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Perverse sheaves and Milnor fibers over singular varieties

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Abstract.

We review some recent applications of perverse sheaves (intersection cohomologies) in singularity theory. Milnor fibers over general complete intersection varieties will be treated. We also give a proof of a result announced in [31].

§1. Introduction

The aim of this note is to introduce some recent applications of perverse sheaves (intersection cohomologies) to the study of complex hypersurface singularities. In the last two decades the theory of Milnor fibrations (see for example, Milnor [29], Dimca [7] etc.) was extended to the Milnor fibers over singular varieties. In particular, for any holomorphic function f with (stratified) isolated singularity on any complete intersection variety, Lê [21], Siersma [34] and Tibar [35] proved that the Milnor fiber of f admits a bouquet decomposition. This result is of course a vast generalization of Milnor's result, but the fact that the cohomological type of the Milnor fiber of f is the same as that of a bouquet of spheres can be easily deduced from the theory of perverse sheaves (see Theorem 2.2 below). It seems therefore that the above mentioned authors studied the topological or homotopy types of Milnor fibers motivated by this cohomological result obtained by perverse sheaves. This example shows that a general result in the theory of perverse sheaves sometimes can become a good guide principle in the study of singularity theory. In this short note, we explain some new topological constraints of general hypersurface singularities obtained by perverse sheaves. We hope that these results will help our understanding of non-isolated hypersurface singularities.

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§2. Milnor fibers over complete intersection varieties

In this section we review some recent results on the topology of Milnor fibers over singular varieties. Let X be an irreducible analytic subset (or an algebraic subvariety) of $\mathbb{C}^N$ of dimension $n+1$ containing the origin $0 \in \mathbb{C}^N$. Throughout this note, unless otherwise stated, we assume moreover that X is locally a complete intersection (we write it CI for short) in the ambient affine space $\mathbb{C}^N$. This weak assumption is necessary because we use the fact that the shifted constant sheaf $\mathbb{C}_X[n+1]$ on a CI variety X is a perverse sheaf (see Section 4). Now let $f\colon X \longrightarrow \mathbb{C}$ be a (non-constant and reduced) holomorphic function on X satisfying the condition $0 \in Y = \{z \in X \mid f(z) = 0\}$. Then we have a topological fibration over a sufficiently small punctured disk $D_\eta^* = \{t \in \mathbb{C} \mid 0 < |t| < \eta\} \subset \mathbb{C}$:

$$f\colon f^{-1}(D_\eta^*) \cap B_\varepsilon \longrightarrow D_\eta^*,$$

where $B_\varepsilon = \{z \in \mathbb{C}^N \cap X \mid \|z\| < \varepsilon\}$ is a small open neighborhood of $0 \in X$ in X and $0 < \eta << \varepsilon$. The general fiber $F_0 = f^{-1}(t) \cap B_\varepsilon$ $(0 < |t| < \eta)$ is called the Milnor fiber of $f\colon X \longrightarrow \mathbb{C}$ at 0. Note that when X is not smooth the Milnor fiber may have singularities. Nevertheless we have now a nice bouquet decomposition theorem for the Milnor fibers of functions $f\colon X \longrightarrow \mathbb{C}$ which have stratified isolated singularities at $0 \in X$ in the following sense. First take a Whitney stratification $X = \sqcup_{\alpha \in A} X_\alpha$ of X and denote it by $\mathcal{S}$. Then the stratified singular locus $\mathrm{sing}^{\mathcal{S}}(f)$ of $f\colon X \longrightarrow \mathbb{C}$ w.r.t. $\mathcal{S}$ is defined by $\mathrm{sing}^{\mathcal{S}}(f) = \sqcup_{\alpha \in A} \mathrm{sing}(f\ |_{X_\alpha})$. Using the Whitney conditons of $\mathcal{S}$ we can easily check that $\mathrm{sing}^{\mathcal{S}}(f)$ is a closed analytic subset of X. It is also easy to see (essentially by the curve selection lemma) that $\mathrm{sing}^{\mathcal{S}}(f)$ is contained in the complex hypersurface $Y = \{z \in X \mid f(z) = 0\} \subset X$ in an open neighborhood of Y in X (see for example Proposition 1.3 of Massey [24]). Now we say that a holomorphic function $f\colon X \longrightarrow \mathbb{C}$ has a stratified isolated singular point at $0 \in X$ w.r.t. $\mathcal{S}$ if the dimension of $\mathrm{sing}^{\mathcal{S}}(f)$ at $0 \in X$ is zero. Then Milnor's bouquet decomposition theorem over non-singular varieties can be generalized as follows.

Theorem 2.1 (Lê [21], Siersma [34], Tibar [35]). *Let $f\colon X \longrightarrow \mathbb{C}$ be a holomorphic function having a stratified isolated singular point at*

$0 \in X$ w.r.t. a Whitney stratification $\mathcal{S}$ of X. Then the Milnor fiber F_0 of f at 0 has the homotopy type of a bouquet of n-dimensional spheres:

$$F_0 \sim_h S^n \vee S^n \vee \cdots \vee S^n.$$

This theorem was obtained by developing the so-called polar curve method, which dates back to the work of Lê-Perron in [22]. By the same method we can also explicitly construct the handle decomposition of the Milnor fiber F_0 when X is smooth. Namely for smooth X we can completely determine the topological type of F_0, though it might be still difficult to compute the Betti numbers of F_0 if Y has non-isolated singularities at 0. For these important results we recommend the reader to see a series of papers by Lê or the recent book [25] by Massey etc. Note also that Massey's paper [24] gives also a method to compute the number of spheres in the above bouquet decomposition (i.e. the generalized Milnor number of f at 0). Now let us consider the general case where f does not necessarily have a stratified isolated singular point at 0. Then we have the following cohomological result.

Theorem 2.2 (the generalized Kato-Matsumoto's theorem). *Assume that the dimension of the stratified singular locus* $\mathrm{sing}^{\mathcal{S}}(f)$ *of f at $0 \in X$ is $s \geq 0$. Then for the reduced cohomology groups $\widetilde{H}^j(F_0; \mathbb{C})$ of F_0 we have*

$$\widetilde{H}^j(F_0; \mathbb{C}) = 0 \quad \textit{for} \quad {}^\forall j \notin [n-s, n].$$

In Section 5 we will show that this theorem can be easily deduced from some well-known properties of perverse sheaves. To end this section we define the complex link $CL(X;0)$ of X at 0, which is an important example of Milnor fibers over singular varieties. Recall that X is embedded in a smooth affine space $\mathbb{C}^N$. We take a linear form $l \colon \mathbb{C}^N \longrightarrow \mathbb{C}$ ($l(0) = 0$) on $\mathbb{C}^N$ and consider its restriction $l|_X$ to $X \subset \mathbb{C}^N$. Then we can show that for a sufficiently generic linear form l the dimension of the stratified singular locus $\mathrm{sing}^{\mathcal{S}}(l|_X)$ of $l|_X$ at $0 \in X$ is zero. Therefore if we define the complex link $CL(X;0)$ of X at 0 to be the Milnor fiber of such a function $l|_X \colon X \longrightarrow \mathbb{C}$ at 0, then we obtain a bouquet decomposition

$$CL(X;0) \sim_h S^n \vee S^n \vee \cdots \vee S^n$$

by Theorem 2.1. Note that the topological type of the complex link does not depend on the choice of linear forms $l \colon \mathbb{C}^N \longrightarrow \mathbb{C}$ nor embeddings $X \hookrightarrow \mathbb{C}^N$. This notion plays an important role also in stratified Morse theory (see Goresky-MacPherson [13]).

§3. Some results and their generalizations

In this section we introduce some results obtained in Nang-T [30], [31] and Dimca [8]. Recall that X is a CI variety (or a CI analytic set) of dimension $n+1$. Then for a non-constant holomorphic function $f\colon X \to \mathbb{C}$ satisfying $f(0)=0$ $(0 \in X)$ the dimensions of $Y = \{z \in X \mid f(z)=0\} \subset X$ and the Milnor fiber F_0 are n. We thus have the monodromy operators

$$T(j)_0\colon H^j(F_0;\mathbb{C}) \xrightarrow{\sim} H^j(F_0;\mathbb{C})$$

for $j=0,\, n-s,\, n-s+1, \dots, n-1,\, n$ at $0 \in Y \subset X$ $(s = \dim_0 \operatorname{sing}^{\mathcal{S}}(f))$. Since the lower dimensional monodromy operators

$$T(j)_0\colon H^j(F_0;\mathbb{C}) \xrightarrow{\sim} H^j(F_0;\mathbb{C}) \quad (j=0,\, n-s, \dots, n-1)$$

are relatively simple as we shall see in Section 5 (in particular

$$T(0)_0\colon H^0(F_0;\mathbb{C}) \xrightarrow{\sim} H^0(F_0;\mathbb{C})$$

is the identity map of $\mathbb{C}$), here we focus our attention on the top dimensional monodromy operator $T(n)_0\colon H^n(F_0;\mathbb{C}) \xrightarrow{\sim} H^n(F_0;\mathbb{C})$.

Definition 3.1. For a complex number $a \in \mathbb{C}$, we denote by N_a the number of Jordan blocks with the eigenvalue a in the monodromy operator $T(n)_0\colon H^n(F_0;\mathbb{C}) \xrightarrow{\sim} H^n(F_0;\mathbb{C})$.

Then we have the following result which gives an upper bound for the multiplicities of eigenvalues of the monodromy

$$T(n)_0\colon H^n(F_0;\mathbb{C}) \xrightarrow{\sim} H^n(F_0;\mathbb{C}).$$

Note that as an upper bound for the sizes of Jordan blocks in the monodromy operators we have the famous monodromy theorem (see the references cited in the paper [10]). For a topological space W we denote by $b_j(W)$ the j-th Betti number of W.

Theorem 3.2 (Nang-T [30], [31] and Dimca [8]). *For any non-zero complex number $a \in \mathbb{C}$ we have*

$$N_a \leq b_{n-1}(CL(Y;0)) + b_n(CL(X;0)).$$

In particular if X is smooth (e.g. $X = \mathbb{C}^{n+1}$) then $N_a \leq b_{n-1}(CL(Y;0))$. Namely N_a's are bounded by the number of $(n-1)$-dimensional spheres in the bouquet decomposition $CL(Y;0) \sim_h S^{n-1} \vee \cdots \vee S^{n-1}$ of the complex link of Y.

This theorem was first obtained by Nang-T [30] for $X = \mathbb{C}^{n+1}$ and the complex numbers $a \neq 0$ satisfying a technical condition. Then Proposition 6.4.17 of Dimca [8] generalized it to the case of Milnor fibers over singular varieties assuming the same condition on $a \neq 0$. Finally Nang-T [31] removed this technical assumption. The proof of Theorem 3.2 will be given in Section 4. Note that if X is $\mathbb{C}^{n+1}$ and f is a quasi-homogeneous polynomial then the monodromy operators are periodic ($\Longrightarrow$ semisimple) and hence N_a is nothing but the multiplicity of the eigenvalue a in the map $T(n)_0 \colon H^n(F_0;\mathbb{C}) \xrightarrow{\sim} H^n(F_0;\mathbb{C})$. Even in such simplest cases Theorem 3.2 seems to be new, because for $Y = \{z \in X \mid f(z) = 0\}$ with a non-isolated singular point at 0 it is in general very difficult to compute the monodromy operators. For general hypersurface singularities we can compute only the monodromy zeta function

$$Z_f(\lambda) = \prod_{j=0}^{n} \det(\mathrm{Id} - \lambda T(j)_0)^{(-1)^j}$$

by constructing an embedded resolution of singularities (see Bierstone-Milman [3] for an algorithm to construct embedded resolutions) of each given complex hypersurface Y in $X = \mathbb{C}^{n+1}$ (A'Campo [1]). If the hypersurface $Y \subset X = \mathbb{C}^{n+1}$ has an isolated singular points at 0, Varchenko's formula ([36]) for the characteristic polynomial of $T(n)_0$ obtained by this monodromy zeta function (and a result of Kouchnirenko [20]) is very useful. However to use his formula, the defining function f of Y must satisfy the so-called Newton non-degeneracy condition. Hence it would be difficult to prove Theorem 3.2 along this line even for all complex hypersurfaces $Y \subset X = \mathbb{C}^{n+1}$ having isolated singular points at 0. For such hypersurfaces we have the following corollary. Let $L(Y;0)$ be the real link of Y at $0 \in Y$. Namely we set

$$L(Y;0) = Y \cap S_\varepsilon \quad (0 < \varepsilon << 1),$$

where $S_\varepsilon \subset \mathbb{C}^N$ is a small sphere centered at 0 with radius ε.

Corollary 3.3. *Assume that $X = \mathbb{C}^{n+1}$ and the complex hypersurface $Y = \{z \in X \mid f(z) = 0\} \subset X$ has an isolated singular point at $0 \in Y$. Then we have*

$$b_{n-1}(L(Y;0)) \leq b_{n-1}(CL(Y;0)).$$

Under the assumptions of this corollary we can easily prove

$$N_1 = b_{n-1}(L(Y;0))$$

by Alexander duality. Therefore Corollary 3.3 immediately follows from Theorem 3.2. In order to understand the topological meaning of Corollary 3.3, recall that if Y has an isolated singular point at $0 \in Y$ then the real link $L(Y;0)$ is a smooth compact orientable $(2n-1)$-manifold whose non-zero Betti numbers are b_0, b_{n-1}, b_n, b_{2n-1}. Since $b_0 = b_{2n-1} = 1$ and $b_{n-1} = b_n$ by Poincaré duality, the only interesting number among them is $b_{n-1}(L(Y;0))$. On the other hand, as we saw in Section 2 the complex link $CL(Y;0)$ has only two non-zero Betti numbers $b_0 = 1$, b_{n-1}. So the inequality $b_{n-1}(L(Y;0)) \leq b_{n-1}(CL(Y;0))$ means that the most interesting invariant of the topology of the real link and that of the complex link are related each other. Finally we remark that this result was generalized in Proposition 6.1.22 and Corollary 6.1.24 of [8] to the case where Y is higher-codimensional in $X = \mathbb{C}^{n+1}$.

§4. Proof of Theorem 3.2

In this section we quickly review the theory of perverse sheaves and give a proof of Theorem 3.2. For the detail of the theory of perverse sheaves and constructible sheaves, we refer to Beilinson-Bernstein-Deligne [2], Dimca [8], Hotta-T-Tanisaki [14], Kashiwara-Schapira [18] and Schürmann [33] etc. Now let X be a complex analytic set or an algebraic variety (endowed with the classical topology). As usual we denote by $\mathbf{D}^b(X)$ the derived category of bounded complexes of sheaves of $\mathbb{C}_X$-modules on X. The category of perverse sheaves is a full abelian subcategory of $\mathbf{D}^b(X)$ which correponds to that of regular holonomic $\mathcal{D}_X$-modules (when X is smooth) through the Riemann-Hilbert correspondence (see Kashiwara [17] and Hotta-T-Tanisaki [14] etc.). In order to recall the definition of perverse sheaves by Beilinson-Bernstein-Deligne [2], denote by $\mathbf{D}^b_c(X)$ the full subcategory of $\mathbf{D}^b(X)$ consisting of complexes of sheaves with $\mathbb{C}$-constructible cohomology sheaves.

Definition 4.1 ([2]). Let $\mathcal{F}^\cdot \in \mathbf{D}^b_c(X)$. Then $\mathcal{F}^\cdot$ is a perverse sheaf on X if the following two conditions are satisfied.

(i) For any $i \in \mathbb{Z}$, we have $\dim[\operatorname{supp} H^i\mathcal{F}^\cdot] \leq -i$.
(ii) The Verdier dual $D(\mathcal{F}^\cdot)$ of $\mathcal{F}^\cdot$ satisfies the condition

$$\dim[\operatorname{supp} H^i D(\mathcal{F}^\cdot)] \leq -i \quad \text{for any} \quad i \in \mathbb{Z}.$$

We denote by $\mathrm{Perv}(\mathbb{C}_X)$ the full subcategory of $\mathbf{D}^b_c(X)$ consisting of perverse objects.

As a special class of perverse sheaves we have the following.

Theorem 4.2. *Assume that X is pure-dimensional and locally a CI. Then the shifted constant sheaf $\mathbb{C}_X[\dim X] \in \mathbf{D}_c^b(X)$ is a perverse sheaf on X. Moreover for any local system (i.e. a locally constant sheaf of finite rank over $\mathbb{C}_X$) $\mathcal{L}$, we have $\mathcal{L}[\dim X] \in \mathrm{Perv}(\mathbb{C}_X)$.*

For the proof, see for example Sorite 1.8 (page 15) of Brylinski [5] etc. The proof of [5] uses $\mathcal{D}$-modules. A purely topological proof can be found in Theorem 5.1.20 of [8].
Now let us prove Theorem 3.2. For the sake of simplicity let X be an $(n+1)$-dimensional CI variety in $\mathbb{C}^N$ containing the origin 0. Recall that $f\colon X \to \mathbb{C}$ is a holomorphic function s.t. $0 \in Y = \{z \in X \mid f(z) = 0\}$. We prove the theorem by constructing a special perverse sheaf $\mathcal{G}^{\cdot}$ on $\mathbb{C}^N$ which contains the information of the top dimensional monodromy operator $T(n)_0\colon H^n(F_0;\mathbb{C}) \xrightarrow{\sim} H^n(F_0;\mathbb{C})$. To begin with, for the given complex number $a \neq 0$ we define a local system $\mathcal{L}_a$ on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ by the representation

$$\begin{array}{ccc} \pi_1(\mathbb{C}^*) \simeq \mathbb{Z} & \longrightarrow & GL(1,\, \mathbb{C}) = \mathbb{C}^* \\ n & \longmapsto & a^n. \end{array}$$

Next consider a local system $\widetilde{\mathcal{L}_a}$ on $X \setminus Y$ obtained by taking the inverse image of $\mathcal{L}_a$ by $f\colon X \setminus Y \to \mathbb{C}^*$. Then by Theorem 4.2 the complex of sheaves $\widetilde{\mathcal{L}_a}[n+1]$ is a perverse sheaf on $X \setminus Y$. Now let us set $j\colon X \setminus Y \hookrightarrow X$, $j_0\colon X \setminus Y \hookrightarrow X \setminus \{0\}$ and $j_1\colon X \setminus \{0\} \hookrightarrow X$ ($j = j_1 \circ j_0$). We will extend the perverse sheaf $\widetilde{\mathcal{L}_a}[n+1]$ to the whole $\mathbb{C}^N$ in three steps. First, since j_0 is a Stein map, the direct image $\mathrm{R}j_{0*}\widetilde{\mathcal{L}_a}[n+1]$ is a perverse sheaf on $X \setminus \{0\}$ by M. Artin's theorem (see for example Corollary 5.2.17 of [8]). Next we define a perverse sheaf $\mathcal{F}^{\cdot}$ on X by $\mathcal{F}^{\cdot} = j_{1!}(\mathrm{R}j_{0*}\widetilde{\mathcal{L}_a}[n+1])$. Here $j_{1!}(*)$ stands for the so-called Deligne-Goresky-MacPherson extension functor. By using the truncation functor $\tau^{\leq -1}(*)$ we can rewrite it as $j_{1!}(*) \simeq (\tau^{\leq -1} \circ \mathrm{R}j_{1*})(*)$. Finally we define a perverse sheaf $\mathcal{G}^{\cdot}$ on $\mathbb{C}^N$ by $\mathcal{G}^{\cdot} = \iota_*\mathcal{F}^{\cdot}$, where we set $\iota\colon X \hookrightarrow \mathbb{C}^N$. By the Riemann-Hilbert correspondence there exists a unique regular holonomic $\mathcal{D}_{\mathbb{C}^N}$-module $\mathcal{M}$ which corresponds to the perverse sheaf $\mathcal{G}^{\cdot}$. Then, just as the proof of Theorem 5.4 of Nang-T [30] (or Proposition 6.4.17 of Dimca [8]), Theorem 3.2 can be proved by calculating the multiplicity $m \in \mathbb{Z}_{\geq 0}$ of $\mathcal{M}$ along the conormal bundle $T^*_{\{0\}}\mathbb{C}^N \subset T^*X$. Namely we obtain the equality

$$m = -N_a + \{b_{n-1}(CL(Y;0)) + b_n(CL(X;0)\}$$

and the theorem follows from the non-negativity of the multiplicity m.

Q.E.D.

§5. Some other consequences of perversity

In this section, using the notations of previous sections, we introduce some other important consequences of perversity in the topology of complex hypersurface singularities. In particular we show that the lower dimensional monodromy operators $T(j)_0\colon H^j(F_0;\mathbb{C}) \xrightarrow{\sim} H^j(F_0;\mathbb{C})$ for $j = 0, n-s, \cdots, n-1$ ($s = \dim_0 \mathrm{sing}^{\mathcal{S}}(f)$) are usually much simpler than the top dimensional one. First of all, for a given holomorphic function $f\colon X \to \mathbb{C}$ we associate to it the shifted vanishing cycle functor

$$ {}^p\phi_f(*)\colon \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(Y) $$

satisfying the condition

$$ H^j({}^p\phi_f(\mathbb{C}_X))_x \simeq \widetilde{H}^{j-1}(F_x;\mathbb{C}) $$

for any $x \in Y = \{z \in X \mid f(z) = 0\}$ and $j \in \mathbb{Z}$. Here F_x is the Milnor fiber of f at $x \in Y$. Then it is well-known that this functor preserves the perversity. For the proof, see for example, Corollary 10.3.13 of Kashiwara-Schapira [18] and Theorem 6.0.2 of Schürmann [33] etc. This important result was first obtained by [2] in the algebraic case. The proof for the analytic case was given by Kashiwara [16] in his study of vanishing cycle functors for $\mathcal{D}$-modules (see also Goresky-MacPherson [12] for a topological approach to this problem). Now we can easily deduce Theorem 2.2 (the generalized Kato-Matsumoto's theorem) from this very general result. Indeed, applying it to the perverse sheaf $\mathbb{C}_X[n+1]$ (we assume that X is locally a CI) we see that the vanishing cycle $\mathcal{G}^{\cdot} = {}^p\phi_f(\mathbb{C}_X[n+1])$ is a perverse sheaf whose support is contained in the stratified singular locus $\mathrm{sing}^{\mathcal{S}}(f)$ of f. Then it remains to apply the following very elemetary property of perserse sheaves to $\mathcal{G}^{\cdot}$.

Lemma 5.1. *Let $\mathcal{G}^{\cdot}$ be a perverse sheaf on an analytic set X whose support is contained in an s-dimensional analytic subset S of X. Then we have $H^j(\mathcal{G}^{\cdot})_x \simeq 0$ for any $x \in X$ and $j \notin [-s, 0]$.*

Namely we obtain $\widetilde{H}^{n+j}(F_0;\mathbb{C}) \simeq H^{n+j+1}({}^p\phi_f(\mathbb{C}_X))_0 \simeq H^j(\mathcal{G}^{\cdot})_0 \simeq 0$ for $j \notin [-s, 0]$ ($s = \dim_0 \mathrm{sing}^{\mathcal{S}}(f)$). This completes the proof of Theorem 2.2. By refining this proof, we can obtain also the following interesting results on the propagation of monodromy eigenvalues up to the center $0 \in Y$. Let us consider the monodromy operators $T(j)_x\colon H^j(F_x;\mathbb{C}) \xrightarrow{\sim} H^j(F_x;\mathbb{C})$ at points $x \in Y$ outside the origin. Then we have

Theorem 5.2. *Let $a \in \mathbb{C}$ be a complex number.*

(i) (Corollary 6.1.7 *of Dimca* [8]) *Assume that a is an eigenvalue of a lower dimensional monodromy $T(j)_0 \colon H^j(F_0;\mathbb{C}) \xrightarrow{\sim} H^j(F_0;\mathbb{C})$ $(j \leq n-1)$ at 0. Then for any open neighborhood U of 0 in Y there exists a point $x \neq 0$ in $U \setminus \{0\}$ such that a is an eigenvalue of $T(k)_x \colon H^k(F_x;\mathbb{C}) \xrightarrow{\sim} H^k(F_x;\mathbb{C})$ for some k.*

(ii) (Theorem 0.4 *of Dimca-Saito* [9]) *Assume that a lower dimensional monodromy $T(j)_0 \colon H^j(F_0;\mathbb{C}) \xrightarrow{\sim} H^j(F_0;\mathbb{C})$ $(j \leq n-1)$ at 0 has a Jordan block with the eigenvalue a of size m. Then there exist points $x_k \neq 0$ sufficiently close to 0 for $k \leq j$ such that the monodromy $T(k)_{x_k} \colon H^k(F_{x_k};\mathbb{C}) \xrightarrow{\sim} H^k(F_{x_k};\mathbb{C})$ at x_k has a Jordan block with the eigenvalue a of size m_k and $\sum_{k \leq j} m_k \geq m$.*

To prove (i) of this theorem we use the direct sum decomposition

$$ {}^p\phi_f(\mathbb{C}_X[n+1]) \simeq \bigoplus_{a \in \mathbb{C}} [{}^p\phi_f(\mathbb{C}_X[n+1])]_a $$

in the category $\mathrm{Perv}(\mathbb{C}_Y)$. Here $[{}^p\phi_f(\mathbb{C}_X[n+1])]_a$ denotes the generalized eigenspace for the eigenvalue a of the monodromy map

$$ T \colon {}^p\phi_f(\mathbb{C}_X[n+1]) \to {}^p\phi_f(\mathbb{C}_X[n+1]) $$

in $\mathrm{Perv}(\mathbb{C}_Y)$. Namely $[{}^p\phi_f(\mathbb{C}_X[n+1])]_a$ is the kernel of $(a\,\mathrm{Id} - T)^k$ for $k >> 0$. Then we can easily prove (i) by considering the supports of perverse sheaves $[{}^p\phi_f(\mathbb{C}_X[n+1])]_a$ as in the proof of Theorem 2.2. To prove (ii) we use the restriction of ${}^p\phi_f(\mathbb{C}_X[n+1])$ to the real link $L(Y;0)$ of Y and a spectral sequence. See [9] for the precise proof. Q.E.D.

Roughly speaking, Theorem 5.2 asserts that some important parts of the lower dimensional monodromy operators $T(j)_0$ $(j \leq n-1)$ at 0 are determined by the monodromy operators at points $x \in Y$, $x \neq 0$. We can observe a similar phenomenon also in Randell's theorem for two-dimensional complex hypersurfaces in $\mathbb{C}^3$ obtained by deprojectivizing plane curves (see Oka [32] for a survey of this subject and related results). Theorem 5.2 (i) in particular implies that if the singularity of Y is normal crossing outside the origin then all the eigenvalues of the lower dimensional monodromy operators $T(j)_0$ $(j \leq n-1)$ at the origin are 1 (see Example 6.1.8 of [8]). In this case, if an embedded resolution of $Y \subset X = \mathbb{C}^{n+1}$ is given, using the monodromy zeta function obtained by the methods of [1] we can determine the multiplicities of the eigenvalues $a \neq 1$ in the top dimensional monodromy $T(n)_0 \colon H^n(F_0;\mathbb{C}) \xrightarrow{\sim} H^n(F_0;\mathbb{C})$.

Finally let us list up some related subjects that we could not explain precisely in this short note. As applications of perverse sheaves (intersection cohomologies) in singularity theory we have also the following results.

(i) We can generalize vanishing theorems (obtained by Esnault-Schechtman-Viehweg, Kohno and Schechtman-Terao-Varchenko etc.) for twisted cohomology groups of the complements to hyperplane arrangements. See Cohen-Dimca-Orlik [6] etc.
(ii) Recently using the theory of perverse sheaves Maxim [28] found a new construction of Alexander modules of hypersurface complements (see [23] and [32] for the definition) and generalized the results of Libgober [23] to the case where the hypersurface has non-isolated singularities.
(iii) The classical theory of projective duality (i.e. the study of dual varieties in projective geometry) was reformuled in terms of constructible sheaves. After the fundamental work by Brylinski [5], Ernström proved that the topological Radon transform of the Euler obstruction of a projective variety V is that of the dual variety V^* modulo constant functions (see also [26] and [27] etc. for its generalizations).

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