N-1} \delta_{(\varphi_m, \varphi_n)} \}_{N=1}^\infty$  to  $\lambda^2$ . It is well-known that the weak convergence is equivalent to the following condition (cf. [10, § 3.1]): For any closed set  $K \subset \widehat{R}^2$ , it holds that

$$(12) \quad \limsup_{N \rightarrow \infty} \frac{1}{N^2} \sum_{m,n=0}^{N-1} \delta_{(\varphi_m, \varphi_n)}(K) \leq \lambda^2(K).$$

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<sup>4</sup> $\delta_{(\varphi_m, \varphi_n)}$  denotes the  $\delta$ -measure at  $(\varphi_m, \varphi_n) \in \widehat{R}^2$ .

§4. Proof of density theorem

Although Theorem 1 could be proved in an elementary way, we here prove it in the light of probability theory by means of the adelic formulation. This section is an analogue of Kubota-Sugita [5, § 6].

If the function  $X(f, g)$  were continuous on  $\widehat{R}^2$ , Corollary 1 would imply Theorem 1. However it is not continuous. Indeed,

$$B := X^{-1}(\{1\}) = \bigcap_{p \in \mathcal{P}} (\widehat{R}^2 \setminus (p\widehat{R})^2) \subset \widehat{R}^2$$

is surely a closed set, but we can show  $B = \partial B$ , which means that in any neighborhood of any point of  $B$ , there exists a point for which  $X = 0$ . Thus  $X$  is not continuous. That  $B = \partial B$  is shown in the following way: Take any  $(f, g) \in B$  and any  $\varepsilon > 0$ . Then choose  $l, m \in \mathbb{N}$  so large that  $d\left(0, \prod_{i=1}^l p_i^m\right) < \varepsilon$ . Now find  $h_1, h_2 \in R$  such that

$$\begin{cases} f \bmod p_{l+1} + h_1 \prod_{i=1}^l p_i^m \equiv 0 \pmod{p_{l+1}}, \\ g \bmod p_{l+1} + h_2 \prod_{i=1}^l p_i^m \equiv 0 \pmod{p_{l+1}}. \end{cases}$$

In fact, since  $\prod_{i=1}^l p_i^m$  and  $p_{l+1}$  are coprime, there exists  $k \in R$  such that  $k \prod_{i=1}^l p_i^m \equiv 1 \pmod{p_{l+1}}$ , so that  $h_1 = k(p_{l+1} - f \bmod p_{l+1})$  and  $h_2 = k(p_{l+1} - g \bmod p_{l+1})$  are required ones. Then it is easily seen that  $d(f, f + h_1 \prod_{i=1}^l p_i^m) < \varepsilon$ ,  $d(g, g + h_2 \prod_{i=1}^l p_i^m) < \varepsilon$ , and that  $(f + h_1 \prod_{i=1}^l p_i^m, g + h_2 \prod_{i=1}^l p_i^m) \notin B$ . Thus  $B \subset \partial B$ .

Let us begin to prove (2) in Theorem 1. For each monic polynomial  $h \in R$ , we set

$$hB := \{(hf, hg) \in \widehat{R}^2; (f, g) \in B\}.$$

Since  $hB \cap R^2 = \{(f, g) \in R^2; \gcd(f, g) = h\}$ , it is easy to see that

$$(13) \quad \sum_{h \in R: \text{monic}} \delta_{(\varphi_m, \varphi_n)}(hB) = \begin{cases} 1, & (m, n) \in \{0, 1, 2, \dots\}^2 \setminus \{(0, 0)\}, \\ 0, & (m, n) = (0, 0). \end{cases}$$

According to Lemma 5,  $\lambda^2(B) = \int_{\widehat{R}^2} X(f, g) \lambda^2(df dg) = (q-1)/q$ . Hence by Lemma 4,

$$\lambda^2(hB) = \frac{1}{q^{2 \deg h}} \cdot \frac{q-1}{q}.$$

Since  $hB$  is a closed set, (12) implies

$$(14) \quad \frac{1}{q^{2 \deg h}} \cdot \frac{q-1}{q} = \lambda^2(hB) \geq \limsup_{N \rightarrow \infty} \frac{1}{N^2} \sum_{m,n=0}^{N-1} \delta_{(\varphi_m, \varphi_n)}(hB).$$

Note that by (6), (7) and (9) with  $s = 2$ , we have

$$(15) \quad \sum_{h \in R: \text{monic}} \frac{1}{q^{2 \deg h}} = \frac{q}{q-1}.$$

Also, since, for  $\nu \geq 0$  and  $\varphi \in R$

$$-\infty \leq \deg \varphi \leq \nu \iff \varphi \in \{\varphi_m; 0 \leq m \leq q^{\nu+1} - 1\},$$

(<)

we see that for  $N \in \mathbb{N} \cap [2, \infty)$ , taking  $\nu \in \mathbb{N} \cup \{0\}$  so that  $q^\nu \leq N-1 < q^{\nu+1}$ ,

$$\begin{aligned} \frac{1}{N^2} \sum_{m,n=1}^{N-1} \delta_{(\varphi_m, \varphi_n)}(hB) &\leq \frac{1}{N^2} \sum_{m,n=1}^{N-1} \delta_{(\varphi_m, \varphi_n)}(h\widehat{R}^2) \\ &\leq \frac{1}{(q^\nu + 1)^2} \sum_{m,n=1}^{q^{\nu+1}-1} \delta_{(\varphi_m, \varphi_n)}(hR \times hR) \\ &= \left( \frac{1}{q^\nu + 1} \sum_{m=1}^{q^{\nu+1}-1} \delta_{\varphi_m}(hR) \right)^2 \\ &= \left( \frac{\#\{1 \leq m \leq q^{\nu+1} - 1; h \mid \varphi_m\}}{q^\nu + 1} \right)^2 \\ &= \left( \frac{\#\{\varphi \in R; -\infty < \deg \varphi \leq \nu, h \mid \varphi\}}{q^\nu + 1} \right)^2 \\ &= \left( \frac{\#\{k \in R \setminus \{0\}; \deg(hk) \leq \nu\}}{q^\nu + 1} \right)^2 \\ &= \left( \frac{\#\{k \in R; -\infty < \deg k \leq \nu - \deg h\}}{q^\nu + 1} \right)^2 \\ &= \begin{cases} \left( \frac{q^{\nu - \deg h + 1} - 1}{q^\nu + 1} \right)^2, & \nu \geq \deg h, \\ 0, & \nu < \deg h \end{cases} \end{aligned}$$

$$\leq \frac{q^2}{q^2 \deg h}.$$

Here the last expression is summable in  $h \in R$ , monic. Then it follows from (15), (14) and the Lebesgue-Fatou theorem that

$$\begin{aligned} (16) \quad 1 - \frac{q-1}{q} &= \sum_{h \in R; \deg h \geq 1, \text{ monic}} \frac{q-1}{q} \cdot \frac{1}{q^2 \deg h} \\ &\geq \sum_{h \in R; \deg h \geq 1, \text{ monic}} \limsup_{N \rightarrow \infty} \frac{1}{N^2} \sum_{m,n=0}^{N-1} \delta_{(\varphi_m, \varphi_n)}(hB) \\ &\geq \sum_{h \in R; \deg h \geq 1, \text{ monic}} \limsup_{N \rightarrow \infty} \frac{1}{N^2} \sum_{m,n=1}^{N-1} \delta_{(\varphi_m, \varphi_n)}(hB) \\ &\geq \limsup_{N \rightarrow \infty} \sum_{h \in R; \deg h \geq 1, \text{ monic}} \frac{1}{N^2} \sum_{m,n=1}^{N-1} \delta_{(\varphi_m, \varphi_n)}(hB) \\ &= \limsup_{N \rightarrow \infty} \frac{1}{N^2} \sum_{m,n=1}^{N-1} \sum_{h \in R; \deg h \geq 1, \text{ monic}} \delta_{(\varphi_m, \varphi_n)}(hB). \end{aligned}$$

Subtracting each side of (16) from 1 and noting (13), we have

$$\begin{aligned} (17) \quad \frac{q-1}{q} &\leq \liminf_{N \rightarrow \infty} \left( 1 - \frac{1}{N^2} \sum_{m,n=1}^{N-1} \sum_{h \in R; \deg h \geq 1, \text{ monic}} \delta_{(\varphi_m, \varphi_n)}(hB) \right) \\ &= \liminf_{N \rightarrow \infty} \left( \frac{1}{N^2} \sum_{m,n=1}^{N-1} \left( 1 - \sum_{h \in R; \deg h \geq 1, \text{ monic}} \delta_{(\varphi_m, \varphi_n)}(hB) \right) \right. \\ &\quad \left. + \frac{1}{N^2} \sum_{\substack{0 \leq m,n \leq N-1; \\ m=0 \text{ or } n=0}} 1 \right) \\ &= \liminf_{N \rightarrow \infty} \left( \frac{1}{N^2} \sum_{m,n=0}^{N-1} \delta_{(\varphi_m, \varphi_n)}(B) \right. \\ &\quad \left. + \frac{1}{N^2} \sum_{\substack{0 \leq m,n \leq N-1; \\ m=0 \text{ or } n=0}} \left( 1 - \delta_{(\varphi_m, \varphi_n)}(B) \right) \right) \\ &= \liminf_{N \rightarrow \infty} \frac{1}{N^2} \sum_{m,n=0}^{N-1} \delta_{(\varphi_m, \varphi_n)}(B). \end{aligned}$$

Finally, (14) with  $h(x) \equiv 1$  and (17) imply that

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{m,n=0}^{N-1} \delta_{(\varphi_m, \varphi_n)}(B) = \frac{q-1}{q},$$

which is equivalent to (2).

Next, let us prove (3) in Theorem 1. Take arbitrary  $f, g \in R$  with  $\deg f \vee \deg g \geq 0$ , and set  $\varphi'_m := f + \varphi_m$  and  $\varphi''_n := g + \varphi_n$ . Then it is easy to see that the sequence of probability measures  $\{\frac{1}{N^2} \sum_{m,n=0}^{N-1} \delta_{(\varphi'_m, \varphi''_n)}\}_N$  weakly converges to  $\lambda^2$ . Furthermore, we have

$$(18) \quad \sum_{h \in R: \text{monic}} \delta_{(\varphi'_m, \varphi''_n)}(hB) = \begin{cases} 1, & (\varphi'_m, \varphi''_n) \neq (0, 0), \\ 0, & (\varphi'_m, \varphi''_n) = (0, 0). \end{cases}$$

By these facts, we can deduce that

$$(19) \quad \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{m,n=0}^{N-1} \delta_{(\varphi'_m, \varphi''_n)}(B) = \frac{q-1}{q},$$

similarly as the case where  $(f, g) = (0, 0)$ .

**Remark 1.** If  $f, g \in \widehat{R}$  fail to belong to  $R$ , (19) may not be true. The following is one of such examples: Let  $\tau : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  be a bijective mapping. For each  $N \in \mathbb{N}$ , we consider a system of equations

$$\begin{aligned} (f + \varphi_m) \bmod p_{\tau(m,n)} &= 0, \\ (g + \varphi_n) \bmod p_{\tau(m,n)} &= 0, \end{aligned} \quad m, n = 1, 2, \dots, N,$$

with unknown variable  $(f, g) \in \widehat{R}^2$ . By the Chinese remainder theorem, the solution  $(f, g)$ , say  $(f_N, g_N) \in R^2$ , exists. Since  $\widehat{R}^2$  is compact,  $\{(f_N, g_N)\}_{N=1}^\infty$  has a limit point, say  $(f_\infty, g_\infty) \in \widehat{R}^2$ . Then since for each  $p \in \mathcal{P}$ ,  $p\widehat{R}$  is a closed ball, it holds that

$$\begin{aligned} (f_\infty + \varphi_m) \bmod p_{\tau(m,n)} &= 0, \\ (g_\infty + \varphi_n) \bmod p_{\tau(m,n)} &= 0, \end{aligned} \quad m, n \in \mathbb{N}.$$

Clearly, we have  $X(f_\infty + \varphi_m, g_\infty + \varphi_n) = 0$ ,  $m, n \in \mathbb{N}$ , and hence

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{m,n=0}^{N-1} \delta_{(f_\infty + \varphi_m, g_\infty + \varphi_n)}(B) = 0.$$

## §5. Proof of strong law of large numbers

### 5.1. Maximal ergodic inequality

Basically, we adopt the method used in Stroock [10, § 5.3]. We begin with the definition of classical maximal function.

**Definition 6.** For  $f \in L^1(\mathbb{R}^l \rightarrow \mathbb{R})$ , we define Hardy-Littlewood's maximal function  $Mf$  by

$$Mf(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy, \quad x \in \mathbb{R}^l,$$

where the sup is taken for all cubes  $Q$  of the form

$$Q = \prod_{j=1}^l [a_j, a_j + r), \quad a = (a_1, \dots, a_l) \in \mathbb{R}^l, \quad r > 0$$

such that  $Q \ni x$ , and

$$|Q| := \text{the Lebesgue measure of } Q.$$

**Lemma 9** (The Hardy-Littlewood inequality). ([10, § 5.3]) *For any  $0 < \alpha < \infty$ , it holds that*

$$|\{x \in \mathbb{R}^l; Mf(x) \geq \alpha\}| \leq \frac{12^l}{\alpha} \int_{\mathbb{R}^l} |f(y)| dy.$$

**Definition 7.** For each  $m, n = 0, 1, 2, \dots$ , there exists a unique  $k \in \mathbb{N} \cup \{0\}$  such that  $\varphi_m(x) + \varphi_n(x) = \varphi_k(x)$ . This  $k$  will be denoted by  $m \cdot n$ , that is,

$$m \cdot n := \sum_{i=1}^{\infty} \left( (d_i^{(q)}(m) + d_i^{(q)}(n)) \bmod q \right) q^{i-1}.$$

As is easily seen,  $m \cdot n \neq m + n$  in general. Therefore the method used in Stroock [10, § 5.3] does not work to derive the maximal ergodic inequality. In this paper, we adopt a modification of Stroock's method due to Miki [8].

**Lemma 10.** ([8]) *Let  $m, n, l = 0, 1, 2, \dots$*

- (i)  $m \cdot 0 = m$ ,  $m \cdot n = n \cdot m$ ,  $(l \cdot m) \cdot n = l \cdot (m \cdot n)$ .
- (ii) *The mapping  $\mathbb{N} \cup \{0\} \ni k \mapsto m \cdot k \in \mathbb{N} \cup \{0\}$  is bijective.*
- (iii)  $(m \vee n) - (q - 1)(m \wedge n) \leq m \cdot n \leq m + n$ .



*Proof.* (i) and (ii) are obvious. We here check (iii). Since, for  $a, b \in \{0, 1, \dots, q-1\}$

$$(a+b) \bmod q = \begin{cases} a+b, & \text{if } a+b < q, \\ a+b-q, & \text{if } a+b \geq q, \end{cases}$$

it follows that

$$(a+b) \bmod q \leq a+b,$$

$$(a+b) \bmod q + (q-1)a = \begin{cases} a+b+(q-1)a \\ = b+qa, & \text{if } a+b < q, \\ a+b-q+(q-1)a \\ = b+q(a-1), & \text{if } a+b \geq q > b \end{cases} \geq b.$$

Hence, for  $0 \leq m \leq n$

$$m \cdot n = \sum_{i=1}^{\infty} \left( (d_i^{(q)}(m) + d_i^{(q)}(n)) \bmod q \right) q^{i-1}$$

$$\begin{cases} \leq \sum_{i=1}^{\infty} (d_i^{(q)}(m) + d_i^{(q)}(n)) q^{i-1} = m+n, \\ \geq \sum_{i=1}^{\infty} (d_i^{(q)}(n) - (q-1)d_i^{(q)}(m)) q^{i-1} = n - (q-1)m. \end{cases} \quad \square$$

**Lemma 11.** For any square array  $\{a_{k_1, k_2}\}_{k_1, k_2 \in \{0, 1, 2, \dots\}} \subset [0, \infty)$  with  $\sum_{k_1, k_2=0}^{\infty} a_{k_1, k_2} < \infty$ , the following inequality holds: For any  $\alpha > 0$ ,

$$\# \left\{ (k_1, k_2) \in \{0, 1, 2, \dots\}^2; \sup_{n \geq 1} \left( \frac{1}{qn} \right)^2 \sum_{j_1, j_2=0}^{n-1} a_{k_1+j_1, k_2+j_2} \geq \alpha \right\}$$

$$\leq \frac{12^2}{\alpha} \sum_{k_1, k_2=0}^{\infty} a_{k_1, k_2}.$$

*Proof.* Put

$$f(x) := \sum_{k_1, k_2=0}^{\infty} a_{k_1, k_2} \mathbf{1}_{C(k_1, k_2)}(x), \quad x \in \mathbb{R}^2,$$

where

$$C(k_1, k_2) := [k_1, k_1+1) \times [k_2, k_2+1).$$

Then clearly we have

$$(20) \quad \int_{\mathbb{R}^2} f(x) dx = \sum_{k_1, k_2=0}^{\infty} a_{k_1, k_2} < \infty,$$

and maximal function  $Mf$  becomes

$$(21) \quad \begin{aligned} Mf(x) &= \sup_{Q \ni x} \frac{1}{|Q|} \int_Q f(y) dy \\ &= \sup_{Q \ni x} \frac{1}{|Q|} \sum_{l_1, l_2 \geq 0} a_{l_1, l_2} |C(l_1, l_2) \cap Q|. \end{aligned}$$

Now suppose that  $x \in C(k_1, k_2)$  ( $k_1, k_2 \in \{0, 1, 2, \dots\}$ ),  $n \in \mathbb{N}$ , and  $0 \leq j_1, j_2 \leq n-1$ . If we take  $Q = [k_1 - (q-1)n, k_1 + n) \times [k_2 - (q-1)n, k_2 + n)$ , then  $Q \ni x$  and

$$(22) \quad Q \supset C(k_1 \cdot j_1, k_2 \cdot j_2)$$

holds. Because Lemma 10(iii) implies

$$\begin{aligned} k_1 \cdot j_1 &\geq k_1 - (q-1)n, \\ k_2 \cdot j_2 &\geq k_2 - (q-1)n \end{aligned}$$

and

$$\begin{aligned} k_1 \cdot j_1 &\leq k_1 + j_1 \leq k_1 + n - 1, \\ k_2 \cdot j_2 &\leq k_2 + j_2 \leq k_2 + n - 1, \end{aligned}$$

we see

$$\begin{aligned} [k_1 \cdot j_1, k_1 \cdot j_1 + 1) &\subset [k_1 - (q-1)n, k_1 + n), \\ [k_2 \cdot j_2, k_2 \cdot j_2 + 1) &\subset [k_2 - (q-1)n, k_2 + n), \end{aligned}$$

and hence (22) holds.

If we take this  $Q$  for (21), we have for  $x \in C(k_1, k_2)$ ,  $n \in \mathbb{N}$  that

$$\begin{aligned} Mf(x) &\geq \frac{1}{|Q|} \sum_{j_1, j_2=0}^{n-1} a_{k_1 \cdot j_1, k_2 \cdot j_2} |C(k_1 \cdot j_1, k_2 \cdot j_2) \cap Q| \\ &= \left(\frac{1}{qn}\right)^2 \sum_{j_1, j_2=0}^{n-1} a_{k_1 \cdot j_1, k_2 \cdot j_2}. \end{aligned}$$

Taking sup in  $n$ ,

$$Mf(x) \geq \sum_{k_1, k_2=0}^{\infty} \left( \sup_{n \in \mathbb{N}} \left( \frac{1}{qn} \right)^2 \sum_{j_1, j_2=0}^{n-1} a_{k_1 \cdot j_1, k_2 \cdot j_2} \right) \mathbf{1}_{C(k_1, k_2)}(x).$$

Then for  $0 < \alpha < \infty$ ,

$$\begin{aligned} & \left\{ x \in [0, \infty)^2; Mf(x) \geq \alpha \right\} \\ & \supset \left\{ x \in [0, \infty)^2; \right. \\ & \quad \left. \sum_{k_1, k_2=0}^{\infty} \left( \sup_{n \in \mathbb{N}} \left( \frac{1}{qn} \right)^2 \sum_{j_1, j_2=0}^{n-1} a_{k_1 \cdot j_1, k_2 \cdot j_2} \right) \mathbf{1}_{C(k_1, k_2)}(x) \geq \alpha \right\} \\ & = \bigcup_{\substack{k_1, k_2 \geq 0; \\ \sup_{n \in \mathbb{N}} \left( \frac{1}{qn} \right)^2 \sum_{j_1, j_2=0}^{n-1} a_{k_1 \cdot j_1, k_2 \cdot j_2} \geq \alpha}} C(k_1, k_2). \end{aligned}$$

Therefore Lemma 9 and (20) imply

$$\begin{aligned} & \frac{12^2}{\alpha} \sum_{k_1, k_2=0}^{\infty} a_{k_1, k_2} \\ & = \frac{12^2}{\alpha} \int_{\mathbb{R}^2} f(x) dx \\ & \geq \left| \left\{ x \in \mathbb{R}^2; Mf(x) \geq \alpha \right\} \right| \\ & \geq \left| \left\{ x \in [0, \infty)^2; Mf(x) \geq \alpha \right\} \right| \\ & \geq \sum_{k_1, k_2=0}^{\infty} \mathbf{1}_{\sup_{n \in \mathbb{N}} \left( \frac{1}{qn} \right)^2 \sum_{j_1, j_2=0}^{n-1} a_{k_1 \cdot j_1, k_2 \cdot j_2} \geq \alpha} \\ & = \# \left\{ (k_1, k_2) \in \{0, 1, 2, \dots\}^2; \sup_{n \in \mathbb{N}} \left( \frac{1}{qn} \right)^2 \sum_{j_1, j_2=0}^{n-1} a_{k_1 \cdot j_1, k_2 \cdot j_2} \geq \alpha \right\}. \square \end{aligned}$$

**Lemma 12** (Maximal ergodic inequality). *Let  $F : \widehat{R}^2 \rightarrow [0, \infty)$  be a Borel measurable function such that*

$$\mathbb{E}^{\lambda^2}[F] := \int_{\mathbb{R}^2} F(f, g) \lambda^2(df dg) < \infty.$$

Then for any  $0 < \alpha < \infty$ , it holds that

$$\lambda^2 \left( \sup_{N \geq 1} \frac{1}{N^2} \sum_{j_1, j_2=0}^{N-1} F(f + \varphi_{j_1}, g + \varphi_{j_2}) \geq q^2 \alpha \right) \leq \frac{24^2}{\alpha} \mathbb{E} \lambda^2 [F].$$

*Proof.* Fix  $M \in \mathbb{N}$  and  $(f, g) \in \widehat{R}^2$ . For each  $k_1, k_2 \in \{0, 1, 2, \dots\}$ , we define

$$a_{k_1, k_2}(f, g) := \begin{cases} F(f + \varphi_{k_1}, g + \varphi_{k_2}), & \text{if } 0 \leq k_1, k_2 \leq 2M - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then Lemma 11 implies that

$$\begin{aligned} & \# \left\{ k_1, k_2 \geq 0; \sup_{N \geq 1} \left( \frac{1}{qN} \right)^2 \sum_{j_1, j_2=0}^{N-1} a_{k_1 \cdot j_1, k_2 \cdot j_2}(f, g) \geq \alpha \right\} \\ & \leq \frac{12^2}{\alpha} \sum_{k_1, k_2=0}^{\infty} a_{k_1, k_2}(f, g) \\ & = \frac{12^2}{\alpha} \sum_{0 \leq k_1, k_2 \leq 2M-1} F(f + \varphi_{k_1}, g + \varphi_{k_2}), \quad 0 < \alpha < \infty. \end{aligned}$$

Noting that

$$\begin{aligned} & 0 \leq k_1, k_2 \leq M, \quad 0 \leq j_1, j_2 < N, \quad 1 \leq N \leq M \\ & \Rightarrow 0 \leq k_1 \cdot j_1 \leq k_1 + j_1 \leq M + N - 1 \leq 2M - 1, \\ & \quad 0 \leq k_2 \cdot j_2 \leq k_2 + j_2 \leq M + N - 1 \leq 2M - 1 \\ & \Rightarrow a_{k_1 \cdot j_1, k_2 \cdot j_2}(f, g) = F(f + \varphi_{k_1 \cdot j_1}, g + \varphi_{k_2 \cdot j_2}) \\ & \quad = F(f + \varphi_{k_1} + \varphi_{j_1}, g + \varphi_{k_2} + \varphi_{j_2}), \end{aligned}$$

we have

$$\begin{aligned} & \# \left\{ (k_1, k_2) \in \{0, 1, 2, \dots, M\}^2; \right. \\ & \quad \left. \max_{1 \leq N \leq M} \left( \frac{1}{qN} \right)^2 \sum_{j_1, j_2=0}^{N-1} F(f + \varphi_{k_1} + \varphi_{j_1}, g + \varphi_{k_2} + \varphi_{j_2}) \geq \alpha \right\} \\ & \leq \frac{12^2}{\alpha} \sum_{k_1, k_2=0}^{2M-1} F(f + \varphi_{k_1}, g + \varphi_{k_2}), \quad 0 < \alpha < \infty. \end{aligned}$$

Therefore taking the expectation  $\mathbb{E}^{\lambda^2}$  of both sides,

$$\begin{aligned} \sum_{k_1, k_2=0}^M \lambda^2 \left( \max_{1 \leq N \leq M} \left( \frac{1}{qN} \right)^2 \sum_{j_1, j_2=0}^{N-1} F(f + \varphi_{k_1} + \varphi_{j_1}, g + \varphi_{k_2} + \varphi_{j_2}) \geq \alpha \right) \\ \leq \frac{12^2}{\alpha} \sum_{k_1, k_2=0}^{2M-1} \mathbb{E}^{\lambda^2} \left[ F(f + \varphi_{k_1}, g + \varphi_{k_2}) \right], \quad 0 < \alpha < \infty. \end{aligned}$$

Since  $\lambda^2$  is shift-invariant, the above inequality reduces to

$$\begin{aligned} \lambda^2 \left( \max_{1 \leq N \leq M} \left( \frac{1}{qN} \right)^2 \sum_{j_1, j_2=0}^{N-1} F(f + \varphi_{j_1}, g + \varphi_{j_2}) \geq \alpha \right) \\ \leq \frac{12^2}{\alpha} \left( \frac{2M}{M+1} \right)^2 \mathbb{E}^{\lambda^2} [F], \quad 0 < \alpha < \infty. \end{aligned}$$

Finally, letting  $M \rightarrow \infty$ , the assertion of the lemma follows. □

### 5.2. Proof of Theorem 2

For simplicity, we here prove Theorem 2 for  $l = 2$  only. The same method works for general  $l$ , too. Namely, what we prove is as follows:

For any  $F \in L^1(\widehat{R}^2, \lambda^2)$ ,

$$(23) \quad \frac{1}{N^2} \sum_{m, n=0}^{N-1} F(f + \varphi_m, g + \varphi_n) \rightarrow \mathbb{E}^{\lambda^2} [F] \quad \lambda^2\text{-a.e.}(f, g).$$

*Proof.* Take sequence of continuous functions  $\{F_k\}_{k=1}^\infty$  so that

$$(24) \quad \|F_k - F\|_{L^1} \leq \frac{1}{k^2}, \quad k \in \mathbb{N}.$$

By Corollary 1, it holds for each  $k \in \mathbb{N}$  that

$$(25) \quad \frac{1}{N^2} \sum_{m, n=0}^{N-1} F_k(f + \varphi_m, g + \varphi_n) \rightarrow \mathbb{E}^{\lambda^2} [F_k] \quad \text{as } N \rightarrow \infty, \quad (f, g) \in \widehat{R}^2.$$

By Lemma 12, it holds for  $0 < \alpha < \infty$  that

$$\begin{aligned} \lambda^2 \left( \sup_{N \geq 1} \frac{1}{N^2} \sum_{j_1, j_2=0}^{N-1} |F_k(f + \varphi_{j_1}, g + \varphi_{j_2}) - F(f + \varphi_{j_1}, g + \varphi_{j_2})| \geq q^2 \alpha \right) \\ \leq \frac{24^2}{\alpha} \mathbb{E}^{\lambda^2} [|F_k - F|] \end{aligned}$$

$$\leq \frac{24^2}{\alpha} \cdot \frac{1}{k^2}.$$

From this, it follows that

$$\begin{aligned} & \sum_{k=1}^{\infty} \lambda^2 \left( (f, g) \in \widehat{R}^2; \right. \\ & \left. \sup_{N \geq 1} \frac{1}{N^2} \sum_{j_1, j_2=0}^{N-1} |F_k(f + \varphi_{j_1}, g + \varphi_{j_2}) - F(f + \varphi_{j_1}, g + \varphi_{j_2})| \geq \frac{q^2}{\sqrt{k}} \right) \\ & \leq \sum_{k=1}^{\infty} 24^2 \sqrt{k} \frac{1}{k^2} < \infty, \end{aligned}$$

which means that

$$\lim_{k \rightarrow \infty} \sup_{N \geq 1} \frac{1}{N^2} \sum_{j_1, j_2=0}^{N-1} |F_k(f + \varphi_{j_1}, g + \varphi_{j_2}) - F(f + \varphi_{j_1}, g + \varphi_{j_2})| = 0, \text{ a.s.}$$

Consequently, by (24) and (25), we see that

$$\begin{aligned} (26) \quad & \left| \frac{1}{N^2} \sum_{j_1, j_2=0}^{N-1} F(f + \varphi_{j_1}, g + \varphi_{j_2}) - \mathbb{E}^{\lambda^2} [F] \right| \\ & = \left| \frac{1}{N^2} \sum_{j_1, j_2=0}^{N-1} \left( F(f + \varphi_{j_1}, g + \varphi_{j_2}) - F_k(f + \varphi_{j_1}, g + \varphi_{j_2}) \right) \right. \\ & \quad + \frac{1}{N^2} \sum_{j_1, j_2=0}^{N-1} F_k(f + \varphi_{j_1}, g + \varphi_{j_2}) - \mathbb{E}^{\lambda^2} [F_k] \\ & \quad \left. + \mathbb{E}^{\lambda^2} [F_k] - \mathbb{E}^{\lambda^2} [F] \right| \\ & \leq \frac{1}{N^2} \sum_{j_1, j_2=0}^{N-1} \left| F(f + \varphi_{j_1}, g + \varphi_{j_2}) - F_k(f + \varphi_{j_1}, g + \varphi_{j_2}) \right| \\ & \quad + \left| \frac{1}{N^2} \sum_{j_1, j_2=0}^{N-1} F_k(f + \varphi_{j_1}, g + \varphi_{j_2}) - \mathbb{E}^{\lambda^2} [F_k] \right| \\ & \quad + \mathbb{E}^{\lambda^2} [F_k - F] \\ & \leq \sup_{M \geq 1} \frac{1}{M^2} \sum_{j_1, j_2=0}^{M-1} \left| F(f + \varphi_{j_1}, g + \varphi_{j_2}) - F_k(f + \varphi_{j_1}, g + \varphi_{j_2}) \right| \end{aligned}$$

$$\begin{aligned}
 & + \left| \frac{1}{N^2} \sum_{j_1, j_2=0}^{N-1} F_k(f + \varphi_{j_1}, g + \varphi_{j_2}) - \mathbb{E}^{\lambda^2} [F_k] \right| \\
 & + \frac{1}{k^2} \\
 & \rightarrow 0 \quad \text{a.s. (first } N \rightarrow \infty, \text{ secondly } k \rightarrow \infty). \quad \square
 \end{aligned}$$

**Remark 2.** If  $F \in L^p(\widehat{R}^2, \lambda^2)$  for some  $1 \leq p < \infty$ , the convergence in (23) is in fact an  $L^p$ -convergence. Indeed, for any  $\varepsilon > 0$ , there exists a bounded measurable function  $F_\varepsilon : \widehat{R}^2 \rightarrow \mathbb{R}$  such that

$$\|F - F_\varepsilon\|_{L^p} < \varepsilon.$$

A similar estimate as (26) can be done in  $L^p$ -norm in the following way:

$$\begin{aligned}
 & \left\| \frac{1}{N^2} \sum_{m, n=0}^{N-1} F(f + \varphi_m, g + \varphi_n) - \mathbb{E}^{\lambda^2} [F] \right\|_{L^p} \\
 & \leq \frac{1}{N^2} \sum_{m, n=0}^{N-1} \|F(f + \varphi_m, g + \varphi_n) - F_\varepsilon(f + \varphi_m, g + \varphi_n)\|_{L^p} \\
 & \quad + \left\| \frac{1}{N^2} \sum_{m, n=0}^{N-1} F_\varepsilon(f + \varphi_m, g + \varphi_n) - \mathbb{E}^{\lambda^2} [F_\varepsilon] \right\|_{L^p} + \|F_\varepsilon - F\|_{L^p} \\
 & < \left\| \frac{1}{N^2} \sum_{m, n=0}^{N-1} F_\varepsilon(f + \varphi_m, g + \varphi_n) - \mathbb{E}^{\lambda^2} [F_\varepsilon] \right\|_{L^p} + 2\varepsilon \\
 & \rightarrow 0 \quad (\text{first } N \rightarrow \infty, \text{ secondly } \varepsilon \rightarrow 0).
 \end{aligned}$$

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