Advanced Studies in Pure Mathematics 46, 2007
Singularities in Geometry and Topology 2004
pp. 95–104

Stably hyperbolic polynomials

Vladimir Petrov Kostov

Abstract.

A real polynomial in one real variable is *hyperbolic* if all its roots are real. Denote the set of monic hyperbolic polynomials of degree n by Π_n. Suppose that for a real polynomial $P(x)$ of degree n there exists $k \in \mathbf{N}$ and a polynomial $Q(x)$ of degree $\leq k-1$ such that $x^k P + Q \in \Pi_{n+k}$. Denote the set of such polynomials P by $\Pi_n(k)$. Call the set $\Pi_n(\infty) = \overline{\cup_{k=0}^{\infty} \Pi_n(k)}$ the domain of *stably hyperbolic polynomials of degree n*. In the present paper we explore the geometric properties of the set $\Pi_4(\infty)$.

§1. Introduction

Consider the family of polynomials $P(x, a) = x^n + a_1 x^{n-1} + \cdots + a_n$, $a_i, x \in \mathbf{R}$.

Definition 1. Call a polynomial from the family P *hyperbolic* (resp. *strictly hyperbolic*) if it has only real (resp. real and distinct) roots. Denote by Π_n the *hyperbolicity domain* of the family P, i.e. the subset of $\mathbf{R}^n$ consisting of the values of the n-tuple of coefficients $(a_1, \ldots, a_n)$ for which P is hyperbolic. Geometric properties of the hyperbolicity domain are given in papers [Ko1], [Ko2], [Me1] and [Me2]. In the proofs in the first two of them the results of the papers [Ar] and [Gi] are used.

Notice that $\Pi_n \cap \{a_1 = 0, a_2 > 0\} = \emptyset$ and $\Pi_n \cap \{a_1 = 0, a_2 = 0\} = 0 \in \mathbf{R}^n$. Indeed, if a polynomial is hyperbolic, then such are its nonconstant derivatives as well. For $a_1 = 0$ one has $P^{(n-2)} = (n!/2)x^2 + (n-2)!a_2$ which is hyperbolic only if $a_2 \leq 0$. If one has $a_1 = a_2 = 0$, then one has $P^{(n-3)} = (n!/6)x^3 + (n-3)!a_3$ which is hyperbolic only if $a_3 = 0$, and in a similar way one must have $a_4 = \cdots = a_n = 0$. Therefore

Received September 21, 2004
Revised May 27, 2005
2000 *Mathematics Subject Classification.* 12D10.
Key words and phrases. (stably) hyperbolic polynomial, multiplicity vector, hyperbolicity domain.

in what follows we set once for all $a_1 = 0$ (this can be achieved by the shift $x \mapsto x - a_1/n$) and $a_2 = -1$ (recall that Π_n is invariant for the one-parameter group of stretchings $a_j \mapsto e^{jt}a_j$).

Notation 2. Set $\Pi_n(0) = \Pi_n$. Denote for $k \in \mathbf{N}$ by $\Pi_n(k)$ the set of polynomials P for which there exist polynomials Q of degree $\leq k-1$ such that $R(x) := x^kP + Q \in \Pi_{n+k}$. Hence, one has $\Pi_n(k+1) \supset \Pi_n(k)$ because if $P \in \Pi_{n+k}$, then $xP \in \Pi_{n+k+1}$. Set $\Pi_n(\infty) = \overline{\cup_{k=0}^{\infty}\Pi_n(k)}$. Notice that for a polynomial from $\partial\Pi_n(\infty)$, the boundary of $\Pi_n(\infty)$, one cannot find k and Q as above.

Definition 3. We call the set $\Pi_n(\infty)$ the *domain of stably hyperbolic polynomials* of degree n.

Proposition 4. *For any $n \in \mathbf{N}$, $n \geq 2$, the set $\Pi_n(\infty)$ (with $a_1 = 0$, $a_2 = -1$) is bounded.*

Proof. Denote by $x_1 \geq \cdots \geq x_{n+k}$ the roots of the polynomial R, see the above notation. One has $x_1 + \cdots + x_{n+k} = 0$, $\sum_{1\leq i<j\leq n+k} x_ix_j = -1$, hence, $\sum_{i=1}^{n+k} x_i^2 = 2$. This means that one can have $|x_i| \geq 1$ only for one value of i, say, for $i = n+k$.

Hence, for each $n \in \mathbf{N}^*$, $n \geq 2$, and for $k \geq 0$ one has $|\sum_{i=1}^{n+k} x_i^m| \leq 2^{m/2} + 2$. Indeed, one has $|x_{n+k}| \leq \sqrt{2}$ and $|x_{n+k}^m| \leq 2^{m/2}$. For $i \neq n+k$ one has $|x_i^m| \leq |x_i^2| = x_i^2$, hence, $|\sum_{i=1}^{n+k-1} x_i^m| \leq \sum_{i=1}^{n+k-1} x_i^2 \leq 2$.

The Vieta symmetric functions σ_l of $x_1, \ldots, x_{n+k}$ (where $\sigma_l = \sum_{1\leq i_1<\cdots<i_l\leq n+k} x_{i_1} \cdots x_{i_l}$) can be expressed as polynomials of the Newton symmetric functions $\varphi_l = \sum_{i=1}^{n+k} x_i^l$. Recall that there exist polynomials M_ν, M_ν^* such that

$$\varphi_l = (-1)^{l-1}l\sigma_l + M_l(\sigma_1, \ldots, \sigma_{l-1}),$$
$$(-1)^{l-1}l\sigma_l = \varphi_l + M_l^*(\varphi_1, \ldots, \varphi_{l-1}) \tag{1}$$

i.e. the passage from the Newton to the Vieta functions and its inverse are described by "triangular" formulas.

Hence, the first n Vieta functions, i.e. the first n coefficients a_m up to a sign of the polynomial R, are bounded by constants not depending on k (but only on n). Q.E.D.

Notation 5. In what follows we set $a_3 = a$, $a_4 = b$, and we denote by Π_n' the projections of the sets Π_n on the space of the variables (a, b). Notice that one has $\Pi_n' = \Pi_4(n-4) \cap \{a_1 = 0, a_2 = -1\}$.

Remarks 6. 1) Proposition 4 and Theorem 14 can be given shorter proofs if one uses the results of papers [Ko3] and [Ko4] concerning the so-called *very hyperbolic*[1] polynomials. We prefer to make the present text self-contained, therefore we do not use these results and we give direct proofs instead. Moreover, the proofs contain an explicit parametrization of the set $\partial\Pi'_n$, the boundary of Π'_n.

2) It is shown in [Ko3] that the mapping

$$\tau\colon a_j \mapsto \beta_j a_j \quad \text{where} \quad \beta_j = (n(n-1))^{j/2}/n(n-1)\cdots(n-j+1)$$

defines a diffeomorphism between the set $\Pi_n(\infty)$ and the set $V\Pi_n$ of very hyperbolic polynomials. Set $\beta_j = ((n(n-1))^{n/2}/n!)((n-j)!/(n(n-1))^{(n-j)/2}$. This allows one to view the mapping τ as a superposition of the mappings $\Phi\colon a_j \mapsto ((n(n-1))^{n/2}/n!)a_j$ (multiplication with a non-zero constant), $\Psi\colon a_j \mapsto a_j/(n(n-1))^{(n-j)/2}$ (change of the scale of the x-axis) and $\Xi\colon a_j \mapsto (n-j)!a_j$.

The latter mapping is related to the *Laplace transform* which transforms the monomial x^k into $\int_0^\infty t^k e^{-\xi t}dt = k!/\xi^{k+1}$ (the formula is meaningful for $\mathrm{Re}\,\xi > 0$). Therefore the mapping Ξ is the Laplace transform followed by $\xi \mapsto 1/x$ and by a division by x.

The mapping Ξ^{-1} results from the *Borel transform* which maps the formal power series $\sum a_k x^k$ into the series $\sum a_k x^k/k!$ (this accelerates the convergence). We call its inverse the *anti-Borel* transform. Thus the Borel (the anti-Borel) transform maps stably hyperbolic (very hyperbolic) polynomials into very hyperbolic (into stably hyperbolic) ones.

Comments 7. The following lines were communicated to the author by B.Z. Shapiro and J. Borcea. Stably hyperbolic polynomials are interesting to study for the following reasons. Consider a linear operator T acting on the space of polynomials of degree $\leq n$ which does not increase the degree of the polynomials. More exactly, suppose that it is "triangular": $T(x^k) = x^k + R_k$ where R_k is a polynomial of degree $\leq k-1$, $k = 0, 1, \ldots, n$. A theorem of Carnicer, Peña and Pinkus (see [CaPePi]) states that if the operator T preserves hyperbolicity, then it is a differential one, i.e. of the form $1 + c_1 D + \cdots + c_n D^n$ $(*)$, $c_j \in \mathbf{C}$, $D := d/dx$. This result has been recently generalized in [BoSh]. It is shown in [Bo] (see also [BoSh]) that an operator of the form $(*)$ (with $c_i \in \mathbf{R}$) preserves hyperbolicity if and only if the polynomial $T(x^n)$ is hyperbolic. In this case a partially proved conjecture due to J. Borcea and B.Z. Shapiro claims that the polynomial $1+c_1x+\cdots+c_nx^n$ is stably hyperbolic.

[1]i.e. hyperbolic and having hyperbolic primitives of all orders.

§2. Properties of the set of stably hyperbolic polynomials

Definition 8. We stratify the sets Π_n and Π'_n the strata being defined by the *multiplicity vectors (MVs)* of the polynomials. A MV is a vector whose components are the multiplicities of the distinct roots of the polynomial given in decreasing order. Example: if $n = 4$ and if one has $x_1 = x_2 > x_3 > x_4$, then the MV of the polynomial is (2, 1, 1). We identify the strata with their MVs.

Comments 9. Recall that (see [Ko2]) the sets Π'_n look as shown on Fig. 1. The picture is symmetric w.r.t. Ob, the tangent lines and their limits at the strata of the form $(k, n-k)$ are nowhere vertical.

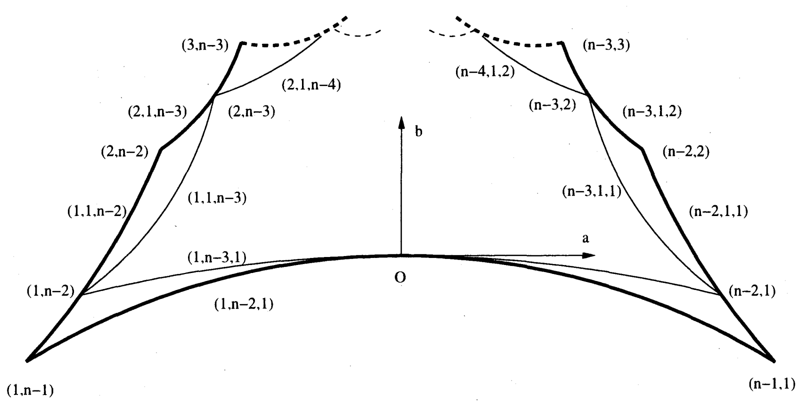

Fig. 1.

Hence, the sets Π'_n and Π'_{n-1} together look as shown on Fig. 1. First of all, it is clear that $\Pi'_n \supset \Pi'_{n-1}$ because if $P \in \Pi_{n-1}$, then $xP \in \Pi_n$. The set $\partial\Pi'_n$ consists of the closures of all strata with MVs of the form $(l, 1, n-l-1)$ and $(1, n-2, 1)$. No point X of a stratum $S = (l, 1, n-l-2) \subset \Pi'_{n-1}$ lies on the boundary $\partial\Pi'_n$ of Π'_n. Indeed, if the middle root (which is a simple one) of a polynomial $P \in S$ is not 0, then the MV of the polynomial xP would be of the form $(l, 1, 1, n-l-2)$ (the left or the right root of P is not 0 because one has $a_1 = 0$). This is not the MV of a stratum of $\partial\Pi'_n$. If the middle root of P is 0, then the MV of xP must be $(l, 2, n-l-2)$ which is not the MV of a stratum of $\partial\Pi'_n$ either.

On the other hand, there exists a single point from the stratum $(s, 1, n-s-1) \subset \partial\Pi'_n$ or $(1, n-2, 1) \subset \partial\Pi'_n$ for which the middle root equals 0 (we leave the proof for the reader). Hence, this point is the stratum $(s, n-s-1) \subset \partial\Pi'_{n-1}$ (resp. a point from the stratum $(1, n-3, 1) \subset \partial\Pi'_{n-1}$; clearly, this must be the point $(0, 0) \in Oab$).

Using the above comments one can draw the sets Π'_n for $n = 4, 5, \ldots$ together, see Fig. 2.

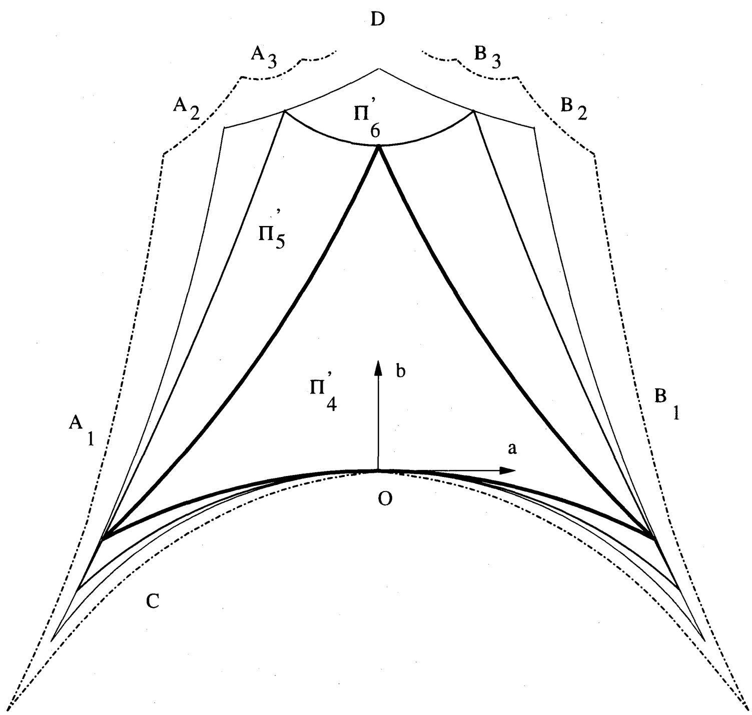

Fig. 2.

Proposition 10. *The limits of the strata* $(n-s-1, 1, s)$ *and* $(s, 1, n-s-1)$ *of* $\partial\Pi'_n$ *exist (for* s *fixed and* $n \to \infty$*) as well as the limit for* $n \to \infty$ *of the stratum* $(1, n-2, 1)$*. These limits are algebraic arcs (denoted by* A_s*,* B_s *and* C*, see* Fig. 2*).*

Proof. The closure of the stratum $(n-s-1, 1, s)$ can be parametrized by the three roots $\xi \geq \eta \geq \zeta$ for which one has

(2) $$(n-s-1)\xi + \eta + s\zeta = 0, \quad (n-s-1)\xi^2 + \eta^2 + s\zeta^2 = 2$$

These two equations define an ellipse in $\mathbf{R}^3$. Adding the inequalities $\xi \geq \eta \geq \zeta$ means cutting off an arc of the ellipse. Hence, Vieta's formulas imply

$$\begin{aligned} -a &= C_{n-s-1}^3\xi^3 + C_{n-s-1}^2\xi^2\eta + C_{n-s-1}^2 s\xi^2\zeta + (n-s-1)s\xi\eta\zeta \\ &\quad + (n-s-1)C_s^2\xi\zeta^2 + C_s^2\eta\zeta^2 + C_s^3\zeta^3 \\ b &= C_{n-s-1}^4\xi^4 + \cdots \end{aligned}$$

Set $\xi = \varphi/n$. Hence, for $n \to \infty$ equations (2) look like this:

$$\varphi + \eta + s\zeta = 0, \quad \eta^2 + s\zeta^2 = 2 \tag{3}$$

Indeed, the second of equations (2) implies that the quantities η and ζ are uniformly bounded in $n \in \mathbf{N}$. The first of these equations implies that then φ is uniformly bounded as well. Hence, the term $(n-s-1)\xi^2 = (n-s-1)\varphi^2/n^2$ in the second of equations (2) tends to 0 when $n \to \infty$.

Equations (3) are again a couple of equations defining an ellipse in $\mathbf{R}^3$. If $\eta > 0$, then for $n \to \infty$ the inequality $\varphi \geq n\eta$ implies that φ cannot be chosen such that (φ, η, ζ) belong to the ellipse. Hence, one must have $0 \geq \eta \geq \zeta$ (and there is no restriction upon φ other than the first of equations (3)). For $n \to \infty$ one has

$$-a = \frac{\varphi^3}{6} + \frac{\varphi^2\eta}{2} + \frac{s\varphi^2\zeta}{2} + s\varphi\eta\zeta + C_s^2\varphi\zeta^2 + C_s^2\eta\zeta^2 + C_s^3\zeta^3 + O\Big(\frac{1}{n}\Big),$$

i.e. for $n \to \infty$ the limit of the quantity a is a homogeneous polynomial of degree 3 in φ, η and ζ which satisfy conditions (3) and the inequalities $0 \geq \eta \geq \zeta$. In the same way one shows that the limit of b is such a polynomial of degree 4. This proves the proposition for the arcs A_s, for the arcs B_s and C the proof is analogous. The reader can find the parametrization of the arc C in 7^0 of the proof of Theorem 14. Q.E.D.

Remark 11. One checks directly that neither of the arcs A_s, B_s and C is a line segment. As each stratum $(n-s-1, 1, s)$, $(1, n-2, 1)$ and $(s, 1, n-s-1)$ of $\partial\Pi'_n$ has a curvature of constant sign (see [Me1] or [Ko2]) such that the concavity is towards the interior of Π'_n, this is also the case of the arcs A_s, B_s and C w.r.t. $\Pi_4(\infty)$.

Notation 12. Denote by D the point from $\Pi_4(\infty)$ lying on the b-axis and with greatest b-coordinate.

Remark 13. The point D is the common limit of the right endpoints of the arcs A_s or of the left endpoints of the arcs B_s when $s \to \infty$. It can be computed also as the limit of the strata $(k, k) \subset \Pi'_{2k}$ for $k \to \infty$. The computation gives $D = (0, 1/2)$.

Theorem 14. 1) *The tangent lines to the arcs A_s, B_s and C are never vertical. Their limits at the endpoints of these arcs exist and are not vertical either.*

2) *The slopes of these tangent lines (together with their limits at the endpoints) are uniformly bounded. These slopes (and their limits at the endpoints) are positive for the arcs A_s and negative for the arcs B_s.*

3) *At the common endpoint of two arcs A_s, A_{s+1} or B_s, B_{s+1} the slopes of the two limits of tangent lines (from left and right) are different.*

4) *At the common endpoints of the arcs A_1 and C and of B_1 and C the two limits of tangent lines are the same.*

5) *The limit of the slope of the tangent lines exists when the point from $\partial\Pi_4(\infty)$ tends to D; this limit equals 0.*

Remarks 15. 1) The boundary of the set $\Pi_4(\infty)$ consists of countably many arcs whose endpoints accumulate towards the point D. These points are singular points for $\Pi_4(\infty)$, see 3) of the theorem. Hence, the set $\Pi_4(\infty)$ is not semi-algebraic.

2) It is decidable whether a point $U = (a^0, b^0) \in Oab$ represents a polynomial from $\Pi_4(\infty)$ (in particular, from $\partial\Pi_4(\infty)$) or not. This follows from the fact that one knows explicit parametrizations of the arcs A_s, B_s and C and the coordinates of the point D.

Indeed, denote by (α_s, β_s) (resp. by (α_s^*, β_s^*)) the left (resp. the right) endpoint of the arc A_s (resp. B_s). By 2^0 of the proof of Theorem 14, see below, one has $(\alpha_s, \beta_s) = ((-2/3)\sqrt{2/s}, 1/2-1/s)$. One has first to check whether $a^0 \in [\alpha_1, \alpha_1^*]$ or not. If not, then $U \notin \Pi_4(\infty)$. If yes, then one has to check whether $a^0 = 0$ or not. If yes, then $U \in \Pi_4(\infty)$ if and only if $b^0 \in [0, 1/2]$. If $a^0 \neq 0$, then one checks for which s one has $a^0 \in [\alpha_s, \alpha_{s+1})$ or $a^0 \in (\alpha_{s+1}^*, \alpha_s^*]$ (and which of these two conditions holds). After this one has to compare b^0 with the b-coordinate of the points of the arcs A_s, C or B_s, C whose a-coordinates equal a^0.

Proof of Theorem 14.

1^0. We use the notation from the proof of Proposition 10. Our first aim is to give explicit parametrization of the arc A_s. The one of the arc B_s is given by analogy and the one of the arc C is given in 7^0. Consider first the stratum $(n-s-1, 1, s) \subset \partial\Pi_n'$. Instead of operating with Vieta's functions a_j (in the variables $\xi \geq \eta \geq \zeta$, of multiplicities $n-s-1$, 1 and s), we use the sums b_j of the j-th powers of these variables (taking their multiplicities into account). Recall that (see formulas (1))

$$b_3 = 3a_3 + \alpha a_1 a_2 + \beta a_1^3, \quad b_4 = -4a_4 + \gamma a_1^4 + \delta a_1^2 a_2 + \varepsilon a_2^2 + \theta a_1 a_3$$

for some $\alpha, \beta, \gamma, \delta, \varepsilon, \theta \in \mathbf{R}$. As $a_1 = 0$, $a_2 = -1$, we have $b_3 = 3a_3$, $b_4 = -4a_4 + \varepsilon$. By computing the values of the symmetric functions for the quadruple $1/\sqrt{2}$, $1/\sqrt{2}$, $-1/\sqrt{2}$, $-1/\sqrt{2}$ one finds that $\varepsilon = 2$. Thus the stratum $(n-s-1,\ 1,\ s)$ is parametrized (in the variables φ, η, ζ) in the following form:

$$\begin{aligned} &\varphi + \eta + s\zeta + O(1/n) = 0 \\ &\eta^2 + s\zeta^2 + O(1/n) = 2 \\ &a = a_3 = (1/3)(\eta^3 + s\zeta^3 + O(1/n)) \\ &b = a_4 = (-1/4)(\eta^4 + s\zeta^4) + 1/2 + O(1/n) \end{aligned}$$

(see (2)) and after deleting the terms $O(1/n)$ one obtains a parametrization of the arc A_s.

2^0. Set $\eta = \sqrt{2}\cos t$, $\zeta = \sqrt{2/s}\sin t$. Recall that $0 \geq \eta \geq \zeta$ (see the proof of Proposition 10). The endpoints of the arc A_s are such that either $(\eta,\ \zeta) = (0,\ -\sqrt{2/s})$ (and one has $(a_3,\ a_4) = ((-2/3)\sqrt{2/s},\ 1/2 - 1/s)$, this is the left endpoint of A_s) or $\eta = \zeta = -\sqrt{2/(s+1)}$ (and one has $(a_3,\ a_4) = ((-2/3)\sqrt{2/(s+1)},\ 1/2 - 1/(s+1))$, this is the right endpoint of A_s).

In the new parametrization of the arc A_s one has

$$\begin{aligned} a = a_3 &= (2/3)\sqrt{2}\cos^3 t + (2/3)\sqrt{2/s}\sin^3 t, \\ b = a_4 &= 1/2 - \cos^4 t + (-1/s)\sin^4 t\ . \end{aligned}$$

One has

$$db/da = (db/dt)/(da/dt) = -(\sqrt{2}\cos t + \sqrt{2/s}\sin t) = -\eta - \zeta \tag{4}$$

This expression depends continuously on t and is uniformly bounded (both in s and t). In the case of arcs A_s we have $0 \geq \eta \geq \zeta$ (and one cannot have both equalities at the same time), hence, $db/da > 0$. This proves parts 1) and 2) of the theorem for the arcs A_s (for the arcs B_s the proof is analogous).

3^0. Recall that one has $0 \geq \eta \geq -\sqrt{2/(s+1)}$, $-\sqrt{2/(s+1)} \geq \zeta \geq -\sqrt{2/s}$. Hence, for $s \to \infty$ the sum $-\eta - \zeta$ (see (4)) tends to 0 uniformly in t. This proves part 5) of the theorem for the arcs A_s (in the same way one proves it for the arcs B_s).

4^0. To prove part 3) of the theorem it suffices to compute the two values of db/da obtained for η, ζ corresponding to the right endpoint of A_s and to the left endpoint of A_{s+1}, see 2^0. These values are $2/\sqrt{s+1}$ and $2/\sqrt{s+2}$. Hence, they are different. For the arcs B_s the proof is analogous.

5^0. Part 4) of the theorem can be proved either by direct computation or by observing that the common endpoints in question are the limits of the strata $(n-1, 1)$ and $(1, n-1)$ of the sets Π'_n where the limits of the tangent lines to the strata $(n-2, 1, 1)$, $(1, n-2, 1)$ and $(1, 1, n-2)$, $(1, n-2, 1)$ coincide, see [Ko2]. We leave the details for the reader.

6^0. To extend the proof of parts 1) and 2) of the theorem to the arc C it suffices to observe that the slope of the tangent line to this arc is comprised between its limit values at the common endpoints with A_1 and B_1 due to the constant sign of the curvature, see Remark 11.

7^0. Give the parametrization of the arc C. For a point of the closure of the stratum $(1, n-2, 1) \subset \partial\Pi'_n$ defined by the roots $\xi \geq \eta \geq \zeta$, of multiplicities $1, n-2, 1$, one has

$$\begin{aligned}
&\xi + (n-2)\eta + \zeta = 0 \\
&\xi^2 + (n-2)\eta^2 + \zeta^2 = 2 \\
&-a = (n-2)\xi\eta\zeta + C^2_{n-2}(\xi\eta^2 + \zeta\eta^2) + C^3_{n-2}\eta^3 \\
&b = C^2_{n-2}\xi\eta^2\zeta + C^3_{n-2}\eta^3(\xi + \zeta) + C^4_{n-2}\eta^4
\end{aligned}$$

Set $\eta = \psi/n$. Hence, when $n \to \infty$ (and the given point tends to a point from C) one has $\xi \geq 0 \geq \zeta$ and

$$\begin{aligned}
&\xi + \psi + \zeta = 0 \\
&\xi^2 + \zeta^2 = 2 \\
&-a = \xi\psi\zeta + (\xi\psi^2 + \zeta\psi^2)/2 + \psi^3/6 \\
&b = \xi\psi^2\zeta/2 + (\xi + \zeta)\psi^3/6 + \psi^4/24
\end{aligned}$$

These formulas provide the parametrization of the arc C. Q.E.D.

Acknowledgement. The author's attention to stably hyperbolic polynomials is due to the fruitful discussions with B.Z. Shapiro and J. Borcea from the University of Stockholm to whom the author expresses his most sincere gratitude.

References

[Ar] V. I. Arnol'd, Hyperbolic polynomials and Vandermonde mappings, Funct. Anal. Appl., **20** (1986), 125–127; transl. from Funktsional. Anal. i Prilozhen., **20** (1986), 52–53.

[Bo] J. Borcea, Monoids of linear differential operators and the distribution of zeros of real polynomials, in preparation.

[BoSh] J. Borcea and B. Z. Shapiro, The Weyl algebra and hyperbolicity-preserving linear operators in finite degree, in preparation.

[CaPePi] J. M. Carnicer, J. M. Peña and A. Pinkus, On some zero-increasing operators, Acta Math. Hungar., **94** (2002), 173–190.

[Gi] A. B. Givental, Moments of random variables and equivariant Morse lemma, Russian Math. Surveys, **42** (1987), 275–276; transl. from Uspekhi Mat. Nauk, **42** (1987), 221–222.

[Ko1] V. P. Kostov, On the geometric properties of Vandermonde's mapping and on the problem of moments, Proc. Roy. Soc. Edinburgh Sect. A **112**, 3-4 (1989), 203–211.

[Ko2] V. P. Kostov, On the hyperbolicity domain of the polynomial $x^n + a_1x^{n-1} + \cdots + a_n$, Serdica Math. J., **25** (1999), 47–70.

[Ko3] V. P. Kostov, Very hyperbolic and stably hyperbolic polynomials, C. R. A. S. Paris, Sér. I, **339** (2004), 157–162.

[Ko4] V. P. Kostov, Very hyperbolic polynomials, Funct. Anal. Appl., **39** (2005), 229–232; translation from Funkc. Anal. Prilozh., **39** (2005), 80–84.

[Me1] I. Meguerditchian, Géométrie du discriminant réel et des polynômes hyperboliques, Thèse de doctorat, Univ. de Rennes I, soutenue le 24. 01. 1991.

[Me2] I. Meguerditchian, A theorem on the escape from the space of hyperbolic polynomials, Math. Z., **211** (1992), 449–460.

Université de Nice
Laboratoire de Mathématiques
Parc Valrose, 06108 Nice Cedex 2
France
tel: (0033) 4 92 07 62 67
fax: (0033) 4 93 51 79 74
kostov@math.unice.fr