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# Bilinear estimates for the transport equations

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**Abstract.**

This note consists of the several types of bilinear estimates for the solutions of transport equations in one space dimension. As applications, the time local wellposedness results for the systems of these equations with quadratic interaction terms are shown.

## §1. Introduction

We consider the following Cauchy problem for the transport equations in one space dimension,

$$\partial_t u - \partial_x u = F, \quad u(0,x) = u_0(x), \tag{1}$$

$$\partial_t v + \partial_x v = G, \quad v(0,x) = v_0(x), \tag{2}$$

where $u$ and $v$ are functions from $\mathbb{R}^2$ to $\mathbb{C}$ and initial data $u_0$ and $v_0$ are functions from $\mathbb{R}$ to $\mathbb{C}$. Inhomogeneous terms $F$ and $G$ are complex valued functions of $t, x, u$ and $v$. In this note we consider the bilinear estimates on $uv$, $uu$ or $vv$ for the solutions of (1) and (2).

The first motivation of these studies is to investigate the following Dirac equation with some nonlinearities in one space dimension [17, 18].

$$\mathcal{D}\mathfrak{u} + m\mathfrak{u} = \mathbf{F}(\mathfrak{u}), \tag{3}$$

where $m \geq 0$ is mass, $\mathfrak{u}$ is a function from $\mathbb{R}^2$ to $\mathbb{C}^2$ such that

$$\mathfrak{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1(t,x) \\ u_2(t,x) \end{pmatrix},$$

the Dirac operator $\mathcal{D}$ is defined by $\mathcal{D} = i\gamma^0\partial_t + i\gamma^1\partial_x$ with the Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4}$$

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It is well known that $\mathcal{D}^2 = -\partial_t^2 + \partial_x^2$ from (4), and this leads Dirac equations to wave equations. We denote the nonlinear term by $\mathbf{F}(\mathfrak{u})$ which is the function from $\mathbb{C}^2$ to $\mathbb{C}^2$. We prescribe the initial data $\phi$ using a function from $\mathbb{R}$ to $\mathbb{C}^2$,

$$\mathfrak{u}(0,x) = \phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}. \tag{5}$$

We have studied the Cauchy problem on (3)–(5).

In this note we deal with (1)–(2) which are essential parts of (3). The first approach to this problem is so called Fourier restriction norm method, which has been developed by Bourgain [2, 3], and after that, refined by Kenig, Ponce and Vega [12, 13, 14]. These methods have been applied for various nonlinear dispersive equations. For example, nonlinear Schrödinger equations [14, 21], KdV equations [12, 13], higher ordered nonlinear Schrödinger equations [20, 22] which are respectively second order, third order and more higher order derivative equations. In our case, transport equations are one order derivative equation and are dispersive. Generally speaking, we have the stronger smoothing effect for higher ordered equations. For instance, it is reported in [20] that the fourth order Schrödinger equation has a powerful smoothing effect to control even the nonlinear term (roughly expressed) $u^2\partial_x^2 u$. We could not expect such properties as to transport equations. Indeed there seems not to exist the study on transport equations by the Fourier restriction norm method. There are results on the coupled system of transport equation with Schrödinger equation [1] in which the norm (6) below was used. But the proof of bilinear estimates for that system rely mainly on the norms regarding Schrödinger equation.

We introduce our Fourier restriction norms and another notation.

$$\begin{aligned} \|f\|_{X_{\pm}^{s,b}} :&= \|U_{\pm}(-t)f\|_{H_t^b H_x^s} \\ &= \left( \int_{\mathbb{R}^2} \langle \tau \pm \xi \rangle^{2b} \langle \xi \rangle^{2s} |\widetilde{f}(\tau,\xi)|^2 d\xi d\tau \right)^{1/2}, \end{aligned} \tag{6}$$

where $U_{\pm}(t) = e^{\mp t\partial_x}$ are free propagators, that is, $U_-(t)u_0(x)$ and $U_+(t)v_0(x)$ solves (1) with $F = 0$ and (2) with $G = 0$ respectively. $\widetilde{f}$ denotes the Fourier transform with respect to variables $x$ and $t$, $\widetilde{f} = \int e^{-ix\xi - it\tau} f(t,x)dtdx$. We use $\langle x \rangle = 1 + |x|$. We set a cut–off function $\psi \in C_0^\infty$ such that $0 \le \psi \le 1$,

$$\psi(t) = \begin{cases} 1, & |t| \le 1, \\ 0, & |t| \ge 2. \end{cases}$$

We study the following type of bilinear estimates,

$$\|uv\|_{X^{s,b-1}} \lesssim \|u\|_{X^{s,b}} \|v\|_{X^{s,b}}$$

where three $X$s are $X_+$ or $X_-$.

For the second approach to the problem on (1)–(2), we study the solutions of inhomogeneous transport equations. We employ the arguments in the series of papers by Fang and Fang–Grillakis [7, 8, 9] for the study on the Cauchy problem of Dirac–Klein–Gordon equation. We study the solutions of transport equations in Fourier space, variables $(\tau, \xi)$. We take a notice of that the Fourier transform of solutions for transport equations are described mainly by variables for the light lines $\tau - \xi$ or $\tau + \xi$ respectively. We estimate on solution of integral equation corresponding to the inhomogeneous transport equations and derive the bilinear estimates which work with these variables. By Plancherel's theorem and suitable cut-off function with respect to variable $t$, we do these investigation in $L^2_\tau L^2_\xi$. We estimate the solutions in the neighborhood of light cone and in otherwise separately as in [7]. In the weaker case [17], we do not take Fourier transform for the solutions and investigate in $L^2_t L^2_x$ to obtain the similar estimate.

The rest of this paper is organized as follows. In section 2, we give the bilinear estimates which are related to the Fourier restriction norms. In section 3, we present the bilinear estimates for the solutions of the inhomogeneous transport equations.

## §2. Bilinear estimates type 1

In this section we consider the some bilinear estimates on the Fourier restriction norm.

From the following lemma, it suffices to derive the bilinear estimate that we obtain the time local wellposedness for the corresponding equations.

**Lemma 1.** *Let $-1/2 < b' \le 0 \le b \le b'+1$, $0 < T \le 1$. The following estimates hold,*

$$\|\psi(t)U_\pm(t)a(x)\|_{X_\pm^{s,b}} \sim \|a\|_{H^s},$$

$$\left\|\psi(t/T)\int_0^t U_\pm(t-t')f(t',x)dt'\right\|_{X_\pm^{s,b}} \lesssim T^{1-b+b'}\|f\|_{X_\pm^{s,b'}}.$$

Now we give the results [17].

**Proposition 2.** *The following estimates hold for any $s > -1/8$ and suitable $b$, and fail for any $s < -1/6$, $b \in \mathbb{R}$,*

$$\|uv\|_{X_\pm^{s,b-1}} \lesssim \|u\|_{X_+^{s,b}}\|v\|_{X_-^{s,b}}. \tag{7}$$

**Proposition 3.** *The following estimates hold for any $s, b > 1/2$, and fail for any $s < 1/2$, $b \in \mathbb{R}$,*

$$\|uv\|_{X_\pm^{s,b-1}} \lesssim \|u\|_{X_\pm^{s,b}}\|v\|_{X_\pm^{s,b}}.$$

**Proposition 4.** *The following estimates hold for any $s > 1/4$, $1/2 < b \le 3/4$, and fail for any $b, s$ satisfying* $\max\{4s, 0\} < 2b-1$,

$$\|uv\|_{X_\pm^{s,b-1}} \lesssim \|u\|_{X_\mp^{s,b}}\|v\|_{X_\mp^{s,b}}.$$

From Proposition 2, we have that the following system of transport equations

$$\begin{cases} \partial_t u - \partial_x u = uv, & u(0,x) = u_0(x), \\ \partial_t v + \partial_x v = uv, & v(0,x) = v_0(x) \end{cases} \tag{8}$$

is time locally wellposed in $H^s$, $s > -1/8$.

For the proof of these estimates, we have utilized following elemental inequalities.

**Lemma 5.** *Let $p, q$ satisfy $p + q > 1$ and* $\max(p,q) \neq 1$. *Set $\kappa =$* $\min(p, q, p+q-1)$. *The following estimates hold,*

$$\int_{-\infty}^{\infty} \frac{dx}{\langle x-a\rangle^p \langle x-b\rangle^q} \sim \frac{1}{\langle a-b\rangle^\kappa}. \tag{9}$$

*Proof of Proposition 2.* See [17] for details. By setting $f(\tau,\xi) = \langle \tau+\xi\rangle^b\langle\xi\rangle^s\hat{u}$, $g(\tau,\xi) = \langle\tau-\xi\rangle^b\langle\xi\rangle^s\hat{v}$ we have estimated the following for the norm of left hand side of (7) by the use of Lemma 5,

(10)

$$\sup_{\|\varphi\|_{L^2_\tau L^2_\xi}=1}\left|\int_{\mathbb{R}^4} \frac{\langle\xi\rangle^s f(\sigma,\eta)g(\tau-\sigma,\xi-\eta)\varphi(\tau,\xi)}{\langle\tau\pm\xi\rangle^{1-b}\langle\sigma+\eta\rangle^b\langle\eta\rangle^s\langle\tau-\sigma-\xi+\eta\rangle^b\langle\xi-\eta\rangle^s} d\sigma d\eta d\tau d\xi\right|.$$

We have followed the arguments of the proof of bilinear estimates on the Fourier restriction norms by Kenig, Ponce and Vega in [13, 14] and so on. Q.E.D.

## §3. Bilinear estimates type 2

In this section we give the bilinear estimate for the solutions of inhomogeneous transport equations (1) and (2). We obtain the following estimate [18].

**Proposition 6.** *Let $s > -1/4$. Let $u$ and $v$ be solutions to* (1) *and* (2) *respectively. Then*

$$
\begin{aligned}
&\|\langle\tau+\xi\rangle^s\langle\tau-\xi\rangle^s(\hat{\psi}*\widetilde{uv})\|_{L^2_\tau L^2_\xi} \\
&\qquad \lesssim \left(\|u_0\|_{H^s}+\left\|\langle\tau+\xi\rangle^s\langle\tau-\xi\rangle^s\widetilde{F}\right\|_{L^2_\tau L^2_\xi}\right) \\
&\qquad\qquad \times\left(\|v_0\|_{H^s}+\left\|\langle\tau+\xi\rangle^s\langle\tau-\xi\rangle^s\widetilde{G}\right\|_{L^2_\tau L^2_\xi}\right).
\end{aligned}
$$

From Proposition 6, we have that the system of transport equations (8) is time locally wellposed in $H^s$, $s > -1/4$. Here the number $-1/4$ is less than $-1/6$ which is the number for failure of Proposition 2.

*Proof.* See [18] for details. We estimate on the following expression of solutions.

$$
\begin{aligned}
\text{(11)}\quad \widetilde{u}(\tau,\xi) =&\delta(\tau-\xi)\hat{u_0}(\xi) \\
&-i\sum_{k=1}^{\infty}\frac{1}{k!}\mathcal{F}_t(it)^k *_\tau \delta(\tau-\xi)\int(\sigma-\xi)^{k-1}\widetilde{F_1}(\sigma,\xi)d\sigma \\
&-i\frac{\widetilde{F_2}(\tau,\xi)}{\tau-\xi}+i\delta(\tau-\xi)\int\frac{\widetilde{F_2}(\sigma,\xi)}{\sigma-\xi}d\sigma,
\end{aligned}
$$

where $F = F_1 + F_2$ such that

$$
\widetilde{F_1}(\tau,\xi)=\psi(\tau-\xi)\widetilde{F}(\tau,\xi), \qquad \widetilde{F_2}(\tau,\xi)=(1-\psi(\tau-\xi))\widetilde{F}(\tau,\xi).
$$

And

$$
\begin{aligned}
\text{(12)}\quad \widetilde{v}(\tau,\xi) =&\delta(\tau+\xi)\hat{v_0}(\xi) \\
&-i\sum_{k=1}^{\infty}\frac{1}{k!}\mathcal{F}_t(it)^k *_\tau \delta(\tau+\xi)\int(\sigma+\xi)^{k-1}\widetilde{G_1}(\sigma,\xi)d\sigma \\
&-i\frac{\widetilde{G_2}(\tau,\xi)}{\tau+\xi}+i\delta(\tau+\xi)\int\frac{\widetilde{G_2}(\sigma,\xi)}{\sigma+\xi}d\sigma,
\end{aligned}
$$

where $G = G_1 + G_2$ such that

$$\widetilde{G_1}(\tau,\xi) = \psi(\tau+\xi)\widetilde{G}(\tau,\xi), \qquad \widetilde{G_2}(\tau,\xi) = (1-\psi(\tau+\xi))\widetilde{G}(\tau,\xi).$$

We investigate the solutions in the neighborhood of $\sigma = \pm\xi$ and otherwise separately. We have to estimate all terms in $\widetilde{uv} = \widetilde{u}*\widetilde{v}$. For example, we here estimate the convolution of the first term in (11) and the forth term in (12). We change the variables $((\xi+\tau)/2, (\xi-\tau)/2) \to (\tau,\xi)$,

$$\begin{aligned}\Big\|\langle\tau+\xi\rangle^s\langle\tau-\xi\rangle^s\Big(\delta(\tau-\xi)\hat{u_0}(\xi) * \delta(\tau+\xi)\int\frac{\widetilde{G_2}(\sigma,\xi)}{\sigma+\xi}d\sigma\Big)\Big\|_{L^2_\tau L^2_\xi}&\\ \sim \|\langle\xi\rangle^s\hat{u_0}\|_{L^2_\xi}\Big\|\langle\xi\rangle^s\int\frac{\widetilde{G_2}(\sigma,\xi)}{\sigma+\xi}d\sigma\Big\|_{L^2_\xi}&.\end{aligned}$$

We estimate from Hölder's inequality and Lemma 5,

$$\begin{aligned}&\Big\|\langle\xi\rangle^s\int\frac{\widetilde{G}_2(\sigma,\xi)}{\sigma+\xi}d\sigma\Big\|_{L^2_\xi}\\ &\le \Big\|\langle\xi\rangle^s\Big(\int\Big|\frac{\langle\sigma-\xi\rangle^{-s}}{\langle\sigma+\xi\rangle^{1+s}}\Big|^2d\sigma\Big)^{1/2}\Big\|_{L^\infty_\xi}\Big\|\langle\tau-\xi\rangle^s\langle\tau+\xi\rangle^s\widetilde{G}_2(\tau,\xi)\Big\|_{L^2_\tau L^2_\xi}\\ &\lesssim \Big\|\langle\tau-\xi\rangle^s\langle\tau+\xi\rangle^s\widetilde{G}_2(\tau,\xi)\Big\|_{L^2_\tau L^2_\xi}.\end{aligned}$$

In the case $s=0$, we do not need Fourier transform. We estimate on the following expression of solutions in $L^2_tL^2_x$, see [17].

$$\begin{aligned}u(t,x) &= u_0(x+t) + \int_0^t F(s,x+t-s)ds,\\ v(t,x) &= v_0(x-t) + \int_0^t G(s,x-t+s)ds.\end{aligned}$$

Q.E.D.

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