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On weighted-degrees for algebraic local cohomologies associated with semiquasihomogeneous singularities

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Abstract.

In this paper, a notion of a weighted-degree is introduced to algebraic local cohomology classes associated with a semiquasihomogeneous function. Utilizing weighted-degrees, computations of a dual basis of Milnor algebra and membership problems are considered as applications.

§1. Introduction

Let X be a neighbourhood of the origin O of n-dimensional affine space $\mathbb{C}^n$. Let f be a holomorphic function on X and S the hypersurface defined by the function f, i.e., $S = \{x \in X \mid f(x) = 0\}$. We assume that the function f has an isolated singularity at the origin, i.e.,

$$\{x \in X \mid f_{x_1}(x) = \cdots = f_{x_n}(x) = 0\} \cap S = \{O\},$$

where $f_{x_j} = \partial f/\partial x_j$ and $x = (x_1, \ldots, x_n)$. Let $\mathcal{J}$ be Jacobi ideal in $\mathcal{O}_{X,O}$ of the function f, and $\mathcal{H}_f$ the set of algebraic local cohomology classes annihilated by $\mathcal{J}$, i.e.,

$$\begin{aligned} \mathcal{J} &= \mathcal{O}_{X,O}\langle f_{x_1}, \ldots, f_{x_n}\rangle \subset \mathcal{O}_{X,O}, \\ \mathcal{H}_f &= \{\eta \in \mathcal{H}^n_{[O]}(\mathcal{O}_X) \mid g\eta = 0, \ \forall g \in \mathcal{J}\} \end{aligned}$$

where $\mathcal{O}_{X,O}$ is the stalk at O of the sheaf $\mathcal{O}_X$ of germs of holomorphic functions and $\mathcal{H}^n_{[O]}(\mathcal{O}_X)$ is the sheaf of n-th algebraic local cohomology groups, supported at the origin. Then, $\mathcal{H}_f$ and $\mathcal{O}_{X,O}/\mathcal{J}$ are finite dimensional vector spaces of the same dimension, i.e., Milnor number. $\mathcal{H}_f$

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is isomorphic to $\mathcal{E}xt^n_{\mathcal{O}_{X,O}}(\mathcal{O}_{X,O}/\mathcal{J}, \mathcal{O}_{X,O})$ and thus there exists non-degenerate pairing,

$$\operatorname{res}_O(\,\cdot\,,\,\cdot\,)\colon O_{X,O}/\mathcal{J} \times \mathcal{H}_f \to \mathbb{C} \tag{1.1}$$

between them defined by Grothendieck local residues ([3], [4]).

As $\mathcal{H}_f$ is the dual space of $\mathcal{O}_{X,O}/\mathcal{J}$, the space $\mathcal{H}_f$ reflects properties of a given singularity. Furthermore, these algebraic local cohomology classes in $\mathcal{H}_f$ and the associated holonomic systems exhibit some characteristic features of the singularity ([5], [6]). This indicates therefore that further studies of $\mathcal{H}_f$ in the context of D-modules would be of interest. In this paper, we study basic properties of $\mathcal{H}_f$ for semiquasihomogeneous function and give a method for constructing a basis of $\mathcal{H}_f$. As applications, we consider a membership problem and a computation of standard basis. We show that Grothendieck local duality provides us with an effective method for these problem. Note that the approach and the results presented in this paper have applications to the study of holonomic systems attached to semiquasihomogeneous isolated singularities.

In Section 2, we define a notion of weighted-degrees of algebraic local cohomologies and study their properties. For semiquasihomogeneous functions f, we clarify relations of these properties for $\mathcal{H}_f$ to Poincaré polynomial. In Section 3, by combining Grothendieck duality (1.1) and the notion of the weighted-degree with respect to the weight vector of the function f, we derive a method for constructing a basis of the space $\mathcal{H}_f$ that gives rise to a dual basis of $\mathcal{O}_{X,O}/\mathcal{J}$. In Section 4, as applications, we study a membership problem for the ideal $\mathcal{J}$ and a computation of a standard basis of $\mathcal{J}$. By making the most of the dual basis, we give an effective method for solving a membership problem for the ideal $\mathcal{J}$ and illustrate a method for computing a standard basis of $\mathcal{J}$ with examples.

§2. Algebraic local cohomologies

We introduce a notion of weighted-degrees for algebraic local cohomology classes and study the dual space of $\mathcal{O}_{X,O}/\mathcal{J}$ associated with semiquasihomogeneous singularities.

2.1. Definition of weighted-degrees

Let us fix a weight vector $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{N}^n_+$ for a fixed coordinate system $x = (x_1, \ldots, x_n) \in X$. Put $|\mathbf{w}| = w_1 + \cdots + w_n$ and $\langle \mathbf{w}, \lambda \rangle = \lambda_1 w_1 + \cdots + \lambda_n w_n$ for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$.

Any algebraic local cohomology class η in $\mathcal{H}^n_{[O]}(\mathcal{O}_X)$ can be expressed in terms of a relative Čech cohomology.

$$\eta = \left[\sum_{\lambda \in \Lambda_\eta} c_\lambda \frac{1}{x^\lambda} \right]$$

where $c_\lambda \in \mathbb{C}$, $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ with $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda_\eta$, Λ_η is a finite subset of $\mathbb{N}^n_+$. Then, we define the weighted-degree of an algebraic local cohomology class $[1/x^\lambda]$ to be $-\langle \mathbf{w}, \lambda \rangle$. We call algebraic local cohomology classes, represented by a single term, monomials. An algebraic local cohomology class $\eta \in \mathcal{H}^n_{[O]}(\mathcal{O}_X)$ can be written in the form

$$\eta = \left[\sum_{\lambda \in \Lambda_\eta} c_\lambda \frac{1}{x^\lambda} \right] \quad \text{where} \quad c_\lambda \in \mathbb{C},\ x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$$
$$\text{with} \quad \lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda_\eta,$$

Λ_η is a finite subset of $\mathbb{N}^n_+$ and $[\,\cdot\,]$ stands for a relative Čech cohomology.

Definition. We define the weighted-degree $\deg_{\mathbf{w}}(\eta)$ of a cohomology class

$$\eta = \left[\sum_{\lambda \in \Lambda_\eta} c_\lambda \frac{1}{x^\lambda} \right] \tag{2.1}$$

by the smallest degree of monomials $[1/x^\lambda]$ for $\lambda \in \Lambda_\eta$:

$$\deg_{\mathbf{w}}(\eta) = \min\{-\langle \mathbf{w}, \lambda \rangle \mid \lambda \in \Lambda_\eta\},$$

where Λ_η is a set of all exponents $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n_+$ with non-zero coefficients c_λ in the above expression (2.1) of the cohomology class η.

2.2. Basic properties

Let $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{N}^n_+$ be a weight vector. A polynomial $f(x)$ is said to be weighted homogeneous of degree d with weight $\mathbf{w}$ if $f(x)$ is a sum of weighted homogeneous monomials of weighted degree d, i.e.,

$$f(x) = \sum_{\langle \mathbf{w}, \kappa \rangle = d} c_\kappa x^\kappa$$

where $c_\kappa \in \mathbb{C}$, $x^\kappa = x_1^{k_1} \cdots x_n^{k_n}$ and $\langle \mathbf{w}, \kappa \rangle = w_1 k_1 + \cdots + w_n k_n$ for $\kappa = (k_1, \dots, k_n) \in \mathbb{N}^n$. We define a weighted degree of a holomorphic function $h(x)$ to be the smallest degree of monomials x^κ constituting $h(x)$;

$$\deg_{\mathbf{w}}(h) = \min\{\langle \mathbf{w}, \kappa\rangle \mid h(x) = \sum_{c_\kappa \neq 0} c_\kappa x^\kappa, \ c_\kappa \neq 0\}.$$

Definition. A polynomial f is said to be semiquasihomogeneous of weighted degree d if f is of the form $f = f_0 + g$ where f_0 is a weighted homogeneous polynomial of weighted degree d defining an isolated singularity at the origin and g is a function of weighted degree strictly greater than d.

Let f be a semiquasihomogeneous function with respect to a weight vector $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{N}_+^n$. Let $\mathcal{J}_0$ denote Jacobi ideal of the weighted homogeneous part of the function f and E_0 the canonical monomial basis of $\mathcal{O}_{X,O}/\mathcal{J}_0$. It is known ([1]) that E_0 also gives a monomial basis of $\mathcal{O}_{X,O}/\mathcal{J}$. We use the notation E when we regard E_0 as a monomial basis of $\mathcal{O}_{X,O}/\mathcal{J}$.

Let us recall the following result (see [1]):

Lemma 2.1. *There exists exactly one basis monomial in E of degree $n \cdot \deg_{\mathbf{w}}(f) - 2|\mathbf{w}|$. Monomials x^κ belong to the ideal $\mathcal{J}$ if $\langle \mathbf{w}, \kappa\rangle > n \cdot \deg_{\mathbf{w}}(f) - 2|\mathbf{w}|$.*

Based on these results, we have the following:

Proposition 2.1. *Any cohomology class $\eta \in \mathcal{H}_f \setminus \{0\}$ satisfies the following inequality:*

$$-n \cdot \deg_{\mathbf{w}}(f) + |\mathbf{w}| \leq \deg_{\mathbf{w}}(\eta) \leq -|\mathbf{w}|.$$

And there exist cohomology classes in $\mathcal{H}_f$ of degree $-|\mathbf{w}|$ and $-n \cdot \deg_{\mathbf{w}}(f) + |\mathbf{w}|$.

Proof. Since the set Λ_η of exponents for the cohomology class η is a subset of $\mathbb{N}_+^n$, we have $\deg_{\mathbf{w}}(\eta) \leq -\langle \mathbf{w}, \mathbf{1}\rangle = -|\mathbf{w}|$ where $\mathbf{1} = (1, \dots, 1) \in \mathbb{N}_+^n$. The equality holds if and only if $\Lambda_\eta = \{\mathbf{1}\}$ which corresponds to Dirac's delta function $\delta = [1/(x_1 \cdots x_n)]$. It is evident that δ is in $\mathcal{H}_f$.

Assume, for the moment, that there exists a cohomology class $\eta \in \mathcal{H}_f$ satisfying $\deg_{\mathbf{w}}(\eta) < -n \cdot \deg_{\mathbf{w}}(f) + |\mathbf{w}|$. Put $\deg_{\mathbf{w}}(\eta) = -n \cdot \deg_{\mathbf{w}}(f) + |\mathbf{w}| - r$ for some positive integer $r \in \mathbb{N}_+$. Then there exists $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda_\eta$ such that $\langle \mathbf{w}, \lambda\rangle = n \cdot \deg_{\mathbf{w}}(f) - |\mathbf{w}| + r$. Then, for an exponent $\kappa = \lambda - \mathbf{1} = (\lambda_1 - 1, \dots, \lambda_n - 1) \in \mathbb{N}^n$, we have $x^\kappa \eta = c\delta$

where c is a non-zero constant. On the other hand, since

$$\begin{aligned}\deg_{\mathbf{w}}(x^{\kappa}) &= \langle \mathbf{w}, \kappa\rangle \\ &= \langle \mathbf{w}, \lambda\rangle - \langle \mathbf{w}, \mathbf{1}\rangle \\ &= n\cdot\deg_{\mathbf{w}}(f) - |\mathbf{w}| + r - |\mathbf{w}| \\ &= n\cdot\deg_{\mathbf{w}}(f) - 2|\mathbf{w}| + r > n\cdot\deg_{\mathbf{w}}(f) - 2|\mathbf{w}|,\end{aligned}$$

we have $x^{\kappa} \in \mathcal{J}$, i.e., $x^{\kappa}\eta = 0$, which is a contradiction.

Let $e \in E$ be a monomial with the weighted-degree $n\cdot\deg_{\mathbf{w}}(f) - 2|\mathbf{w}|$. There exists a cohomology class $\tau \in \mathcal{H}_f$ such that $e\tau \in \mathcal{H}_f \setminus \{0\}$. Then we have

$$\begin{aligned}\deg_{\mathbf{w}}(e) + \deg_{\mathbf{w}}(\tau) &= n\cdot\deg_{\mathbf{w}}(f) - 2|\mathbf{w}| + \deg_{\mathbf{w}}(\tau) \\ &\le \deg_{\mathbf{w}}(e\tau) \\ &\le -|\mathbf{w}|.\end{aligned}$$

Thus $\deg_{\mathbf{w}}(\tau) \le -n\cdot\deg_{\mathbf{w}}(f) + |\mathbf{w}|$. On the other hand, since $\tau \in \mathcal{H}_f$, $\deg_{\mathbf{w}}(\tau) \ge -n\cdot\deg_{\mathbf{w}}(f) + |\mathbf{w}|$. Therefore we have $\deg_{\mathbf{w}}(\tau) = -n\cdot\deg_{\mathbf{w}}(f) + |\mathbf{w}|$. Q.E.D.

Proposition 2.2. *Let η be an algebraic local cohomology class belonging to $\mathcal{H}_f$. Then the following conditions are equivalent:*

(1) $\deg_{\mathbf{w}}(\eta) = -n\cdot\deg_{\mathbf{w}}(f) + |\mathbf{w}|$.
(2) *η generates $\mathcal{H}_f$ over $\mathcal{O}_{X,O}$.*

Proof. It is obvious that a generator of $\mathcal{H}_f$ over $\mathcal{O}_{X,O}$ has a degree $-n\cdot\deg_{\mathbf{w}}(f) + |\mathbf{w}|$ since a degree of any holomorphic function in $\mathcal{O}_{X,O}$ is greater than or equal to 0 and the smallest degree of classes in $\mathcal{H}_f$ is $-n\cdot\deg_{\mathbf{w}}(f) + |\mathbf{w}|$. Let σ be a generator of $\mathcal{H}_f$. There exists a function $h = h(z) \in \mathcal{O}_{X,O}$ satisfying $\eta = h\sigma$. Since both η and σ are of degree $-n\cdot\deg_{\mathbf{w}}(f) + |\mathbf{w}|$, we have $\deg_{\mathbf{w}}(h) = 0$ equivalently $h(0) \neq 0$. Thus, the function $1/h$ is in $\mathcal{O}_{X,O}$ and σ can be represented by $\sigma = (1/h)\eta$. This completes the proof. Q.E.D.

Corollary 2.1. *For any basis monomial $e \in E$, there exists a cohomology class $\eta \in \mathcal{H}_f$ satisfying the following condition:*

(i) *$e\eta = c\delta$, where δ is the delta function with support at the origin and c is a non zero constant.*
(ii) $\deg_{\mathbf{w}}(\eta) = -|\mathbf{w}| - \deg_{\mathbf{w}}(e)$.

Proof. Put $d = n\cdot\deg_{\mathbf{w}}(f) - 2|\mathbf{w}|$. Let $e \in E$ be a monomial with the weighted-degree w. It is known that the number of the basis monomial of weighted-degree w is the same with that of $d - w$. Let

$e' \in E$ be a monomial with weighted-degree $d - w$. Then a monomial ee' is in $\mathcal{O}_{X,0}/\mathcal{J}$. For a generator σ of $\mathcal{H}_f$ over $\mathcal{O}_{X,O}$, we have

$$\begin{aligned}\deg_{\mathbf{w}}(ee'\sigma) &= d + \deg_{\mathbf{w}}(\sigma)\\ &= (n \cdot \deg_{\mathbf{w}}(f) - 2|\mathbf{w}|) + (-n \cdot \deg_{\mathbf{w}}(f) + |\mathbf{w}|)\\ &= -|\mathbf{w}|.\end{aligned}$$

Since the only element in $\mathcal{H}_f$ with the weighted-degree $-|\mathbf{w}|$ is the delta function δ, we have $ee'\sigma = c\delta$ with some non-zero constant c. Thus the algebraic local cohomology class $\eta = e'\sigma$ satisfies the conditions (i) and (ii). Q.E.D.

The next corollary is obvious from Corollary 2.1.

Corollary 2.2. *Let $\chi_{\mathcal{J}}(t) = \sum_{j=1}^{p} \mu_{d_j} t^{d_j}$ be Poincaré polynomial of $\mathcal{J}_0$ where $\mu_{d_j} \in \mathbb{N}$, $j = 1, \ldots, \ell$. There is a basis of $\mathcal{H}_f$ consisting of μ_{d_j} classes of the degree $-d_j - |\mathbf{w}|$.*

For instance, for any generator σ over $\mathcal{O}_{X,O}$ of $\mathcal{H}_f$, the set $\{e_1\sigma, \ldots, e_\mu\sigma\}$ with $e_j \in E$ gives a basis of $\mathcal{H}_f$ enjoying Corollary 2.2.

In general, weighted-degrees of basis monomials of $\mathcal{O}_{X,O}/\mathcal{J}$ depend on representatives and thus some monomial bases do not meet the condition of Poincaré polynomial. In contrast, the semiquasihomogeneity of f always warrants the existence of a basis of $\mathcal{H}_f$ as claimed in Corollary 2.2.

§3. Computation of the dual basis

In this section, we give a method for constructing relative Čech cohomologies that constitute the dual basis of E with respect to Grothendieck local residues.

Let f_0 be the quasihomogeneous part of the semiquasihomogeneous function $f \in \mathcal{O}_{X,O}$. Let K_0 be the set of exponents κ of basis monomials in E_0;

$$K_0 = \{\kappa \in \mathbb{N}^n \mid x^\kappa \in E_0\}.$$

For an exponent $\kappa \in \mathbb{N}^n$, let Γ_κ be a set of multi indices $\lambda \in \mathbb{N}^n_+$ satisfying $\lambda - \mathbf{1} \notin K_0$ and $\langle \mathbf{w}, \lambda \rangle = \langle \mathbf{w}, \kappa \rangle + |\mathbf{w}|$;

$$\Gamma_\kappa = \{\lambda \in \mathbb{N}^n_+ \mid \lambda - \mathbf{1} \notin K_0,\ \langle \mathbf{w}, \lambda \rangle = \langle \mathbf{w}, \kappa \rangle + |\mathbf{w}|\}.$$

We have the following:

Proposition 3.1.

(1) *For every exponent $\kappa \in K_0$, there exists an algebraic local cohomology class $\eta_{0,\kappa}$ in $\mathcal{H}_{f_0}$ of the form*

$$\eta_{0,\kappa} = \left[\frac{1}{x^{\kappa+1}} + \sum_{\lambda \in \Gamma_\kappa} c_\lambda \frac{1}{x^\lambda}\right]. \tag{3.1}$$

(2) *For $K_0 = \{\kappa_1, \ldots, \kappa_\mu\}$, algebraic local cohomology classes η_{0,κ_1}, $\ldots$, η_{0,κ_μ} in (3.1) constitute the dual basis of E_0 with respect to Grothendieck local duality between $\mathcal{O}_{X,O}/\mathcal{J}_0$ and $\mathcal{H}_{f_0}$.*

Let $H_0 = \{\eta_{0,\kappa_1}, \ldots, \eta_{0,\kappa_\mu}\}$ be the dual basis given in Proposition 3.1 of E_0. Since the basis E_0 for $\mathcal{O}_{X,O}/\mathcal{J}_0$ also gives a basis E of $\mathcal{O}_{X,O}/\mathcal{J}$, we have the following:

Proposition 3.2. *For each $\eta_{0,\kappa} \in H_0$, there exists algebraic local cohomology class τ_κ satisfying $\Lambda_{\tau_\kappa} \cap \Gamma_\kappa = \emptyset$ and $\deg_{\mathbf{w}}(\tau_\kappa) > \deg_{\mathbf{w}}(\eta_{0,\kappa})$ or $\tau_\kappa = 0$ such that the algebraic local cohomology class*

$$\eta_\kappa = \eta_{0,\kappa} + \tau_\kappa \tag{3.2}$$

belongs to $\mathcal{H}_f$.

Let us discuss a method for constructing algebraic local cohomology classes η_κ based on the above proposition.

There are monomials η_κ in $\mathcal{H}_f$ that are determined by the conditions $f_{x_j}\eta_\kappa = 0$ for all $j = 1, \ldots, n$. Note that such monomials also belong to H_0. Denote the set of these monomials η_κ in H_0 by H_M:

$$H_M = \left\{ \left[\frac{1}{x^\lambda}\right] \in H_0 \;\middle|\; f_{x_j}\left[\frac{1}{x^\lambda}\right] = 0,\ \forall j = 1, \ldots, n \right\}.$$

Let Λ_0 be the set of multi indices defined by $\{\lambda \in \mathbb{N}_+^n \mid \lambda - \mathbf{1} \in K_0\}$. Let $L_\eta = \{\lambda \in \mathbb{N}_+^n \mid \lambda \notin \Lambda_0,\ \langle \mathbf{w}, \lambda\rangle \leq -\deg_{\mathbf{w}}(\eta)\}$ for an algebraic local cohomology class $\eta \in H_0$. Then, in order for η_κ given in (3.2) to constitute the dual basis of E, we may take τ_κ for $\eta_{0,\kappa} \in H_0 \setminus H_M$ by a linear combination of monomials $[1/x^\lambda]$ with $\lambda \in L_{\eta_{0,\kappa}}$.

We give a procedure for constructing the dual basis of E with respect to Grothendieck duality among $\mathcal{O}_{X,O}/\mathcal{J}$ and $\mathcal{H}_f$. Put $\Lambda_{\eta, x_j} = \Lambda_{f_{x_j}\eta}$. Let R_η denote a set of multi indices defined by

$$R_\eta = \{\nu \in N \mid \exists j \in \{1, \ldots, n\}, \text{ s.t., } \Lambda_{\eta,x_j} \cap \Lambda_{[1/x^\nu], x_j} \neq \emptyset\}$$

where N is a given set of multi indices.

Procedure 1. Put $H = H_M$. For each $\eta_0 = \eta_{0,k} \in H_0 \setminus H_M$,

(1) Put $\eta = \eta_0$ and $N = L_{\eta_0}$.

(2) Until $R_\eta = \emptyset$,
compute R_η,
put $\eta := \eta + \sum_{\nu \in R_\eta} c_\nu [1/x^\nu]$ with undetermined coefficients c_ν and
put $N := N \setminus R_\eta$.

(3) Determine coefficients c_ν in η by the condition
$f_{x_j}\eta = 0$ for all $j = 1, \ldots, n$.

(4) Put $H = H \cup \{\eta\}$.

Theorem 3.1. *The set* H *of algebraic local cohomology classes constructed by* Procedure 1 *gives rise to the dual basis of* E.

Proof. It is obvious that each η_{κ_j} constructed by the above procedure satisfies the condition

$$\mathrm{res}_0(\eta_{\kappa_j}, x^{\kappa_i}) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

It completes the proof. Q.E.D.

Example 1. Let $f = x^3y + y^6 + axy^5$ with a parameter a. This is a normal form of Z_{13} type semiquasihomogeneous function with the weighted-degree $\deg_{\mathbf{w}}(f) = 18$ with respect to the weight vector $\mathbf{w} = (5, 3) \in \mathbb{N}_+^2$. The quasihomogeneous part of the function f is $f_0 = x^3y + y^6$ and thus

$$E = \{1, y, x, y^2, xy, y^3, x^2, xy^2, xy^3, y^5, xy^4, xy^5\}.$$

We have

$$\left[\frac{1}{xy}\right], \left[\frac{1}{xy^2}\right], \left[\frac{1}{x^2y}\right], \left[\frac{1}{xy^3}\right], \left[\frac{1}{x^2y^2}\right], \left[\frac{1}{xy^4}\right], \left[\frac{1}{x^3y}\right], \left[\frac{1}{x^2y^3}\right],$$
$$\left[\frac{1}{xy^5}\right], \left[\frac{1}{x^2y^4}\right], \left[\frac{1}{xy^6} - 6\frac{1}{x^4y}\right], \left[\frac{1}{x^2y^5}\right], \left[\frac{1}{x^2y^6} - 6\frac{1}{x^5y}\right] \in \mathcal{H}_{f_0}.$$

The partial derivatives of the function f are $f_x = 3x^2y + ay^5$ and $f_y = x^3 + 6y^5 + 5axy^4$. Then, the set H_M is given by the following ten monomials:

$$\left[\frac{1}{xy}\right], \left[\frac{1}{xy^2}\right], \left[\frac{1}{x^2y}\right], \left[\frac{1}{xy^3}\right], \left[\frac{1}{x^2y^2}\right], \left[\frac{1}{xy^4}\right],$$
$$\left[\frac{1}{x^3y}\right], \left[\frac{1}{x^2y^3}\right], \left[\frac{1}{xy^5}\right], \left[\frac{1}{x^2y^4}\right].$$

Thus, in order to construct the dual basis of E, it suffices to compute other three cohomology classes in $\mathcal{H}_f$ with quasihomogeneous parts

$$\left[\frac{1}{xy^6} - 6\frac{1}{x^4y}\right], \left[\frac{1}{x^2y^5}\right], \left[\frac{1}{x^2y^6} - 6\frac{1}{x^5y}\right].$$

(1) Let $\eta_0 = [(1/xy^6) - 6(1/x^4y)]$. Then $L_{\eta_0} = \{(3, 2), (4, 1)\}$.
Put $\eta = \eta_0$. Then $\Lambda_{\eta, x} = \Lambda_{\eta, y} = \{(1, 1)\}$.
Put $N = \{(3, 2), (4, 1)\}$.
Since $R_\eta = \{(3, 2)\}$, put $\eta = \eta_0 + c[1/x^3y^2]$.
By the condition $f_x\eta = f_y\eta = 0$, we have $c = -(1/3)a$.

(2) Let $\eta_0 = [1/x^2y^5]$. Then $L_{\eta_0} = \{(3, 2), (4, 1), (3, 3)\}$.
Put $\eta = \eta_0$. Then $\Lambda_{\eta, x} = \emptyset$ and $\Lambda_{\eta, y} = \{(1, 1)\}$.
Put $N = \{(3, 2), (4, 1), (3, 3)\}$.
Since $R_\eta = \{(4, 1)\}$, put $\eta = \eta_0 + c[1/x^4y]$.
By the condition $f_x\eta = f_y\eta = 0$, we have $c = -5a$.

(3) Let $\eta_0 = [(1/x^2y^6) - 6(1/x^5y)]$. Then

$$L_{\eta_0} = \{(3, 2), (4, 1), (3, 3), (1, 7), (4, 2), (3, 4), (5, 1)\}.$$

Put $\eta = \eta_0$. Then $\Lambda_{\eta, x} = \{(2, 1)\}$, $\Lambda_{\eta, y} = \{(2, 1), (1, 2)\}$.
Put $N = L_{\eta_0}$.
Since $R_\eta = \{(1, 7), (4, 2), (3, 3)\}$, put

$$\eta = \eta_0 + s[1/xy^7] + t[1/x^4y^2] + u[1/x^3y^3].$$

Then $\Lambda_{\eta, x} = \{(2, 1), (1, 2)\}$ and $\Lambda_{\eta, y} = \{(2, 1), (1, 2)\}$.
Put $N = \{(3, 2), (4, 1), (3, 4), (5, 1)\}$.
Then, $R_\eta = \emptyset$.
Now, $\eta = [(1/x^2y^6) - 6(1/x^5y) + s(1/xy^7) + t(1/x^4y^2) + u(1/x^3y^3)]$.
By the condition $f_x\eta = f_y\eta = 0$, we have $s = -(7/9)a$, $t = -(1/3)a$ and $u = (7/27)a^2$.

Thus, the dual basis of E with respect to Grothendieck pairing between $\mathcal{O}_{X,O}/\mathcal{J}$ and $\mathcal{H}_f$ is given by

$$\left\{\left[\frac{1}{xy}\right], \left[\frac{1}{xy^2}\right], \left[\frac{1}{x^2y}\right], \left[\frac{1}{xy^3}\right], \left[\frac{1}{x^2y^2}\right], \left[\frac{1}{xy^4}\right], \left[\frac{1}{x^3y}\right], \left[\frac{1}{x^2y^3}\right],\right.$$

$$\left[\frac{1}{xy^5}\right], \left[\frac{1}{x^2y^4}\right], \left[\frac{1}{xy^6} - 6\frac{1}{x^4y} - \frac{1}{3}a\frac{1}{x^3y^2}\right], \left[\frac{1}{x^2y^5} - 5a\frac{1}{x^4y}\right],$$

$$\left.\left[\frac{1}{x^2y^6} - 6\frac{1}{x^5y} - \frac{7}{9}a\frac{1}{xy^7} - \frac{1}{3}a\frac{1}{x^4y^2} + \frac{7}{27}a^2\frac{1}{x^3y^3}\right]\right\}.$$

§4. Applications

We give two applications of results in Section 3; one is a method for solving a membership problem for Jacobi ideal $\mathcal{J}$, the other is a method for computing a standard basis of $\mathcal{J}$.

4.1. A membership problem

Let us recall the following result which immediately follows from Grothendieck local duality (1.1):

Proposition 4.1. *Let $p(x) \in \mathcal{O}_{X,O}$. Then, $\mathrm{res}_O(p(x), \eta) = 0$ for all $\eta \in \mathcal{H}_f$ is a necessary and sufficient condition for $p(x)$ to be in the ideal $\mathcal{J}$.*

By using the dual basis H of E constructed by Procedure 1, we can find whether a given $p(x)$ is in $\mathcal{J}$ based on Proposition 4.1. For the dual basis H of E, let

$$K = \cup_{\eta \in H}\{\kappa \in \mathbb{N}^n \mid \kappa + \mathbf{1} \in \Lambda_\eta\}$$

and $K_M = \{\kappa \in K \mid [1/x^{\kappa+\mathbf{1}}] \in H_M\}$. Then,

(1) if there are monomials x^κ in $p(x)$ with $\kappa \in K_M$, $p(x)$ does not belong to the ideal $\mathcal{J}$.

On the other hand, Proposition 4.1 assures that

(2) linear combinations of monomials x^κ with exponents κ satisfying $\kappa \notin K$ belong to $\mathcal{J}$.

Let $K(p) = \{\kappa \in \mathbb{N}^n \mid p(x) = \sum a_\kappa x^\kappa,\ a_\kappa \neq 0\}$ for a function $p(x)$ and $K' = K \setminus K_M$. Then, after testing the above two conditions (1) and (2), it suffices to find if the part $q(x)$ of a given function satisfying $K(q) \subset K'$ belongs to $\mathcal{J}$ or not. By following the procedure below, one can solve the membership problem for the ideal $\mathcal{J}$.

Procedure 2. For a given function $p(x)$,

If $K(p) \cap K_M \neq \emptyset$, then $p(x) \notin \mathcal{J}$.
Else, let $q(x)$ be the part of $p(x)$ given by the linear combination of monomials x^κ with $\kappa \in K'$, i.e., $p(x) = q(x) + \sum_{\kappa \notin K'} c_\kappa x^\kappa$.
  if $q(x)$ satisfies $\mathrm{res}_O(q(x), \eta) = 0$ for all $\eta \in H \setminus H_M$, then $p(x) \in \mathcal{J}$.
  else, $p(x) \notin \mathcal{J}$.

4.2. A standard basis

Making use of the dual basis of E constructed by the above procedure, we can compute a standard basis of the ideal $\mathcal{J}$. Note that the method described below is also applicable to the case where the given function f contains parameters. Let us illustrate a procedure for computing a standard basis of $\mathcal{J}$ by using examples. Following notations will be used in examples,

$$K_\eta = \{\kappa \in \mathbb{N}^n \mid \kappa + \mathbf{1} \in \Lambda_\eta\}, \quad \Delta = \sum_{\eta \in H \setminus H_M} K_\eta.$$

Let $\succ$ be the lexicographical ordering, and $\succ_{\mathbf{w}}$ defined by

$$x^\alpha \succ_{\mathbf{w}} x^\beta \Leftrightarrow (\langle \mathbf{w}, \alpha\rangle < \langle \mathbf{w}, \beta\rangle) \text{ or } (\langle \mathbf{w}, \alpha\rangle = \langle \mathbf{w}, \beta\rangle \text{ and } x^\alpha \succ x^\beta).$$

Example 2. Let us compute a standard basis of $\mathcal{J}$ for the same function f with Example 1. As seen in Example 1, three algebraic local cohomology classes

$$\eta_1 = \left[\frac{1}{xy^6} - 6\frac{1}{x^4y} - \frac{1}{3}a\frac{1}{x^3y^2}\right], \; \eta_2 = \left[\frac{1}{x^2y^5} - 5a\frac{1}{x^4y}\right]$$

and

$$\eta_3 = \left[\frac{1}{x^2y^6} - 6\frac{1}{x^5y} - \frac{7}{9}a\frac{1}{xy^7} - \frac{1}{3}a\frac{1}{x^4y^2} + \frac{7}{27}a^2\frac{1}{x^3y^3}\right]$$

together with H_M constitute the dual basis of E. Then,

$$K_{\eta_1} = \{(2, 1), (3, 0), (0, 5)\}, \; K_{\eta_2} = \{(3, 0), (1, 4)\},$$
$$K_{\eta_3} = \{(2, 2), (3, 1), (4, 0), (0, 6), (1, 5)\}$$

and

$$\Delta = \{(2, 1), (3, 0), (2, 2), (3, 1), (4, 0), (0, 5), (1, 4), (0, 6), (1, 5)\}.$$

Put $G = \Delta$.

(1) The exponent $(2, 1)$ is the smallest one in Δ with respect to $\succ_{\mathbf{w}}$.
Since $(2, 1)$ is only in K_{η_1}, take the biggest one $(0, 5)$ from $(K_{\eta_1} \cap G) \setminus \{(2, 1)\}$.
Since $(2, 1) \succ_{\mathbf{w}} (0, 5)$, put $p(x, y) = x^2y + sy^5$.
By the conditions $\mathrm{res}_O(p(x, y), \eta_1) = 0$, we have $p(x, y) = x^2y + (1/3)ay^5 \in \mathcal{J}$.
Put $G = G \setminus (G \cap \{(i, j) \mid i \geq 2, j \geq 1\})$
$= \{(3, 0), (4, 0), (0, 5), (1, 4), (0, 6), (1, 5)\}$.

(2) The exponent $(3, 0)$ appears both in K_{η_1} and K_{η_2}.
Take the biggest ones $(1, 4)$ from $(K_{\eta_1} \cap G) \setminus \{(3, 0)\}$ and $(0, 5)$ from $(K_{\eta_2} \cap G) \setminus \{(3, 0)\}$ respectively.
Since $x^3 \succ_{\mathbf{w}} y^5 \succ_{\mathbf{w}} xy^4$, put $q(x, y) = x^3 + sy^5 + txy^4$.

By the conditions $\mathrm{res}_0(q(x, y), \eta_1) = \mathrm{res}_0(q(x, y), \eta_2) = 0$, we have $q(x, y) = x^3 + 6y^5 + 5axy^4 \in \mathcal{J}$.
Put $G = G \setminus (G \cap \{(i, j) \mid i \geq 3, j \geq 0\})$
$= \{(0, 5), (1, 4), (0, 6), (1, 5)\}$.

(3) While the exponent $(0, 5) \in K_{\eta_1}$ is the smallest one in G, y^5 can not become the leading term of any generator of $\mathcal{J}$ because $(K_{\eta_1} \cap G) \setminus \{(0, 5)\} = \emptyset$.
Put $G = G \setminus \{(0, 5)\} = \{(1, 4), (0, 6), (1, 5)\}$.

(4) While the exponent $(1, 4) \in K_{\eta_2}$ is the smallest one in G, xy^4 can not become the leading term of any generator of $\mathcal{J}$ because $(K_{\eta_2} \cap G) \setminus \{(1, 4)\} = \emptyset$.
Put $G = G \setminus \{(1, 4)\} = \{(0, 6), (1, 5)\}$.

(5) The exponent $(0, 6) \in K_{\eta_3}$ is the smallest one in Δ and the other exponent $(1, 5)$ in G is also belong to K_{η_3}.
Since $(0, 6) \succ_{\mathbf{w}} (1, 5)$, put $r(x, y) = y^6 + sxy^5$.
By the condition $\mathrm{res}_0(r(x, y), \eta_3) = 0$, we have $r(x, y) = y^6 + (7/9)ax^5$.

By the condition of the weighted-degrees, we have $y^7 \in \mathcal{J}$. Then, we have constructed a standard basis

$$\{y^7, y^6 + (7/9)axy^5, x^3 + 6y^5 + 5axy^4, x^2y + (1/3)ay^5\}$$

of the ideal $\mathcal{J}$ with respect to $\succ_{\mathbf{w}}$.

Example 3. Let us consider a plane curve defined by $x = t^5$ and $y = t^{16} + t^{54}$. The defining equation of this curve is

$$f(x, y) = x^{16} - y^5 + 5x^{14}y^3 - 5x^{28}y + x^{54}.$$

This is a semiquasihomogeneous function with a weighted homogeneous part $f_0(x, y) = x^{16} - y^5$ of the weighted-degree 80 with respect to the weight vector $\mathbf{w} = (5, 16)$. Then, the dual basis H_{f_0} of $E_0 = \{x^iy^j \mid 0 \leq i \leq 14, 0 \leq j \leq 3\}$ is given by monomials $\{[1/x^ky^l] \mid 1 \leq k \leq 15, 1 \leq l \leq 4\}$.

Since, $H_M = H_{f_0} \setminus \{[1/x^{15}y^3], [1/x^{14}y^4], [1/x^{15}y^4]\}$, in order to construct the dual basis of $E(= E_0)$, it suffices to find cohomology classes with terms $[1/x^{15}y^3]$, $[1/x^{14}y^4]$, $[1/x^{15}y^4]$ respectively. By direct computations, we obtain algebraic local cohomology classes $[(1/x^{15}y^3) + 3(1/xy^5)]$ and $[(1/x^{14}y^4) - (35/8)(1/x^{16}y)]$ that belong to H_f. It is easy to verify that $[(1/x^{15}y^4) - (35/8)(1/x^{17}y) + 3(1/xy^6)]$ belongs to H_f.

Then, cohomology classes

$$\eta_1 = \left[\frac{1}{x^{15}y^3} + 3\frac{1}{xy^5}\right],\ \eta_2 = \left[\frac{1}{x^{14}y^4} - \frac{35}{8}\frac{1}{x^{16}y}\right],$$
$$\eta_3 = \left[\frac{1}{x^{15}y^4} - \frac{35}{8}\frac{1}{x^{17}y} + 3\frac{1}{xy^6}\right]$$

together with H_M constitute the dual basis of E.

In order to construct a standard basis of $\mathcal{J}$, it suffices to use η_1 and η_2. It is easy to see that $35x^{13}y^3 + 8x^{15}$ and $3x^{14}y^2 - y^4$ constitute a standard basis of $\mathcal{J}$ with respect to the total lexicographic ordering.

References

[1] V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko, Singularities of Differentiable Maps Volume *I*, Monographs in Mathematics, **82**, Birkhäuser, Boston, 1985.

[2] A. M. Dickenstein and C. Sheja, Duality methods for the membership problem, Effective Methods in Algebraic Geometry, (eds. T. Mora and C. Traverso), Progress in Math., **94**, Birkhäuser, Boston, 1991, pp. 89–103.

[3] A. Grothendieck, Théorème de dualité pour les faisceaux algébriques cohérents, Séminaire Bourbaki, **149**, Paris, 1957.

[4] R. Hartshorne, Residues and Duality, Lecture Notes in Math., **20**, Springer, Berlin, 1966.

[5] Y. Nakamura and S. Tajima, Unimodal singularities and differential operators, Séminaires et Congrès, Société Mathématique de France, 2005, 191–208.

[6] S. Tajima and Y. Nakamura, Algebraic local cohomology classes attached to quasi-homogeneous isolated hypersurface singularities, Publ. of RIMS, Kyoto Univ., **41** (2005), 1–10.

[7] S. Tajima, Grothendieck duality and Hermite-Jacobi formulas, Proceedings of International Conference on finite or infinite dimensional Complex Analysis, Dekker, 2000, 503–509.

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