

On Q -multiplicative functions having a positive upper-meanvalue

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Abstract.

A classical approach to study properties of Q -multiplicative functions $f(n)$ is to associate to the mean $\frac{1}{x} \sum_{0 \leq n \leq x} f(n)$ the product $\prod_{0 \leq j \leq k} \frac{1}{q_j} \sum_{0 \leq a \leq q_j - 1} f(aQ_j)$. We discuss its validity in the case of non-negative Q -multiplicative functions $f(n)$ with a positive upper meanvalue, defined via a Cantor numeration system.

§1. Introduction and notations

1.1. Numeration systems and associated additive functions

Let N be the set of non-negative integers, and $Q = (Q_k)_{k \geq 0}$, $Q_0 = 1$, be an increasing sequence of positive integers. Using the greedy algorithm to every element n of N , one can associate a representation

$$n = \sum_{k=0}^{+\infty} \varepsilon_k(n) Q_k,$$

which is unique if for every K ,

$$\sum_{k=0}^{K-1} \varepsilon_k(n) Q_k < Q_K.$$

Such a condition provides a numeration scale and in this case, we can define on N a complex-valued arithmetic function $f(n)$ by $f(0.Q_k) = 1$

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and $f(n) = \prod_{k \geq 0} f(\varepsilon_k(n)Q_k)$, and it will be called a Q -multiplicative function.

Simple examples of numeration scales are the q -adic scale, where $Q_k = q^k$, q integer, $q \geq 2$, and its generalization, the Cantor scale $Q_{k+1} = q_k Q_k$, $Q_0 = 1$, $q_k \geq 2$, $k \geq 0$.

A classical approach to study properties of Q -multiplicative functions $f(n)$ is to associate to the mean $\frac{1}{x} \sum_{0 \leq n < x} f(n)$ the product

$$\prod_{0 \leq j \leq k} \frac{1}{q_j} \sum_{0 \leq a \leq q_j - 1} f(aQ_j),$$

and in fact, this correspondence essentially explains a natural underlying probabilistic structure.

Now, although the q -adic scale and its generalization, the Cantor scale, seem very similar, basic differences may exist between them. More precisely, if a Cantor system is such that there exists some uniform bound B of the q_k , there is practically no differences, and this is due essentially to this uniformity condition. Otherwise, if we allow the q_k to be unbounded, the situation is not so simple. An example was given in [4], where the case of the mean-value of unimodular Q -multiplicative functions is considered.

§2. Results

In the simple case of non-negative Q -multiplicative functions, the existence of some essential difference can be shown. In fact, we have the following result:

Theorem 1. 1) For a given Cantor scale with uniformly bounded q_k and for any non-negative q -multiplicative function f , the condition

$$\limsup_{x \rightarrow +\infty} \frac{1}{x} \sum_{0 \leq n < x} f(n) \text{ exists and is positive}$$

is equivalent to the condition

$$\limsup_{k \rightarrow +\infty} \prod_{0 \leq j \leq k} \frac{1}{q_j} \sum_{0 \leq a \leq q_j - 1} f(aQ_j) \text{ exists and is positive.}$$

2) There exist Cantor scales (Q) with not uniformly bounded q_k and non-negative Q -multiplicative functions f such that the condition

$$\limsup_{x \rightarrow +\infty} \frac{1}{x} \sum_{0 \leq n < x} f(n) \text{ exists and is positive}$$

will not imply the condition

$$\limsup_{k \rightarrow +\infty} \prod_{0 \leq j \leq k} \frac{1}{q_j} \sum_{0 \leq a \leq q_j - 1} f(aQ_j) \text{ exists and is positive,}$$

and non-negative Q -multiplicative functions f such that the condition

$$\limsup_{k \rightarrow +\infty} \prod_{0 \leq j \leq k} \frac{1}{q_j} \sum_{0 \leq a \leq q_j - 1} f(aQ_j) \text{ exists and is positive}$$

will not imply the condition

$$\limsup_{x \rightarrow +\infty} \frac{1}{x} \sum_{0 \leq n < x} f(n) \text{ exists and is positive.}$$

In this article, we shall consider the case of non-negative Q -multiplicative functions with a positive upper meanvalue defined via an unbounded Cantor system.

Given an arbitrary arithmetical function f , we set

$$\begin{aligned} S_N(f) &= \sum_{0 \leq n < N} f(n), \\ \varpi_k(f) &= q_k^{-1} \sum_{0 \leq n \leq q_k - 1} f(aQ_k), \\ \prod_{k-}(f) &= \prod_{0 \leq r \leq k-1} \varpi_r(f). \end{aligned}$$

For our convenience, the result of a summation (resp. a product) on an empty set will be 0 (resp.1).

Now, for a given f of non-negative Q -multiplicative function, we define a sequence of arithmetical functions $f_{k-}(x)$ on Z_Q (resp. $f_{k-}^*(x)$) by $f_{k-}(x) = \prod_{0 \leq j < k} f(a_j Q_j)$ (resp. $f_{k-}^*(x) = \prod_{0 \leq j < k} f(a_j Q_j) \cdot \varpi_j(f)^{-1}$), where x being written in base Q as $x = \sum_{j=0}^{+\infty} a_j Q_j$. For simplicity, we shall also use the notations $f_j(x) = f(aQ_j)$ and $f^*(aQ_j) = f(aQ_j) \cdot \varpi_j(f)^{-1}$.

We denote by Z_Q the compact group $Z_Q = \lim_{k \rightarrow +\infty} Z/Q_k Z$ equipped with the natural Haar measure μ , and we shall identify it with the compact space $\prod_k Z/q_k Z$ equipped with the measure $\mu = \otimes_k \mu_{q_k}$, where μ_{q_k} is the uniform measure on $Z/q_k Z$. An element a of Z_Q can be written as $a = (a_0, a_1, \dots)$, $0 \leq a_k \leq q_k - 1$, $0 \leq k$, and an integer is an element of Z_Q which has only a finite number of digits different from zero. For

$a = (a_0, a_1, \dots)$ in Z_Q , we denote by $x_{k-}(a)$ the sequence of random variables defined by $x_{k-}(a) = \{a_j\}_{0 \leq j \leq k-1}$, and by $x_{k+}(a)$ the sequence of random variables defined by $x_{k+}(a) = \{a_j\}_{k \leq j}$. We shall use also the notation x_k for an integer $x_k = \sum_{j=0}^{k-1} a_j Q_k$ when $x = \sum_{j=0}^{+\infty} a_j Q_k$.

We have the following result:

Theorem 2. *Let (Q) be an unbounded Cantor system, and $f(n)$ be a non-negative Q -multiplicative function such that*

$$\limsup_{x \rightarrow +\infty} \frac{1}{x} \sum_{0 \leq n \leq x-1} f(n)$$

exists and is positive. Then, there are two possibilities:

1) $\sum_{1 \leq k} q_k^{-1} \sum_{0 \leq a \leq q_k-1} (1 - f(aQ_k)^{1/2} \varpi_k(f^{1/2})^{-1})^2$ is bounded, and in this case, for any $r, 0 \leq r \leq 1$, we have μ -almost surely

$$\frac{1}{x_k} \sum_{0 \leq n \leq x_k-1} f(n)^r = \left(\prod_{0 \leq j \leq k} \frac{1}{q_j} \sum_{0 \leq a \leq q_j-1} f(aQ_j)^r \right) \cdot (1 + o(1)),$$

as $x_k \rightarrow x$.

2) $\sum_{1 \leq k} q_k^{-1} \sum_{0 \leq a \leq q_k-1} (1 - f(aQ_k)^{1/2} \varpi_k(f^{1/2})^{-1})^2$ is not bounded, and in this case, for any $r, 0 < r < 1$, we have

$$\frac{1}{x} \sum_{0 \leq n \leq x-1} f(n)^r = o(1), \quad \text{as } x \rightarrow +\infty.$$

§3. Proof of the results

3.1. Proof of Theorem 1

1) We begin with a proof of assertion 1).

Proof. Assume that $S = \limsup_{x \rightarrow +\infty} x^{-1} \sum_{0 \leq n < x} f(n)$ exists and is positive.

Let x_i be a sequence such that

$$\frac{1}{2}S \leq x_i^{-1} \sum_{0 \leq n < x_i} f(n).$$

A fortiori, if $\kappa(x_i)$ denotes the maximal index k for which $a_k(x_i)$ is different from zero, then we have

$$\frac{1}{2}S \leq x_i^{-1} \sum_{0 \leq n < Q_{\kappa(x_i)+1}} f(n),$$

and so

$$\left(\frac{Q_{\kappa(x_i)+1}}{x_i}\right)^{-1} \times \left(\frac{1}{2}S\right) \leq \left(\frac{1}{Q_{\kappa(x_i)+1}} \sum_{0 \leq n < Q_{\kappa(x_i)+1}} f(n)\right).$$

Since $\left(\frac{Q_{\kappa(x_i)+1}}{x_i}\right)^{-1} \geq \frac{1}{\max(q_k)}$ and $\max(q_k)$ is bounded, this gives us that there is some $S' \geq \frac{1}{2 \cdot \max(q_k)}S$, hence > 0 , such that

$$0 < S' \leq \limsup_{k \rightarrow +\infty} \frac{1}{Q_k} \sum_{0 \leq n \leq Q_k-1} f(n) < +\infty.$$

Conversely, if there exists some positive S'' such that

$$\limsup_{k \rightarrow +\infty} \frac{1}{Q_k} \sum_{0 \leq n \leq Q_k-1} f(n) = S'' < +\infty,$$

then by using the same notations as above, we remark that, since

$$\sum_{0 \leq n \leq Q_{\kappa(x)}} f(n) \leq \sum_{0 \leq n \leq x} f(n) \leq \sum_{0 \leq n < Q_{\kappa(x)+1}} f(n),$$

we have

$$(x^{-1}Q_{\kappa(x)}) \left(Q_{\kappa(x)}^{-1} \sum_{0 \leq n < Q_{\kappa(x)}} f(n) \right) \leq x^{-1} \sum_{0 \leq n \leq x} f(n)$$

and

$$x^{-1} \sum_{0 \leq n \leq x} f(n) \leq (x^{-1}Q_{\kappa(x)+1}) \left(Q_{\kappa(x)+1}^{-1} \sum_{0 \leq n < Q_{\kappa(x)+1}} f(n) \right).$$

Hence we get that

$$0 < \frac{1}{\max(q_k)} S'' \leq \limsup_{x \rightarrow +\infty} x^{-1} \sum_{0 \leq n \leq x} f(n),$$

for $(x^{-1}Q_{\kappa(x)}) \geq \frac{1}{\max(q_k)} > 0$, and

$$\limsup_{x \rightarrow +\infty} x^{-1} \sum_{0 \leq n \leq x} f(n) \leq \max(q_k) S'' < +\infty,$$

since

$$(x^{-1}Q_{\kappa(x)+1}) \leq \max(q_k) < +\infty,$$

and so

$$\limsup_{x \rightarrow +\infty} x^{-1} \sum_{0 \leq n \leq x} f(n)$$

exists and its value is positive.

Q.E.D.

2) We prove now assertion 2).

Proof. We consider the following (with indexation shifted for convenience of notations) Q -system, satisfying $\limsup(q_k) = +\infty$:

$$q_k = k, \quad k \geq 2,$$

and the Q -multiplicative function f defined by

$$f(aQ_k) = 1 \quad \text{if } k \neq 2^r \text{ and } 0 \leq a \leq q_k - 2,$$

$$f((q_k - 1)Q_k) = 0 \quad \text{if } k \neq 2^r,$$

$$f(Q_{2^r}) = 2^r - 1,$$

$$f(aQ_{2^r}) = 0 \quad \text{if } 2 \leq a \leq 2^r - 1.$$

We have

$$\begin{aligned} & \prod_{2 \leq j \leq k} \frac{1}{q_j} \sum_{0 \leq a \leq q_j - 1} f(aQ_j) \\ &= \left(\prod_{2 \leq j \leq k, j \neq 2^r} \frac{1}{j} (j-1) \right) \left(\prod_{2 \leq j \leq k, j = 2^r} \frac{1}{2^r} (1 + (2^r - 1)) \right) \end{aligned}$$

and

$$\begin{aligned} \prod_{2 \leq j \leq k, j \neq 2^r} \frac{1}{j} (j-1) &= \left(\prod_{2 \leq j \leq k} \frac{1}{j} (j-1) \right) \left(\prod_{2 \leq j \leq k, j = 2^r} \frac{1}{2^r} (2^r - 1) \right)^{-1} \\ &= ((k-1)!/k!) \left(\prod_{2 \leq j \leq k, j = 2^r} \frac{1}{2^r} (2^r - 1) \right)^{-1} \\ &= \frac{1}{k} \prod_{2 \leq j \leq k, j = 2^r} \left(1 - \frac{1}{2^r} \right)^{-1}, \end{aligned}$$

and so, since $\prod_{2 \leq r} \left(1 - \frac{1}{2^r} \right)^{-1}$ is convergent, we have

$$\lim_{k \rightarrow +\infty} \prod_{2 \leq j \leq k} \frac{1}{q_j} \sum_{0 \leq a \leq q_j - 1} f(aQ_j) = 0.$$

Now, for $x = 2Q_{2^k} - 1$, we have

$$\begin{aligned} \frac{1}{x+1} \sum_{0 \leq n \leq x} f(n) &= \frac{1}{2Q_{2^k}} \sum_{0 \leq n \leq 2Q_{2^k} - 1} f(n) \\ &= \left(\frac{1}{2} (f(0 \cdot Q_{2^k}) + f(1 \cdot Q_{2^k})) \right) \times \left(\prod_{r=2}^{2^k-1} \frac{1}{q_r} \sum_{a=0}^{q_r-1} f(aQ_r) \right) \\ &= \left(\frac{1}{2} 2^k \right) \times \left(\frac{1}{2^k - 1} \prod_{2 \leq r \leq k-1} \left(1 - \frac{1}{2^r} \right)^{-1} \right) \\ &\geq \frac{1}{2} \\ &> 0. \end{aligned}$$

As a consequence, the condition

$$0 < \limsup_{x \rightarrow +\infty} \frac{1}{x} \sum_{0 \leq n < x} f(n) < +\infty$$

will not imply

$$0 < S' = \limsup_{k \rightarrow +\infty} \prod_{2 \leq j \leq k} \frac{1}{q_j} \sum_{0 \leq a \leq q_j - 1} f(aQ_j) < +\infty$$

for some S' .

In a similar way, it is possible, using the same kind of approach as above, to provide an example of Q -multiplicative function such that the condition

$$\limsup_{k \rightarrow +\infty} \prod_{0 \leq j \leq k} \frac{1}{q_j} \sum_{0 \leq a \leq q_j - 1} f(aQ_j) < +\infty$$

will not imply the condition

$$\limsup_{x \rightarrow +\infty} \frac{1}{x} \sum_{0 \leq n < x} f(n) < +\infty.$$

It is sufficient to consider the following (again with indexation shifted for convenience of notations) Q -system, satisfying $\limsup(q_k) = +\infty$:

$$q_k = k, \quad k \geq 2,$$

and the Q -multiplicative function f defined by

$$f(aQ_k) = 1 \quad \text{if } k \neq 2^r,$$

$$f(Q_{2^r}) = 2^r - 1,$$

$$f(aQ_{2^r}) = 0 \text{ if } 2 \leq a \leq 2^r - 1.$$

We have

$$\prod_{2 \leq j \leq k} \frac{1}{q_j} \sum_{0 \leq a \leq q_j - 1} f(aQ_j)$$

$$= \left(\prod_{2 \leq j \leq k, j \neq 2^r} \frac{1}{j} \sum_{0 \leq a \leq j - 1} 1 \right) \left(\prod_{2 \leq j \leq k, j = 2^r} \frac{1}{2^r} (1 + (2^r - 1)) \right) = 1.$$

Now, for $x = 2Q_{2^k} - 1$, we have

$$\frac{1}{x + 1} \sum_{0 \leq n \leq x} f(n) = \frac{1}{2Q_{2^k}} \sum_{0 \leq n \leq 2Q_{2^k} - 1} f(n)$$

$$= \left(\frac{1}{2} (f(0 \cdot Q_{2^k}) + f(1 \cdot Q_{2^k})) \right) \times \left(\prod_{r=2}^{2^k - 1} \frac{1}{q_r} \sum_{a=0}^{q_r - 1} f(aQ_r) \right)$$

$$= \left(\frac{1}{2} 2^k \right)$$

$$= 2^{k-1}.$$

Q.E.D.

3.2. Proof of theorem 2

3.2.1. Method of proof

The method is as follows:

- i) We associate to f a Radon measure ν_f on Z_Q .
- ii) We prove that ν_f is absolutely continuous with respect to μ if

$$\sum_{1 \leq k} q_k^{-1} \sum_{0 \leq a \leq q_k - 1} \left(1 - f(aQ_k)^{1/2} \varpi_k(f^{1/2})^{-1} \right)^2$$

is bounded, and orthogonal to μ if

$$\sum_{1 \leq k} q_k^{-1} \sum_{0 \leq a \leq q_k - 1} \left(1 - f(aQ_k)^{1/2} \varpi_k(f^{1/2})^{-1} \right)^2$$

is not bounded.

Remark that this dichotomy leaves no other eventuality.

- iii) We prove part 1) of Theorem 2 in the case $r = 1$ as a simple consequence of the absolute continuity of ν_f .

iv) We show that to f^r , $0 < r < 1$, one can associate a Radon measure which is absolutely continuous with respect to μ . As a consequence, with iii), this gives the proof of part 1) of Theorem 2.

v) We prove directly part 2) of Theorem 2.

3.2.2.

We denote by $(a, k(a))$ an arithmetical progression $\{a + Q_{k(a)}n\}_{n \in N}$, where a is in N , $k(a)$ is a positive integer such that $Q_{k(a)} > a$. Let $I_{a,k(a)}$ be its characteristic function. Remark that $I_{a,k(a)}$ is the restriction to N of the characteristic function, still denoted $I_{a,k(a)}$, of the open subset $O_{(a,k(a))}$ of Z_Q defined by $O_{(a,k(a))} = (x_{k(a)-}(a), \prod_{k \geq k(a)} Z/q_k Z)$, and that this function is continuous, which implies that

$$\lim \frac{1}{x} \sum_{0 \leq n < x} I_{a,k(a)}(n) = \mu(O_{(a,k(a))}).$$

i) Radon measure associated to f .

Let $f(n)$ be a nonnegative Q -multiplicative function with a positive bounded upper mean-value $\overline{M}(f)$. Since $\overline{M}(f)$ exists, the series $\sum_{n \in N} f(n)x^n$ converges for $|x| < 1$ and can be written as

$$\sum_{n \in N} f(n)x^n = \lim_{k \rightarrow +\infty} \sum_{0 \leq n \leq Q_k - 1} f(n)x^n = \prod_{0 \leq k} \left(\sum_{0 \leq b \leq q_k - 1} f(bQ_k)x^{bQ_k} \right).$$

Moreover, since $f(n)$ is non-negative for all n in N , as a consequence of a theorem of Hardy and Littlewood ([1], theorem 4), we get that there exists some $L > 0$ and a sequence $(x_k)_{k \in N}$ such that $\lim_{k \rightarrow +\infty} x_k = 1$ and $\lim_{k \rightarrow +\infty} (1 - x_k)^{-1} \sum_{n \in N} f(n)x_k^n = L$.

In fact if not, then,

$$\lim_{x \rightarrow 1^-} (1 - x)^{-1} \sum_{n \in N} f(n)x^n = 0,$$

which implies that the mean value of $f(n)$ is equal to zero, a contradiction with our hypothesis that $f(n)$ has a positive bounded upper mean-value $\overline{M}(f)$.

Now, we remark that

$$\sum_{n \in N} f(n)I_{a,k(a)}(n)x^n = \sum_{n \in N, n \equiv a \pmod{Q_{k(a)}}} f(n)x^n$$

and, since the function $f_{k(a)}(n)$ defined by $f_{k(a)}(n) = f(Q_{k(a)}n)$ can be regarded as a Q -multiplicative function for the Cantor system defined

by $q'_k = q_{k+k(a)}$, $k \geq 0$, we get that

$$\begin{aligned}
 \sum_{n \in \mathbb{N}} f(n) I_{a,k(a)}(n) x^n &= \sum_{m \in \mathbb{N}} f(a + Q_{k(a)} m) x^{a + Q_{k(a)} m} \\
 &= f(a) x^a \sum_{m \in \mathbb{N}} f(Q_{k(a)} m) x^{Q_{k(a)} m} \\
 &= f(a) x^a \prod_{k \geq k(a)} \left(\sum_{0 \leq b \leq q_k - 1} f(b Q_k) x^{b Q_k} \right) \\
 &= f(a) x^a \left(\left(\prod_{0 \leq k \leq k(a) - 1} \left(\sum_{0 \leq b \leq q_k - 1} f(b Q_k) x^{b Q_k} \right) \right)^{-1} \right. \\
 &\quad \left. \times \left(\prod_{0 \leq k} \left(\sum_{0 \leq b \leq q_k - 1} f(b Q_k) x^{b Q_k} \right) \right) \right) \\
 &= \left(f(a) x^a \left(\prod_{0 \leq k \leq k(a) - 1} \left(\sum_{0 \leq b \leq q_k - 1} f(b Q_k) x^{b Q_k} \right) \right)^{-1} \right) \times \left(\sum_{n \in \mathbb{N}} f(n) x^n \right).
 \end{aligned}$$

Since $f(n)$ is non-negative and $f(0 \cdot Q_k) = 1$, the function $F_{a,k(a)}(x)$ defined by

$$F_{a,k(a)}(x) = \left(f(a) x^a \left(\prod_{0 \leq k \leq k(a) - 1} \left(\sum_{0 \leq b \leq q_k - 1} f(b Q_k) x^{b Q_k} \right) \right)^{-1} \right)$$

is analytic on a neighborhood of 1, and as a consequence of the relation

$$\sum_{n \in \mathbb{N}} f(n) x_k^n \sim (1 - x_k) L \quad \text{as } k \rightarrow +\infty,$$

we get that

$$\sum_{n \in \mathbb{N}} f(n) I_{a,k(a)}(n) x_k^n \sim (1 - x_k) L F_{a,k(a)}(1) \quad \text{as } k \rightarrow +\infty,$$

i.e.

$$\begin{aligned}
 &\lim_{k \rightarrow +\infty} (1 - x_k)^{-1} \sum_{n \in \mathbb{N}} f(n) I_{a,k(a)}(n) x_k^n \\
 &= L f(a) \left(\prod_{0 \leq k \leq k(a) - 1} \left(\sum_{0 \leq b \leq q_k - 1} f(b Q_k) \right) \right)^{-1} \quad \text{as } k \rightarrow +\infty.
 \end{aligned}$$

And so, we shall define $\nu_f(I_{a,k(a)})$ by

$$\nu_f(I_{a,k(a)}) = f(a) \left(\prod_{0 \leq k \leq k(a)-1} \left(\sum_{0 \leq b \leq q-1} f(bq^k) \right) \right)^{-1}.$$

Now, we check that ν_f is a Radon measure. (For the definition, properties of the Radon measures, see [3], ch2, p.57 et seq.). To do that, we consider the set \mathcal{A} of complex-valued continuous functions defined on Z_Q by

$$\mathcal{A} = \left\{ h = \sum_{l_a \in L} l_a \cdot I_{a,k(a)}, L \text{ finite, } l_a \text{ complex numbers} \right\}.$$

This is an algebra of step functions, and by the Stone-Weierstrass theorem ([2], p. 101, note 1.a), \mathcal{A} is dense with respect to the uniform topology in the set of the complex-valued continuous functions defined on Z_Q . If h is in \mathcal{A} , we define $\nu_f(h)$ by $\nu_f(h) = \sum_{l_a \in L} l_a \cdot \nu_f(I_{a,k(a)})$. It is a simple remark that we have

$$\nu_f(h) = L^{-1} \lim_{k \rightarrow +\infty} (1 - x_k)^{-1} \sum_{n \in N} f(n) h(n) x_k^n.$$

Since $\nu_f(1) = 1$, for a given $\varepsilon > 0$, if h and h' are in \mathcal{A} and satisfy $\sup_{t \in Z_Q} |h'(t) - h(t)| \leq \varepsilon$, then we have $|\nu_f(h' - h)| \leq \varepsilon$, since $|\nu_f(h' - h)| \leq \nu_f(1) \cdot \sup_{t \in Z_Q} |h'(t) - h(t)| \leq 1 \cdot \varepsilon$, and so ν_f defines a continuous linear form on the set of the complex-valued continuous functions defined on Z_Q . By Riesz representation theorem ([2], p. 129, (11.37)), this gives us that ν_f is a positive Radon measure on Z_Q .

ii) Characterization of the absolute continuity (resp. orthogonality) of ν_f with respect to μ .

For K in N , we have

$$\begin{aligned} & 1 - f_{K-}(t)^{1/2} \prod_{K-} (f^{1/2})^{-1} \\ &= \sum_{1 \leq k \leq K} \left(f_{(k-1)-}(t)^{1/2} \prod_{(k-1)-} (f^{1/2})^{-1} - f_{k-}(t)^{1/2} \prod_{k-} (f^{1/2})^{-1} \right) \\ &= \sum_{1 \leq k \leq K} \left(f_{(k-1)-}(t)^{1/2} \prod_{(k-1)-} (f^{1/2})^{-1} \right) \left(1 - f_{k-1}(t)^{1/2} \varpi_{k-1} (f^{1/2})^{-1} \right). \end{aligned}$$

We remark that

$$\int \left(1 - f_{k-1}(t)^{1/2} \varpi_{k-1} (f^{1/2})^{-1} \right) d\mu(t) = 0,$$

$$\int \left(1 - f_k(t)^{1/2} \varpi_k(f^{1/2})^{-1}\right) \left(1 - f_l(t)^{1/2} \varpi_l(f^{1/2})^{-1}\right) d\mu(t)$$

$$\begin{cases} = 0 & \text{if } k \neq l, \text{ and} \\ = q_k^{-1} \sum_{0 \leq a \leq q_k + 1 - 1} \left(1 - f(aQ_k)^{1/2} \varpi_k(f^{1/2})^{-1}\right)^2 & \text{if } k = l. \end{cases}$$

As a consequence of these orthogonality relations, we get that

$$\int \left(1 - f_{K-}(t)^{1/2} \prod_{K-}(f^{1/2})^{-1}\right)^2 d\mu(t)$$

$$= \sum_{1 \leq k \leq K} \int \left(f_{(k-)-(t)}^{1/2} \prod_{(k-)-(f^{1/2})^{-1}}\right)^2 d\mu(t)$$

$$\times \int \left(1 - f_{k-1}(t)^{1/2} \varpi_{k-1}(f^{1/2})^{-1}\right)^2 d\mu(t).$$

Now, since we have

$$\int \left(f_{(k-)-(t)}^{1/2} \prod_{(k-)-(f^{1/2})^{-1}}\right)^2 d\mu(t)$$

$$= \prod_{(k-)-(f)} \times \prod_{(k-)-(f^{1/2})^{-2}},$$

we obtain that

$$\int \left(1 - f_{K-}(t)^{1/2} \prod_{K-}(f^{1/2})^{-1}\right)^2 d\mu(t)$$

$$= \prod_{K-}(f) \times \prod_{K-}(f^{1/2})^{-2} - 1$$

$$= \sum_{1 \leq k \leq K-1} \left(\prod_{(k-)-(f)} \times \prod_{(k-)-(f^{1/2})^{-2}} \right)$$

$$\times \left(q_k^{-1} \sum_{0 \leq a \leq q_k - 1} \left(1 - f(aQ_k)^{1/2} \varpi_k(f^{1/2})^{-1}\right)^2 \right),$$

and if we are in the situation such that $\lim_{k \rightarrow +\infty} \left(\prod_{k-}(f) \times \prod_{k-}(f^{1/2})^{-2}\right)$ exists and is > 0 , we get that the series

$$\sum_{1 \leq k} q_k^{-1} \sum_{0 \leq a \leq q_k - 1} \left(1 - f(aQ_k)^{1/2} \varpi_k(f^{1/2})^{-1}\right)^2$$

is convergent.

Assuming that we are in the case where

$$\lim_{K \rightarrow +\infty} \prod_{K-}(f)^{-1} \times \prod_{K-}(f^{1/2})^2 = 0,$$

we consider the equality

$$\begin{aligned} & \prod_{K-}(f) \times \prod_{K-}(f^{1/2})^{-2} - 1 \\ &= \sum_{1 \leq k \leq K-1} \left(\prod_{(k-1)-}(f) \times \prod_{(k-1)-}(f^{1/2})^{-2} \right) \\ & \quad \times \left(q_k^{-1} \sum_{0 \leq a \leq q_k + 1 - 1} \left(1 - f(aQ_k)^{1/2} \varpi_k(f^{1/2})^{-1} \right)^2 \right). \end{aligned}$$

We multiply each member of this equality by $\prod_{K-}(f)^{-1} \times \prod_{K-}(f^{1/2})^2$, and we get that

$$\begin{aligned} & 1 - \prod_{K-}(f)^{-1} \times \prod_{K-}(f^{1/2})^2 \\ &= \sum_{1 \leq k \leq K-1} A(K, k) \times \left(q_k^{-1} \sum_{0 \leq a \leq q_k - 1} \left(1 - f(aQ_k)^{1/2} \varpi_k(f^{1/2})^{-1} \right)^2 \right), \end{aligned}$$

where $A(K, k)$ is defined by

$$A(K, k) = \prod_{(k-1)-}(f) \times \prod_{(k-1)-}(f^{1/2})^{-2} \times \prod_{K-}(f)^{-1} \times \prod_{K-}(f^{1/2})^2.$$

Now, we remark that if

$$\lim_{K \rightarrow +\infty} \prod_{K-}(f)^{-1} \times \prod_{K-}(f^{1/2})^2 = 0,$$

then, for a fixed k , we have

$$\lim_{K \rightarrow +\infty} A(K, k) = 0.$$

Since we have

$$\lim_{K \rightarrow +\infty} (1 - \prod_{K-}(f)^{-1} \times \prod_{K-}(f^{1/2})^2) = 1,$$

we get that the series of general term $q_k^{-1} \sum_{0 \leq a \leq q_k - 1} (1 - f(aQ_k)^{1/2} \varpi_k(f^{1/2})^{-1})^2$ is not convergent, i.e.

$$\limsup_{K \rightarrow +\infty} \sum_{1 \leq k \leq K} q_k^{-1} \sum_{0 \leq a \leq q_k - 1} \left(1 - f(aQ_k)^{1/2} \varpi_k(f^{1/2})^{-1} \right)^2 = +\infty.$$

This proves that the measure ν_f is continuous with respect to μ (resp. orthogonal to μ) if and only if the series of general term $q_k^{-1} \sum_{0 \leq a \leq q_k - 1} (1 - f(aQ_k)^{1/2} \varpi_k(f^{1/2})^{-1})^2$ is convergent (resp. divergent).

iii) Part 1) of Theorem 2 in the case $r = 1$ is a simple consequence of the absolute continuity of ν_f .

Proof. We shall apply to the present situation the method of proof given in [4].

1) First we prove

Lemma 1. *There exists a subset F_∞ of \mathbf{Z}_Q such that $\mu(F_\infty) = 1$ and for every $x = (a_0(x), a_1(x), \dots)$ in F_∞ , we have*

$$\lim_{\substack{k \rightarrow +\infty \\ a_k(x) \neq 0}} \frac{1}{a_k(x)} \sum_{0 \leq a < a_k(x)} (1 - f^*(aQ_k)) = 0.$$

2) This is a consequence of the following result:

Lemma 2. *There exists a subset F_∞ of \mathbf{Z}_Q such that $\mu(F_\infty) = 1$ and for every $x = (a_0(x), a_1(x), \dots)$ in F_∞ , we have*

$$\lim_{\substack{k \rightarrow +\infty \\ a_k(x) \neq 0}} \frac{1}{a_k(x)} \sum_{0 \leq a < a_k(x)} \left(1 - f^*(aQ_k)^{1/2}\right)^2 = 0.$$

Proof. 2) \Rightarrow 1).

We have

$$\left(1 - f^*(aQ_k)^{1/2}\right)^2 = 2 \cdot (1 - f^*(aQ_k)^{1/2}) - (1 - f^*(aQ_k)),$$

which gives us that

$$(1 - f^*(aQ_k)) = 2 \cdot (1 - f^*(aQ_k)^{1/2}) - \left(1 - f^*(aQ_k)^{1/2}\right)^2.$$

As a consequence, we get that

$$\begin{aligned} & \sum_{0 \leq a < a_k(x)} (1 - f^*(aQ_k)) \\ &= \sum_{0 \leq a < a_k(x)} 2 \cdot (1 - f^*(aQ_k)^{1/2}) - \sum_{0 \leq a < a_k(x)} \left(1 - f^*(aQ_k)^{1/2}\right)^2, \end{aligned}$$

which gives that

$$\begin{aligned} & \left| \sum_{0 \leq a < a_k(x)} (1 - f^*(aQ_k)) \right| \\ & \leq 2 \left| \sum_{0 \leq a < a_k(x)} \left(1 - f^*(aQ_k)^{1/2}\right) \right| + \sum_{0 \leq a < a_k(x)} \left(1 - f^*(aQ_k)^{1/2}\right)^2. \end{aligned}$$

By the Cauchy inequality, we have

$$\left| \sum_{0 \leq a < a_k(x)} (1 - f^*(aQ_k)^{1/2}) \right| \leq a_k(x)^{1/2} \cdot \left(\sum_{0 \leq a < a_k(x)} (1 - f^*(aQ_k)^{1/2})^2 \right)^{1/2},$$

and so we get that

$$\begin{aligned} & \left| \frac{1}{a_k(x)} \sum_{0 \leq a < a_k(x)} (1 - f^*(aQ_k)) \right| \\ & \leq 2 \cdot \left(\frac{1}{a_k(x)} \sum_{0 \leq a < a_k(x)} (1 - f^*(aQ_k)^{1/2})^2 \right)^{1/2} \\ & \quad + \frac{1}{a_k(x)} \sum_{0 \leq a < a_k(x)} (1 - f^*(aQ_k)^{1/2})^2. \end{aligned}$$

Hence we have

$$\lim_{\substack{k \rightarrow +\infty \\ a_k(x) \neq 0}} \frac{1}{a_k(x)} \sum_{0 \leq a < a_k(x)} (1 - f^*(aQ_k)) = 0.$$

Q.E.D.

3) We prove that there exists a subset F_∞ of \mathbf{Z}_Q such that $\mu(F_\infty) = 1$ and for every $x = (a_0(x), a_1(x), \dots)$ in F_∞ , we have

$$\lim_{\substack{k \rightarrow +\infty \\ a_k(x) \neq 0}} \frac{1}{a_k(x)} \sum_{0 \leq a < a_k(x)} (1 - f(aQ_k)^{1/2} \varpi_k(f^{1/2})^{-1})^2 = 0.$$

Proof. Since the series

$$\sum_{1 \leq k} q_k^{-1} \sum_{0 \leq a \leq q_k - 1} (1 - f(aQ_k)^{1/2} \varpi_k(f^{1/2})^{-1})^2$$

is convergent, let σ_k be defined by $\sigma_k = \frac{1}{q_k} \sum_{a=0}^{q_k-1} (1 - f(aQ_k)^{1/2} \varpi_k(f^{1/2})^{-1})^2$. For x in \mathbf{Z}_Q , we write $x = (a_0(x), a_1(x), \dots)$, $0 \leq a_k(x) \leq$

$q_k - 1, 0 \leq k$ and we remark that, on the sequence of the $a_k(x)$ different from 0, one has

$$\begin{aligned} & \frac{1}{a_k(x)} \sum_{0 \leq a < a_k(x)} \left(1 - f(aQ_k)^{1/2} \varpi_k(f^{1/2})^{-1}\right)^2 \\ & \leq \frac{1}{a_k(x)} \sum_{0 \leq a < q_k} \left(1 - f(aQ_k)^{1/2} \varpi_k(f^{1/2})^{-1}\right)^2 \\ & \leq \frac{q_k}{a_k(x)} \frac{1}{q_k} \sum_{0 \leq a < q_k} \left(1 - f(aQ_k)^{1/2} \varpi_k(f^{1/2})^{-1}\right)^2. \end{aligned}$$

Since $\sum_k \sigma_k < +\infty$, there exists an increasing positive function h tending to infinity as k tends to infinity such that $\sum_k \sigma_k h(k) < +\infty$ and $\prod_{k=0}^{+\infty} (1 - \sigma_k h(k)) > 0$. We consider the set $F(h)$ of points x in \mathbf{Z}_Q such that for all k , the inequality

$$[q_k \sigma_k h(k)] \leq a_k(x) \leq q_k - 1$$

holds, where $[\cdot]$ denotes the integer part function. This set $F(h)$ is closed, and its measure $\mu(F(h))$ is equal to

$$\prod_{k=0}^{+\infty} \frac{1}{q_k} (q_k - [q_k \sigma_k h(k)]),$$

and we have

$$\mu F(h) \geq \prod_{k=0}^{+\infty} \frac{1}{q_k} (q_k - q_k \sigma_k h(k)).$$

Now, we remark that this last product can be written as $\prod_{k=0}^{+\infty} (1 - \sigma_k h(k))$ and so, $\mu F(h) \neq 0$. For an x in $F(h)$, we consider the condition $[q_k \sigma_k h(k)] \leq a_k(x) \leq q_k - 1$, for $a_k(x) \neq 0$. If $[q_k \sigma_k h(k)]$ is not 0, then we have

$$\begin{aligned} \frac{q_k}{a_k(x)} \sigma_k & \leq \frac{q_k}{[q_k \sigma_k h(k)]} \sigma_k \\ & \leq \frac{q_k \sigma_k h(k)}{[q_k \sigma_k h(k)]} \cdot \frac{q_k}{q_k \sigma_k h(k)} \sigma_k \leq \frac{q_k \sigma_k h(k)}{[q_k \sigma_k h(k)]} \frac{1}{h(k)} \leq \frac{2}{h(k)} \end{aligned}$$

and in this case, we get $\lim_{k \rightarrow +\infty} \frac{q_k}{a_k(x)} \sigma_k = 0$. Now the remaining case is that $[q_k \sigma_k h(k)] = 0$. We have $0 \leq q_k \sigma_k h(k) < 1$, i.e. $q_k \sigma_k < 1/h(k)$. Hence

$$\frac{q_k}{a_k(x)} \sigma_k \leq \frac{q_k}{1} \sigma_k \leq q_k \sigma_k \leq \frac{1}{h(k)} = o(1), \quad k \rightarrow +\infty.$$

To obtain the result, we remark that the sequence of functions h_r indexed by positive integers r and defined by $h_r(n) = h(n)$ if $n > r$ and $h(n)r^{-1}$ otherwise, satisfies the same requirements as h . Now, the sequence of closed sets $F(h_r)$ is increasing with r and $\lim_{r \rightarrow +\infty} \mu(F(h_r)) = 1$. This gives that F_∞ , the union of the $F(h_r)$, is a measurable set of measure 1. Now, if x belongs to F_∞ , it belongs to some $F(h_r)$ and as a consequence, along the sequence k such that $a_k(x) \neq 0$, we have

$$\begin{aligned} & \frac{1}{a_k(x)} \sum_{0 \leq a < a_k(x)} \left(1 - f(aQ_k)^{1/2} \varpi_k(f^{1/2})^{-1}\right)^2 \\ & \leq \frac{q_k}{a_k(x)} \sigma_k \\ & \leq q_k \sigma_k \\ & \leq \frac{2}{h_r(k)} = o(1), \quad k \rightarrow +\infty. \end{aligned}$$

Q.E.D.

4) We shall need the following result:

Lemma 3. *There exists a subset E_∞ of Z_Q such that $\mu(E_\infty) = 1$ and for every $x = (a_0(x), a_1(x), \dots)$ in E_∞ and $\varepsilon > 0$, there exists a positive integer $K(x)$ such that for $s \geq r \geq K(x)$, and we have*

$$\left| \left(\prod_{s \geq r \geq K(x)} f(aQ_j) \varpi_j(f)^{-1} \right) - 1 \right| \leq \varepsilon.$$

Proof. We consider the sequence of real-valued functions $f_{(k+1)-}^*$ defined on Z_Q by $x \mapsto f_{(k+1)-}^*(x) = \prod_{0 \leq j \leq k} f(a_j(x)Q_j) \varpi_j(f)^{-1}$, $x = (a_0(x), a_1(x), \dots)$. Kakutani's Theorem ([5], p. 109) gives us that $f_{(k+1)-}^*(x)$ converges μ -a.s. and in $L^1(Z_Q, d\mu)$. Hence we get that $f_\infty^*(x) = \prod_{0 \leq j} f(a_j(x)Q_j) \varpi_j(f)^{-1}$ exists μ -a.s. and is in $L^1(Z_Q, d\mu)$. Now, as a consequence of Jessen's Theorem [5, p.108],

$$\lim_{k \rightarrow +\infty} \int f_\infty^*(x) \otimes_{0 \leq j \leq k} d\mu_j(x) = \int f_\infty^* d\mu = 1 \quad \mu\text{-a.s.},$$

i.e.

$$\lim_{k \rightarrow +\infty} \prod_{k \leq j} f(a_j(x)Q_j) \varpi_j(f)^{-1} = 1 \quad \mu\text{-a.s.},$$

and as a consequence, by Cauchy's criterion, we get our result.

Q.E.D.

5) End of the proof

We consider the intersection of the sets E_∞ and F_∞ . We shall prove that, for every ξ in $E_\infty \cap F_\infty$ which is not an integer, we have

$$\frac{1}{x_k(\xi)} \sum_{n < x_k(\xi)} f(n) = \left(\prod_{0 \leq j \leq k} \frac{1}{q_j} \sum_{0 \leq a \leq q_j - 1} f(aQ_j) \right) \cdot (1 + o(1)), \quad \text{as } k \rightarrow +\infty.$$

Let $\xi = (a_0, a_1, a_2, \dots)$ be an element of $E_\infty \cap F_\infty$ and abbreviate $x_k(\xi)$ by x_k . We have:

$$S_{x_k}(f) = \left(\sum_{0 \leq a < a_k} f(aQ_k) \right) \left(\prod_{r=0}^{k-1} \sum_{a=0}^{q_r-1} f(aQ_r) \right) + (f(a_k Q_k)) S_{x_{k-1}}(f)$$

and by iteration

$$\begin{aligned} S_{x_k}(f) &= \sum_{j=0}^k \left(\prod_{j+1 \leq r \leq k} f(a_r Q_r) \right) \left(\sum_{0 \leq a < a_j(\xi)} f(aQ_j) \right) \left(\prod_{r=0}^{j-1} \sum_{a=0}^{q_r-1} f(aQ_r) \right) \\ &= \sum_{j=0}^k \left(\prod_{j+1 \leq r \leq k} f(a_r Q_r) \right) \left(\sum_{0 \leq a < a_j(\xi)} f(aQ_j) \right) \left(\prod_{r=0}^{j-1} \sum_{a=0}^{q_r-1} f(aQ_r) \right). \end{aligned}$$

We remark now that this equality can be written as

$$\begin{aligned} S_{x_k}(f) &\left(\prod_{r=0}^k q_r^{-1} \sum_{a=0}^{q_r-1} f(aQ_r) \right)^{-1} \\ &= \sum_{j=0}^k \left[\left(\prod_{j+1 \leq r \leq k} f^*(a_r Q_r) \right) \left(\sum_{0 \leq a < a_j(\xi)} f^*(aQ_j) \right) \left(\prod_{r=0}^{j-1} \sum_{a=0}^{q_r-1} f^*(aQ_r) \right) \right]. \end{aligned}$$

Since

$$\sum_{a=0}^{q_r-1} f^*(aQ_r) = q_r,$$

we have

$$\left(\prod_{r=0}^{j-1} \sum_{a=0}^{q_r-1} f^*(aQ_r) \right) = \left(\prod_{r=0}^{j-1} q_r \right) = Q_j.$$

The choice of ξ in F_∞ implies that

$$\sum_{0 \leq a < a_j(\xi)} f^*(aQ_r) = a_j(\xi)(1 + \varepsilon_j),$$

with $\varepsilon_j = o(1)$ as j tends to infinity. The choice of ξ in E_∞ implies that

$$\prod_{j+1 \leq r \leq k} f^*(a_r Q_r) = 1 + \varepsilon'_j,$$

with $\varepsilon'_j = o(1)$ as j tends to infinity.

This gives us that

$$S_{x_k}(f) \left(\prod_{r=0}^k q_r^{-1} \sum_{a=0}^{q_r-1} f(aQ_r) \right)^{-1} = \sum_{j=0}^k a_j(\xi) Q_j (1 + \varepsilon_j)(1 + \varepsilon'_j),$$

as $j \rightarrow +\infty$,

and so, since

$$\sum_{j=0}^k a_j(\xi) Q_j = x_k,$$

we remark that we have

$$\begin{aligned} & \lim_{k \rightarrow +\infty} \left(\sum_{j=0}^k a_j(\xi) Q_j \right)^{-1} \left(\sum_{j=0}^k a_j(\xi) Q_j (1 + \varepsilon_j)(1 + \varepsilon'_j) \right) \\ &= \lim_{k \rightarrow +\infty} (x_k)^{-1} \left(\sum_{j=0}^k a_j(\xi) Q_j (1 + \varepsilon_j)(1 + \varepsilon'_j) \right) \\ &= 1 \end{aligned}$$

and as a consequence, we obtain that

$$S_{x_k}(f) x_k^{-1} = \left(\prod_{r=0}^k q_r^{-1} \sum_{a=0}^{q_r-1} f(aQ_r) \right) (1 + o(1)), \quad \text{as } k \rightarrow +\infty.$$

Q.E.D.

iv) To f^r , $0 < r < 1$, one can associate a Radon measure absolutely continuous with respect to μ .

By 3) above, this will give the end of the proof of part 1) of Theorem 2.

We consider the sequence of real-valued functions f_k^* defined on Z_Q by $x \mapsto f_{k-}^*(x) = \prod_{0 \leq j < k} f(a_j(x) Q_j) \varpi_j(f)^{-1}$, $x = (a_0(x), a_1(x), \dots)$. Kakutani's Theorem ([5], p. 109) gives us that $f_{k-}^*(x)$ converges μ -a.s. and in $L^1(Z_Q, d\mu)$. As a consequence, we get that $(f_{k-}^*(x))^r$ converges

$\mu - a.s.$ and in $L^{1/r}(Z_Q, d\mu)$. This implies that

$$\lim_{K \rightarrow +\infty} \left(\prod_{r=0}^{K-1} (1/q_r) \cdot \sum_{a=0}^{q_r-1} f^r(aQ_r) \right) \left(\prod_{r=0}^{K-1} (1/q_r) \cdot \sum_{a=0}^{q_r-1} f(aQ_r) \right)^{-r}$$

exists, and the value is less or equal to 1, but is not zero.

Hence we get that the sequence of functions

$$\left(\left(\prod_{r=0}^{k-1} (1/q_r) \cdot \sum_{a=0}^{q_r-1} f^r(aQ_r) \right) \left(\prod_{r=0}^{k-1} (1/q_r) \cdot \sum_{a=0}^{q_r-1} f(aQ_r) \right)^{-r} \right)^{-1} (f_{k-}^*(x))^r$$

converges μ -a.s. and in $L^{1/r}(Z_Q, d\mu)$, i.e.

$$(f_{k-}(x))^r \left(\prod_{r=0}^{k-1} (1/q_r) \cdot \sum_{a=0}^{q_r-1} f^r(aQ_r) \right)^{-1}$$

converges μ -a.s. and in $L^{1/r}(Z_Q, d\mu)$.

As a consequence, since $L^1(Z_Q, d\mu) \supset L^{1/r}(Z_Q, d\mu)$, this product defines a measure absolutely continuous with respect to μ .

Q.E.D.

v) We prove directly part 2) of Theorem 2.

1) Assume that $\lim_{k \rightarrow +\infty} \int (f_{k-}^*)^{1/2} d\mu = 0$. Then, we have

$$\lim_{x \rightarrow +\infty} \frac{1}{x} S_x(f^{1/2}) = 0.$$

Proof. If $x = \sum_{k=0}^K a_k Q_k$ and K denotes the maximal index k for which $a_k(x)$ is different from zero, we have

$$a_K Q_K \leq x \leq (a_K + 1) Q_K,$$

and so,

$$((a_K + 1) Q_K)^{-1} \leq x^{-1}.$$

But

$$((a_K + 1) Q_K)^{-1} = ((a_K Q_K) \times ((a_K + 1) Q_K)^{-1}) \times (a_K Q_K)^{-1},$$

and since

$$(a_K Q_K) \times ((a_K + 1) Q_K)^{-1} = (a_K) \times (a_K + 1)^{-1}$$

and $a_K \geq 1$, we get that

$$(a_K) \times (a_K + 1)^{-1} \geq 1/2.$$

This implies that

$$\begin{aligned} & ((a_K + 1)Q_K)^{-1} \\ &= ((a_K Q_K) \times ((a_K + 1)Q_K)^{-1}) \times (a_K Q_K)^{-1} \geq (1/2) \times (a_K Q_K)^{-1}, \end{aligned}$$

and as a consequence, since

$$((a_K + 1)Q_K)^{-1} \leq x^{-1},$$

we get that

$$(1/2) \times (a_K Q_K)^{-1} \leq x^{-1}.$$

Similarly, since we have $x^{-1} \leq (a_K Q_K)^{-1}$, we get that $x^{-1} \leq 2 \times ((a_K + 1)Q_K)^{-1}$.

Now, if $g(n)$ is any non-negative Q -multiplicative function, from the inequality

$$a_K Q_K \leq x \leq (a_K + 1)Q_K,$$

we obtain that

$$S_{a_K Q_K}(g) \leq S_x(g) \leq S_{(a_K+1)Q_K}(g)$$

i.e.

$$x^{-1} S_{a_K Q_K}(g) \leq x^{-1} S_x(g) \leq x^{-1} S_{(a_K+1)Q_K}(g)$$

and so, using the above inequalities, we get that

$$(1/2) \times \left((a_K Q_K)^{-1} S_{a_K Q_K}(g) \right) \leq x^{-1} S_{a_K Q_K}(g) \leq x^{-1} S_x(g),$$

i.e.,

$$(1/2) \times \left((a_K Q_K)^{-1} S_{a_K Q_K}(g) \right) \leq x^{-1} S_x(g)$$

and similarly,

$$x^{-1} S_x(g) \leq x^{-1} S_{(a_K+1)Q_K}(g) \leq 2 \times \left(((a_K + 1)Q_K)^{-1} S_{(a_K+1)Q_K}(g) \right),$$

i.e.,

$$x^{-1} S_x(g) \leq 2 \times \left(((a_K + 1)Q_K)^{-1} S_{(a_K+1)Q_K}(g) \right).$$

Replacing g by f , since $\limsup_{x \rightarrow +\infty} \frac{1}{x} S_x(f) = L > 0$, we have, if K is large enough,

$$S_{a_K Q_K}(f) \leq 2L a_K Q_K,$$

$$S_{(a_K+1)Q_K}(f) \leq 2L(a_K+1)Q_K.$$

Now, replacing g by $f^{1/2}$, we have

$$x^{-1}S_x(f^{1/2}) \leq 2 \times \left(((a_K+1)Q_K)^{-1} S_{(a_K+1)Q_K}(f^{1/2}) \right)$$

with

$$S_{(a_K+1)Q_K}(f^{1/2}) = \left(\sum_{0 \leq a \leq a_K} f^{1/2}(aQ_K) \right) \left(\prod_{r=0}^{K-1} \sum_{a=0}^{q_r-1} f^{1/2}(aQ_r) \right),$$

and by Cauchy's inequality, we get that

$$\begin{aligned} & S_{(a_K+1)Q_K}(f^{1/2}) \\ & \leq \left((a_K+1) \left(\sum_{0 \leq a \leq a_K} f(aQ_K) \right) \right)^{1/2} \left(\prod_{r=0}^{K-1} \sum_{a=0}^{q_r-1} f^{1/2}(aQ_r) \right). \end{aligned}$$

This gives us that

$$\begin{aligned} x^{-1}S_x(f^{1/2}) & \leq 2 \times ((a_K+1)Q_K)^{-1} \\ & \quad \times \left((a_K+1) \left(\sum_{0 \leq a \leq a_K} f(aQ_K) \right) \right)^{1/2} \left(\prod_{r=0}^{K-1} \sum_{a=0}^{q_r-1} f^{1/2}(aQ_r) \right), \end{aligned}$$

and we write the right member of this inequality as

$$\begin{aligned} & 2 \times \left(((a_K+1)Q_K)^{-1} \left(\sum_{0 \leq a \leq a_K} f(aQ_K) \right) \right)^{1/2} \left(\prod_{r=0}^{K-1} \sum_{a=0}^{q_r-1} f(aQ_r) \right) \\ & \times \left\{ \left((Q_K)^{-1/2} \right) \times \left(\prod_{r=0}^{K-1} \sum_{a=0}^{q_r-1} f^{1/2}(aQ_r) \right) \left(\prod_{r=0}^{K-1} \sum_{a=0}^{q_r-1} f(aQ_r) \right)^{-1/2} \right\}, \end{aligned}$$

i.e.,

$$\begin{aligned} & 2 \times \left[((a_K+1)Q_K)^{-1} S_{(a_K+1)Q_K}(f) \right]^{1/2} \times \\ & \left[\left(\prod_{r=0}^{K-1} (1/q_r) \cdot \sum_{a=0}^{q_r-1} f^{1/2}(aQ_r) \right) \left(\prod_{r=0}^{K-1} (1/q_r) \cdot \sum_{a=0}^{q_r-1} f(aQ_r) \right)^{-1/2} \right], \end{aligned}$$

and so we have

$$x^{-1}S_x(f^{1/2})$$

$$\leq 2 \times \left[((a_K + 1)Q_K)^{-1} S_{(a_K+1)Q_K}(f) \right]^{1/2} \\ \times \left[\left(\prod_{r=0}^{K-1} (1/q_r) \cdot \sum_{a=0}^{q_r-1} f^{1/2}(aQ_r) \right) \left(\prod_{r=0}^{K-1} (1/q_r) \cdot \sum_{a=0}^{q_r-1} f(aQ_r) \right)^{-1/2} \right].$$

Since

$$((a_K + 1)Q_K)^{-1} S_{(a_K+1)Q_K}(f) \leq 2L$$

and

$$\left(\prod_{r=0}^{K-1} (1/q_r) \cdot \sum_{a=0}^{q_r-1} f^{1/2}(aQ_r) \right) \left(\prod_{r=0}^{K-1} (1/q_r) \cdot \sum_{a=0}^{q_r-1} f(aQ_r) \right)^{-1/2} \\ = o(1), \quad \text{as } K \rightarrow +\infty,$$

we get that $\lim_{x \rightarrow +\infty} x^{-1} S_x(f^{1/2}) = 0$.

Q.E.D.

2) For any r in $]0, 1[$, we have $\lim_{x \rightarrow +\infty} \frac{1}{x} \sum_{0 \leq n \leq x} f(n)^r = 0$.

Proof. Since

$$\lim_{x \rightarrow +\infty} \frac{1}{x} \sum_{0 \leq n \leq x} f(n)^{1/2} = 0,$$

we get that

$$\lim_{x \rightarrow +\infty} \frac{1}{x} \sum_{0 \leq n \leq x, f(n)^{1/2} \geq 1} f(n)^{1/2} = 0,$$

i.e.

$$\lim_{x \rightarrow +\infty} \frac{1}{x} \sum_{0 \leq n \leq x, f(n) \geq 1} f(n)^{1/2} = 0,$$

and as a consequence

$$\lim_{x \rightarrow +\infty} \frac{1}{x} \sum_{0 \leq n \leq x, f(n) \geq 1} 1 = 0,$$

which implies that

$$\lim_{x \rightarrow +\infty} \frac{1}{x} \sum_{0 \leq n \leq x, f(n) \leq 1} 1 = 1.$$

If r is in $]0, 1[$, we have

$$\sum_{0 \leq n \leq x} f(n)^r = \sum_{0 \leq n \leq x, f(n) \geq 1} f(n)^r + \sum_{0 \leq n \leq x, f(n) \leq 1} f(n)^r.$$

Using Hölder's inequality, we get that

$$\sum_{0 \leq n \leq x, f(n) \geq 1} f(n)^r \leq \left(\sum_{0 \leq n \leq x, f(n) \geq 1} 1 \right)^{1-r} \cdot \left(\sum_{0 \leq n \leq x, f(n) \geq 1} f(n) \right)^r.$$

Since

$$\limsup_{x \rightarrow +\infty} x^{-1} \sum_{0 \leq n \leq x} f(n) = L,$$

we get that

$$\limsup_{x \rightarrow +\infty} x^{-1} \sum_{0 \leq n \leq x, f(n) \geq 1} f(n) \leq L,$$

and since

$$\lim_{x \rightarrow +\infty} x^{-1} \sum_{0 \leq n \leq x, f(n) \geq 1} 1 = 0,$$

we obtain that

$$\begin{aligned} & \limsup_{x \rightarrow +\infty} x^{-1} \sum_{0 \leq n \leq x, f(n) \geq 1} f(n)^r \\ & \leq \left(\limsup_{x \rightarrow +\infty} x^{-1} \sum_{0 \leq n \leq x, f(n) \geq 1} 1 \right)^{1-r} \cdot \left(\limsup_{x \rightarrow +\infty} x^{-1} \sum_{0 \leq n \leq x, f(n) \geq 1} f(n) \right)^r \\ & \leq \left(\limsup_{x \rightarrow +\infty} x^{-1} \sum_{0 \leq n \leq x, f(n) \geq 1} 1 \right)^{1-r} \cdot L^r \\ & = 0. \end{aligned}$$

Now, we remark that as above, we have

$$\begin{aligned} & x^{-1} \sum_{0 \leq n \leq x, f(n) \leq 1} f(n)^r \\ & \leq \left(x^{-1} \sum_{0 \leq n \leq x, f(n) \leq 1} 1 \right)^{1-r} \cdot \left(x^{-1} \sum_{0 \leq n \leq x, f(n) \leq 1} f(n) \right)^r \end{aligned}$$

and similarly,

$$\begin{aligned} & \limsup_{x \rightarrow +\infty} x^{-1} \sum_{0 \leq n \leq x, f(n) \leq 1} f(n)^r \\ & \leq \left(\limsup_{x \rightarrow +\infty} x^{-1} \sum_{0 \leq n \leq x, f(n) \leq 1} 1 \right)^{1-r} \cdot \left(\limsup_{x \rightarrow +\infty} x^{-1} \sum_{0 \leq n \leq x, f(n) \leq 1} f(n) \right)^r \\ & \leq 1 \cdot \left(\limsup_{x \rightarrow +\infty} x^{-1} \sum_{0 \leq n \leq x, f(n) \leq 1} f(n) \right)^r. \end{aligned}$$

But if $0 \leq f(n) \leq 1$, then the inequality $0 \leq f(n) \leq f(n)^{1/2}$ holds, and as a consequence, we get that

$$\sum_{0 \leq n \leq x, f(n) \leq 1} f(n) \leq \sum_{0 \leq n \leq x, f(n) \leq 1} f(n)^{1/2}$$

and a fortiori,

$$\sum_{0 \leq n \leq x, f(n) \leq 1} f(n) \leq \sum_{0 \leq n \leq x} f(n)^{1/2}.$$

Now, since

$$\lim_{x \rightarrow +\infty} x^{-1} \sum_{0 \leq n \leq x} f(n)^{1/2} = 0,$$

we get that

$$\lim_{x \rightarrow +\infty} x^{-1} \sum_{0 \leq n \leq x, f(n) \leq 1} f(n) = 0$$

and so, we have

$$\limsup_{x \rightarrow +\infty} x^{-1} \sum_{0 \leq n \leq x, f(n) \geq 1} f(n)^r = 0.$$

This proves that for any r in $]0, 1[$, we have

$$\lim_{x \rightarrow +\infty} x^{-1} \sum_{0 \leq n \leq x} f(n)^r = 0.$$

Q.E.D.

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THEORIE DES NOMBRES

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