Advanced Studies in Pure Mathematics 44, 2006
Potential Theory in Matsue
pp. 403–413

On the behavior at infinity for non-negative superharmonic functions in a cone

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Abstract.

This paper shows that a positive superharmonic function on a cone behaves regularly outside an a-minimally thin set in a cone. This fact is known for a half space which is a special cone.

§1. Introduction

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of all real numbers and the set of all positive real numbers, respectively. We denote by $\mathbf{R}^n$ $(n \geq 2)$ the n-dimensional Euclidean space. A point in $\mathbf{R}^n$ is denoted by $P = (X, y)$, $X = (x_1, x_2, \ldots, x_{n-1})$. The Euclidean distance of two points P and Q in $\mathbf{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$ with the origin O of $\mathbf{R}^n$ is simply denoted by $|P|$. The boundary and the closure of a set S in $\mathbf{R}^n$ are denoted by ∂S and $\bar{S}$, respectively.

We introduce spherical coordinates (r, Θ), $\Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, in $\mathbf{R}^n$ which are related to cartesian coordinates $(x_1, x_2, \ldots, x_{n-1}, y)$ by

$$x_1 = r(\Pi_{j=1}^{n-1} \sin\theta_j) \ \ (n \geq 2), \qquad y = r\cos\theta_1,$$

and if $n \geq 3$, then

$$x_{n+1-k} = r(\Pi_{j=1}^{k-1} \sin\theta_j)\cos\theta_k \quad (2 \leq k \leq n-1),$$

where $0 \leq r < +\infty$, $-\frac{1}{2}\pi \leq \theta_{n-1} < \frac{3}{2}\pi$, and if $n \geq 3$, then $0 \leq \theta_j \leq \pi$ $(1 \leq j \leq n-2)$.

The unit sphere and the upper half unit sphere are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}_+^{n-1}$, respectively. For simplicity, a point $(1, \Theta)$ on $\mathbf{S}^{n-1}$ and the

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Received May 30, 2005.
Revised December 1, 2005.
2000 *Mathematics Subject Classification.* Primary 31B05; Secondary 31B20.
Key words and phrases. cone, superharmonic functions, a-minimally thin sets.

set $\{\Theta;(1,\Theta)\in\Omega\}$ for a set Ω, $\Omega\subset\mathbf{S}^{n-1}$, are often identified with Θ and Ω, respectively. For two sets $\Lambda\subset\mathbf{R}_+$ and $\Omega\subset\mathbf{S}^{n-1}$, the set $\{(r,\Theta)\in\mathbf{R}^n;\ r\in\Lambda,\ (1,\Theta)\in\Omega\}$ in $\mathbf{R}^n$ is simply denoted by $\Lambda\times\Omega$. In particular, the half-space $\mathbf{R}_+\times\mathbf{S}_+^{n-1}=\{(X,y)\in\mathbf{R}^n;\ y>0\}$ will be denoted by $\mathbf{T}_n$. By $C_n(\Omega)$, we denote the set $\mathbf{R}_+\times\Omega$ in $\mathbf{R}^n$ with the domain Ω on $\mathbf{S}^{n-1}(n\geq 2)$ having smooth boundary. We call it a cone. Then $\mathbf{T}_n$ is a special cone obtained by putting $\Omega=\mathbf{S}_+^{n-1}$.

Let Ω be a domain on $\mathbf{S}^{n-1}(n\geq 2)$ with smooth boundary. Consider the Dirichlet problem

$$\begin{aligned}(\Lambda_n+\tau)f&=0\quad\text{on }\Omega\\ f&=0\quad\text{on }\partial\Omega,\end{aligned}$$

where Λ_n is the spherical part of the Laplace operator Δ_n

$$\Delta_n=\frac{n-1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}+r^{-2}\Lambda_n.$$

We denote the least positive eigenvalue of this boundary value problem by τ_Ω and the normalized positive eigenfunction corresponding to τ_Ω by $f_\Omega(\Theta)$; $\int_\Omega f_\Omega^2(\Theta)d\sigma_\Theta=1$, where $d\sigma_\Theta$ is the surface element on $\mathbf{S}^{n-1}$. We denote the solutions of the equation $t^2+(n-2)t-\tau_\Omega=0$ by $\alpha_\Omega,-\beta_\Omega$ $(\alpha_\Omega,\beta_\Omega>0)$. If $\Omega=\mathbf{S}_+^{n-1}$, then $\alpha_\Omega=1$, $\beta_\Omega=n-1$ and $f_\Omega(\Theta)=(2ns_n^{-1})^{1/2}\cos\theta_1$, where s_n is the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\mathbf{S}^{n-1}$.

In the following, we shall assume that if $n\geq 3$, then Ω is a $C^{2,\alpha}$-domain $(0<\alpha<1)$ on $\mathbf{S}^{n-1}$ (e.g. see Gilbarg and Trudinger [4] for the definition of $C^{2,\alpha}$-domain).

It is known that the Martin boundary of $C_n(\Omega)$ is the set $\partial C_n(\Omega)\cup\{\infty\}$, each of which is a minimal Martin boundary point. When we denote the Martin kernel by $\tilde{K}(P,Q)$ $(P\in C_n(\Omega),Q\in\partial C_n(\Omega)\cup\{\infty\})$ with respect to a reference point chosen suitably, we know

$$\tilde{K}(P,\infty)=r^{\alpha_\Omega}f_\Omega(\Theta),\quad \tilde{K}(P,O)=\kappa r^{-\beta_\Omega}f_\Omega(\Theta)\quad (P\in C_n(\Omega)),$$

where κ is a positive constant (Yoshida [8, p.292]).

Let $u(P)$ be a non-negative superharmonic function on $\mathbf{T}_n$, and let $c(u)=\inf_{P=(X,y)\in\mathbf{T}_n}u(P)/y$. Aikawa [1] introduced the notion of a-minimal thinness $(0\leq a\leq 1)$, which is identical to minimal thinness when $a=1$ and which is identical to rarefiedness when $a=0$, and showed that

$$\lim_{|P|\to\infty,\ P\in\mathbf{T}_n\setminus E}\frac{u(P)-c(u)y}{y^a|P|^{1-a}}=0, \tag{1.1}$$

with a set E in $\mathbf{T}_n$ which is a-minimally thin at ∞. Aikawa also showed that if $E \subset \mathbf{T}_n$ is unbounded and a-minimally thin at ∞ in $\mathbf{T}_n$, then there exists a non-negative superharmonic function u on $\mathbf{T}_n$ such that

$$(1.2) \qquad \lim_{|P|\to\infty,\ P\in E} \frac{u(P)-c(u)y}{y^a|P|^{1-a}} = +\infty,$$

and showed that (1.1) is the best possible as to the size of the exceptional set. The cases of $a = 1$ in (1.1) and (1.2) give the result of Lelong-Ferrand [6, pp. 134-143], and the cases of $a = 0$ in (1.1) and (1.2) give the result of Essén and Jackson [3, Theorem 4.6].

For a non-negative superharmonic function in a cone, the results corresponding to $a = 1$ of (1.1) and (1.2) are showed by the Fatou boundary limit theorem for Martin space (Miyamoto and Yoshida [7, Remark 2]). In detail, for a non-negative superharmonic function u on $C_n(\Omega)$, there exists a set $E \subset C_n(\Omega)$ which is minimally thin at ∞ such that

$$(1.3) \qquad \lim_{|P|\to+\infty,\ P\in C_n(\Omega)\setminus E} \frac{u(P)-c_\infty(u)\tilde{K}(P,\infty)}{\tilde{K}(P,\infty)} = 0,$$

where we put $c_\infty(u) = \inf_{P\in C_n(\Omega)} \frac{u(P)}{\tilde{K}(P,\infty)}$. On the other hand, Miyamoto and Yoshida [7, Theorem 3] introduced the notion of rarefiedness at ∞ with respect to $C_n(\Omega)$, and showed that for a non-negative superharmonic function u on $C_n(\Omega)$, there exists a set $E \subset C_n(\Omega)$ which is rarefied at ∞ such that

$$(1.4) \qquad \lim_{|P|\to+\infty,\ P\in C_n(\Omega)\setminus E} \frac{u(P)-c_\infty(u)\tilde{K}(P,\infty)}{|P|^{\alpha_\Omega}} = 0.$$

(1.4) gives the extension of the case $a = 0$ in (1.1).

From these results, in this paper we shall introduce the notion of a-mimal thinness ($0 \le a \le 1$) at ∞ with respect to a cone and extend the above results for a cone ((1.3) and (1.4)). We shall also extend the results (1.1) and (1.2) bacause our main result contains (1.1) and (1.2) as the case $\Omega = \mathbf{S}_+^{n-1}$. The results of this paper are proved by modifying the methods of Aikawa [1] and Essén and Jackson [3].

I would like to thank Professor Ikuko Miyamoto and Professor Hidenobu Yoshida for their help in preparing this paper.

§2. Preliminaries

We denote by $G(P,Q)$ $(P \in C_n(\Omega), Q \in C_n(\Omega))$ the Green function of $C_n(\Omega)$, and let $G\mu(P) = \int_{C_n(\Omega)} G(P,Q)d\mu(Q)$ be the Green potential at $P \in C_n(\Omega)$ of a positive Radon measure μ.

Let $S_n(\Omega)$ be the set $\partial C_n(\Omega) \setminus \{O\}$. Now we shall define the Martin type kernel $K(P,Q)$ $(P = (r,\Theta) \in C_n(\Omega), Q = (t,\Phi) \in \overline{C_n(\Omega)} \cup \{\infty\})$ as follows:

$$K(P,Q) = \begin{cases} \dfrac{G(P,Q)}{t^{\alpha_\Omega} f_\Omega(\Phi)} & \text{on } C_n(\Omega) \times C_n(\Omega) \\ \dfrac{\partial G(P,Q)}{\partial n_Q} \left\{ t^{\alpha_\Omega - 1} \dfrac{\partial}{\partial n_\Phi} f_\Omega(\Phi) \right\}^{-1} & \text{on } C_n(\Omega) \times S_n(\Omega) \\ r^{\alpha_\Omega} f_\Omega(\Theta) & \text{on } C_n(\Omega) \times \{\infty\} \\ \kappa r^{-\beta_\Omega} f_\Omega(\Theta) & \text{on } C_n(\Omega) \times \{O\}, \end{cases}$$

where $\partial/\partial n_Q$ denotes the differentiation at Q along the inward normal into $C_n(\Omega)$. We note that $K_P(Q) = K(P,Q)$ is continuous in the extended sence on $C_n(\Omega) \cup S_n(\Omega)$. Following Brelot [2, p.31], we let $K^*(P,Q) = K(Q,P)$ be the associated kernel of K on $(\overline{C_n(\Omega)} \cup \{\infty\}) \times C_n(\Omega)$.

If μ is a measure on $\overline{C_n(\Omega)} \cup \{\infty\}$, we abbreviate $\int_{\overline{C_n(\Omega)} \cup \{\infty\}} K(P,Q)d\mu(Q)$ to $K\mu(P)$ and also $\int_{C_n(\Omega)} K^*(P,Q)d\nu(Q)$ to $K^*\nu(P)$ for a measure ν on $C_n(\Omega)$.

Let u be a non-negative superharmonic function on $C_n(\Omega)$ and put $c_O(u) = \inf_{P \in C_n(\Omega)} \frac{u(P)}{K(P,O)}$. Then from Miyamoto and Yoshida [7, Lemma 3], we see that there exists a unique measure μ_u on $\overline{C_n(\Omega)} \cup \{\infty\}$ such that $u = K\mu_u$. When we denote by μ'_u the restriction of the measure μ_u on $C_n(\Omega)$, we have $u(P) = c_\infty(u)K(P,\infty) + c_O(u)K(P,O) + K\mu'_u(P)$.

For a number a, $0 \le a \le 1$, we define the positive superharmonic function g_a by $g_a(P) = (K(P,\infty))^a$ $(P \in C_n(\Omega))$.

For a non-negative function v on $C_n(\Omega)$ and $E \subset C_n(\Omega)$, let $\hat{R}_v^E$ be the regularized reduced function of v relative to E (Helms [5, p.116]).

Let E be a bounded subset of $C_n(\Omega)$. We define the a-mass of E by $\lambda_E^a(\overline{C_n(\Omega)})$ for $0 \le a \le 1$, where λ_E^a is the measure on $\overline{C_n(\Omega)}$ such that $K\lambda_E^a = \hat{R}_{g_a}^E$.

Let $E \subset C_n(\Omega)$ be bounded. Then there exists a unique measure λ_E on $C_n(\Omega)$ such that $\hat{R}_{K(\cdot,\infty)}^E = G\lambda_E$ on $C_n(\Omega)$. If $0 < a \le 1$, then following Yoshida [8, Corollary 5.3] we see the greatest harmonic minorant of $\hat{R}^E_{g_a}$ is zero, so that $\lambda^a_E(\partial C_n(\Omega)) = 0$. Then according to the proof of Aikawa [1, Lemma 2.1] we can similarly have

$$\lambda^a_E(\overline{C_n(\Omega)}) = \int_{C_n(\Omega)} g_a d\lambda_E. \tag{2.1}$$

In particular $\lambda^1_E(\overline{C_n(\Omega)}) = \int G\lambda_E d\lambda_E$ and $\lambda^0_E(\overline{C_n(\Omega)}) = \lambda_E(C_n(\Omega))$.

Let E be a subset of $C_n(\Omega)$ and $E_k = E \cap I_k$, where

$$I_k = \{P \in C_n(\Omega);\ 2^k \leq |P| < 2^{k+1}\} \quad (k = 0, 1, 2, \ldots).$$

We say that $E \subset C_n(\Omega)$ is a-minimally thin at ∞ in $C_n(\Omega)$ if

$$\sum_{k=0}^{\infty} \lambda^a_{E_k}(\overline{C_n(\Omega)}) 2^{-k(a\alpha_\Omega + \beta_\Omega)} < +\infty.$$

Remark 2.1. From Theorems 1 and 2 of Miyamoto and Yoshida [7] and (2.1), we see that the notion of a-minimal thinness contains the notions of minimal thinness and rarefiedness.

In the following we set

$$C_n(\Omega;\ a,\ b) = \{P = (r, \Theta) \in C_n(\Omega);\ a < r < b\} \qquad (0 < a < b \leq +\infty),$$
$$S_n(\Omega;\ a,\ b) = \{P = (r, \Theta) \in S_n(\Omega);\ a < r < b\} \qquad (0 < a < b \leq +\infty).$$

As far as we are concerned with a-minimal thinness in the following, we shall restrict a subset E of $C_n(\Omega)$ to the set located in $C_n(\Omega; 1, +\infty)$, because the part of E separated from ∞ is unessential to a-minimal thinness.

§3. Statements of results

Let η be a real number satisfying $(2-n)\frac{1}{\alpha_\Omega} - 1 < \eta \leq 1$. We define the positive superharmonic function h_η on $C_n(\Omega)$ by $h_\eta(P) = K(P, \infty)|P|^{\{(2-n)\frac{1}{\alpha_\Omega} - 1 - \eta\}\alpha_\Omega}$. Since $K(P, \infty)$ is a minimal harmonic function on $C_n(\Omega)$, we see that there exists a measure ν_η on $C_n(\Omega)$ such that $G\nu_\eta(P) = \min(K(P, \infty), h_\eta(P))$.

Let $\mathfrak{F}_\eta$ be the class of all non-negative superharmonic functions u on $C_n(\Omega)$ such that $c_\infty(u) = 0$ and

$$\int_{C_n(\Omega;1,+\infty) \cup S_n(\Omega;1,+\infty)} |Q|^{\{(2-n)\frac{1}{\alpha_\Omega} - 1 - \eta\}\alpha_\Omega} d\mu_u(Q) < +\infty. \tag{3.1}$$

Remark 3.1. If $P \in C_n(\Omega)$, then $K^*\nu_\eta(P) = G\nu_\eta(P)/K(P,\infty)$. If $P \in S_n(\Omega)$, then $K^*\nu_\eta(P) = \liminf_{Q\to P, Q\in C_n(\Omega)} K^*\nu_\eta(Q)$ (cf. Essén and Jackson [3, p.240]). Hence for a point $P \in C_n(\Omega) \cup S_n(\Omega)$, we have

$$(3.2) \qquad K^*\nu_\eta(P) = \begin{cases} 1 & \text{for } 0 < |P| < 1, \\ |P|^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} & \text{for } |P| \geq 1. \end{cases}$$

Let $u \in \mathfrak{F}_\eta$. From (3.2) we see that (3.1) is equivalent to the following condition;

$$\int_{C_n(\Omega)} \{u(P) - c_O(u)K(P,O)\} d\nu_\eta(P) < +\infty.$$

If u_1, $u_2 \in \mathfrak{F}_\eta$ and c is a positive constant, then $u_1 + u_2$, $cu_1 \in \mathfrak{F}_\eta$.

Let $v \in \mathfrak{F}_\eta$ such that $c_O(v) = 0$, and let u be a non-negative superharmonic function such that $c_O(u) = 0$. Then $0 \leq u \leq v$ on $C_n(\Omega)$ implies $u \in \mathfrak{F}_\eta$ (cf. Aikawa [1, Lemma 3.1]).

We define the function $h_{\eta,a}(P) = K(P,\infty)^a |P|^{(\eta-a)\alpha_\Omega}$ $(P \in C_n(\Omega))$.

Theorem 3.1. *If $u(P) \in \mathfrak{F}_\eta$, then there exists a set $E \subset C_n(\Omega)$ which is a-minimally thin at ∞ with respect to $C_n(\Omega)$ such that*

$$\lim_{|P|\to+\infty,\ P\in C_n(\Omega)\setminus E} \frac{u(P)}{h_{\eta,a}(P)} = 0.$$

Conversely, if E is unbounded and a-minimally thin at ∞ with respect to $C_n(\Omega)$, then there exists $u(P) \in \mathfrak{F}_\eta$ such that

$$\lim_{|P|\to+\infty,\ P\in E} \frac{u(P)}{h_{\eta,a}(P)} = +\infty.$$

When $\Omega = \mathbf{S}_+^{n-1}$, we obtain the result of Aikawa [1, Theorem 3.2].

Let $u(P)$ be a non-negative superharmonic function on $C_n(\Omega)$. Since $u_1(P) = u(P) - c_\infty(u)K(P,\infty)$ belongs to $\mathfrak{F}_1$, we obtain the following Corollary 3.1 by applying Theorem 3.1 of the case $\eta = 1$ to u_1.

Corollary 3.1. *Let $u(P)$ be a non-negative superharmonic function on $C_n(\Omega)$. Then there exists a set $E \subset C_n(\Omega)$ which is a-minimally thin at ∞ with respect to $C_n(\Omega)$ such that*

$$\lim_{|P|\to+\infty,\ P\in C_n(\Omega)\setminus E} \frac{u(P) - c_\infty(u)K(P,\infty)}{K(P,\infty)^a |P|^{(1-a)\alpha_\Omega}} = 0.$$

Conversely, if E is unbounded and a-minimally thin at ∞ with respect to $C_n(\Omega)$, then there exists a non-negative superharmonic function $u(P)$

such that

$$\lim_{|P|\to+\infty,\ P\in E} \frac{u(P)-c_\infty(u)K(P,\infty)}{K(P,\infty)^a|P|^{(1-a)\alpha_\Omega}} = +\infty.$$

The case $a = 0$ in Corollary 3.1 gives the result of Miyamoto and Yoshida [7, Theorem 3].

§4. Proof of Theorem 3.1

We remark that

$$\text{(4.1)} \qquad G(P,Q) \le M_1 r^{\alpha_\Omega} t^{-\beta_\Omega} f_\Omega(\Theta) f_\Omega(\Phi)$$

$$\text{(4.2)} \qquad (\text{resp. } G(P,Q) \le M_2 t^{\alpha_\Omega} r^{-\beta_\Omega} f_\Omega(\Theta) f_\Omega(\Phi))$$

for any $P = (r,\Theta) \in C_n(\Omega)$ and any $Q = (t,\Phi) \in C_n(\Omega)$ satisfying $0 < \frac{r}{t} \le \frac{1}{2}$ (resp. $0 < \frac{t}{r} \le \frac{1}{2}$), where M_1 (resp. M_2) is a positive constant. From (4.1) and (4.2) we have the following inequalities:

$$\text{(4.3)} \qquad \frac{\partial G(P,Q)}{\partial n_Q} \le M_3 r^{\alpha_\Omega} t^{-\beta_\Omega-1} f_\Omega(\Theta) \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi)$$

$$\text{(4.4)} \qquad (\text{resp. } \frac{\partial G(P,Q)}{\partial n_Q} \le M_4 t^{\alpha_\Omega-1} r^{-\beta_\Omega} f_\Omega(\Theta) \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi))$$

for any $P = (r,\Theta) \in C_n(\Omega)$ and any $Q = (t,\Phi) \in S_n(\Omega)$ satisfying $0 < \frac{r}{t} \le \frac{1}{2}$ (resp. $0 < \frac{t}{r} \le \frac{1}{2}$), where M_3 (resp. M_4) is a positive constant and $\partial/\partial n_\Phi$ denotes the differntiation at $\Phi \in \partial\Omega$ along the inward normal into Ω (Miyamoto and Yoshida [7]).

For two positive functions u and v, we shall write $u \approx v$ if and only if there exist constants A, B, $0 < A \le B$, such that $Av \le u \le Bv$ everywhere on $C_n(\Omega)$.

Lemma 4.1. *$E \subset C_n(\Omega; 1, +\infty)$ is a-minimally thin at ∞ if and only if $\sum_{k=0}^{\infty} \hat{R}_{h_{\eta,a}}^{E_k} \in \mathfrak{F}_\eta$.*

Proof. We note that for every $k = 0, 1, 2, \ldots$,

$$\hat{R}_{g_a}^{E_k} \approx 2^{-k(\eta-a)\alpha_\Omega} \hat{R}_{h_{\eta,a}}^{E_k},$$

$$\lambda_{E_k}^a(\overline{C_n(\Omega)}) \approx 2^{-k\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} \int_{C_n(\Omega)\cup S_n(\Omega)} |Q|^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} d\lambda_{E_k}^a(Q),$$

where the constants of comparison are independent of k. Since

$$\int_{C_n(\Omega)} \hat{R}_{g_a}^{E_k}(P)d\nu_\eta(P) = \int_{C_n(\Omega)} K\lambda_{E_k}^a(P)d\nu_\eta(P)$$

$$= \int_{C_n(\Omega)\cup S_n(\Omega)} K^*\nu_\eta(Q)d\lambda_{E_k}^a(Q) = \int_{C_n(\Omega)\cup S_n(\Omega)} |Q|^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} d\lambda_{E_k}^a(Q),$$

we have $2^{k(-a\alpha_\Omega-\beta_\Omega)}\lambda_{E_k}^a(\overline{C_n(\Omega)}) \approx \int_{C_n(\Omega)} \hat{R}_{h_{\eta,a}}^{E_k}(P)d\nu_\eta(P)$ where the constants of comparison are independent of k, which gives the conclusion. □

Lemma 4.2. *Let E be a set in $C_n(\Omega;1,+\infty)$. If $\hat{R}_{h_{\eta,a}}^{E} \in \mathfrak{F}_\eta$, then E is a-minimally thin at ∞.*

Proof. Since $h_{\eta,a}(P)$ satisfies

$$\liminf_{|P|\to\infty} \frac{h_{\eta,a}(P)}{K(P,\infty)|P|^{(\eta-1)\alpha_\Omega}} > 0,$$

we find a positive constant C' and a natural number N_1 such that $h_{\eta,a}(P) \geq C'K(P,\infty)|P|^{(\eta-1)\alpha_\Omega}$ for $|P| > 2^{N_1}$. Let $C_1 = M_1/C'$, $C_2 = M_2/C'$, $C_3 = M_3/C'$ and $C_4 = M_4/C'$. And put $C = \max_{1\leq i\leq 4}\{C_i\}$.

Let $\hat{R}_{h_{\eta,a}}^{E} = K\mu$, where μ satisfies (3.1). Noting (3.1), we put $A = \int_{C_n(\Omega;1,+\infty)\cup S_n(\Omega;1,+\infty)} |Q|^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} d\mu(Q) < +\infty$. We take a natural number N_2 such that $4AC < 2^{-N_2\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega}$. Then there exists a natural number k_0 such that

$$C\int_{\{Q\in C_n(\Omega)\cup S_n(\Omega);\ |Q|\geq 2^{k+N_2+1}\}} |Q|^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} d\mu(Q) < \frac{1}{4}$$

for $k \geq k_0$. Let $N = \max\{N_1, N_2, k_0\}$. Hence it is sufficient to prove $\sum_{k>N} \hat{R}_{h_{\eta,a}}^{E_k} \in \mathfrak{F}_\eta$ beacause $\sum_{k=0}^{N} \hat{R}_{h_{\eta,a}}^{E_k} \leq (N+1)\hat{R}_{h_{\eta,a}}^{E} \in \mathfrak{F}_\eta$. We set $J_k = I_{k-N_2} \cup \cdots \cup I_k \cup \cdots \cup I_{k+N_2}$. Let $k > N$ and let $P = (r,\Theta) \in E_k$. If $Q \in C_n(\Omega)$ and $|Q| \leq 2^{k-N_2}$, then from (4.2) we have

$$K(P,Q) = \frac{G(P,Q)}{t^{\alpha_\Omega} f_\Omega(\Phi)} \leq M_2 r^{-\beta_\Omega} f_\Omega(\Theta).$$

Hence

$$\begin{aligned}\int_{\{Q\in C_n(\Omega);\ |Q|\leq 2^{k-N_2}\}} K(P,Q)d\mu(Q) &\leq C_2 h_{\eta,a}(P) r^{-(\eta\alpha_\Omega+\beta_\Omega)} \int_{1\leq|Q|\leq 2^{k-N_2}} d\mu(Q) \\ &\leq C_2 h_{\eta,a}(P) \int_{1\leq|Q|\leq 2^{k-N_2}} |P|^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} d\mu(Q).\end{aligned}$$

On the other hand, if $Q \in C_n(\Omega)$ and $|Q| \geq 2^{k+N_2+1}$, then from (4.1) we have

$$\int_{\{Q\in C_n(\Omega);\ |Q|\geq 2^{k+N_2+1}\}} K(P,Q)d\mu(Q) \leq C_1 h_{\eta,a}(P) r^{-(\eta-1)\alpha_\Omega} \int_{|Q|\geq 2^{k+N_2+1}} |Q|^{-(\alpha_\Omega+\beta_\Omega)} d\mu(Q)$$

$$\leq C_1 h_{\eta,a}(P) \int_{|Q|\geq 2^{k+N_2+1}} |Q|^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} d\mu(Q).$$

If $Q \in S_n(\Omega)$ and $|Q| \leq 2^{k-N_2}$ or $Q \in S_n(\Omega)$ and $|Q| \geq 2^{k+N_2+1}$, then from (4.4) or (4.3) we have similar inequalities. From these inequalities, we have

$$C^{-1}\int_{\overline{C_n(\Omega)}\setminus \bar{J}_k} K(P,Q)d\mu(Q) \leq h_{\eta,a}(P) \int_{|Q|\leq 2^{k-N_2}} |P|^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} d\mu(Q)$$

$$+h_{\eta,a}(P) \int_{|Q|\geq 2^{k+N_2+1}} |Q|^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} d\mu(Q).$$

Since $4AC < 2^{-N_2\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega}$, we see that

$$C\int_{|Q|\leq 2^{k-N_2}} |P|^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} d\mu(Q) \leq \frac{1}{4A} \int_{|Q|\leq 2^{k-N_2}} \left(\frac{|P|}{2^{N_2}}\right)^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} d\mu(Q)$$

$$\leq \frac{1}{4A} \int_{|Q|\leq 2^{k-N_2}} |Q|^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} d\mu(Q) \leq \frac{1}{4}.$$

So we have $\int_{\overline{C_n(\Omega)}\setminus \bar{J}_k} K(P,Q)d\mu(Q) \leq \frac{1}{2}h_{\eta,a}(P)$ on E_k, which implies that

$$h_{\eta,a}(P) \leq \hat{R}^E_{h_{\eta,a}}(P) \leq \int_{\bar{J}_k} K(P,Q)d\mu(Q) + \frac{1}{2}h_{\eta,a}(P)$$

q.e. on E_k. Hence $h_{\eta,a}(P) \leq 2\int_{\bar{J}_k} K(P,Q)d\mu(Q)$ q.e. on E_k. Therefore $\hat{R}^{E_k}_{h_{\eta,a}}(P) \leq 2\int_{J_k} K(P,Q)d\mu(Q)$ on $C_n(\Omega)$, by the definition of $\hat{R}^{E_k}_{h_{\eta,a}}$. If we sum up $\hat{R}^{E_k}_{h_{\eta,a}}$ over $k > N$, we obtain $\sum_{k>N}^{\infty} \hat{R}^{E_k}_{h_{\eta,a}} \leq 2(2N_2+1)\hat{R}^{E}_{h_{\eta,a}}$. By Remark 3.1 we see $\sum_{k>N} \hat{R}^{E_k}_{h_{\eta,a}} \in \mathfrak{F}_\eta$. Thus the lemma follows from Lemma 4.1. □

Proof of Theorem 3.1. Let $u_1(P) = u(P) - c_O(u)K(P,O)$ ($P \in C_n(\Omega)$), then we see $u_1 \in \mathfrak{F}_\eta$. For each non-negative integer j, we set $A_j = \{P \in C_n(\Omega; 1, +\infty);\ u_1(P)/h_{\eta,a}(P) \geq (j+1)^{-1}\}$. Since $\hat{R}^{A_j}_{h_{\eta,a}} \leq$

$(j+1)u_1 \in \mathfrak{F}_\eta$, we see from Remark 3.1 that $\hat{R}^{A_j}_{h_{\eta,a}} \in \mathfrak{F}_\eta$, and then A_j is a-minimally thin by Lemma 4.2. Following Aikawa [1, Lemma 3.4], we can similarly find an increasing sequence $\{m(j)\}$ of natural numbers such that $\sum_j \hat{R}^{\cup_{k\geq m(j)}(A_j\cap I_k)}_{h_{\eta,a}} \in \mathfrak{F}_\eta$. Set $\cup_{j=0}^{\infty} \cup_{k\geq m(j)} (A_j \cap I_k) = E$. Since $\hat{R}^E_{h_{\eta,a}} \leq \sum_j \hat{R}^{\cup_{k\geq m(j)}(A_j\cap I_k)}_{h_{\eta,a}}$, E is a-minimally thin by Lemma 4.2. If $P \notin E$, then $P \notin \cup_{k\geq m(j)}(A_j \cap I_k)$ for every j. It follows that if $|P| \geq 2^{m(j)}$, then $P \notin A_j$. This implies that $u_1(P)/h_{\eta,a}(P) < (j+1)^{-1}$. Hence we have $u_1(P)/h_{\eta,a}(P) \to 0$ as $|P| \to \infty$, $P \in C_n(\Omega) \setminus E$. On the other hand, we see $K(P,O)/h_{\eta,a}(P) = \kappa r^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} f_\Omega(\Theta)^{1-a} \to 0$ as $|P| \to \infty$. Thus we have

$$\frac{u(P)}{h_{\eta,a}(P)} = \frac{u_1(P) + c_O(u)K(P,O)}{h_{\eta,a}(P)} \to 0 \quad (|P| \to \infty, P \in C_n(\Omega) \setminus E).$$

For the converse we take an unbounded and a-minimally thin set E. As in the proof of Aikawa [1, Lemma 2.4 (iv)], we see that if U is bounded, then $\lambda^a_U(\overline{C_n(\Omega)}) = \inf\{\lambda^a_O(\overline{C_n(\Omega)}); U \subset O, O \text{ is open}\}$. By applying the above property to E_k $(k = 0, 1, 2, \ldots,)$, we obtain an open set $O \supset E$ such that O is a-minimally thin. By Lemma 4.1 we have $\sum_{k=0}^{\infty} \hat{R}^{O_k}_{h_{\eta,a}}(P) \in \mathfrak{F}_\eta$, where $O_k = O \cap I_k$, which implies $\sum_k \int \hat{R}^{O_k}_{h_{\eta,a}}(P)d\nu_\eta(P) < +\infty$. We find an increasing sequence $\{c_k\}$ of positive numbers such that $c_k \nearrow \infty$ and $\sum_k c_k \int \hat{R}^{O_k}_{h_{\eta,a}}(P)d\nu_\eta(P) < +\infty$. Set $u(P) = \sum_{k=0}^{\infty} c_k \hat{R}^{O_k}_{h_{\eta,a}}(P)$. By Lebesgue's monotone convergence theorem, we see that $u \in \mathfrak{F}_\eta$. Since O_k is included in the interior of $O_{k-1} \cup O_k$,

$$\hat{R}^{O_{k-1}}_{h_{\eta,a}}(P) + \hat{R}^{O_k}_{h_{\eta,a}}(P) \geq \hat{R}^{O_{k-1}\cup O_k}_{h_{\eta,a}}(P) \geq h_{\eta,a}(P)$$

for $P \in O_k$. Hence, if $P \in E_k \subset O_k$, then

$$u(P) \geq c_{k-1}\hat{R}^{O_{k-1}}_{h_{\eta,a}}(P) + c_k \hat{R}^{O_k}_{h_{\eta,a}}(P) \geq c_{k-1}h_{\eta,a}(P).$$

Therefore

$$\lim_{|P|\to+\infty,\ P\in E} \frac{u(P)}{h_{\eta,a}(P)} = +\infty.$$

References

[1] H. Aikawa, On the behavior at infinity of non-negative superharmonic functions in a half space, Hiroshima Math. J., **11** (1981), 425–441, MR0620547.

[2] M. Brelot, On topologies and boundaries in potential theory, Lect. Notes in Math., **175**, Springer-Verlag, 1971, MR0281940.

[3] M. Essén and H. L. Jackson, On the covering properties of certain exceptional sets in a half-space, Hiroshima Math. J., **10** (1980), 233–262, MR0577853.

[4] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, Springer Verlag, Berlin, 2001, MR1814364.

[5] L. L. Helms, Introduction to potential theory, Wiley, New York, 1969, MR0261018.

[6] J. Lelong-Ferrand, Étude au voisinage de la frontière des fonctions surharmoniques positives dans un demi-espace, Ann. Sci. Ecole Norm. Sup., **66** (1949), 125–159, MR0031603.

[7] I. Miyamoto and H. Yoshida, Two criterions of Wiener type for minimally thin sets and rarefied sets in a cone, J. Math. Soc. Japan, **54** (2002), 487–512, MR21900954.

[8] H. Yoshida, Nevanlinna norm of a subharmonic function on a cone or on a cylinder, Proc. London Math. Soc., **54** (1987), 267–299, MR0872808.

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