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Moduli spaces of twisted sheaves on a projective variety

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Dedicated to Masaki Maruyama on the occation of his 60th birthday

Appendix by Daniel Huybrechts and Paolò Stellari

§0. Introduction

Let X be a smooth projective variety over $\mathbb{C}$. Let $\alpha := \{\alpha_{ijk} \in H^0(U_i \cap U_j \cap U_k, \mathcal{O}_X^\times)\}$ be a 2-cocycle representing a torsion class $[\alpha] \in H^2(X, \mathcal{O}_X^\times)$. An α-twisted sheaf $E := \{(E_i, \varphi_{ij})\}$ is a collection of sheaves E_i on U_i and isomorphisms $\varphi_{ij} : E_{i|U_i \cap U_j} \to E_{j|U_i \cap U_j}$ such that $\varphi_{ii} = \mathrm{id}_{E_i}$, $\varphi_{ji} = \varphi_{ij}^{-1}$ and $\varphi_{ki} \circ \varphi_{jk} \circ \varphi_{ij} = \alpha_{ijk}\,\mathrm{id}_{E_i}$. We assume that there is a locally free α-twisted sheaf, that is, α gives an element of the Brauer group $\mathrm{Br}(X)$. A twisted sheaf naturally appears if we consider a non-fine moduli space M of the usual stable sheaves on X. Indeed the transition functions of the local universal families satisfy the patching condition up to the multiplication by constants and gives a twisted sheaf. If the patching condition is satisfied, i.e., the moduli space M is fine, than the universal family defines an integral functor on the bounded derived categories of coherent sheaves $\mathbf{D}(M) \to \mathbf{D}(X)$. Assume that X is a $K3$ surface and $\dim M = \dim X$. Then Mukai, Orlov and Bridgeland showed that the integral functor is the Fourier-Mukai functor, i.e., it is an equivalence of the categories. In his thesis [C2], Căldăraru studied the category of twisted sheaves and its bounded derived category. In particular, he generalized Mukai, Orlov and Bridgeland's results on the Fourier-Mukai transforms to non-fine moduli spaces on a $K3$ surface. For the usual derived category, Orlov [Or] showed that the equivalence class is described in terms of the Hodge structure of the Mukai lattice. Căldăraru conjectured that a similar result also holds for the derived

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category of twisted sheaves. Recently this conjecture was modified and proved by Huybrechts and Stellari, if $\rho(X) \geq 12$ in [H-St]. As is well-known, twisted sheaves also appear if we consider a projective bundle over X.

In this paper, we define a notion of the stability for a twisted sheaf and construct the moduli space of stable twisted sheaves on X. We also construct a projective compactification of the moduli space by adding the S-equivalence classes of semi-stable twisted sheaves. In particular if $H^1(X, \mathcal{O}_X) = 0$ (e.g. X is a $K3$ surface), then the moduli space of locally free twisted sheaves is the moduli space of projective bundles over X. Thus we compactify the moduli space of projective bundles by using twisted sheaves. The idea of the construction is as follows. We consider a twisted sheaf as a usual sheaf on the Brauer-Severi variety. Instead of using the Hilbert polynomial associated to an ample line bundle on the Brauer-Severi variety, we use the Hilbert polynomial associated to a line bundle coming from X in order to define the stability. Then the construction is a modification of Simpson's construction of the moduli space of usual sheaves (cf. [Y3]). M. Lieblich informed us that our stability condition coincides with Simpson's stability for modules over the associated Azumaya algebra via Morita equivalence. Hence the construction also follows from Simpson's moduli space [S, Thm. 4.7] and the valuative criterion for properness.

In section 3, we consider the moduli space of twisted sheaves on a $K3$ surface. We generalize known results on the moduli space of usual stable sheaves to the moduli spaces of twisted stable sheaves (cf. [Mu2], [Y1]). In particular, we consider the non-emptyness, the deformation type and the weight 2 Hodge structure. Then we can consider twisted version of the Fourier-Mukai transform by using 2 dimensional moduli spaces, which is done in section 4. As an application of our results, Huybrechts and Stellari prove Căldăraru's conjecture generally (see Appendix).

Since our main example of twisted sheaves are those on $K3$ surfaces or abelian surfaces, we consider twisted sheaves over $\mathbb{C}$. But some of the results (e.g., subsection 2.2) also hold over any field.

E. Markman and D. Huybrechts communicated to the author that M. Lieblich independently constructed the moduli of twisted sheaves. In his paper [Li], Lieblich developed a general theory of twisted sheaves in terms of algebraic stack and constructed the moduli space intrinsic way. He also studied the moduli spaces of twisted sheaves on surfaces. There are also some overlap with a paper by N. Hoffmann and U. Stuhler [Ho-St]. They also constructed the moduli space of rank 1 twisted sheaves and studied the symplectic structure of the moduli space.

§1. Twisted sheaves

Notation: For a locally free sheaf E on a variety X, $\mathbb{P}(E) \to X$ denotes the projective bundle in the sense of Grothendieck, that is, $\mathbb{P}(E) = \operatorname{Proj}(\bigoplus_{n=0}^{\infty} S^n(E))$.

Let X be a smooth projective variety over $\mathbb{C}$. Let $\alpha := \{\alpha_{ijk} \in H^0(U_i \cap U_j \cap U_k, \mathcal{O}_X^\times)\}$ be a 2-cocycle representing a torsion class $[\alpha] \in H^2(X, \mathcal{O}_X^\times)$. An α-twisted sheaf $E := \{(E_i, \varphi_{ij})\}$ is a collection of sheaves E_i on U_i and isomorphisms $\varphi_{ij} : E_{i|U_i \cap U_j} \to E_{j|U_i \cap U_j}$ such that $\varphi_{ii} = \mathrm{id}_{E_i}$, $\varphi_{ji} = \varphi_{ij}^{-1}$ and $\varphi_{ki} \circ \varphi_{jk} \circ \varphi_{ij} = \alpha_{ijk} \, \mathrm{id}_{E_i}$. If all E_i are coherent, then we say that E is coherent. Let $\mathrm{Coh}(X, \alpha)$ be the category of coherent α-twisted sheaves on X.

If E_i are locally free for all i, then we can glue $\mathbb{P}(E_i^\vee)$ together and get a projective bundle $p : Y \to X$ with $\delta([Y]) = [\alpha]$, where $[Y] \in H^1(X, PGL(r))$ is the corresponding cohomology class of Y and $\delta : H^1(X, PGL(r)) \to H^2(X, \mathcal{O}_X^\times)$ is the connecting homomorphism induced by the exact sequence

$$1 \to \mathcal{O}_X^\times \to GL(r) \to PGL(r) \to 1.$$

Thus $[\alpha]$ belongs to the Brauer group $\mathrm{Br}(X)$. If X is a smooth projective surface, then $\mathrm{Br}(X)$ coincides with the torsion part of $H^2(X, \mathcal{O}_X^\times)$. Let $\mathcal{O}_{\mathbb{P}(E_i^\vee)}(\lambda_i)$ be the tautological line bundle on $\mathbb{P}(E_i^\vee)$. Then, φ_{ij} induces an isomorphism $\widetilde{\varphi}_{ij} : \mathcal{O}_{\mathbb{P}(E_i^\vee)}(\lambda_i)_{|p^{-1}(U_i \cap U_j)} \to \mathcal{O}_{\mathbb{P}(E_j^\vee)}(\lambda_j)_{|p^{-1}(U_i \cap U_j)}$. $\mathcal{L}(p^*(\alpha^{-1})) := \{(\mathcal{O}_{\mathbb{P}(E_i^\vee)}(\lambda_i), \widetilde{\varphi}_{ij})\}$ is an $p^*(\alpha^{-1})$-twisted line bundle on Y.

1.1. Sheaves on a projective bundle

In this subsection, we shall interpret twisted sheaves as usual sheaves on a Brauer-Severi variety. Let $p : Y \to X$ be a projective bundle. Let $X = \cup_i U_i$ be an analytic open covering of X such that $p^{-1}(U_i) \cong U_i \times \mathbb{P}^{r-1}$. We set $Y_i := p^{-1}(U_i)$. We fix a collection of tautological line bundles $\mathcal{O}_{Y_i}(\lambda_i)$ on Y_i and isomorphisms $\phi_{ji} : \mathcal{O}_{Y_i \cap Y_j}(\lambda_j) \to \mathcal{O}_{Y_i \cap Y_j}(\lambda_i)$. We set $G_i := p_*(\mathcal{O}_{Y_i}(\lambda_i))^\vee$. Then G_i are vector bundles on U_i and $p^*(G_i)(\lambda_i)$ defines a vector bundle G of rank r on Y. We have the Euler sequence

$$0 \to \mathcal{O}_Y \to G \to T_{Y/X} \to 0.$$

Thus G is a non-trivial extension of $T_{Y/X}$ by $\mathcal{O}_Y$.

Lemma 1.1. $\mathrm{Ext}^1(T_{Y/X}, \mathcal{O}_Y) = \mathbb{C}$. *Thus G is characterized as a non-trivial extension of $T_{Y/X}$ by $\mathcal{O}_Y$. In particular, G does not depend on the choice of the local trivialization of p.*

Proof. Since $\mathbf{R}p_*(G^\vee) = 0$, the Euler sequence inplies that

$$\operatorname{Ext}^1(T_{Y/X}, \mathcal{O}_Y) \cong H^0(Y, \mathcal{O}_Y) \cong \mathbb{C}.$$

Q.E.D.

Definition 1.1. For a projective bundle $p : Y \to X$, let $\epsilon(Y)(:= G)$ be a vector bundle on Y which is a non-trivial extension

$$0 \to \mathcal{O}_Y \to \epsilon(Y) \to T_{Y/X} \to 0.$$

By the exact sequence $0 \to \mu_r \to SL(r) \to PGL(r) \to 1$, we have a connecting homomorphism $\delta' : H^1(X, PGL(r)) \to H^2(X, \mu_r)$. Let $o : H^2(X, \mu_r) \to H^2(X, \mathcal{O}_X^\times)$ be the homomorphism induced by the inclusion $\mu_r \hookrightarrow \mathcal{O}_X^\times$. Then we have $\delta = o \circ \delta'$.

Definition 1.2. For a $\mathbb{P}^{r-1}$-bundle $p : Y \to X$ corresponding to $[Y] \in H^1(X, PGL(r))$, we set $w(Y) := \delta'([Y]) \in H^2(X, \mu_r)$.

Lemma 1.2 ([C1],[H-Sc]). *If $p : Y \to X$ is a $\mathbb{P}^{r-1}$-bundle associated to a vector bundle E on X, i.e., $Y = \mathbb{P}(E^\vee)$, then*

$$w(Y) = [c_1(E) \mod r].$$

Lemma 1.3. $[c_1(G) \mod r] = p^*(w(Y)) \in H^2(Y, \mu_r)$.

Proof. There is a line bundle L on $Y \times_X Y$ such that $L_{|Y_i \times_{U_i} Y_i} \cong p_1^*(\mathcal{O}_{Y_i}(-\lambda_i)) \otimes p_2^*(\mathcal{O}_{Y_i}(\lambda_i))$, where $p_i : Y \times_X Y \to Y$, $i = 1, 2$ are i-th projections. By the definition of G, $p_{1*}(L) \cong G^\vee$. Hence $p_1 : Y \times_X Y \to Y$ is the projective bundle $\mathbb{P}(G^\vee) \to Y$. Then we get

$$-[c_1(G^\vee) \mod r] = w(Y \times_X Y) = p^*(w(Y)).$$

Q.E.D.

Lemma 1.4. *Let $p : Y \to X$ be a $\mathbb{P}^{r-1}$-bundle. Then the following conditions are equivalent.*

(1) *$Y = \mathbb{P}(E^\vee)$ for a vector bundle on X.*
(2) *$w(Y) \in \mathrm{NS}(X) \otimes \mu_r$.*
(3) *There is a line bundle L on Y such that $L_{|p^{-1}(x)} \cong \mathcal{O}_{p^{-1}(x)}(1)$.*

Proof. (2) $\Rightarrow$ (3): If $w(Y) = [D \mod r]$, $D \in \mathrm{NS}(X)$, then $c_1(\epsilon(Y)) - p^*(D) \equiv 0 \mod r$. We take a line bundle L on Y with $c_1(\epsilon(Y)) - p^*(D) = rc_1(L)$. (3) $\Rightarrow$ (1): We set $E^\vee := p_*(L)$. Then $Y = \mathbb{P}(E^\vee)$. Q.E.D.

Definition 1.3. $\mathrm{Coh}(X,Y)$ is a subcategory of $\mathrm{Coh}(Y)$ such that $E \in \mathrm{Coh}(X,Y)$ if and only if

$$E_{|Y_i} \cong p^*(E_i) \otimes \mathcal{O}_{Y_i}(\lambda_i)$$

for $E_i \in \mathrm{Coh}(U_i)$. For simplicity, we call $E \in \mathrm{Coh}(X,Y)$ a Y-sheaf.

By this definition, $\{(U_i, E_i)\}$ gives a twisted sheaf on X. Thus we have an equivalence

$$\begin{array}{rcccc} \Lambda^{\mathcal{L}(p^*(\alpha^{-1}))}: & \mathrm{Coh}(X,Y) & \cong & \mathrm{Coh}(X,\alpha) \\ & E & \mapsto & p_*(E \otimes L^\vee), \end{array} \tag{1.1}$$

where $\mathcal{L}(p^*(\alpha^{-1})) := \{(\mathcal{O}_{Y_i}(\lambda_i), \phi_{ij})\}$ is a twisted line bundle on Y and $\alpha_{ijk}^{-1} \operatorname{id}_{\mathcal{O}_{Y_i}(\lambda_i)} = \phi_{ki} \circ \phi_{jk} \circ \phi_{ij}$.

We have the following relations:

$$\begin{aligned} p_*(G^\vee \otimes E)_{|U_i} &= p_*(p^*(G_i^\vee) \otimes \mathcal{O}_{Y_i}(-\lambda_i) \otimes p^*(E_i) \otimes \mathcal{O}_{Y_i}(\lambda_i)) \\ &= p_* p^*(G_i^\vee \otimes E_i) = G_i^\vee \otimes E_i, \end{aligned}$$

$$\begin{aligned} p_*(E)_{|U_i} &= p_*(p^*(E_i) \otimes \mathcal{O}_{Y_i}(\lambda_i)) \\ &= E_i \otimes p_*(\mathcal{O}_{Y_i}(\lambda_i)) = G_i^\vee \otimes E_i. \end{aligned}$$

Lemma 1.5. *A coherent sheaf E on Y belongs to* $\mathrm{Coh}(X,Y)$ *if and only if $\phi : p^*p_*(G^\vee \otimes E) \to G^\vee \otimes E$ is an isomorphism. In particular $E \in \mathrm{Coh}(X,Y)$ is an open condition.*

Proof. $\phi_{|Y_i}$ is the homomorphism

$$p^*G_i^\vee \otimes p^*p_*(E(-\lambda_i)) \to p^*G_i^\vee \otimes E(-\lambda_i).$$

Hence $\phi_{|Y_i}$ is an isomorphism if and only if $p^*p_*(E(-\lambda_i)) \to E(-\lambda_i)$ is an isomorphism, which is equivalent to $E \in \mathrm{Coh}(X,Y)$. Q.E.D.

Lemma 1.6. *Assume that $H^3(X,\mathbb{Z})_{tor} = 0$. Then $H^*(Y,\mathbb{Z}) \cong H^*(X,\mathbb{Z})[x]/(f(x))$, where $f(x) \in H^*(X,\mathbb{Z})[x]$ is a monic polynomial of degree r. In particular, $H^2(X,\mathbb{Z}) \otimes \mu_{r'} \to H^2(Y,\mathbb{Z}) \otimes \mu_{r'}$ is injective for all r'.*

Proof. $R^2p_*\mathbb{Z}$ is a local system of rank 1. Since $c_1(K_{Y/X})$ is a section of this local system, $R^2p_*\mathbb{Z} \cong \mathbb{Z}$. Let h be the generator. Then $R^{2i}p_*\mathbb{Z} \cong \mathbb{Z}h^i$. Since $H^3(X,\mathbb{Z})_{\mathrm{tor}} = 0$, by the Leray spectral sequence, we get a surjective homomorphism $H^2(Y,\mathbb{Z}) \to H^0(X, R^2p_*\mathbb{Z})$. Let $x \in H^2(Y,\mathbb{Z})$ be a lifting of h. Then x^i is a lifting of $h^i \in H^0(X, R^{2i}p_*\mathbb{Z})$. Therefore the Leray-Hirsch theorem implies that

$$H^*(Y,\mathbb{Z}) \cong H^*(X,\mathbb{Z})[x]/(f(x)).$$

Q.E.D.

Lemma 1.7. *Assume that* $o(w(Y)) = o(w(Y'))$.

(i) *Then there is a line bundle* L *on* $Y' \times_X Y$ *such that*

$$L_{|p'^{-1}(x) \times p^{-1}(x)} \cong \mathcal{O}_{p'^{-1}(x)}(1) \boxtimes \mathcal{O}_{p^{-1}(x)}(-1)$$

for all $x \in X$. *If* $L' \in \operatorname{Pic}(Y' \times_X Y)$ *also satisfies the property, then* $L' = L \otimes q^*(P)$, $P \in \operatorname{Pic}(X)$, *where* $q : Y' \times_X Y \to X$ *is the projection.*

(ii) *We have an equivalence*

$$\begin{array}{cccc} \Xi^L_{Y \to Y'}: & \operatorname{Coh}(X, Y) & \to & \operatorname{Coh}(X, Y') \\ & E & \mapsto & p_{Y'*}(p'^*_Y(E) \otimes L), \end{array}$$

where $p_{Y'} : Y' \times_X Y \to Y'$ *and* $p'_Y : Y' \times_X Y \to Y$ *are projections.*

Remark 1.1. We also see that E is a Y-sheaf if and only if $p'^*_Y(E) \otimes L \cong p^*_{Y'}(E')$ for a sheaf E' on Y'.

Definition 1.4. Assume that $H^3(X, \mathbb{Z})_{tor} = 0$. For a Y-sheaf E of rank r', $[c_1(E) \mod r'] \in H^2(Y, \mu_{r'})$ belongs to $p^*(H^2(X, \mu_{r'}))$. We set

$$w(E) := (p^*)^{-1}([c_1(E) \mod r']) \in H^2(X, \mu_{r'}).$$

By Lemmas 1.3 and 1.7, we see that

Lemma 1.8. (i) *By the functor* $\Xi^L_{Y \to Y'}$ *in Lemma 1.7,*

$$w(\Xi^L_{Y \to Y'}(E)) = w(E), \quad \textit{for } E \in \operatorname{Coh}(X, Y).$$

(ii) $w(\epsilon(Y)) = w(Y)$.

§2. Moduli of twisted sheaves

2.1. Definition of the stability

Let $(X, \mathcal{O}_X(1))$ be a pair of a projective scheme X and an ample line bundle $\mathcal{O}_X(1)$ on X. Let $p : Y \to X$ be a projective bundle over X.

Definition 2.1. A Y-sheaf E is of dimension d, if $p_*(E)$ is of dimension d.

For a coherent sheaf F of dimension d on X, we define $a_i(F) \in \mathbb{Z}$ by the coefficient of the Hilbert polynomial of F:

$$\chi(F(m)) = \sum_{i=0}^{d} a_i(F) \binom{m+i}{i}.$$

Let G be a locally free Y-sheaf. For a Y-sheaf E of dimension d, we set $a_i^G(E) := a_i(p_*(G^\vee \otimes E))$. Thus we have

$$\chi(G, E \otimes p^*\mathcal{O}_X(m)) = \chi(p_*(G^\vee \otimes E)(m)) = \sum_{i=0}^{d} a_i^G(E)\binom{m+i}{i}.$$

Definition 2.2. Let E be Y-sheaf of dimension d. Then E is (G-twisted) semi-stable (with respect to $\mathcal{O}_X(1)$), if

(i) E is of pure dimension d,

(ii)

$$\frac{\chi(p_*(G^\vee \otimes F)(m))}{a_d^G(F)} \leq \frac{\chi(p_*(G^\vee \otimes E)(m))}{a_d^G(E)}, m \gg 0 \tag{2.1}$$

for all subsheaf $F \neq 0$ of E.

If the inequality in (2.1) is strict for all proper subsheaf $F \neq 0$ of E, then E is (G-twisted) stable with respect to $\mathcal{O}_X(1)$.

Theorem 2.1. *Let $p : Y \to X$ be a projective bundle. There is a coarse moduli scheme $\overline{M}_{X/\mathbb{C}}^h$ parametrizing S-equivalence classes of G-twisted semi-stable Y-sheaves E with the G-twisted Hilbert polynomial h. $\overline{M}_{X/\mathbb{C}}^h$ is a projective scheme.*

Remark 2.1. The construction also works for a projective bundle $Y \to X$ over any field and also for a family of projective bundles, by the fundamental work of Langer [L].

Lemma 2.2. *Let $p' : Y' \to X$ be a projective bundle with $o(w(Y')) = o(w(Y))$ and $\Xi_{Y\to Y'}^L$ the correspondence in Lemma 1.7. Then a Y-sheaf E is G-twisted semi-stable if and only if $\Xi_{Y\to Y'}^L(E) \in \mathrm{Coh}(X, Y')$ is $\Xi_{Y\to Y'}^L(G)$-twisted semi-stable. In particular, we have an isomorphism of the corresponding moduli spaces.*

Indeed, since $\Xi_{Y\times S\to Y'\times S}^{L\boxtimes\mathcal{O}_S}(*)_s = \Xi_{Y\to Y'}^L(* \otimes k(s))$, if we have a flat family of Y-sheaves $\{\mathcal{E}_s\}_{s\in S}$, $\mathcal{E} \in \mathrm{Coh}(Y \times S)$, then $\{\mathcal{E}'_s\}_{s\in S}$ is also a flat family of Y'-sheaves, where $\mathcal{E}' := \Xi_{Y\times S\to Y'\times S}^{L\boxtimes\mathcal{O}_S}(\mathcal{E})$.

Remark 2.2. For a locally free Y-sheaf G, we have a projective bundle $Y' \to X$ with $\epsilon(Y') = \Xi_{Y\to Y'}^L(G)$. Hence it is sufficient to study the $\epsilon(Y)$-twisted semi-stability.

Remark 2.3. This definition is the same as in [C1]. If $Y = \mathbb{P}(G^\vee)$ for a vector bundle G on X, then $\mathrm{Coh}(X, Y)$ is equivalent to $\mathrm{Coh}(X)$ and G-twisted stability is nothing but the twisted semi-stability in [Y3].

Definition 2.3. Let λ be a rational number. Let E be a Y-sheaf of dimension d. Then E is of type λ with respect to the G-twisted semi-stability, if

(i) E is of pure dimension d,

(ii)
$$\frac{a_{d-1}^G(F)}{a_d^G(F)} \leq \frac{a_{d-1}^G(E)}{a_d^G(E)} + \lambda$$
for all subsheaf F of E.

If $\lambda = 0$, then E is μ-semi-stable.

2.2. Construction of the moduli space

From now on, we assume that $G = \epsilon(Y)$ (cf. Remark 2.2). Let $P(x)$ be a numerical polynomial. We shall construct the moduli space of G-twisted semi-stable Y-sheaves E with $\chi(p_*(G^\vee \otimes E)(n)) = P(n)$.

2.2.1. *Boundedness* Let E be a Y-sheaf. Then

$$p^*p_*(G^\vee \otimes E) \otimes G \to E$$

is surjective. Indeed $p^*p_*(G^\vee \otimes E) \to G^\vee \otimes E$ is an isomorphism and $G \otimes G^\vee \to \mathcal{O}_Y$ is surjective.

We take a surjective homomorphism $\mathcal{O}_X(-n_G)^{\oplus N} \to p_*(G^\vee \otimes G)$, $n_G \gg 0$. Then we have a surjective homomorphism $p^*(\mathcal{O}_X(-n_G))^{\oplus N} \to G^\vee \otimes G$.

Lemma 2.3. *Let E be a Y-sheaf of pure dimension d. If*

$$a_{d-1}^G(F) \geq a_d^G(F)\left(\frac{a_{d-1}^G(E)}{a_d^G(E)} - \nu\right) \tag{2.2}$$

for all quotient $E \to F$, then $a_{d-1}(F') \geq a_d(F')\left(\frac{a_{d-1}^G(E)}{a_d^G(E)} - \nu - n_G\right)$ for all quotient $p_*(G^\vee \otimes E) \to F'$. In particular

$$S_\nu := \left\{ E \in \operatorname{Coh}(X,Y) \;\middle|\; \begin{array}{l} E \text{ satisfies (2.2) and} \\ \chi(p_*(G^\vee \otimes E)(nH)) = P(n) \end{array} \right\}$$

is bounded.

Proof. Since $p^*p_*(G^\vee \otimes E) \cong G^\vee \otimes E$, we have a surjective homomorphism

$$p^*(\mathcal{O}_X(-n_GH))^{\oplus N} \otimes E \to G \otimes p^*p_*(G^\vee \otimes E) \to G \otimes p^*(F').$$

By our assumption, we get

$$\begin{aligned} & a_{d-1}(p_*(G^\vee \otimes G) \otimes F') \\ \geq & a_d(p_*(G^\vee \otimes G) \otimes F') \left(\frac{a_{d-1}(p_*(G^\vee \otimes E))}{a_d(p_*(G^\vee \otimes E))} - n_G - \nu \right). \end{aligned}$$

Since $a_{d-1}(p_*(G^\vee \otimes G) \otimes F') = \mathrm{rk}(G)^2 a_{d-1}(F')$ and $a_d(p_*(G^\vee \otimes G) \otimes F') = \mathrm{rk}(G)^2 a_d(F')$, we get our claim. The boundedness of S_ν follows from the boundedness of $\{p_*(G^\vee \otimes E) | E \in S_\nu\}$ and Lemma 2.4 below. Q.E.D.

Lemma 2.4. *Let S be a bounded subset of* $\mathrm{Coh}(X)$. *Then* $T := \{E \in \mathrm{Coh}(X, Y) | p_*(G^\vee \otimes E) \in S\}$ *is also bounded.*

Proof. For $E \in T$, we set $I(E) := \ker(p^* p_*(G^\vee \otimes E) \otimes G \to E)$. We shall show that $T' := \{I(E) | E \in T\}$ is bounded. We note that $I(E) \in \mathrm{Coh}(X, Y)$ and we have an exact sequence

$$0 \to p_*(G^\vee \otimes I(E)) \to p_*(G^\vee \otimes E) \otimes p_*(G \otimes G^\vee) \to p_*(G^\vee \otimes E) \to 0.$$

Since $p_*(G^\vee \otimes E) \in S$, $\{p_*(G^\vee \otimes I(E)) | E \in T\}$ is also bounded. Since $p^* p_*(G^\vee \otimes I(E)) \otimes G \to I(E)$ is surjective and $I(E)$ is a subsheaf of $p^* p_*(G^\vee \otimes E) \otimes G$, T' is bounded. Q.E.D.

Corollary 2.5. *Under the same assumption* (2.2), *there is a rational number ν' which depends on ν such that*

$$a_{d-1}(F') \leq a_d(F') \left(\frac{a^G_{d-1}(E)}{a^G_d(E)} + \nu' \right)$$

for a subsheaf $F' \subset p_*(G^\vee \otimes E)$.

Combining this with Langer's important result [L, Cor. 3.4], we have the following

Lemma 2.6. *Under the same assumption* (2.2),

$$\frac{h^0(G, E)}{a^G_d(E)} \leq \left[\frac{1}{d!} \left(\frac{a^G_{d-1}(E)}{a^G_d(E)} + \nu' + c \right)^d \right]_+,$$

where c depends only on $(X, \mathcal{O}_X(1))$, G, d and $a^G_d(E)$.

2.2.2. *A quot-scheme* Since $p_*(G^\vee \otimes E)(n)$, $n \gg 0$ is generated by global sections,

$$H^0(G^\vee \otimes E \otimes p^*\mathcal{O}_X(n)) \otimes G \to E \otimes p^*\mathcal{O}_X(n)$$

is surjective. Since $R^ip_*(G^\vee \otimes E) = 0$ for $i > 0$, we also see that $H^i(G^\vee \otimes E \otimes p^*\mathcal{O}_X(n)) = 0$, $i > 0$ and $n \gg 0$.

We fix a sufficiently large integer n_0. We set $N := \chi(p_*(G^\vee \otimes E)(n_0)) = P(n_0)$. We set $V := \mathbb{C}^N$. We consider the quot-scheme $\mathfrak{Q}$ parametrizing all quotients

$$\phi : V \otimes G \to E$$

such that $E \in \mathrm{Coh}(X, Y)$ and $\chi(p_*(G^\vee \otimes E)(n)) = P(n_0 + n)$. By Lemma 2.4, $\mathfrak{Q}$ is bounded, in particular, it is a quasi-projective scheme.

Lemma 2.7. *$\mathfrak{Q}$ is complete.*

Proof. We prove our claim by using the valuative criterion. Let R be a discrete valuation ring and K the quotient field of R. Let $\phi : V_R \otimes G \to \mathcal{E}$ be a R-flat family of quotients such that $\mathcal{E} \otimes_R K \in \mathrm{Coh}(X, Y)$, where $V_R := V \otimes_{\mathbb{C}} R$. We set $\mathcal{I} := \ker \phi$. We have an exact and commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & p^*p_*(\mathcal{I} \otimes G^\vee) & \to & V_R \otimes G \otimes G^\vee & \to & p^*p_*(\mathcal{E} \otimes G^\vee) & \to & 0 \\
 & & \downarrow & & \| & & \downarrow \psi & & \\
0 & \to & \mathcal{I} \otimes G^\vee & \to & V_R \otimes G \otimes G^\vee & \to & \mathcal{E} \otimes G^\vee & \to & 0
\end{array}$$

We shall show that ψ is an isomorphism. Obviously ψ is surjective. Since $\mathcal{E}$ is R-flat, $\mathcal{E}$ has no R-torsion, which implies that $p^*p_*(\mathcal{E} \otimes G^\vee)$ is a torsion free R-module. Hence $\ker \psi$ is also torsion free. On the other hand, our choice of $\mathcal{E}$ implies that $\psi \otimes K$ is an isomorphism. Therefore $\ker \psi = 0$. Q.E.D.

Since $\ker \phi \in \mathrm{Coh}(X, Y)$, we have a surjective homomorphism

$$V \otimes \mathrm{Hom}(G, G \otimes p^*\mathcal{O}_X(n)) \to \mathrm{Hom}(G, E \otimes p^*\mathcal{O}_X(n))$$

for $n \gg 0$. Thus we can embed $\mathfrak{Q}$ as a subscheme of an Grassmann variety $Gr(V \otimes W, P(n_0 + n))$, where $W = \mathrm{Hom}(G, G \otimes p^*\mathcal{O}_X(n))$. Since all semi-stable Y-sheaf are pure, we may replace $\mathfrak{Q}$ by the closure of the open subset parametrizing pure quotient Y-sheaves. The same arguments in [Y3] imply that $\mathfrak{Q}/\!/GL(V)$ is the moduli space of G-twisted semi-stable sheaves. The details are left to the reader. For the proof, we also use the following.

Let $(R, \mathfrak{m})$ be a discrete valuation ring R and the maximal ideal $\mathfrak{m}$. Let K be the fractional field and k the residue field. Let $\mathcal{E}$ be a R-flat family of $Y \otimes R$-sheaves such that $\mathcal{E} \otimes_R K$ is pure.

Lemma 2.8. *There is a R-flat family of coherent $Y \otimes R$-sheaves $\mathcal{F}$ and a homomorphism $\psi : \mathcal{E} \to \mathcal{F}$ such that $\mathcal{F} \otimes_R k$ is pure, ψ_K is an isomorphism and ψ_k is an isomorphic at generic points of* $\operatorname{Supp}(\mathcal{F} \otimes_R k)$.

By using [S, Lem. 1.17] or [H-L, Prop. 4.4.2], we first construct $\mathcal{F}$ as a usual family of sheaves. Then the very construction of it, $\mathcal{F}$ becomes a $Y \otimes R$-sheaf.

2.3. A family of Y-sheaves on a projective bundle over $M^h_{X/\mathbb{C}}$

Assume that $\mathfrak{Q}^{ss}$ consists of stable points. Then $\mathfrak{Q}^{ss} \to \overline{M}^h_{X/\mathbb{C}}$ is a principal $PGL(N)$-bundle. For a scheme S, $f_S : Y \times S \to S$ denotes the projection. Let $\mathcal{Q}$ be the universal quotient sheaf on $Y \times \mathfrak{Q}^{ss}$. $V := \operatorname{Hom}_{f_{\mathfrak{Q}^{ss}}}(G \boxtimes \mathcal{O}_{\mathfrak{Q}^{ss}}, \mathcal{Q})$ is a locally free sheaf on $\mathfrak{Q}^{ss}$. We consider the projective bundle $\mathfrak{q} : \mathbb{P}(V) \to \mathfrak{Q}^{ss}$. Since $\mathcal{Q}$ is $GL(N)$-linearized, V is also $GL(N)$-linearized. Then we have a quotient $\psi : \mathbb{P}(V) \to \mathbb{P}(V)/PGL(N)$ with the commutative diagram:

$$\begin{array}{ccc} \mathbb{P}(V) & \xrightarrow{\mathfrak{q}} & \mathfrak{Q}^{ss} \\ \downarrow & & \downarrow \\ \widetilde{\overline{M}^h_{X/\mathbb{C}}} := \mathbb{P}(V)/PGL(N) & \xrightarrow{q} & \overline{M}^h_{X/\mathbb{C}} \end{array}$$

Since $(1_Y \times \mathfrak{q})^*(\mathcal{Q}) \otimes f^*_{\mathbb{P}(V)}(\mathcal{O}_{\mathbb{P}(V)}(-1))$ is $PGL(N)$-linearlized, we have a family of G-twisted stable Y-sheaves $\mathcal{E}$ on $Y \times \widetilde{\overline{M}^h_{X/\mathbb{C}}}$ with

$$(1_Y \times \psi)^*(\mathcal{E}) = (1_Y \times \mathfrak{q})^*(\mathcal{Q}) \otimes f^*_{\mathbb{P}(V)}(\mathcal{O}_{\mathbb{P}(V)}(-1)).$$

Hence $\mathcal{E}^\vee \in \operatorname{Coh}(Y \times \overline{M}^h_{X/\mathbb{C}}, Y \times \widetilde{\overline{M}^h_{X/\mathbb{C}}})$ (if $\mathcal{E}$ is locally free). Let W be a locally free sheaf on $\widetilde{\overline{M}^h_{X/\mathbb{C}}}$ such that $\psi^*(W) = \mathfrak{q}^*(V)(-1)$. Then we also have $W^\vee = \epsilon(\widetilde{\overline{M}^h_{X/\mathbb{C}}}) \in \operatorname{Coh}(\overline{M}^h_{X/\mathbb{C}}, \widetilde{\overline{M}^h_{X/\mathbb{C}}})$ and $\mathcal{E} \otimes f^*_{\widetilde{\overline{M}^h_{X/\mathbb{C}}}}(W^\vee)$ descends to a sheaf on $Y \times \overline{M}^h_{X/\mathbb{C}}$.

Remark 2.4. There is also a family of G-twisted stable Y-sheaves $\mathcal{E}'$ on $Y \times \mathbb{P}(V^\vee)/PGL(N)$ such that

$$\mathcal{E}' \in \operatorname{Coh}(Y \times \overline{M}^h_{X/\mathbb{C}}, Y \times \mathbb{P}(V^\vee)/PGL(N)).$$

§3. Twisted sheaves on a projective $K3$ surface

3.1. Basic properties

Let X be a projective $K3$ surface and $p : Y \to X$ a projective bundle.

Lemma 3.1. *For a locally free Y-sheaf E of rank r,*

$$c_2(\mathbf{R}p_*(E^\vee \otimes E)) \equiv -(r-1)(w(E)^2) \mod 2r.$$

Proof. First we note that $(r-1)(D^2) \mod 2r$ is well-defined for $D \in H^2(Z, \mu_r)$, $Z = X, Y$. We take a representative $\alpha \in H^2(X, \mathbb{Z})$ of $w(E)$. Then $c_1(E) \equiv p^*(\alpha) \mod r$. Hence $c_2(p^*(\mathbf{R}p_*(E^\vee \otimes E))) = 2rc_2(E) - (r-1)(c_1(E)^2) \equiv -(r-1)(p^*(\alpha^2)) \mod 2r$. Since $H^4(X, \mathbb{Z})$ is a direct summand of $H^4(Y, \mathbb{Z})$,

$$c_2(\mathbf{R}p_*(E^\vee \otimes E)) \equiv -(r-1)(\alpha^2) \mod 2r.$$

Q.E.D.

Let $K(X, Y)$ be the Grothendieck group of Y-sheaves.

Lemma 3.2. (1) *There is a locally free Y-sheaf E_0 such that*

$$\operatorname{rk} E_0 = \min\{\operatorname{rk} E > 0 | E \in \operatorname{Coh}(X, Y)\}.$$

(2) *$K(X, Y) = \mathbb{Z}E_0 \oplus K(X, Y)_{\leq 1}$, where $K(X, Y)_{\leq 1}$ is the submodule of $K(X, Y)$ generated by $E \in \operatorname{Coh}(X, Y)$ of $\dim E \leq 1$.*

Proof. (1) Let F be a Y-sheaf such that $\operatorname{rk} F = \min\{\operatorname{rk} E > 0 | E \in \operatorname{Coh}(X, Y)\}$. Then $E_0 := F^{\vee\vee}$ satisfies the required properties. (2) We shall show that the image of $E \in \operatorname{Coh}(X, Y)$ in $K(X, Y)$ belongs to $\mathbb{Z}E_0 \oplus K(X, Y)_{\leq 1}$ by the induction of $\operatorname{rk} E$. We may assume that $\operatorname{rk} E > 0$. Let T be the torsion submodule of E. Then $E = T + E/T$ in $K(X, Y)$. Since $\operatorname{Hom}(E_0(-nH), E/T) \neq 0$ for $n \gg 0$, we have a non-zero homomorphism $\varphi : E_0(-nH) \to E/T$. By our choice of E_0, φ is injective. Since $E_0(-nH) = E_0 - E_{0|nH}$ in $K(X, Y)$, $E = ((E/T)/E_0 + E_0) + (T - E_{0|nH})$. Since $\operatorname{rk}(E/T)/E_0 < \operatorname{rk} E$, we get $(E/T)/E_0 \in \mathbb{Z}E_0 \oplus K(X, Y)_{\leq 1}$, and hence E also belongs to $\mathbb{Z}E_0 \oplus K(X, Y)_{\leq 1}$. Q.E.D.

Remark 3.1. $\operatorname{rk} E_0$ is the order of the Brauer class of Y.

Let $\langle\ ,\ \rangle$ be the Mukai pairing on $H^*(X, \mathbb{Z})$:

$$\langle x, y \rangle = -\int_X x^\vee y, \quad x, y \in H^*(X, \mathbb{Z}).$$

Definition 3.1. Let G be a locally free Y-sheaf. For a Y-sheaf E, we define a Mukai vector of E as

$$
\begin{aligned}
v_G(E) :=& \frac{\operatorname{ch}(\mathbf{R}p_*(E \otimes G^\vee))}{\sqrt{\operatorname{ch}(\mathbf{R}p_*(G \otimes G^\vee))}} \sqrt{\operatorname{td}_X} \\
=&(\operatorname{rk}(E), \zeta, b) \in H^*(X, \mathbb{Q}),
\end{aligned} \tag{3.1}
$$

where $p^*(\zeta) = c_1(E) - \operatorname{rk}(E)\frac{c_1(G)}{\operatorname{rk} G}$ and $b \in \mathbb{Q}$. More generally, for $G \in \operatorname{Coh}(X, Y)$ with $\operatorname{rk} G > 0$, we define $v_G(E)$ by (3.1).

Since

$$
\mathbf{R}p_*(E_1 \otimes G^\vee) \otimes \mathbf{R}p_*(E_2 \otimes G^\vee)^\vee = \mathbf{R}p_*(E_1 \otimes E_2^\vee) \otimes \mathbf{R}p_*(G \otimes G^\vee),
$$

$$
\begin{aligned}
\langle v_G(E_1), v_G(E_2) \rangle =& -\int_X \frac{\operatorname{ch}(\mathbf{R}p_*(E_1 \otimes G^\vee)) \operatorname{ch}(\mathbf{R}p_*(E_2 \otimes G^\vee))^\vee}{\operatorname{ch}(\mathbf{R}p_*(G \otimes G^\vee))} \operatorname{td}_X \\
=& -\int_X \operatorname{ch}(\mathbf{R}p_*(E_1 \otimes E_2^\vee)) \operatorname{td}_X \\
=& -\chi(E_2, E_1).
\end{aligned}
$$

We define an integral structure on $H^*(X, \mathbb{Q})$ such that $v_G(E)$ is integral. This is due to Huybrechts and Stellari [H-St]. For a positive integer r and $\xi \in H^2(X, \mathbb{Z})$, we consider an injective homomorphism

$$
\begin{array}{cccc}
T_{-\xi/r}: & H^*(X, \mathbb{Z}) & \to & H^*(X, \mathbb{Q}) \\
& x & \mapsto & e^{-\xi/r} x.
\end{array}
$$

$T_{-\xi/r}$ preserves the bilinear form $\langle\ ,\ \rangle$.

Lemma 3.3. *We take a representative* $\xi \in H^2(X, \mathbb{Z})$ *of* $w(G) \in H^2(X, \mu_r)$, *where* $\operatorname{rk}(G) = r$. *We set* $(\operatorname{rk}(E), D, a) := e^{\xi/r} v_G(E)$. *Then* $(\operatorname{rk}(E), D, a)$ *belongs to* $H^*(X, \mathbb{Z})$ *and* $[D \mod \operatorname{rk}(E)] = w(E)$.

Proof. We set $\sigma := (c_1(G) - p^*(\xi))/r \in H^2(Y, \mathbb{Z})$. Since $p^*(D) = p^*(\zeta) + \operatorname{rk}(E) p^*(\xi)/\operatorname{rk}(G) = c_1(E) - \operatorname{rk}(E)\sigma \in H^2(Y, \mathbb{Z})$, we get $D \in H^2(X, \mathbb{Z})$. By Lemma 3.1, we see that

$$
\begin{aligned}
\langle e^{\xi/r} v_G(E), e^{\xi/r} v_G(E) \rangle =& \langle v_G(E), v_G(E) \rangle \\
=& c_2(\mathbf{R}p_*(E \otimes E^\vee)) - 2\operatorname{rk}(E)^2 \\
\equiv& (D^2) \mod 2\operatorname{rk}(E).
\end{aligned}
$$

Hence $a \in \mathbb{Z}$. The last claim is obvious. Q.E.D.

Remark 3.2. $e^{\xi/r}v_G(E)$ is the same as the Mukai vector defined by the rational B-field ξ/r in [H-St]. More precisely, there is a topological line bundle L on Y with $c_1(L)=\sigma$ and $E\otimes L^{-1}$ is the pull-back of a topological sheaf $E_{\xi/r}$ on X. Then we see that $e^{\xi/r}v_G(E)=\mathrm{ch}(E_{\xi/r})\sqrt{\mathrm{td}_X}$ (we use $H^i(X,\mathbb{Q})=0$ for $i>4$, or we deform X so that L becomes holomorphic).

Definition 3.2. [H-St] We define a weight 2 Hodge structure on the lattice $(H^*(X,\mathbb{Z}),\langle\ ,\ \rangle)$ as

$$\begin{aligned} H^{2,0}(H^*(X,\mathbb{Z})\otimes\mathbb{C}) :=& T_{-\xi/r}^{-1}(H^{2,0}(X))\\ H^{1,1}(H^*(X,\mathbb{Z})\otimes\mathbb{C}) :=& T_{-\xi/r}^{-1}(\bigoplus_{p=0}^{2} H^{p,p}(X))\\ H^{0,2}(H^*(X,\mathbb{Z})\otimes\mathbb{C}) :=& T_{-\xi/r}^{-1}(H^{0,2}(X)). \end{aligned}$$

We denote this polarized Hodge structure by $(H^*(X,\mathbb{Z}),\langle\ ,\ \rangle,-\frac{\xi}{r})$.

Lemma 3.4. *The Hodge structure $(H^*(X,\mathbb{Z}),\langle\ ,\ \rangle,-\frac{\xi}{r})$ depends only on the Brauer class $\delta'([\xi \mod r])$.*

Proof. If $\delta'([\xi \mod r])=\delta'([\xi' \mod r'])\in H^2(X,\mathcal{O}_X^{\times})$, then we have $r'\xi-r\xi'=L+rr'N$, where $L\in\mathrm{NS}(X)$ and $N\in H^2(X,\mathbb{Z})$. Then we have the following commutative diagram:

$$\begin{tikzcd} H^*(X,\mathbb{Z}) \arrow[r, "e^{-\frac{\xi}{r}}"] \arrow[d, "e^{-N}"'] & H^*(X,\mathbb{Q}) \arrow[d, "e^{\frac{L}{rr'}}"] \\ H^*(X,\mathbb{Z}) \arrow[r, "e^{-\frac{\xi'}{r'}}"'] & H^*(X,\mathbb{Q}). \end{tikzcd}$$

Thus we have an isometry of Hodge structures

$$(H^*(X,\mathbb{Z}),\langle\ ,\ \rangle,-\frac{\xi}{r})\cong(H^*(X,\mathbb{Z}),\langle\ ,\ \rangle,-\frac{\xi'}{r'}).$$

Q.E.D.

Definition 3.3. Let $Y\to X$ be a projective bundle and G a locally free Y-sheaf. Let $\xi\in H^2(X,\mathbb{Z})$ be a lifting of $w(G)\in H^2(X,\mu_r)$, where $r=\mathrm{rk}(G)$.

(i) We define an integral Hodge structure of $H^*(X,\mathbb{Q})$ as

$$T_{-\xi/r}((H^*(X,\mathbb{Z}),\langle\ ,\ \rangle,-\frac{\xi}{r})).$$

(ii) $v := (r, \zeta, b)$ is a Mukai vector, if $v \in T_{-\xi/r}(H^*(X,\mathbb{Z}))$ and $\zeta \in \operatorname{Pic}(X) \otimes \mathbb{Q}$. Moreover if v is primitive in $T_{-\xi/r}(H^*(X,\mathbb{Z}))$, then v is primitive.

Definition 3.4. Let $v := (r, \zeta, b) \in H^*(X, \mathbb{Q})$ be a Mukai vector.

(i) $\overline{M}_H^{Y,G}(r,\zeta,b)$ (resp. $M_H^{Y,G}(r,\zeta,b)$) denotes the coarse moduli space of S-equivalence classes of G-twisted semi-stable (resp. stable) Y-sheaves E with $v_G(E) = v$.

(ii) $\mathcal{M}_H^{Y,G}(r,\zeta,b)^{ss}$ (resp. $\mathcal{M}_H^{Y,G}(r,\zeta,b)^{s}$) denotes the moduli stack of G-twisted semi-stable (resp. stable) Y-sheaves E with $v_G(E) = v$.

Lemma 3.5. *Assume that* $o(w(Y)) = o(w(Y'))$. *Then* $\Xi^L_{Y \to Y'}$ *induces an isomorphism*

$$\mathcal{M}_H^{Y,G}(v)^{ss} \cong \mathcal{M}_H^{Y',G'}(v)^{ss},$$

where $G' := \Xi^L_{Y\to Y'}(G)$. *Moreover if* $\dim Y = \dim Y'$ *and* $w(Y) = w(Y')$, *then* $\mathcal{M}_H^{Y,\epsilon(Y)}(v)^{ss} \cong \mathcal{M}_H^{Y',\epsilon(Y')}(v)^{ss}$.

Proof. We use the notation in Lemma 1.7. For a Y-sheaf E, we set $E' := \Xi^L_{Y\to Y'}(E)$. Then ${p'_Y}^*(E \otimes G^\vee) \cong p^*_{Y'}(E' \otimes {G'}^\vee)$. Hence $v_G(E) = v_{G'}(E')$. If $\dim Y = \dim Y'$ and $w(Y) = w(Y')$, then since $w(\epsilon(Y)) = w(\epsilon(Y'))$, replacing L by $L \otimes q^*(P)$, $P \in \operatorname{Pic}(X)$, we may assume that $c_1(\Xi^L_{Y\to Y'}(\epsilon(Y))) = c_1(\epsilon(Y))$. Thus $\Xi^L_{Y\to Y'}(\epsilon(Y)) = \epsilon(Y) + T$ in $K(X, Y')$, where T is a Y-sheaf with $\dim T = 0$. From this fact, we get $\mathcal{M}_H^{Y',\Xi^L_{Y\to Y'}(\epsilon(Y))}(v)^{ss} = \mathcal{M}_H^{Y',\epsilon(Y')}(v)^{ss}$. Q.E.D.

Let E be a Y-sheaf. Then the Zariski tangent space of the Kuranishi space is $\operatorname{Ext}^1(E,E)$ and the obstruction space is the kernel $\operatorname{Ext}^2(E,E)_0$ of the trace map

$$\operatorname{tr} : \operatorname{Ext}^2(E,E) \to H^2(Y, \mathcal{O}_Y) \cong H^2(X, \mathcal{O}_X).$$

Hence as in the usual sheaves on a $K3$ surfaces [Mu1], we get the following.

Proposition 3.6. *Let E be a simple Y-sheaf. Then the Kuranishi space is smooth of dimension* $\langle v_G(E)^2 \rangle + 2$ *with a holomorphic symplectic form. In particular,* $\langle v_G(E)^2 \rangle \geq -2$.

Corollary 3.7. *Let E be a μ-semi-stable Y-sheaf such that* $E = lE_0 + F \in K(X,Y)$, $F \in K(X,Y)_{\leq 1}$. *Then* $\langle v_G(E)^2 \rangle \geq -2l^2$.

3.1.1. *Wall and Chamber* In this subsection, we generalize the notion of the wall and the chamber for the usual stable sheaves to the twisted case.

Lemma 3.8. *Assume that there is an exact sequence of twisted sheaves*

$$0 \to E_1 \to E \to E_2 \to 0, \tag{3.2}$$

such that E_i, $i = 1, 2$ are μ-semi-stable Y-sheaves. We set $E_i = l_i E_0 + F_i \in K(X, Y)$ with $F_i \in K(X, Y)_{\leq 1}$. Then we have

$$\frac{\langle v_G(E)^2 \rangle}{l} + 2l \geq -\frac{(l_2 v_G(F_1) - l_1 v_G(F_2))^2}{l l_1 l_2}.$$

This lemma easily follows from Corollary 3.7 and the following lemma.

Lemma 3.9. *Let E_0 be a locally free Y-sheaf such that $\operatorname{rk} E_0 = \min\{\operatorname{rk} E > 0 | E \in \operatorname{Coh}(X, Y)\}$. For an exact sequence of twisted sheaves*

$$0 \to E_1 \to E \to E_2 \to 0, \tag{3.3}$$

we have

$$\frac{\langle v_G(E_1)^2 \rangle}{l_1} + \frac{\langle v_G(E_2)^2 \rangle}{l_2} - \frac{\langle v_G(E)^2 \rangle}{l} = \frac{(l_2 v_G(F_1) - l_1 v_G(F_2))^2}{l l_1 l_2},$$

where $E_i = l_i E_0 + F_i$ and $E = l E_0 + F$ in $K(X, Y)$ with $F_i, F \in K(X, Y)_{\leq 1}$.

Proof.

$$\begin{aligned}
&\frac{\langle v_G(E_1)^2 \rangle}{l_1} + \frac{\langle v_G(E_2)^2 \rangle}{l_2} - \frac{\langle v_G(E)^2 \rangle}{l} \\
=& \left(l_1 \langle v_G(E_0)^2 \rangle + 2\langle v_G(E_0), v_G(F_1) \rangle + \frac{\langle v_G(F_1), v_G(F_1) \rangle}{l_1} \right) \\
&+ \left(l_2 \langle v_G(E_0)^2 \rangle + 2\langle v_G(E_0), v_G(F_2) \rangle + \frac{\langle v_G(F_2), v_G(F_2) \rangle}{l_2} \right) \\
&- \left(l \langle v_G(E_0)^2 \rangle + 2\langle v_G(E_0), v_G(F) \rangle + \frac{\langle v_G(F), v_G(F) \rangle}{l} \right) \\
=& \frac{\langle v_G(F_1), v_G(F_1) \rangle}{l_1} + \frac{\langle v_G(F_2), v_G(F_2) \rangle}{l_2} - \frac{\langle v_G(F), v_G(F) \rangle}{l} \\
=& \frac{(l_2 v_G(F_1) - l_1 v_G(F_2))^2}{l l_1 l_2}.
\end{aligned}$$

Q.E.D.

Definition 3.5. We set $v = v_G(lE_0 + F)$, where F is of dimension 1 or 0.

(i) For a $\xi \in \mathrm{NS}(X)$ with $0 < -(\xi^2) \leq \frac{l^2}{4}(2l^2 + \langle v^2 \rangle)$, we define a wall W_ξ as

$$W_\xi := \{L \in \mathrm{Amp}(X) \otimes \mathbb{R} | (\xi, L) = 0\}.$$

(ii) A chamber with respect to v is a connected component of $\mathrm{Amp}(X) \otimes \mathbb{R} \setminus \bigcup_\xi W_\xi$.

(iii) A polarization H is general with respect to v, if H does not lie on any wall.

Remark 3.3. The concept of chambers and walls are determined by $\mathrm{rk}(lE_0 + F)$ and $\langle v^2 \rangle$. Thus they do not depend on the choice of Y and G.

Proposition 3.10. *Keep notation as above.*

(i) *If H and H' belong to the same chamber, then $\mathcal{M}_H^{Y,G}(v)^{ss} \cong \mathcal{M}_{H'}^{Y,G}(v)^{ss}$.*

(ii) *If H is general, then $\mathcal{M}_H^{Y,G}(v_G(F))^{ss} \cong \mathcal{M}_H^{Y,G'}(v_{G'}(F))^{ss}$ for $F \in K(X,Y)$ with $\mathrm{rk}\, F > 0$. Thus $\mathcal{M}_H^{Y,G}(v_G(F))^{ss}$ does not depend on the choice of a Y-sheaf G.*

(iii) *If*

$$\min\{-(D^2) > 0 | D \in \mathrm{NS}(X), (D,H) = 0\} > \frac{l^2}{4}(2l^2 + \langle v^2 \rangle),$$

then H is general with respect to v.

The proof is standard (cf. [H-L]) and is left to the reader. By Proposition 3.10 and Proposition 3.6, we have

Theorem 3.11. *Assume that v is a primitive Mukai vector and H is general with respect to v. Then all G-twisted semi-stable Y-sheaves E with $v_G(E) = v$ are G-twisted stable. In particular $M_H^{Y,G}(v)$ is a projective manifold, if it is not empty.*

In the next subsection, we show the non-emptyness of the moduli space. We also show that $M_H^{Y,G}(v)$ is a $K3$ surface, if $\langle v^2 \rangle = 0$.

Proposition 3.12. *(cf.* [Mu3, Prop. 3.14]*) Assume that* $\mathrm{Pic}(X) = \mathbb{Z}H$. *Let E be a simple twisted sheaf with $\langle v_G(E)^2 \rangle \leq 0$. Then E is stable.*

For the proof, we use Lemma 3.9 and the following:

Lemma 3.13. [Mu3, Cor. 2.8] *If* $\mathrm{Hom}(E_1, E_2) = 0$, *then*

$$\dim \mathrm{Ext}^1(E_1, E_1) + \dim \mathrm{Ext}^1(E_2, E_2) \leq \dim \mathrm{Ext}^1(E, E).$$

3.2. Existence of stable sheaves

In this subsection, we shall show that the moduli space of twisted sheaves is deformation equivalent to the usual one. In particular we show the non-emptyness of the moduli space.

Theorem 3.14. [H-Sc] *$H^1(X, PGL(r)) \to H^2(X, \mu_r)$ is surjective.*

Proposition 3.15. *For a $w \in H^2(X, \mu_r)$, there is a $\mathbb{P}^{r-1}$-bundle $p : Z \to X$ such that $w(Z) = w$ and $\epsilon(Z)$ is μ-stable.*

D. Huybrechts informed us that the claim follows from the proof of Theorem 3.14. Here we give another proof which works for other surfaces.

Proof. Let $p : Y \to X$ be a $\mathbb{P}^{r-1}$-bundle with $w(Y) = w$. We set $E_0 := \epsilon(Y)$. In order to prove our claim, it is sufficient to find a μ-stable locally free Y-sheaf E of rank r with $c_1(E) = c_1(E_0)$. For points $x_1, x_2, \dots, x_n \in X$, let F be a Y-sheaf which is the kernel of a surjection $E_0 \to \bigoplus_{i=1}^n \mathcal{O}_{p^{-1}(x_i)}(1)$. We take a smooth divisor $D \in |mH|$, $m \gg 0$. We set $\widetilde{D} := p^{-1}(D)$. Let $\operatorname{Ext}^i(F, F(-\widetilde{D}))_0$ be the kernel of the trace map

$$\operatorname{Ext}^i(F, F(-\widetilde{D})) \to H^i(Y, \mathcal{O}_Y(-\widetilde{D})) \cong H^i(X, \mathcal{O}_X(-D)).$$

If $n \gg 0$, then by the Serre duality,

$$\operatorname{Ext}^2(F, F(-\widetilde{D}))_0 \cong \operatorname{Hom}(F, F(\widetilde{D}))_0 = 0.$$

Hence $\operatorname{Ext}^1(F, F)_0 \to \operatorname{Ext}^1(F_{|\widetilde{D}}, F_{|\widetilde{D}})_0$ is surjective. Since $F_{|\widetilde{D}}$ deforms to a μ-stable vector bundle on $\widetilde{D}$, F deforms to a Y-sheaf F' such that $F'_{|\widetilde{D}}$ is μ-stable. Then F' is also μ-stable. Then $E := (F')^{\vee\vee}$ satisfies required properties. Q.E.D.

Theorem 3.16. *Let $Y \to X$ be a projective bundle and G a locally free Y-sheaf. Let $v_G := (r, \zeta, b)$ be a primitive Mukai vector with $r > 0$. Then $M_H^{Y,G}(v_G)$ is an irreducible symplectic manifold which is deformation equivalent to $\operatorname{Hilb}_X^{\langle v_G^2 \rangle/2+1}$ for a general polarization H. In particular*

(1) *$M_H^{Y,G}(v_G) \neq \emptyset$ if and only if $\langle v_G^2 \rangle \geq -2$.*
(2) *If $\langle v_G^2 \rangle = 0$, then $M_H^{Y,G}(v_G)$ is a $K3$ surface.*

We divide the proof into several steps.

Step 1 (Reduction to $M_H^{Y,\epsilon(Y)}(r, 0, -a)$) : Let ξ be a lifting of $w(G)$. Then $e^{\xi/\operatorname{rk}(G)} v_G = (r, D, b') \in H^*(X, \mathbb{Z})$. By Theorem 3.14, there is a projective bundle $Y' \to X$ such that $w(Y') = [D \bmod r]$. Since $D/r - \xi/\operatorname{rk}(G) = \zeta/r \in \operatorname{Pic}(X) \otimes \mathbb{Q}$, $o(w(Y')) = o(w(Y))$. Let G' be a locally free Y-sheaf such that $\Xi^L_{Y \to Y'}(G') = \epsilon(Y')$, where we use the notation in Lemma 1.7. By Lemma 1.8, $w(G') = w(\epsilon(Y')) = [D \bmod r]$. Then replacing L by $L \otimes q^*(P)$, $P \in \operatorname{Pic}(X)$, we may assume that $e^{\xi/\operatorname{rk} G} v_G(G') = (r, D, c)$, $c \in \mathbb{Z}$. Hence $v_{G'}(E) = (r, 0, -a)$ for a Y-sheaf E with $v_G(E) = (r, \zeta, b)$. Since H is general with respect to (r, ζ, b), Proposition 3.10 implies that $M_H^{Y,G}(r, \zeta, b) \cong M_H^{Y,G'}(r, 0, -a)$. By Lemma 3.5, $M_H^{Y,G'}(r, 0, -a) \cong M_H^{Y',\epsilon(Y')}(r, 0, -a)$. Therefore replacing (Y, G) by $(Y', \epsilon(Y'))$, we shall prove the assertion for $M_H^{Y,G}(r, 0, -a)$ with $G = \epsilon(Y)$.

Step 2: First we assume that $w(Y) \in \operatorname{NS}(X) \otimes \mu_r \subset H^2(X, \mu_r)$. Then the Brauer class of Y is trivial, that is, $Y = \mathbb{P}(F)$ for a locally free sheaf F on X. Since H is general with respect to $(r, 0, -a)$, Proposition 3.10 (ii) and Lemma 3.5 imply that $M_H^{Y,G}(r, 0, -a) \cong M_H^{X,\mathcal{O}_X}(r, D, c)$ with $2ra = (D^2) - 2rc$. By [Y1, Thm. 8.1], $M_H^{X,\mathcal{O}_X}(r, D, c)$ is deformation equivalent to $\operatorname{Hilb}_X^{ra+1}$.

We next treat the general cases. We shall deform the projective bundle $Y \to X$ to a projective bundle in Step 2.

Step 3: We first construct a local family of projective bundles.

Proposition 3.17. *Let $f : (\mathcal{X}, \mathcal{H}) \to T$ be a family of polarized K3 surfaces. Let $p : Y \to \mathcal{X}_{t_0}$ be a projective bundle associated to a stable Y-sheaf E. Then there is a smooth morphism $U \to T$ whose image contains t_0 and a projective bundle $p : \mathcal{Y} \to \mathcal{X} \times_T U$ such that $\mathcal{Y}_{t_0} \cong Y$.*

Proof. We note that $p_*(K_{Y/\mathcal{X}_{t_0}}^\vee)$ is a vector bundle on $\mathcal{X}_{t_0}$ and we have an embedding $Y \hookrightarrow \mathbb{P}(p_*(K_{Y/\mathcal{X}_{t_0}}^\vee))$. We take an embedding $\mathbb{P}(p_*(K_{Y/\mathcal{X}_{t_0}}^\vee)) \hookrightarrow \mathbb{P}^{N-1} \times \mathcal{X}_{t_0}$ by a suitable quotient $\mathcal{O}_{\mathcal{X}_{t_0}}(-n\mathcal{H}_{t_0})^{\oplus N} \to p_*(K_{Y/\mathcal{X}_{t_0}}^\vee)$. More generally, let $\mathcal{Y}_S \to \mathcal{X} \times_T S$ be a projective bundle and a surjective homomorphism $\mathcal{O}_{\mathcal{X} \times_T S}(-n\mathcal{H})^{\oplus N} \to p_*(K_{\mathcal{Y}_S/\mathcal{X} \times_T S}^\vee)$. Then we have an embedding $\mathcal{Y}_S \hookrightarrow \mathbb{P}^{N-1} \times \mathcal{X} \times_T S$.

Let $\mathfrak{Y}$ be a connected component of the Hilbert scheme $\operatorname{Hilb}_{\mathbb{P}^{N-1} \times \mathcal{X}/T}$ containing Y. Let $\mathcal{Y} \subset \mathbb{P}^{N-1} \times \mathcal{X} \times_T \mathfrak{Y}$ be the universal subscheme. Let $\varphi : \mathcal{Y} \to \mathcal{X} \times_T \mathfrak{Y}$ be the projection. Let $\mathfrak{Y}^0$ be an open subscheme of $\mathfrak{Y}$ such that $\varphi_{|\mathcal{X} \times_T \{t\}}$ is smooth and $H^1(T_{\varphi^{-1}(x,t)}) = 0$ for $(x, t) \in \mathcal{X} \times_T \mathfrak{Y}^0$. Since $Y \in \mathfrak{Y}^0$, it is non-empty. Then φ is locally trivial on $\mathcal{X} \times_T \mathfrak{Y}^0$. Thus $\mathcal{Y} \to \mathcal{X} \times_T \mathfrak{Y}^0$ is a projective bundle.

If Y is a projective bundle associated to a twisted vector bundle E, then the obstruction for the infinitesimal liftings belongs to

$$H^2(\mathcal{E}nd(E)/\mathcal{O}_X) \cong H^0(\mathcal{E}nd(E)_0)^\vee,$$

where $\mathcal{E}nd(E)_0$ is the trace free part of $\mathcal{E}nd(E)$. Hence if E is simple (and $\operatorname{rk} E$ is not divisible by the characteristic), then there is no obstruction for the infinitesimal liftings. In particular $\mathfrak{Y}^0 \to T$ is smooth at Y. Q.E.D.

Step 4 (A relative moduli space of twisted sheaves): Let $f : (\mathcal{X}, \mathcal{H}) \to T$ be a family of polarized $K3$ surfaces and $p : \mathcal{Y} \to \mathcal{X}$ a projective bundle on $\mathcal{X}$. We set $g := f \circ p$. We note that $H^i(\mathcal{Y}_t, \Omega_{\mathcal{Y}_t/\mathcal{X}_t}) = 0$, $i \neq 1$ and $H^1(\mathcal{Y}_t, \Omega_{\mathcal{Y}_t/\mathcal{X}_t}) = \mathbb{C}$ for $t \in T$. Hence $L := \operatorname{Ext}^1_g(T_{\mathcal{Y}/\mathcal{X}}, \mathcal{O}_\mathcal{Y}) \cong R^1 g_*(\Omega_{\mathcal{Y}/\mathcal{X}})$ is a line bundle on T. By the local-global spectral sequence, we have an isomorphism

$$\operatorname{Ext}^1(T_{\mathcal{Y}/\mathcal{X}}, g^*(L^\vee)) \cong H^0(T, \operatorname{Ext}^1_g(T_{\mathcal{Y}/\mathcal{X}}, g^*(L^\vee))) \cong H^0(T, \mathcal{O}_T).$$

We take the extension corresponding to $1 \in H^0(T, \mathcal{O}_T)$:

$$0 \to g^*(L^\vee) \to \mathcal{G} \to T_{\mathcal{Y}/\mathcal{X}} \to 0$$

such that $\mathcal{G}_t = \epsilon(\mathcal{Y}_t)$. Let $v := (r, \zeta, b) \in R^* f_* \mathbb{Q}$ be a family of Mukai vectors with $\zeta \in \operatorname{NS}(\mathcal{X}/T) \otimes \mathbb{Q}$. Then as in the absolute case, we have a family of the moduli spaces of semi-stable twisted sheaves $\overline{M}^{\mathcal{Y},\mathcal{G}}_{(\mathcal{X},\mathcal{H})/T}(v) \to T$ parametrizing $\mathcal{G}_t$-twisted semi-stable $\mathcal{Y}_t$-sheaves E on $\mathcal{X}_t$, $t \in T$ with $v_{\mathcal{G}_t}(E) = v_t$. $\overline{M}^{\mathcal{Y},\mathcal{G}}_{(\mathcal{X},\mathcal{H})/T}(v) \to T$ is a projective morphism. Let E be a $\mathcal{G}_t$-twisted stable $\mathcal{Y}_t$-sheaf. By our choice of ζ, $\det(E)$ is unobstructed under deformations over T, and hence E itself is unobstructed. Therefore $M^{\mathcal{Y},\mathcal{G}}_{(\mathcal{X},\mathcal{H})/T}(v)$ is smooth over T.

Step 5 (A family of $K3$ surfaces): Let $\mathcal{M}_d$ be the moduli space of the polarized $K3$ surfaces (X, H) with $(H^2) = 2d$. $\mathcal{M}_d$ is constructed as a quotient of an open subscheme T of a suitable Hilbert scheme $\operatorname{Hilb}_{\mathbb{P}^N/\mathbb{C}}$. Let $(\mathcal{X}, \mathcal{H}) \to T$ be the universal family. Let Γ be the abstruct $K3$ lattice and h a primitive vector with $(h^2) = 2d$. Let $\mathcal{D}$ be the period domain for polarized $K3$ surfaces (X, H). Let $\tau : \widetilde{T} \to T$ be the universal covering and $\phi_{\tilde{t}} : H^2(\mathcal{X}_{\tau(\tilde{t})}, \mathbb{Z}) \to \Gamma$, $\tilde{t} \in \widetilde{T}$ a trivialization on $\widetilde{T}$. We may assume that $\phi_{\tilde{t}}(\mathcal{H}_{\tau(\tilde{t})}) = h$. Then we have a period map $\mathfrak{p} : \widetilde{T} \to \mathcal{D}$. By the surjectivity of the period map, we can show that $\mathfrak{p}$ is surjective: Let U be a suitable analytic neighborhood of a point $x \in \mathcal{D}$. Then we have a family of polarized $K3$ surfaces $(\mathcal{X}_U, \mathcal{H}_U) \to U$ and an embedding of $\mathcal{X}$ as a subscheme of $\mathbb{P}^N \times U$. Thus we have a morphism $h : U \to T$. The embedding is unique up to the action of $PGL(N+1)$. Moreover if there is a point $\tilde{t}_0 \in \widetilde{T}$ such that $\mathfrak{p}(\tilde{t}_0) \in U$, then we have a lifting $\widetilde{h} : U \to \widetilde{T}$ of $h : U \to T$ such that $\tilde{t}_0 = \widetilde{h}(\mathfrak{p}(\tilde{t}_0))$. Then $U \to \widetilde{T} \to \mathcal{D}$ is the identity. Hence we can construct a lifting of any path on $\mathcal{D}$ intersecting $\mathfrak{p}(\widetilde{T})$. Since $\mathcal{D}$ is connected, we get the assertion.

Step 6 (Reduction to step 2): We take a point $\widetilde{t} \in \widetilde{T}$. We set $(X, H) := (\mathcal{X}_{\tau(\widetilde{t})}, \mathcal{H}_{\tau(\widetilde{t})})$. Let $p : Y \to X$ be a $\mathbb{P}^{r-1}$-bundle. Assume that H is general with respect to $v := (r, 0, -a)$. We take a $D \in \Gamma$ with $[D \mod r] = \overline{\phi}_{\widetilde{t}}(w(Y))$. Let $e_1, e_2, \dots, e_{22}$ be a $\mathbb{Z}$-basis of Γ such that $e_1 = \phi_{\widetilde{t}}(\mathcal{H}_{\tau(\widetilde{t})})$ and $D = ae_1 + be_2$. For an $\eta \in \oplus_{i=3}^{22} \mathbb{Z} e_i$ with $(e_1^2)(\eta^2) - (e_1, \eta)^2 < 0$, we set $\widetilde{\eta} := e_2 + rk\eta \in \Gamma$, $k \gg 0$. Since $\det \begin{pmatrix} (e_1^2) & (e_1, e_2 + rk\eta) \\ (e_1, e_2 + rk\eta) & ((e_2 + rk\eta)^2) \end{pmatrix} \ll 0$ for $k \gg 0$, the signature of the primitive sublattice $L := \mathbb{Z} e_1 \oplus \mathbb{Z}\widetilde{\eta}$ of Γ is of type $(1,1)$. Moreover $e_1^\perp \cap L$ does not contain a (-2)-vector. We take a general $\omega \in L^\perp \cap \Gamma \otimes \mathbb{C}$ with $(\omega, \omega) = 0$ and $(\omega, \bar{\omega}) > 0$. Then $\omega^\perp \cap \Gamma = L$. Replacing ω by its complex conjugate if necessary, we may assume that $\omega \in \mathcal{D}$. Since $\mathfrak{p}$ is surjective, there is a point $\tilde{t}_1 \in \mathfrak{H}$ such that $\mathfrak{p}(\tilde{t}_1) = \omega$. Then $\mathcal{X}_{\tau(\widetilde{t}_1)}$ is a $K3$ surface with $\operatorname{Pic}(\mathcal{X}_{\tau(\widetilde{t}_1)}) = \mathbb{Z}\mathcal{H}_{\tau(\widetilde{t}_1)} \oplus \mathbb{Z}\phi_{\widetilde{t}_1}^{-1}(e_2 + rk\eta)$. Hence $[\phi_{\widetilde{t}_1}^{-1}(D) \mod r] = [\phi_{\widetilde{t}_1}^{-1}(ae_1 + b\widetilde{\eta}) \mod r] \in \operatorname{Pic}(\mathcal{X}_{\tau(\widetilde{t}_1)}) \otimes \mu_r$. Since

$$\min\{-(L^2) | 0 \neq L \in \operatorname{Pic}(\mathcal{X}_{\tau(\widetilde{t}_1)}), (L, \mathcal{H}_{\tau(\widetilde{t}_1)}) = 0\} \gg \frac{r^2}{4}(2r^2 + \langle v^2 \rangle),$$

Proposition 3.10 (iii) implies that $\mathcal{H}_{\tau(\widetilde{t}_1)}$ is a general polarization with respect to v. Then by the following lemma, we can reduce the proof to Step 2. Therefore we complete the proof of Theorem 3.16.

Lemma 3.18. *For $\widetilde{t}_1, \widetilde{t}_2 \in \widetilde{T}$, let $Y^i \to \mathcal{X}_{\tau(\tilde{t}_i)}$, $i = 1, 2$ be $\mathbb{P}^{r-1}$-bundles with $w(Y^i) = [\phi_{\widetilde{t}_i}^{-1}(D) \mod r]$ and $G_i := \epsilon(Y^i)$. Let $v = (r, 0, -a)$ be a primitive Mukai vector. Assume that $\mathcal{H}_{\tau(\tilde{t}_i)}$, $i = 1, 2$ are general polarization. Then $M_{\mathcal{H}_{\tau(\tilde{t}_1)}}^{Y^1, G_1}(r, 0, -a)$ is deformation equivalent to $M_{\mathcal{H}_{\tau(\tilde{t}_2)}}^{Y^2, G_2}(r, 0, -a)$.*

Proof. In order to simplify the notation, we denote $M_{\mathcal{H}_t}^{Y, \epsilon(Y)}(r, 0, -a)$ by $M(Y)$ for a projective bundle Y over $(\mathcal{X}_t, \mathcal{H}_t)$. By Proposition 3.15 and Lemma 3.5, we may assume that $\epsilon(Y^i)$ $(i = 1, 2)$ is μ-stable. Let $\widetilde{\gamma} : [0, 1] \to \widetilde{T}$ be a path from $\tilde{t}_1 = \widetilde{\gamma}(0)$ to $\tilde{t}_2 = \widetilde{\gamma}(1)$ and $\gamma := \tau \circ \widetilde{\gamma}$. Then we have a trivialization $\overline{\phi}_s : H^2(\mathcal{X}_{\gamma(s)}, \mu_r) \to \Gamma \otimes_{\mathbb{Z}} \mu_r$. By Proposition 3.15, there is a projective bundle $Y_s \to \mathcal{X}_{\gamma(s)}$ such that $\overline{\phi}_s(w(Y_s)) = [D$

mod r] and $\epsilon(Y_s)$ is μ-stable for each $s \in [0,1]$. By Proposition 3.17, we have a family of projective bundles $\mathcal{Y}^s \to \mathcal{X} \times_T \mathfrak{Y}^s$ over a T-scheme $\psi^s : \mathfrak{Y}^s \to T$ such that there is a point $y^s \in (\psi^s)^{-1}(\gamma(s)) \subset \mathfrak{Y}^s$ with $Y_s = \mathcal{Y}^s_{y^s}$ and ψ^s is smooth at y^s. Then we have a family of moduli spaces $\overline{M}^{\mathcal{Y}^s, \mathcal{G}^s}_{(\mathcal{X} \times_T \mathfrak{Y}^s, \widetilde{\mathcal{H}})/\mathfrak{Y}^s}(r, 0, -a) \to \mathfrak{Y}^s$, where $\widetilde{\mathcal{H}}$ is the pull-back of $\mathcal{H}$ to $\mathcal{X} \times_T \mathfrak{Y}^s$ (Step 4). Since ψ^s is smooth, $\psi^s(\mathfrak{Y}^s)$ is an open subscheme of T containing $\gamma(s)$. We take an analytic open neighborhood U_s of $\gamma(s)$ such that U_s is contractible and has a section $\sigma_s : U_s \to \mathfrak{Y}^s$ with $\sigma_s(\gamma(s)) = y^s$. Let V_s be a connected neighborhood of s which is contained in $\gamma^{-1}(U_s)$. Since $[0,1]$ is compact, we can take a finite open covering of $[0,1]$: $[0,1] = \cup_{j=1}^n V_{s_j}$, $s_1 < s_2 < \cdots < s_n$. Since $\{t \in T \mid \operatorname{rk}\operatorname{Pic}(\mathcal{X}_t) = 1\}$ is a dense subset of T, there is a point $t_j \in U_{s_j} \cap U_{s_{j+1}}$ such that t_j is sufficiently close to a point $\gamma(s_{j,j+1})$, $s_{j,j+1} \in V_{s_j} \cap V_{s_{j+1}}$ and $\operatorname{Pic}(\mathcal{X}_{t_j}) = \mathbb{Z}\mathcal{H}_{t_j}$. Under the identification $H^2(\mathcal{X}_t, \mu_r) \cong H^2(\mathcal{X}_{\gamma(s)}, \mu_r)$ for $t \in U_s$, we have $w(\mathcal{Y}^{s_j}_{\sigma_j(t_j)}) = w(\mathcal{Y}^{s_j}_{y^j})$ and $w(\mathcal{Y}^{s_{j+1}}_{\sigma_{j+1}(t_j)}) = w(\mathcal{Y}^{s_{j+1}}_{y^{j+1}})$, where we set $\sigma_j := \sigma_{s_j}$ and $y^j := y^{s_j}$. Since t_j is sufficiently close to the point $\gamma(s_{j,j+1})$, we have $w(\mathcal{Y}^{s_j}_{\sigma_j(t_j)}) = w(\mathcal{Y}^{s_{j+1}}_{\sigma_{j+1}(t_j)})$. Hence by Lemma 3.5, $M(\mathcal{Y}^{s_j}_{\sigma_j(t_j)})$ is isomorphic to $M(\mathcal{Y}^{s_{j+1}}_{\sigma_{j+1}(t_j)})$. By Step 4, $M(\mathcal{Y}^{s_j}_{\sigma_j(t_{j-1})})$ is deformation equivalent to $M(\mathcal{Y}^{s_j}_{\sigma_j(t_j)})$. Therefore $M(\mathcal{Y}^{s_1}_{\sigma_1(t_1)})$ is deformation equivalent to $M(\mathcal{Y}^{s_n}_{\sigma_n(t_{n-1})})$. By using Step 4 again, we also see that $M(Y^1) = M(\mathcal{Y}^0_{y^0})$ is deformation equivalent to $M(\mathcal{Y}^{s_1}_{\sigma_1(t_1)})$ and $M(Y^2) = M(\mathcal{Y}^1_{y^1})$ is deformation equivalent to $M(\mathcal{Y}^{s_n}_{\sigma_n(t_{n-1})})$. Therefore our claim holds. Q.E.D.

Remark 3.4. Let $v_G := (r, \zeta, b)$ be a Mukai vector with $r, \langle v_G^2 \rangle > 0$ which is not necessary primitive. By the same proof, we can also show that $\overline{M}^{Y,G}_H(v_G)$ is an irreducible normal variety for a general H (cf. [Y2]).

3.3. The second cohomology groups of moduli spaces

By Theorem 3.16, $M^{Y,G}_H(v_G)$ is an irreducible symplectic manifold, if v_G is primitive and H is general. Then $H^2(M^{Y,G}_H(v_G), \mathbb{Z})$ is equipped with a bilinear form called the Beauville form. In this subsection, we shall describe the Beauville form in terms of the Mukai lattice.

Let $p : Y \to X$ be a projective bundle with $w(Y) = [\xi \mod r]$ and set $G := \epsilon(Y)$. We consider a Mukai lattice with a Hodge structure $(H^*(X, \mathbb{Z}), \langle\ ,\ \rangle, -\frac{\xi}{r})$ in this subsection. We set $w := r(1, 0, \frac{a}{r} - \frac{1}{2}\frac{(\xi^2)}{r^2})$, $a \in \mathbb{Z}$. In this subsection, we assume that w is primitive, that is, $\gcd(r, \xi, a) = 1$. We set $v := we^{\xi/r} = (r, \xi, a) \in H^*(X, \mathbb{Z})$. Then v is algebraic.

Let $q : \widetilde{M_H^{Y,G}(w)} \to M_H^{Y,G}(w)$ be a projective bundle in subsection 2.3 and $\mathcal{E}$ the family of twisted sheaves on $Y \times \widetilde{M_H^{Y,G}(w)}$. We set $W^\vee := \epsilon(\widetilde{M_H^{Y,G}(w)})$. Let $\widetilde{\pi}_{\widetilde{M_H^{Y,G}(w)}} : Y \times \widetilde{M_H^{Y,G}(w)} \to \widetilde{M_H^{Y,G}(w)}$ and $\widetilde{\pi}_Y : Y \times \widetilde{M_H^{Y,G}(w)} \to Y$ be projections. Then $(1_Y \times q)_* \left(\mathcal{E} \otimes \widetilde{\pi}^*_{\widetilde{M_H^{Y,G}(w)}}(W^\vee)\right)$ is a quasi-universal family on $Y \times M_H^{Y,G}(w)$.

Let $\pi_X : X \times M_H^{Y,G}(w) \to X$ be the projection. We define a homomorphism $\theta_v^G : v^\perp \to H^*(M_H^{Y,G}(w), \mathbb{Q})$ by

$$\theta_v^G(u) := \int_X [\mathcal{Q}^\vee \pi_X^*(e^{-\xi/r}u)]_3$$

where $[...]_3$ means the degree 6 part and

$$\begin{aligned}\mathcal{Q} :=&\frac{\sqrt{\operatorname{td}_X}}{\sqrt{\operatorname{ch}(\mathbf{R}p_*(G^\vee \otimes G))}} \frac{\sqrt{\operatorname{td}_{M_H^{Y,G}(w)}}}{\sqrt{\operatorname{ch}(\mathbf{R}q_*(W^\vee \otimes W))}}\\ &\cdot \operatorname{ch}\left(\mathbf{R}(p \times q)_* \left(\widetilde{\pi}_Y^*(G^\vee) \otimes \mathcal{E} \otimes \widetilde{\pi}^*_{\widetilde{M_H^{Y,G}(w)}}(W^\vee)\right)\right)\\ &\in H^*(X \times M_H^{Y,G}(w), \mathbb{Q}).\end{aligned}$$

Remark 3.5. If ξ is algebraic, then Y is isomorphic to the projective bundle $\mathbb{P}(F^\vee)$ and $G = F^\vee \otimes \mathcal{O}_Y(1)$, where F is a vector bundle of rank r on X with $c_1(F) = -\xi$. In this case, $M_H^{Y,G}(w)$ is the usual moduli space of stable sheaves F with the Mukai vector v and $\mathbf{R}(p \times q)_*(\widetilde{\pi}_Y^*(\mathcal{O}_Y(-1)) \otimes \mathcal{E} \otimes \widetilde{\pi}^*_{\widetilde{M_H^{Y,G}(w)}}(W^\vee))$ is a quasi-universal family. Since $\operatorname{ch} F/\sqrt{\operatorname{ch}(F \otimes F^\vee)} = e^{-\xi/r}$, we have

$$\begin{aligned}\mathcal{Q} =&e^{-\frac{\xi}{r}} \sqrt{\operatorname{td}_X} \frac{\sqrt{\operatorname{td}_{M_H^{Y,G}(w)}}}{\sqrt{\operatorname{ch}(\mathbf{R}q_*(W^\vee \otimes W))}}\\ &\cdot \operatorname{ch}\left(\mathbf{R}(p \times q)_* \left(\widetilde{\pi}_Y^*(\mathcal{O}_Y(-1)) \otimes \mathcal{E} \otimes \widetilde{\pi}^*_{\widetilde{M_H^{Y,G}(w)}}(W^\vee)\right)\right).\end{aligned}$$

Hence θ_v^G is the usual Mukai homomorphism, which is defined over $\mathbb{Z}$.

Let $p' : Y' \to X$ be another $\mathbb{P}^{r-1}$-bundle with $w(Y') = w(Y)$. Then by the proof of Lemma 3.5, we see that the following diagram is commutative:

$$\begin{array}{ccc} v^{\perp} & = & v^{\perp} \\ {\scriptstyle \theta_v^G}\downarrow & & \downarrow{\scriptstyle \theta_v^{G'}} \\ H^2(M_H^{Y,G}(w),\mathbb{Q}) & \longrightarrow & H^2(M_H^{Y',G'}(w),\mathbb{Q}), \end{array}$$

where $G' := \Xi^L_{Y \to Y'}(G) = \epsilon(Y')$. Since $\mathcal{Q}$ is algebraic, θ_v^G preserves the Hodge structure. By the deformation argument, Remark 3.5 implies that θ_v^G is defined over $\mathbb{Z}$. Moreover it preserves the bilinear forms.

Theorem 3.19. *For $\xi \in H^2(X,\mathbb{Z})$ with $[\xi \mod r] = w(Y)$, we set $v = we^{\xi/r}$.*

(i) *If $\langle v^2 \rangle > 0$, then $\theta_v^G : v^{\perp} \to H^2(M_H^{Y,G}(w),\mathbb{Z})$ is an isometry of the Hodge structures.*

(ii) *If $\langle v^2 \rangle = 0$, then θ_v^G induces an isometry of the Hodge structures $v^{\perp}/\mathbb{Z}v \to H^2(M_H^{Y,G}(w),\mathbb{Z})$.*

The second claim is due to Mukai [Mu4].

§4. Fourier-Mukai transform

4.1. Integral functor

Let $p : Y \to X$ be a projective bundle such that $\delta([Y]) = [\alpha] \in \mathrm{Br}(X)$ and $p' : Y' \to X'$ a projective bundle such that $\delta([Y']) = [\alpha'] \in \mathrm{Br}(X')$. Let $\pi_X : X' \times X \to X$ and $\pi_{X'} : X' \times X \to X'$ be projections. We also let $\widetilde{\pi}_Y : Y' \times Y \to Y$ and $\widetilde{\pi}_{Y'} : Y' \times Y \to Y'$ be projections. We set $G := \epsilon(Y)$ and $G' := \epsilon(Y')$.

Definition 4.1. Let $\mathrm{Coh}(X' \times X, Y', Y)$ be the subcategory of $\mathrm{Coh}(Y' \times Y)$ such that $Q \in \mathrm{Coh}(Y' \times Y)$ belongs to $\mathrm{Coh}(X' \times X, Y', Y)$ if and only if $(p' \times p)^*(p' \times p)_*(G' \otimes Q \otimes G^{\vee}) \cong G' \otimes Q \otimes G^{\vee}$. In terms of local trivialization of p, p', this is equivalent to

$$Q_{|Y_i' \times Y_j} \cong \mathcal{O}_{Y_i'}(-\lambda_i') \boxtimes \mathcal{O}_{Y_j}(\lambda_j) \otimes (p' \times p)^*(Q_{ij}),$$

$Q_{ij} \in \mathrm{Coh}(U_i' \times U_j)$. $\mathrm{Coh}(X' \times X, Y', Y)$ is equivalent to $\mathrm{Coh}(X' \times X, \alpha'^{-1} \times \alpha)$.

Remark 4.1. We take twisted line bundles $\mathcal{L}(p'^*(\alpha'^{-1}))$ on Y' and $\mathcal{L}(p^*(\alpha^{-1}))$ on Y respectively which give equivalences $\Lambda^{\mathcal{L}(p'^*(\alpha'^{-1}))} : \mathrm{Coh}(X', Y') \cong \mathrm{Coh}(X', \alpha')$ and $\Lambda^{\mathcal{L}(p^*(\alpha^{-1}))} : \mathrm{Coh}(X, Y) \cong \mathrm{Coh}(X, \alpha)$ in (1.1). Then we have an equivalence $\Lambda^{\mathcal{L}(p'^*(\alpha'^{-1}))^{\vee}} \times \Lambda^{\mathcal{L}(p^*(\alpha^{-1}))}$:

$$\begin{array}{ccc} \mathrm{Coh}(X' \times X, Y', Y) & \to & \mathrm{Coh}(X' \times X, \alpha'^{-1} \times \alpha) \\ Q & \mapsto & (p' \times p)_*(\mathcal{L}(p'^*(\alpha'^{-1})) \otimes Q \otimes \mathcal{L}(p^*(\alpha^{-1}))^{\vee}). \end{array}$$

Let $\mathbf{D}(X' \times X, Y', Y) \cong \mathbf{D}(X' \times X, {\alpha'}^{-1} \times \alpha)$ be the bounded derived category of $\operatorname{Coh}(X' \times X, Y', Y)$. For $\mathcal{Q} \in \mathbf{D}(X' \times X, Y', Y)$, we define an integral functor

$$\begin{array}{rccc} \Phi_{X' \to X}^{\widetilde{\mathcal{Q}}}: & \mathbf{D}(X', Y') & \to & \mathbf{D}(X, Y) \\ & x & \mapsto & \mathbf{R}\widetilde{\pi}_{Y*}(\mathcal{Q} \otimes \widetilde{\pi}_{Y'}^*(x)). \end{array}$$

For $\mathcal{Q} \in \mathbf{D}(X' \times X, Y', Y)$ and $\mathcal{R} \in \mathbf{D}(X'' \times X', Y'', Y')$, we have

$$\Phi_{X' \to X}^{\mathcal{Q}} \circ \Phi_{X'' \to X'}^{\mathcal{R}} = \Phi_{X'' \to X}^{\mathcal{S}},$$

where $\mathcal{S} = \mathbf{R}\widetilde{\pi}_{Y'' \times Y*}(\widetilde{\pi}_{Y'' \times Y'}^*(\mathcal{R}) \otimes \widetilde{\pi}_{Y' \times Y}^*(\mathcal{Q}))$ and $\widetilde{\pi}_{(\ )}^* : Y'' \times Y' \times Y \to (\ )$ is the projection.

4.1.1. *Cohomological correspondence* For simplicity, we denote the pull-backs of G and G' to $Y' \times Y$ by the same letters. For example $G' \otimes \mathcal{Q} \otimes G^{\vee}$ implies $\pi_{Y'}^*(G') \otimes \mathcal{Q} \otimes \pi_Y(G^{\vee})$. We note that

$$\mathbf{R}(p' \times p)_*(G' \otimes \mathcal{Q} \otimes G^{\vee}) \in \mathbf{D}(X' \times X)$$

satisfies

$$(p' \times p)^*(\mathbf{R}(p' \times p)_*(G' \otimes \mathcal{Q} \otimes G^{\vee})) = G' \otimes \mathcal{Q} \otimes G^{\vee}.$$

We define a homomorphism

$$\Psi_{X' \to X}^{\mathcal{Q}} : H^*(X', \mathbb{Q}) \to H^*(X, \mathbb{Q})$$

by

$$\begin{aligned} &\Psi_{X' \to X}^{\mathcal{Q}}(y) \\ :=&\pi_{X*} \circ (p' \times p)_* \Bigg((p' \times p)^* \circ \pi_{X'}^*(y) \operatorname{ch}(G') \operatorname{ch}(\mathcal{Q}) \operatorname{ch}(G^{\vee}) \\ &\qquad \cdot \frac{\sqrt{\operatorname{td}_{X'}} \operatorname{td}_{Y'/X'}}{\sqrt{\operatorname{ch}(G'^{\vee} \otimes G')}} \frac{\sqrt{\operatorname{td}_X} \operatorname{td}_{Y/X}}{\sqrt{\operatorname{ch}(G^{\vee} \otimes G)}} \Bigg) \\ =&\pi_{X*} \Bigg(\pi_{X'}^*(y) \frac{\sqrt{\operatorname{td}_{X'}}}{\sqrt{\operatorname{ch}(\mathbf{R}p'_*(G'^{\vee} \otimes G'))}} \frac{\sqrt{\operatorname{td}_X}}{\sqrt{\operatorname{ch}(\mathbf{R}p_*(G^{\vee} \otimes G))}} \\ &\qquad \cdot \operatorname{ch}(\mathbf{R}(p' \times p)_*(G' \otimes \mathcal{Q} \otimes G^{\vee})) \Bigg), \end{aligned}$$

where td_X, $\operatorname{td}_{X'}$,... are identified with their pull-backs.

Lemma 4.1. $\Psi_{X'' \to X}^{\mathcal{S}} = \Psi_{X' \to X}^{\mathcal{Q}} \circ \Psi_{X'' \to X'}^{\mathcal{R}}$.

Proof. $\pi_{(\ )} : X'' \times X' \times X \to (\ )$ denotes the projection to $(\ )$. We note that

$$\begin{aligned}
&\pi^*_{X''\times X}\left(\mathbf{R}(p''\times p')_*(G''\otimes\mathcal{R}\otimes G'^{\vee})\right)\\
&\qquad\otimes\pi^*_{X'\times X}\left(\mathbf{R}(p'\times p)_*(G'\otimes\mathcal{Q}\otimes G^{\vee})\right)\\
&=\mathbf{R}(p''\times p'\times p)_*(G''\otimes\mathcal{R}\otimes\mathcal{Q}\otimes G^{\vee})\otimes\pi^*_{X'}(\mathbf{R}p'_*(G'^{\vee}\otimes G')).
\end{aligned}$$

Then

$$\begin{aligned}
&\pi^*_{X''\times X}\left(\mathrm{ch}\left(\mathbf{R}(p''\times p')_*(G''\otimes\mathcal{R}\otimes G'^{\vee})\right)\right)\cdot\\
&\quad\pi^*_{X'\times X}\left(\mathrm{ch}\left(\mathbf{R}(p'\times p)_*(G'\otimes\mathcal{Q}\otimes G^{\vee})\right)\right)\pi^*_{X'}\left(\frac{\mathrm{td}_{X'}}{\mathrm{ch}(\mathbf{R}p'_*(G'^{\vee}\otimes G'))}\right)\\
&=\mathrm{ch}\left(\mathbf{R}(p''\times p'\times p)_*(G''\otimes\mathcal{R}\otimes\mathcal{Q}\otimes G^{\vee})\right)\pi^*_{X'}(\mathrm{td}_{X'}).
\end{aligned}$$

Since

$$\begin{aligned}
&\pi_{X''\times X*}\left(\mathrm{ch}\left(\mathbf{R}(p''\times p'\times p)_*(G''\otimes\mathcal{R}\otimes\mathcal{Q}\otimes G^{\vee})\right)\pi^*_{X'}(\mathrm{td}_{X'})\right)\\
&=\mathrm{ch}\left(\mathbf{R}\pi_{X''\times X*}\left(\mathbf{R}(p''\times p'\times p)_*(G''\otimes\mathcal{R}\otimes\mathcal{Q}\otimes G^{\vee})\right)\right)\\
&=\mathrm{ch}(\mathbf{R}(p''\times p)_*\circ\mathbf{R}\tilde{\pi}_{Y''\times Y*}(G''\otimes\mathcal{R}\otimes\mathcal{Q}\otimes G^{\vee}))\\
&=\mathrm{ch}(\mathbf{R}(p''\times p)_*(G''\otimes\mathcal{S}\otimes G^{\vee})),
\end{aligned}$$

we get

$$\begin{aligned}
\Psi^{\mathcal{S}}_{X''\to X}(z)=&\pi_{X*}\Bigg(\pi^*_{X''}(z)\,\mathrm{ch}(\mathbf{R}(p''\times p)_*(G''\otimes\mathcal{S}\otimes G^{\vee}))\\
&\qquad\cdot\frac{\sqrt{\mathrm{td}_{X''}}}{\sqrt{\mathrm{ch}(\mathbf{R}p''_*(G''^{\vee}\otimes G''))}}\frac{\sqrt{\mathrm{td}_X}}{\sqrt{\mathrm{ch}(\mathbf{R}p_*(G^{\vee}\otimes G))}}\Bigg)\\
=&\Psi^{\mathcal{Q}}_{X'\to X}\circ\Psi^{\mathcal{R}}_{X''\to X'}(z).
\end{aligned}$$

Q.E.D.

Lemma 4.2. *Assume that the canonical bundles $K_X, K_{X'}$ are trivial. Then*

$$\langle x,\Psi^{\mathcal{Q}}_{X'\to X}(y)\rangle=\langle\Psi^{\mathcal{Q}^{\vee}}_{X\to X'}(x),y\rangle,\quad x\in H^*(X,\mathbb{Q}),\ y\in H^*(X',\mathbb{Q}),$$

where $\langle\ ,\ \rangle$ is the Mukai pairing.

Proof.

$$
\begin{aligned}
&\langle x, \Psi^{\mathcal{Q}}_{X'\to X}(y)\rangle \\
=&-\int_X x\Psi^{\mathcal{Q}}_{X'\to X}(y)^{\vee} \\
=&-\int_{X'\times X} \pi_X^*(x)\Bigg(\pi_{X'}^*(y)\frac{\sqrt{\mathrm{td}_{X'}}}{\sqrt{\mathrm{ch}(\mathbf{R}p'_*(G'^{\vee}\otimes G'))}}\frac{\sqrt{\mathrm{td}_X}}{\sqrt{\mathrm{ch}(\mathbf{R}p_*(G^{\vee}\otimes G))}} \\
&\qquad\qquad\cdot \mathrm{ch}(\mathbf{R}(p'\times p)_*(G'\otimes \mathcal{Q}\otimes G^{\vee}))\Bigg)^{\vee} \\
=&-\int_{X'\times X}\Bigg(\frac{\sqrt{\mathrm{td}_{X'}}}{\sqrt{\mathrm{ch}(\mathbf{R}p'_*(G'^{\vee}\otimes G'))}}\frac{\sqrt{\mathrm{td}_X}}{\sqrt{\mathrm{ch}(\mathbf{R}p_*(G^{\vee}\otimes G))}} \\
&\qquad\qquad\cdot \mathrm{ch}(\mathbf{R}(p'\times p)_*(G'^{\vee}\otimes \mathcal{Q}^{\vee}\otimes G))\pi_X^*(x)\Bigg)\pi_{X'}^*(y^{\vee}) \\
=&-\int_{X'} \Psi^{\mathcal{Q}^{\vee}}_{X\to X'}(x)y^{\vee} \\
=&\langle \Psi^{\mathcal{Q}^{\vee}}_{X\to X'}(x), y\rangle.
\end{aligned}
$$

Q.E.D.

4.2. Fourier-Mukai transform induced by stable twisted sheaves

Let $p: Y \to X$ be a projective bundle over an abelian surface or a $K3$ surface. Let G be a locally free Y-sheaf. Assume that $X' := \overline{M}_H^{Y,G}(v)$ is a surface and consists of stable sheaves. We set $Y' := \widetilde{\overline{M}}_H^{Y,G}(v)$. Let $\mathcal{E}$ be the family on $Y'\times Y$.

We consider integral functors

$$
\begin{array}{rccc}
\Phi^{\mathcal{E}}_{X'\to X}: & \mathbf{D}(X',Y') & \to & \mathbf{D}(X,Y) \\
& x & \mapsto & \mathbf{R}\widetilde{\pi}_{Y*}(\mathcal{E}\otimes\widetilde{\pi}_{Y'}^*(x)),
\end{array}
$$

$$
\begin{array}{rccc}
\Phi^{\mathcal{E}^{\vee}}_{X\to X'}[2]: & \mathbf{D}(X,X) & \to & \mathbf{D}(X',Y') \\
& y & \mapsto & \mathbf{R}\widetilde{\pi}_{Y'*}(\mathcal{E}^{\vee}\otimes\widetilde{\pi}_{Y}^*(y)[2]).
\end{array}
$$

Remark 4.2. Let $\mathcal{L}(p'^*(\alpha^{-1}))$ and $\mathcal{L}(p^*(\alpha^{-1}))$ be twisted line bundles on Y' and Y respectively in (1.1). Then $\Lambda^{\mathcal{L}(p^*(\alpha^{-1}))}\circ\Phi^{\mathcal{E}}_{X'\to X}\circ(\Lambda^{\mathcal{L}(p'^*(\alpha'^{-1}))})^{-1}: \mathbf{D}(X',\alpha')\to\mathbf{D}(X,\alpha)$ is an integral functor with the kernel $\mathbf{R}(p'\times p)_*(\mathcal{L}(p'^*(\alpha'^{-1}))\otimes\mathcal{E}\otimes\mathcal{L}(p^*(\alpha^{-1}))^{\vee})\in\mathbf{D}(X'\times X,\alpha'^{-1}\times\alpha)$.

Căldăraru [C2] developed a theory of derived category of twisted sheaves. In particular, Grothendieck-Serre duality holds. Then we see that $\Phi_{X \to X'}^{\mathcal{E}^\vee}[2]$ is the adjoint of $\Phi_{X' \to X}^{\mathcal{E}}$. As in the usual Fourier-Mukai functor, we see that the following theorem holds (see [Br], [C1]).

Theorem 4.3. $\Phi_{X \to X'}^{\mathcal{E}^\vee}[2] \circ \Phi_{X' \to X}^{\mathcal{E}} \cong 1$ *and* $\Phi_{X' \to X}^{\mathcal{E}} \circ \Phi_{X \to X'}^{\mathcal{E}^\vee}[2] \cong 1$. *Thus* $\Phi_{X' \to X}^{\mathcal{E}}$ *is an equivalence.*

Then we have the following which also follows from a more general statement [H-St, Thm. 0.4].

Corollary 4.4. $\Psi_{X' \to X}^{\mathcal{E}}$ *induces an isometry of the Hodge structures:*

$$(H^*(X', \mathbb{Z}), \langle \ , \ \rangle, -\frac{\xi'}{r}) \cong (H^*(X, \mathbb{Z}), \langle \ , \ \rangle, -\frac{\xi}{r}).$$

Proof. Obviously $\Psi_{X' \to X}^{\mathcal{E}}$ induces an isometry of the Hodge structures over $\mathbb{Q}$. If X is a $K3$ surface such that $w(Y) \in \mathrm{NS}(X) \otimes \mu_r$ and X' is a fine moduli space, then $\Psi_{X' \to X}^{\mathcal{E}}$ is defined over $\mathbb{Z}$. For a general case, we use the deformation arguments. Q.E.D.

We also have the following which is used in [Y4].

Corollary 4.5. *Assume that* X' *consists of locally free* Y*-sheaves. Then* $\mathcal{E}_{|Y' \times \{y\}}^\vee$, $y \in Y$ *is a simple* Y'*-sheaf. If* $\mathrm{NS}(X) \cong \mathbb{Z}H$, *then* $\mathcal{E}_{|Y' \times \{y\}}^\vee$, $y \in Y$ *is a stable* Y'*-sheaf.*

Proof. Since $\Phi_{X \to X'}^{\mathcal{E}^\vee}[2]$ is an equivalence, $\Phi_{X \to X'}^{\mathcal{E}^\vee}(\mathcal{O}_{p^{-1}(p(y))}(1)) = \mathcal{E}_{|Y' \times \{y\}}^\vee$ is a simple Y'-sheaf. If $\mathrm{NS}(X) \cong \mathbb{Z}$, then Proposition 3.12 implies the stability of $\mathcal{E}_{|Y' \times \{y\}}^\vee$. Q.E.D.

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