Advanced Studies in Pure Mathematics 44, 2006
Potential Theory in Matsue
pp. 363–376

The Littlewood-Paley inequalities for Hardy-Orlicz spaces of harmonic functions on domains in $\mathbb{R}^n$

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Abstract.

For the unit disc $\mathbb{D}$ in $\mathbb{C}$, the harmonic Hardy spaces $\mathcal{H}^p$, $1 \leq p < \infty$, are defined as the set of harmonic functions h on $\mathbb{D}$ satisfying

$$\|h\|_p^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\theta})|^p d\theta < \infty.$$

The classical Littlewood-Paley inequalities for harmonic functions [3] in $\mathbb{D}$ are as follows: Let h be harmonic on $\mathbb{D}$. Then there exist positive constants C_1, C_2, independent of h, such that

(a) for $1 < p \leq 2$,

$$\|h\|_p^p \leq C_1 \left[|h(0)|^p + \iint_{\mathbb{D}} (1-|z|)^{p-1} |\nabla h(z)|^p dx\, dy \right].$$

(b) For $p \geq 2$, if $h \in \mathcal{H}^p$, then

$$\iint_{\mathbb{D}} (1-|z|)^{p-1} |\nabla h(z)|^p dx\, dy \leq C_2 \|h\|_p^p.$$

In the paper we consider generalizations of these inequalities to Hardy-Orlicz spaces $\mathcal{H}_\psi$ of harmonic functions on domains $\Omega \subsetneq \mathbb{R}^n$, $n \geq 2$, with Green function G satisfying the following: There exist constants α and β, $0 < \beta \leq 1 \leq \alpha < \infty$, such that for fixed $t_o \in \Omega$, there exist constants C_1 and C_2, depending only on t_o, such that $C_1 \delta(x)^\alpha \leq G(t_o, x)$ for all $x \in \Omega$, and $G(t_0, x) \leq C_2 \delta(x)^\beta$ for all $x \in \Omega \setminus B(t_o, \frac{1}{2}\delta(t_o))$.

Received March 30, 2005.
Revised April 30, 2005.
2000 *Mathematics Subject Classification.* 30D55, 31B05.
Key words and phrases. Green's function, Hardy-Orlicz spaces, harmonic function, harmonic majorant, holomorphic function, subharmonic function.

§1. Introduction

For the unit disc $\mathbb{D}$ in $\mathbb{C}$, the harmonic Hardy spaces $\mathcal{H}^p$, $1 \le p < \infty$, are defined as the set of harmonic functions h on $\mathbb{D}$ satisfying

$$\|h\|_p^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\theta})|^p d\theta < \infty.$$

The classical Littlewood-Paley inequalities for harmonic functions [3] in $\mathbb{D}$ are as follows: Let h be harmonic on $\mathbb{D}$. Then there exist positive constants C_1, C_2, independent of h, such that

(a) for $1 < p \le 2$,

$$\|h\|_p^p \le C_1 \left[|h(0)|^p + \iint_{\mathbb{D}} (1 - |z|)^{p-1} |\nabla h(z)|^p dx\, dy \right]. \tag{1.1}$$

(b) For $p \ge 2$, if $h \in \mathcal{H}^p$, then

$$\iint_{\mathbb{D}} (1 - |z|)^{p-1} |\nabla h(z)|^p dx\, dy \le C_2 \|h\|_p^p. \tag{1.2}$$

In 1956 T. M. Flett [2] proved that for analytic functions inequality (1.1) is valid for all p, $0 < p \le 2$. Hence if $u = \operatorname{Re} h$, h analytic, then since $|\nabla u| = |h'|$ it immediately follows that inequality (1.1) also holds for harmonic functions in $\mathbb{D}$ for all p, $0 < p \le 2$. A short proof of the Littlewood-Paley inequalities for harmonic functions in $\mathbb{D}$ valid for all p, $0 < p < \infty$ has also been given recently by Pavlović in [5]. The Littlewood-Paley inequalities are also known to be valid for harmonic functions in the unit ball in $\mathbb{R}^n$. In fact Stević [7] has recently proved that for $n \ge 3$, inequality (1.1) is valid for all $p \in [\frac{n-2}{n-1}, 1]$. In [10] analogue's of the Littlewood-Paley inequalities have been proved by the author for domains Ω in $\mathbb{R}^n$ for which the Green function satisfies $G(t_o, x) \approx \delta(x)$ for all $x \in \Omega \setminus B(t_o \frac{1}{2}\delta(t_o))$, where $\delta(x)$ denotes the distance from x to the boundary of Ω. In the same paper it was proved that for bounded domains with $C^{1,1}$ boundary the analogue of (1.1) is also valid for all p, $0 < p \le 1$.

In the present paper we extend the Littlewood-Paley inequalities to harmonic functions in the Hardy–Orlicz spaces $\mathcal{H}_\psi$ on domains $\Omega \subsetneq \mathbb{R}^n$, $n \ge 2$, with Green function G satisfying the following conditions: There exist constants α and β, $0 < \beta \le 1 \le \alpha < \infty$, such that for fixed $t_o \in \Omega$, there exist constants C_1 and C_2, depending only on t_o, such that

$$C_1 \delta(x)^\alpha \le G(t_o, x) \qquad \text{for all } x \in \Omega, \text{ and} \tag{1.3}$$

$$G(t_0, x) \le C_2 \delta(x)^\beta \qquad \text{for all } x \in \Omega \setminus B(t_o, \tfrac{1}{2}\delta(t_o))^1. \tag{1.4}$$

Let Ω be an arbitrary domain in $\mathbb{R}^n$, $n \geq 2$, and let ψ be a non-negative increasing convex function on $[0,\infty)$ satisfying $\psi(0) = 0$ and

$$\psi(2x) \leq c\,\psi(x) \tag{1.5}$$

for some positive constant c. We denote by $\mathcal{H}_\psi(\Omega)$ the set of real or complex valued harmonic functions h on Ω for which $\psi(|h|)$ has a harmonic majorant on Ω. Since ψ is convex and increasing, the function $\psi(|h|)$ is subharmonic on Ω. The existence of a harmonic majorant consequently guarantees the existence of a least harmonic majorant. For $h \in \mathcal{H}_\psi$ we denote the least harmonic majorant of $\psi(|h|)$ by H_ψ^h, and for fixed $t_o \in \Omega$ we set

$$N_\psi(h) = H_\psi^h(t_o). \tag{1.6}$$

It is known that $N_\psi(h)$ is given by

$$N_\psi(h) = \lim_{n\to\infty} \int_{\partial\Omega_n} \psi(|h(t)|)\,d\omega_n^{t_o}(t), \tag{1.7}$$

where $\{\Omega_n\}$ is a regular exhaustion of Ω and $\omega_n^{t_o}$ is the harmonic measure on $\partial\Omega_n$ with respect to the point t_o. Here we assume that $t_o \in \Omega_n$ for all n. With $\psi(t) = t^p$, $1 \leq p < \infty$, one obtains the usual Hardy $\mathcal{H}^p$ space of harmonic functions on Ω, with

$$\|h\|_p = \lim_{n\to\infty} \left(\int_{\partial\Omega_n} |h(t)|^p d\omega_n^{t_o}(t) \right)^{1/p}, \tag{1.8}$$

which is the usual norm on $\mathcal{H}^p(\Omega)$, $p \geq 1$.

In the paper we prove the following generalizations of the Littlewood-Paley inequalities.

Theorem 1. *Let $\Omega \subsetneq \mathbb{R}^n$ be a domain with Green function G satisfying inequalities* (1.3) *and* (1.4). *Let $\psi \geq 0$ be an increasing convex C^2 function on $[0,\infty)$ with $\psi(0) = 0$ satisfying* (1.5). *Set $\varphi(t) = \psi(\sqrt{t})$. Then there exist positive constants C_1 and C_2 such that the following hold for all $h \in \mathcal{H}_\psi(\Omega)$.*

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[1]As in [1] [4], if Ω is a bounded k-Lipschitz domain, then such constants α and β exist. If the boundary of Ω is C^2 or $C^{1,1}$, then $\alpha = \beta = 1$, and the inequalities can be established by comparing the Green function G to the Green function of balls that are internally and externally tangent to the boundary of Ω. By the results of Widman [11], the inequalities are also valid with $\alpha = \beta = 1$ for domains with $C^{1,\alpha}$ or Liapunov-Dini boundaries.

(a) *If φ is concave on $[0,\infty)$, then*

$$\mathcal{N}_\psi(h) \le C_1 \left[\psi(|h(t_o)|) + \int_\Omega \delta(x)^{\beta-2}\psi(\delta(x)|\nabla h(x)|)\, dx\right].$$

(b) *If φ is convex on $[0,\infty)$, then*

$$\psi(|h(t_o)|) + \int_\Omega \delta(x)^{\alpha-2}\psi(\delta(x)|\nabla h(x)|)\, dx \le C_2 \mathcal{N}_\psi(h).$$

An immediate consequence of the previous theorem with $\psi(t) = t^p$, $1 \le p < \infty$, is the following:

Theorem 2. *Let $\Omega \subsetneq \mathbb{R}^n$ be a domain with Green function G satisfying inequalities* (1.3) *and* (1.4)*, and let $1 \le p < \infty$. Then there exist positive constants C_1 and C_2 such that the following hold for all $h \in \mathcal{H}^p(\Omega)$.*

(a) *For $1 \le p \le 2$,*

$$\|h\|_p^p \le C_1 \left[|h(t_o)|^p + \int_\Omega \delta(x)^{\beta+p-2}|\nabla h(x)|^p\, dx\right].$$

(b) *For $2 \le p < \infty$,*

$$|h(t_o)|^p + \int_\Omega \delta(x)^{\alpha+p-2}|\nabla h(x)|^p\, dx \le C_2 \|h\|_p^p.$$

§2. Preliminaries

Our setting throughout the paper is $\mathbb{R}^n$, $n \ge 2$, the points of which are denoted by $x = (x_1, ..., x_n)$ with euclidean norm $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$. For $r > 0$ and $x \in \mathbb{R}^n$, set $B_r(x) = B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$ and $S_r(x) = S(x,r) = \{y \in \mathbb{R}^n : |x-y| = r\}$. For convenience we denote the ball $B(0,\rho)$ by B_ρ, and the unit sphere $S_1(0)$ by S. Lebesgue measure in $\mathbb{R}^n$ will be denoted by $d\lambda$ or simply dx, and the normalized surface measure on S by $d\sigma$. The volume of the unit ball B_1 in $\mathbb{R}^n$ will be denoted by ω_n. For an integrable function f on $\mathbb{R}^n$ we have

$$\int_{\mathbb{R}^n} f(x)dx = n\omega_n \int_0^\infty r^{n-1} \int_S f(r\zeta)\, d\sigma(\zeta)\, dr.$$

Finally, for a real (or complex) valued C^1 function f, the gradient of f is denoted by ∇f, and if f is C^2, the Laplacian Δf of f is given by

$$\Delta f = \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j^2}.$$

Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 2$, with $\Omega \subsetneq \mathbb{R}^n$. For $x \in \Omega$, let $\delta(x)$ denote the distance from x to the boundary of Ω, and set

$$B(x) = B(x, \tfrac{1}{2}\delta(x)) = \left\{y \in \Omega : |y - x| < \tfrac{1}{2}\delta(x)\right\}. \tag{2.1}$$

Then for all $y \in B(x)$ we have

$$\tfrac{1}{2}\delta(x) \leq \delta(y) \leq \tfrac{3}{2}\delta(x). \tag{2.2}$$

For the proof of Theorem 1 we require several preliminary lemmas.

Lemma 1. *For $f \in L^1(\Omega)$ and $\gamma \in \mathbb{R}$,*

$$\int_\Omega \delta(x)^\gamma |f(x)|\, dx \approx \int_\Omega \delta(w)^{\gamma-n} \left[\int_{B(w)} |f(x)|\, dx\right] dw.$$

Note. The notation $A \approx B$ means that there exist constants c_1 and c_2 such that $c_1 A \leq B \leq c_2 A$.

Proof. The proof is a straightforward application of Tonelli's theorem, and consequently is omitted. Details may be found in [10]. □

Lemma 2. *For $u \in C^2(\overline{B_\rho})$, $\rho > 0$,*

$$\int_S u(\rho\zeta)\, d\sigma(\zeta) = u(0) + \int_{B_\rho} \Delta u(x) G_\rho(x)\, dx,$$

where

$$G_\rho(x) = \begin{cases} \dfrac{1}{n(n-2)\omega_n}\left[\dfrac{1}{|x|^{n-2}} - \dfrac{1}{\rho^{n-2}}\right], & 0 < |x| \leq \rho, \quad n \geq 3, \\ \dfrac{1}{2\pi}\log\dfrac{\rho}{|x|}, & 0 < |x| \leq \rho, \quad n = 2, \end{cases} \tag{2.3}$$

is the Green function of B_ρ with singularity at 0.

Proof. The proof is an immediate consequence of Green's formula and hence is omitted. □

Lemma 3. *Let φ be an increasing absolutely continuous function on $[0, \infty)$ with $\varphi(0) = 0$.*

(a) *If φ is convex, then $\varphi(x) + \varphi(y) \leq \varphi(x+y)$ for all x, $y \in [0, \infty)$.*

(b) *If φ is concave, then $\varphi(x) + \varphi(y) \geq \varphi(x+y)$ for all x, $y \in [0, \infty)$.*

Proof. (a) Suppose φ is convex. Since φ is absolutely continuous and increasing, $\varphi(x) = \int_0^x \varphi'$ where $\varphi' \geq 0$. Hence

$$\varphi(x+y) = \int_0^{x+y} \varphi' = \varphi(x) + \int_x^{x+y} \varphi'.$$

But

$$\int_x^{x+y} \varphi'(t)dt = \int_0^y \varphi'(x+t)dt.$$

Since φ is convex, φ' is increasing. Thus

$$\int_0^y \varphi'(x+t)dt \geq \int_0^y \varphi'(t)dt = \varphi(y),$$

from which the result follows. The proof of (b) is similar. □

Lemma 4. *Suppose φ is an increasing C^2 function on $(0,\infty)$ with $\varphi(0) = 0$ and*

$$2t\varphi''(t) + \varphi'(t) \geq 0, \quad t > 0. \tag{2.4}$$

Let h be a harmonic function on $\overline{B}_\rho$, $\rho > 0$.

(a) If φ is concave, then

$$\int_{B_{\rho/4}} \rho^2 \Delta\varphi(|h|^2)dx \leq C \int_{B_\rho} \varphi(\rho^2|\nabla h|^2)dx.$$

(b) If φ is convex and satisfies inequality (1.5), then

$$\int_{B_\rho} \rho^2 \Delta\varphi(|h|^2)dx \geq C \int_{B_{\rho/2}} \varphi(\rho^2|\nabla h|^2)dx.$$

Remark. If u is a positive real-valued C^2 function, then

$$\Delta\varphi(u^2) = 2|\nabla u|^2 \left[2\varphi''(u^2)u^2 + \varphi'(u^2)\right] + 2\varphi'(u^2)u\Delta u.$$

Thus the hypothesis $2t\varphi''(t) + \varphi'(t) \geq 0$ guarantees that $\varphi(u^2)$ is subharmonic whenever u is subharmonic. For $\psi_p(t) = t^p$, the function $\varphi_p(t) = \psi_p(\sqrt{t}) = t^{p/2}$ satisfies inequality (2.4) if and only if $p \geq 1$.

Proof. We only prove the Lemma for $n \geq 3$, the special case $n = 2$ is similar. (a) Suppose φ is concave. Set $\epsilon = \rho/4$, $\delta = \rho/2$, and let G_δ be the Green function of B_δ with singularity at 0. For $|x| \leq \epsilon$,

$$\begin{aligned} G_\delta(x) &= \frac{1}{n(n-2)\omega_n}\left[\frac{1}{|x|^{n-2}} - \frac{1}{\delta^{n-2}}\right] \\ &\geq \frac{1}{n(n-2)\omega_n}\left[\frac{4^{n-2}}{\rho^{n-2}} - \frac{2^{n-2}}{\rho^{n-2}}\right] = c_n\rho^{2-n}. \end{aligned}$$

Hence

$$I_1 = \int_{B_\epsilon} \Delta\varphi(|h|^2)dx \le C\rho^{n-2} \int_{B_\delta} \Delta\varphi(|h(x)|^2)G_\delta(x)dx,$$

which by Lemma 2

$$= C\rho^{n-2}\left[\int_S \varphi(|h(\delta\zeta)|^2)d\sigma(\zeta) - \varphi(|h(0)|^2)\right].$$

Since φ is concave, $\int_S \varphi(|h|^2)d\sigma \le \varphi\left(\int_S |h|^2 d\sigma\right)$. Thus

$$I_1 \le C\rho^{n-2}\left[\varphi\left(\int_S |h(\delta\zeta)|^2 d\sigma(\zeta)\right) - \varphi(|h(0)|^2)\right].$$

Since φ is concave and increasing with $\varphi(0) = 0$, by Lemma 3

$$\varphi(b) - \varphi(a) \le \varphi(b-a), \quad 0 < a \le b.$$

Therefore

$$I_1 \le C\rho^{n-2}\varphi\left(\int_S |h(\delta\zeta)|^2 d\sigma(\zeta) - |h(0)|^2\right),$$

which by Green's identity (Lemma 2)

$$= C\rho^{n-2}\varphi\left(2\int_{B_\delta} |\nabla h(x)|^2 G_\delta(x)dx\right).$$

Hence

$$I_1 \le C\rho^{n-2}\varphi\left(2\sup_{x\in B_\delta} |\nabla h(x)|^2 \int_{B_\delta} G_\delta(x)dx\right).$$

But

$$\int_{B_\delta} G_\delta(x)dx = \frac{1}{2n}\delta^2.$$

Therefore since $\delta = \frac{1}{2}\rho$,

$$I_1 \le C\rho^{n-2}\varphi\left(\frac{\rho^2}{4n}\sup_{x\in B_\delta} |\nabla h(x)|^2\right),$$

which since φ is increasing

$$\le C\rho^{n-2}\sup_{x\in B_\delta} \varphi(\rho^2|\nabla h(x)|^2).$$

But since $x \to \varphi(\rho^2|\nabla h(x)|^2)$ is subharmonic,

$$\varphi(\rho^2|\nabla h(x)|^2) \le \frac{C}{\rho^n}\int_{B_\rho} \varphi(\rho^2|\nabla h(y)|^2)dy.$$

for all $x \in B_\delta$. Therefore, combining the above we have

$$\int_{B_{\rho/4}} \rho^2 \Delta\varphi(|h|^2)\, d\lambda \le C\int_{B_\rho} \varphi(\rho^2|\nabla h|^2)\, d\lambda.$$

(b) Suppose φ is convex and satisfies inequality (1.5). By Lemma 2

$$\int_{B_\delta} \Delta\varphi(|h(x)|^2)G_\delta(x)\, dx = \int_S \varphi(|h(\delta\zeta)|^2)\, d\sigma(\zeta) - \varphi(|h(0)|^2),$$

which since φ is convex

$$\ge \varphi\left(\int_S |h(\delta\zeta)|^2 d\sigma(\zeta)\right) - \varphi(|h(0)|^2) = I_2.$$

But by Lemma 3,

$$I_2 \ge \varphi\left(\int_S |h(\delta\zeta)|^2 d\sigma(\zeta) - |h(0)|^2\right).$$

Thus by Lemma 2,

$$\int_{B_\delta} \Delta\varphi(|h(x)|^2)G_\delta(x)\, dx \ge \varphi\left(2\int_{B_\delta} |\nabla h(x)|^2 G_\delta(x)\, dx\right).$$

For $|x| \le \epsilon$ and $n \ge 3$, $G_\delta(x) \ge c_n\rho^{2-n}$, where $c_n = 2^{2n-5}/n(n-2)\omega_n$. Therefore

$$2\int_{B_\delta} |\nabla h(x)|^2 G_\delta(x)\, dx \ge \frac{2^{2n-4}\rho^{2-n}}{n(n-2)\omega_n}\int_{B_\epsilon} |\nabla h(x)|^2 dx,$$

which since $|\nabla h(x)|^2$ is subharmonic and $\epsilon = \rho/4$

$$\ge \frac{1}{2^4 n(n-2)}\rho^2|\nabla h(0)|^2 \ge \frac{1}{2^{n+3}}\rho^2|\nabla h(0)|^2.$$

By inequality (1.5)

$$\varphi\left(\frac{1}{2^{n+3}}\rho^2|\nabla h(0)|^2\right) \ge \frac{1}{c^{n+3}}\varphi(\rho^2|\nabla h(0)|^2),$$

where c is the constant in inequality (1.5). Combining the above gives

$$\varphi(\rho^2|\nabla h(0)|^2) \le c^{n+3} \int_{B_\delta} \Delta\varphi(|h(x)|^2) G_\delta(x)\, dx.$$

Since $G_\delta(x) \le C_n |x|^{2-n}$ we have

$$\varphi(\rho^2|\nabla h(0)|^2) \le C_n \int_{B_\delta} \Delta\varphi(|h(x)|^2)|x|^{2-n}\, dx,$$

where C_n is a constant depending only on n.

For $w \in B_\delta$, set $h_w(x) = h(w+x)$. Thus

$$\varphi(\rho^2|\nabla h(w)|^2) \le C_n \int_{B_\delta} \Delta_x\varphi(|h_w(x)|^2)|x|^{2-n}\, dx,$$

which by the change of variable $y = w + x$

$$= C_n \int_{B_\delta(w)} \Delta\varphi(|h(y)|^2)|y-w|^{2-n} dy.$$

Therefore,

$$\int_{B_\delta} \varphi(\rho^2|\nabla h(w)|^2)\, dw \le C_n \int_{B_\delta}\int_{B_\delta(w)} \Delta\varphi(|h(y)|^2)|y-w|^{2-n}\, dy\, dw,$$

which by Fubini's theorem

$$\le C_n \int_{B_{2\delta}} \Delta\varphi(|h(y)|^2) \left(\int_{B_\delta(y)} |y-w|^{2-n} dw \right) dy.$$

But

$$\int_{B_\delta(y)} |y-w|^{2-n} dw = \int_{B_\delta} |x|^{2-n} dx = n\omega_n \frac{\rho^2}{4}.$$

Therefore,

$$\int_{B_\delta} \varphi(\rho^2|\nabla h|^2)\, d\lambda \le C_n \rho^2 \int_{B_{2\delta}} \Delta\varphi(|h|^2)\, d\lambda,$$

which completes the proof. □

Lemma 5. *Let ψ and φ be as in Theorem 1, and let h be harmonic on Ω. Assume that $\psi(|h|) \in C^2(\Omega)$. Then for $\gamma \in \mathbb{R}$, the following hold:*

(a) If φ is concave, then

$$\int_\Omega \delta(x)^\gamma \Delta\psi(|h(x)|)dx \le C \int_\Omega \delta(x)^{\gamma-2}\psi(\delta(x)|\nabla h(x)|)dx.$$

(b) If φ is convex and satisfies inequality (1.5), then

$$\int_\Omega \delta(x)^\gamma \Delta\psi(|h(x)|)dx \ge C \int_\Omega \delta(x)^{\gamma-2}\psi(\delta(x)|\nabla h(x)|)dx.$$

Proof. (a) By Lemma 1

$$\int_\Omega \delta(x)^\gamma \Delta\psi(|h(x)|)dx \le C \int_\Omega \delta(w)^{\gamma-n}\left[\int_{B(w,\frac{1}{8}\delta(w))} \Delta\psi(|h(y)|)dy\right] dw.$$

Set $\rho = \frac{1}{2}\delta(w)$ and $u(x) = h(w+x)$. Then

$$\int_{B(w,\frac{1}{8}\delta(w))} \Delta\psi(|h(y)|)dy = \int_{B_{\rho/4}} \Delta\psi(|u(x)|)dx,$$

which by Lemma 4

$$\le C\rho^{-2}\int_{B_\rho} \psi(\rho|\nabla u(x)|)dx = C\delta(w)^{-2}\int_{B_\rho(w)} \psi(\tfrac{1}{2}\delta(w)|\nabla h(y)|)dy.$$

But $\frac{1}{2}\delta(w) \le \delta(y)$ for all $y \in B_\rho(w)$. Hence since ψ is increasing, $\psi(\frac{1}{2}\delta(w)|\nabla h(y)|) \le \psi(\delta(y)|\nabla h(y)|)$, and thus

$$\int_{B(w,\frac{1}{8}\delta(w))} \Delta\psi(|h(y)|)dy \le C\delta(w)^{-2}\int_{B(w)} \psi(\delta(y)|\nabla h(y)|)dy.$$

Finally, by Lemma 1,

$$\int_\Omega \delta(w)^{\gamma-n-2}\left[\int_{B(w)} \psi(\delta(y)|\nabla h(y)|)dy\right] dw \le C\int_\Omega \delta(x)^{\gamma-2}\psi(\delta(x)|\nabla h(x)|)dx,$$

which proves (a). The proof of part (b) proceeds in the same manner, except that this case also requires inequality (1.5). □

§3. Proof of Theorem 1

Before proving Theorem 1 we require two preliminary results about subharmonic functions. Let $\mathcal{S}^+(\Omega)$ denote the set of non-negative subharmonic functions on Ω that have a harmonic majorant on Ω. As in the Introduction, for $f \in \mathcal{S}^+(\Omega)$ we let H_f denote the least harmonic majorant of f on Ω. For convenience we will assume that $f \in C^2(\Omega)$. As in [8],[9] we have the following.

Lemma 6. *Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, with Green function G, and let $f \in C^2(\Omega)$. Then $f \in \mathcal{S}^+(\Omega)$ if and only if there exists $t_o \in \Omega$ such that*

$$\int_\Omega G(t_o, x)\Delta f(x)\, dx < \infty. \tag{3.1}$$

If this is the case, then by the Riesz decomposition theorem

$$H_f(x) = f(x) + \int_\Omega G(x, y)\Delta f(y)\, dy \tag{3.2}$$

If the subharmonic function f is not C^2, then the quantity $\Delta f(x)\, dx$ may be replaced by $d\mu_f$, where μ_f is the Riesz measure of the subharmonic function f.

Lemma 7. *Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, with Green function G satisfying* (1.3) *and* (1.4). *Let $t_o \in \Omega$ be fixed, and let α and β be as in inequalities* (1.3) *and* (1.4) *respectively. Then there exists constants C_1 and C_2, depending only on t_o and Ω, such that for all $f \in \mathcal{S}^+(\Omega) \cap C^2(\Omega)$,*

$$C_1 \left[f(t_o) + \int\limits_\Omega \delta(x)^\alpha \Delta f(x) dx \right] \leq H_f(t_o)$$
$$\leq C_2 \left[\int\limits_{B(t_o)} f(x) dx + \int\limits_\Omega \delta(x)^\beta \Delta f(x) dx \right].$$

Proof. The left side of the previous inequality is an immediate consequence of identity (3.2) and inequality (1.3). For the right side, integrating equation (3.2) over $B(t_o)$ gives

$$H_f(t_o) = \frac{1}{\omega_n \rho_o^n} \int_{B(t_o)} f(x)\, dx + \frac{1}{\omega_n \rho_o^n} \int_{B(t_o)} \int_\Omega G(x, y)\Delta f(y)\, dy\, dx,$$

where $\rho_o = \frac{1}{2}\delta(t_o)$. By Fubini's theorem,

$$\frac{1}{\omega_n \rho_o^n}\int_{B(t_o)}\int_\Omega G(x,y)\Delta f(y)\,dy\,dx = \frac{1}{\omega_n \rho_o^n}\int_\Omega \Delta f(y)\int_{B(t_o)} G(x,y)\,dx\,dy.$$

Set

$$I(y) = \frac{1}{\omega_n \rho_o^n}\int_{B(t_o)} G(x,y)\,dx.$$

To complete the proof it remains to be shown that $I(y) \le C\,\delta(y)^\beta$.

If $y \notin B(t_o)$, then since $x \to G(x,y)$ is harmonic on $B(t_o)$ and G satisfies inequality (1.4),

$$I(y) = G(t_o, y) \le C_2\delta(y)^\beta.$$

Suppose $y \in B(t_o)$ and $n \ge 3$. Then since $G(x,y) \le c_n|x-y|^{2-n}$,

$$I(y) \le \frac{c_n}{\omega_n \rho_o^n}\int_{B(t_o)}|x-y|^{2-n}dx \le \frac{c_n}{\omega_n \rho_o^n}\int_{B(y,2\rho_o)}|x-y|^{2-n}dx = 2nc_n\rho_o^{2-n}.$$

But for $y \in B(t_o)$, $\rho_o \le 2\delta(y)$. Thus

$$I(y) \le 2nc_n 2^\beta \delta(y)^\beta \rho_o^{2-n-\beta} = C\delta(y)^\beta,$$

where C is a constant depending only on t_o and Ω. □

Proof of Theorem 1. (a) Let ψ be as in the statement of the theorem, and let h be a real-valued harmonic function on Ω. Set $h_\epsilon(x) = h(x) + i\epsilon$. Then h_ϵ is harmonic on Ω and $\psi(|h_\epsilon|) \in C^2(\Omega)$. Hence by Lemma 7,

$$N_\psi(h_\epsilon) \le C_2\left[\int_{B(t_o)}\psi(|h_\epsilon(x)|)\,dx + \int_\Omega \delta(x)^\beta \Delta\psi(|h_\epsilon(x)|)\,dx\right],$$

which by Lemma 5(a)

$$\le C_2\left[\int_{B(t_o)}\psi(|h_\epsilon(x)|)\,dx + \int_\Omega \delta(x)^{\beta-2}\psi(\delta(x)|\nabla h(x)|)\,dx\right].$$

Letting $\epsilon \to 0^+$ gives

$$N_\psi(h) \le C_2\left[\max_{x\in B(t_o)}\psi(|h(x)|) + \int_\Omega \delta(x)^{\beta-2}\psi(\delta(x)|\nabla h(x)|)\,dx\right].$$

It remains to be shown that

(3.3)
$$\max_{x\in B(t_o)} \psi(|h(x)|) \le C\left[\psi(|h(t_o)|) + \int_\Omega \delta(x)^{\beta-2}\psi(\delta(x)|\nabla h(x)|)\,dx\right].$$

Without loss of generality we take $t_o = 0$. As a consequence of the Fundamental Theorem of Calculus, for all $x \in B(t_o)$,

$$|h(x)| \le |h(0)| + \rho_o \max_{y\in B(t_o)} |\nabla h(y)|.$$

Since ψ is increasing, convex, and continuous, and satisfies property (1.5)

$$\psi(|h(x)|) \le \frac{c}{2}\left[\psi(|h(0)|) + \max_{y\in B(t_o)} \psi(\rho_o|\nabla h(y)|)\right].$$

Also, since $y \to \psi(\rho_o|\nabla h(y)|)$ is subharmonic,

$$\psi(\rho_o|\nabla h(y)|) \le \frac{2^n}{\omega_n \rho_o^n}\int_{B(y,\frac{1}{2}\rho_o)} \psi(\rho_o|\nabla h(x)|)\,dx$$

But $\rho_o \le \delta(y) \le 3\rho_o$ for all $y \in B(t_o)$, and $\frac{1}{2}\delta(y) \le \delta(x) \le \frac{3}{2}\delta(y)$ for all $x \in B(y, \frac{1}{2}\rho_o)$. Thus

$$\psi(\rho_o|\nabla h(y)|) \le C(\rho_o)\int_\Omega \delta(x)^{\beta-2}\psi(\delta(x)|\nabla h(x)|)\,dx,$$

from which inequality (3.3) now follows. This completes the proof of (a). The proof of (b) is an immediate consequence of Lemma 7 and Lemma 5(b).

§4. Remarks

The techniques employed in this paper may also be used to prove analogue's of Theorems 1 and 2 for Hardy-Orlicz spaces of holomorphic functions on a domain $\Omega \subsetneq \mathbb{C}^n$, $n \ge 1$.

In this setting the spaces $\mathcal{H}_\psi$ are traditionally defined as in [6, page 83]. For a non-negative, non-decreasing convex function ψ on $(-\infty, \infty)$ with $\lim_{t\to-\infty} \psi(t) = 0$, the Hardy-Orlicz space $\mathcal{H}_\psi(\Omega)$ is defined as the set of holomorphic functions f on Ω for which $\psi(\log|f|)$ has a harmonic majorant on Ω. As in (1.5) we set $N_\psi(f) = H_\psi^f(t_o)$, where H_ψ^f denotes the least harmonic majorant of $\psi(\log|f|)$. With $\psi(t) = e^{pt}$, $0 < p < \infty$, one obtains the usual Hardy $\mathcal{H}^p$ space of holomorphic functions on Ω.

To obtain the analogue of Theorem 1 one considers the function $\varphi(t) = \psi(\frac{1}{2}\log t)$. In this setting, hypothesis (2.4) can be replaced by

$$x\varphi''(x) + \varphi'(x) \geq 0 \tag{4.1}$$

for all $x \in (0, \infty)$. If the above holds, then it is easily shown that for f holomorphic on Ω, $\varphi(|f|^2)$ is plurisubharmonic on Ω, hence also subharmonic. Clearly $\varphi(x) = \psi(\frac{1}{2}\log x)$ satisfies (4.1) whenever ψ is convex. The details of the statements and proofs of the appropriate theorems are left to the reader.

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