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Hölder continuity of solutions to quasilinear elliptic equations with measure data

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Abstract.

We consider quasi-linear second order elliptic differential equations with measures date on the right hand side. In this talk, we investigate Hölder continuity of solutions of such equations.

§1. Introduction.

Let G be a bounded open set in $\mathbf{R}^N$ $(N \geq 2)$ and $1 < p < N$. Suppose that ν is a signed Radon measure on G. We consider quasi-linear second order elliptic differential equations with measure date of the form

$$(\mathrm{E}_\nu) \qquad -\operatorname{div} \mathcal{A}(x, \nabla u(x)) + \mathcal{B}(x, u(x)) = \nu,$$

where $\mathcal{A}(x,\xi) : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^N$ satisfies structure conditions of p-th order and $\mathcal{B}(x,t) : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ is nondecreasing in t (see section 2 below for more details).

Hölder continuity of a solution to the equation (E_ν) was investigated in [17], [8] and [6]. In these papers, they showed that the solution of (E_ν) is locally Hölder continuous with some exponent if the signed Radon measure ν satisfies the condition that there exist constants $M > 0$ and $0 < \beta < \lambda$ with

$$|\nu|(B(x_0, r)) \leq M\ r^{N-p+\beta(p-1)}$$

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whenever $B(x, 3r) \subset G$, where λ is a number depending on N, p and structure conditions for $\mathcal{A}$ and $\mathcal{B}$. Further, in [7], in the case $\mathcal{B} = 0$ in the equation (E_ν), namely for the equation

$$-\operatorname{div} \mathcal{A}(x, \nabla u(x)) = \nu \tag{1}$$

and ν is a nonnegative Radon measure, Kilpeläinen and Zhong showed that a solution to the equation (1) is Hölder continuous with the same exponent β. In this talk, we extend this result to the case of the equation (E_ν).

Throughout this paper, we use some standard notation without explanation.

§2. Preliminaries.

We assume that $\mathcal{A} : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^N$ and $\mathcal{B} : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ satisfy the following conditions for $1 < p < N$:

(A.1) $x \mapsto \mathcal{A}(x, \xi)$ is measurable on $\mathbf{R}^N$ for every $\xi \in \mathbf{R}^N$ and $\xi \mapsto \mathcal{A}(x, \xi)$ is continuous for a.e. $x \in \mathbf{R}^N$;

(A.2) $\mathcal{A}(x, \xi) \cdot \xi \geq \alpha_1 |\xi|^p$ for all $\xi \in \mathbf{R}^N$ and a.e. $x \in \mathbf{R}^N$ with a constant $\alpha_1 > 0$;

(A.3) $|\mathcal{A}(x, \xi)| \leq \alpha_2 |\xi|^{p-1}$ for all $\xi \in \mathbf{R}^N$ and a.e. $x \in \mathbf{R}^N$ with a constant $\alpha_2 > 0$;

(A.4) $\big(\mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2)\big) \cdot \big(\xi_1 - \xi_2\big) > 0$ whenever $\xi_1,\ \xi_2 \in \mathbf{R}^N$, $\xi_1 \neq \xi_2$, for a.e. $x \in \mathbf{R}^N$;

(B.1) $x \mapsto \mathcal{B}(x, t)$ is measurable on $\mathbf{R}^N$ for every $t \in \mathbf{R}$ and $t \mapsto \mathcal{B}(x, t)$ is continuous for a.e. $x \in \mathbf{R}^N$;

(B.2) For any open set $G \Subset \mathbf{R}^N$, there is a constant $\alpha_3(G) \geq 0$ such that $|\mathcal{B}(x, t)| \leq \alpha_3(G)(|t|^{p-1} + 1)$ for all $t \in \mathbf{R}$ and a.e. $x \in G$;

(B.3) $t \mapsto \mathcal{B}(x, t)$ is nondecreasing on $\mathbf{R}$ for a.e. $x \in \mathbf{R}^N$.

We consider elliptic quasi-linear equations of the form

$$-\operatorname{div} \mathcal{A}(x, \nabla u(x)) + \mathcal{B}(x, u(x)) = 0. \tag{E}$$

For an open subset G of $\mathbf{R}^N$, we consider the Sobolev spaces $W^{1,p}(G)$, $W_0^{1,p}(G)$ and $W_{loc}^{1,p}(G)$.

Let G be an open subset of $\mathbf{R}^N$. A function $u \in W_{loc}^{1,p}(G)$ is said to be a (weak) *solution* of (E) in G if

$$\int_G \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx + \int_G \mathcal{B}(x, u) \varphi \, dx = 0$$

for all $\varphi \in C_0^\infty(G)$.

A continuous solution of (E) in an open subset G of $\mathbf{R}^N$ is called $(\mathcal{A}, \mathcal{B})$-*harmonic* in G.

We can see the following proposition by the proof of [14; Theorem 4.7]. By carefully analyzing the proof of [14; Theorem 4.2 and Theorem 4.7], we can choose constants c and $0 < \lambda \le 1$ independent of the radius R if $R \le 1$.

Proposition 2.1. *Let G be a bounded open set. Then there are constants c and $0 < \lambda \le 1$ such that for $B(x_0, R) \Subset G$ and for every $(\mathcal{A}, \mathcal{B})$-harmonic function h in G with $|h| \le L$ in $B(x_0, R)$,*

$$osc(h, B(x_0, r)) \le c\left(\frac{r}{R}\right)^{\lambda}\big(osc(h, B(x_0, R)) + R\big),$$

whenever $0 < r < R \le 1$. Here c depends only on $N, p, \alpha_1, \alpha_2, \alpha_3(G)$ and L and λ depends only on N, p, α_1, α_2 and $\alpha_3(G)$.

In the case of $\mathcal{A}(x, \xi) = |\xi|^{p-2}\xi$ and $\mathcal{B} = 0$, namely for the p-Laplace equation, we can choose $\lambda = 1$ ([4; Lemma 2.1]).

We recall the following propositions ([13; Theorem 2.2 and putting $k = 0$ in Definition 2.1, and Lemma 3.1]).

Proposition 2.2. *Let G be a bounded open set and $M_0 \ge 0$. Then there is a constant c such that, for every $(\mathcal{A}, \mathcal{B})$-harmonic function h in G, nonnegative $\eta \in C_0^\infty(G)$ and constant M with $|M| \le M_0$,*

$$\begin{aligned}\int_{\{h>M\}} |\nabla h|^p \eta^p \, dx \quad &\le \quad c \int_G \max(h - M, 0)^p \left(\eta^p + |\nabla \eta|^p\right) dx \\ &\quad + c\,(M_0 + 1)^p \int_{\{h>M\}} \eta^p \, dx,\end{aligned}$$

where c depends only on p, α_1, α_2 and $\alpha_3(G)$.

Proposition 2.3. *Let G be a bounded open set, $M_0 \ge 0$, $\gamma \in (0, p]$. Then there is a constant c such that, for every $r \in (0, 1]$ with $B(x_0, r) \Subset G$, an $(\mathcal{A}, \mathcal{B})$-harmonic function h in G and a constant M with $|M| \le M_0$,*

$$\sup_{B(x_0, r/2)} |h - M| \le c\left(\frac{1}{|B(x_0, r)|}\int_{B(x_0, r)} |h - M|^{\gamma} dx\right)^{1/\gamma} + c\, r,$$

where c depends only on p, α_1, α_2, $\alpha_3(G)$, γ and M_0.

Lemma 2.1. *Let G be a bounded open set. Then there is a constant c depending only on p, N, α_1, α_2 and $\alpha_3(G)$ such that for $B(x_0, R) \subset G$*

with $R \le 1$, $u \in W^{1,p}(B(x_0,R))$ and the $(\mathcal{A},\mathcal{B})$-harmonic function h with $h-u \in W_0^{1,p}(B(x_0,R))$

$$\left(\int_{B(x_0,R)} |\nabla h|^p\,dx\right)^{1/p} \le c\left\{\left(\int_{B(x_0,R)} |u|^p\,dx\right)^{1/p} + \left(\int_{B(x_0,R)} |\nabla u|^p\,dx\right)^{1/p} + R^{N/p}\right\}.$$

Proof. Fix $B = B(x_0,R) \subset G$ with $R \le 1$ and let $\|\cdot\|_{p,G}$ denote the usual $L^p(G)$-norm. It follows from (A.2), (A.3), (B.2) and (B.3) that

$$\begin{aligned}\|\nabla h\|_{p,B}^p &\le \alpha_1^{-1}\int_B \mathcal{A}(x,\nabla h)\cdot\nabla h\,dx\\ &= \alpha_1^{-1}\left\{\int_B \mathcal{A}(x,\nabla h)\cdot\nabla u\,dx - \int_B \mathcal{B}(x,h)(h-u)\,dx\right\}\\ &\le \alpha_1^{-1}\alpha_2\,\|\nabla h\|_{p,B}^{p-1}\,\|\nabla u\|_{p,B} - \alpha_1^{-1}\int_B \mathcal{B}(x,u)(h-u)\,dx\\ &\le \alpha_1^{-1}\alpha_2\,\|\nabla h\|_{p,B}^{p-1}\,\|\nabla u\|_{p,B}\\ &\qquad + \alpha_1^{-1}\alpha_3(G)\,\||u|+1\|_{p,B}^{p-1}\,\|u-h\|_{p,B}.\end{aligned}$$

Because $h-u \in W_0^{1,p}(B)$, by the Poincaré inequality we have

$$\|h-u\|_{p,B} \le c\,\|\nabla h - \nabla u\|_{p,B} \le c\,(\|\nabla h\|_{p,B} + \|\nabla u\|_{p,B}),$$

where we can take c depending only on N because $R \le 1$. Also,

$$\||u|+1\|_{p,B}^{p-1} \le c'\,(\|u\|_{p,B}^{p-1} + R^{N(p-1)/p}),$$

with $c' = c'(p) > 0$. Thus, by the above inequalities and Young's inequality we have

$$\begin{aligned}\|\nabla h\|_{p,B}^p &\le c_1\,\|\nabla h\|_{p,B}^{p-1}\,\|\nabla u\|_{p,B}\\ &\qquad + c_2\,(\|u\|_{p,B}^{p-1} + R^{N(p-1)/p})\,(\|\nabla h\|_{p,B} + \|\nabla u\|_{p,B})\\ &\le \frac{1}{2}\,\|\nabla h\|_{p,B}^p + c_3\,(\|\nabla u\|_{p,B}^p + \|u\|_{p,B}^p + R^N).\end{aligned}$$

Hence $\|\nabla h\|_{p,B}^p \le 2c_3\,(\|\nabla u\|_{p,B}^p + \|u\|_{p,B}^p + R^N)$, which implies the desired inequality. □

Lemma 2.2. *Suppose that G is a bounded open set and $B(x_0,R) \Subset G$. There exists a number $\lambda = \lambda(N,p,\alpha_1,\alpha_2,\alpha_3(G)) > 0$ such that for* *every* $0 < r < R \leq 1$ *and* $(\mathcal{A}, \mathcal{B})$*-harmonic function* h *in* G *with* $|h| \leq L$ *in* $B(x_0, R)$ *it holds that*

$$\int_{B(x_0,r)} |\nabla h|^p \, dx \leq c \left(\frac{r}{R}\right)^{N-p+p\lambda} \int_{B(x_0,R)} |\nabla h|^p \, dx + c\, R^N,$$

where $c = c(N, p, \alpha_1, \alpha_2, \alpha_3(G), L) > 0$.

Proof. We may assume that $0 < r < \frac{R}{4}$. From Proposition 2.2 and Proposition 2.1 we obtain

$$\begin{aligned}
&\int_{B(x_0,r)} |\nabla h|^p \, dx \leq \frac{c}{r^p} \int_{B(x_0,2r)} \{(h - \inf_{B(x_0,2r)} h)^p + (L+1)^p r^p\} \, dx \\
&\leq \frac{c}{r^p} \left\{ \left(\sup_{B(x_0,2r)} h - \inf_{B(x_0,2r)} h \right)^p + (L+1)^p r^p \right\} r^N \\
&\leq c\, r^{N-p} \\
&\quad \times \left[\left\{ \left(\frac{r}{R}\right)^\lambda \left(\sup_{B(x_0,R/2)} h - \inf_{B(x_0,R/2)} h + R \right) \right\}^p + (L+1)^p r^p \right] \\
&\leq c\, r^{N-p} \left\{ \left(\frac{r}{R}\right)^{p\lambda} \left(\sup_{B(x_0,R/2)} h - \inf_{B(x_0,R/2)} h \right)^p + R^p \right\}.
\end{aligned}$$

On the other hand, setting

$$h_R = \frac{1}{|B(x_0, R)|} \int_{B(x_0,R)} h \, dx,$$

by Proposition 2.3 and the Poincaré inequality, we have

$$\begin{aligned}
&\left(\sup_{B(x_0,R/2)} h - \inf_{B(x_0,R/2)} h \right)^p \\
&\leq 2 \sup_{B(x_0,R/2)} |h - h_R|^p \\
&\leq \frac{c}{|B(x_0, R)|} \int_{B(x_0,R)} |h - h_R|^p \, dx + c\, R^p \\
&\leq \frac{c\, R^p}{|B(x_0, R)|} \int_{B(x_0,R)} |\nabla h|^p \, dx + c\, R^p.
\end{aligned}$$

Hence,

$$\begin{aligned}\int_{B(x_0,r)} |\nabla h|^p\,dx &\le c\,r^{N-p}\left\{\left(\frac{r}{R}\right)^{p\lambda}\left(\frac{1}{R}\right)^{N-p}\int_{B(x_0,R)}|\nabla h|^p\,dx + R^p\right\}\\ &\le c\left(\frac{r}{R}\right)^{N-p+p\lambda}\int_{B(x_0,R)}|\nabla h|^p\,dx + c\,R^N.\end{aligned}$$

□

§3. Hölder continuity of solutions to $(\mathbf{E}_\nu)$.

In this section, we establish Hölder continuity of solutions to the equation (E_ν). First, we recall the following Adams' inequality ([17; Theorem 3.3]).

Proposition 3.1. *Suppose that ν is a nonnegative Radon measure supported in an open set Ω such that there is a constant M with the property that for all $x \in \mathbf{R}^N$ and $0 < r < \infty$,*

$$\nu(B(x,r)) \le M\ r^a$$

where $a = q(N/p - 1)$, $1 < p < q < \infty$ and $p < N$. If $u \in W_0^{1,p}(\Omega)$, then

$$\left(\int_\Omega |u|^q\,d\nu\right)^{1/q} \le c\,M^{1/q}\left(\int_\Omega |\nabla u|^p\,dx\right)^{1/p},$$

where $c = c(p,q,N)$.

Let G be an open subset in $\mathbf{R}^N$. A function $u : G \to \mathbf{R}\cup\{\infty\}$ is said to be $(\mathcal{A},\mathcal{B})$-*superharmonic* in G if it is lower semicontinuous, finite on a dense set in G and, for each bounded open set U and for $h \in C(\overline{U})$ which is $(\mathcal{A},\mathcal{B})$-harmonic in U, $u \ge h$ on ∂U implies $u \ge h$ in U. $(\mathcal{A},\mathcal{B})$-*subharmonic* functions are similarly defined.

To show Hölder continuity of solutions to the equation (E_ν), we prepare the following lemma.

Lemma 3.1. *Suppose that G is a bounded open set, $B(x_0,R) \Subset G$, $0 < \beta < 1$, ν is a signed Radon measure on G such that*

$$|\nu|(B(x_0,r)) \le c_0\ r^{N-p+\beta(p-1)}$$

for every $0 < r \le R$ and $u \in W_{loc}^{1,p}(G)$ is a solution of (E_ν) in G with $|u| \le L$ in $B(x_0,R)$. Then for every $0 < r \le R \le 1$ and $\varepsilon >$

0, there exist constants $c_1 = c_1(N, p, \alpha_1, \alpha_2, \alpha_3(G), L) > 0$ *and* $c_2 = c_2(N, p, \alpha_1, \alpha_2, \alpha_3(G), \beta, c_0, \varepsilon, L) > 0$ *such that*

$$\int_{B(x_0,r)} |\nabla u|^p\, dx \leq c_1 \left(\left(\frac{r}{R}\right)^{N-p+p\lambda} + \varepsilon\right) \int_{B(x_0,R)} |\nabla u|^p\, dx + c_2\, R^{N-p+p\beta}.$$

where λ *is the constant in Lemma* 2.2.

Proof. We may assume that $0 < r < \frac{R}{2}$. Let h be an $(\mathcal{A}, \mathcal{B})$-harmonic function with $u - h \in W_0^{1,p}(B(x, R))$. First, we will show that

$$|h| \leq L' \tag{3.1}$$

on $B(x, R)$ with $L' = L'(\mathcal{A}, \mathcal{B}, G, L)$. Let B_0 be a ball containing G. There exists an $(\mathcal{A}, \mathcal{B})$-harmonic function h_0 in B_0 belonging to $W_0^{1,p}(B_0)$ (see [10; Theorem 1.4]). Then h_0 is continuous on $\overline{B_0}$ and hence bounded in G. Let $-m_1 \leq h_0 \leq m_2$ in G with $m_1 \geq 0$ and $m_2 \geq 0$. Then, $v_1 = h_0 + m_1 + L$ is $(\mathcal{A}, \mathcal{B})$-superharmonic and $v_1 \geq L$ in G; and $v_2 = h_0 - m_2 - L$ is $(\mathcal{A}, \mathcal{B})$-subharmonic and $v_2 \leq -L$ in G. Since

$$0 \geq \min(0, v_1 - h) \geq \min(0, L - h) \geq \min(0, u - h) \in W_0^{1,p}(B(x, R)),$$

$\min(0, v_1 - h) \in W_0^{1,p}(B(x, R))$. Hence by the comparison principle (see [16; Proposition 5.1.1 and Lemma 2.2.1]), $v_1 \geq h$, so that $h \leq L + m_1 + m_2$. Similarly, we see that $v_2 \leq h$, which shows $h \geq -(L + m_1 + m_2)$. Thus, we have (3.1) with $L' = L + m_1 + m_2$.

Next, we note that $|\nu| \in (W_0^{1,p}(V))^*$ for any $V \Subset G$, that is, $|\nu|$ is in the dual space of $W_0^{1,p}(V)$. Indeed, there exists an $\mathcal{A}$-superharmonic function U in G satisfying

$$-\operatorname{div} \mathcal{A}(x,\, DU(x)) = |\nu|$$

with $\min(U, k) \in W_0^{1,p}(G)$ for all $k > 0$, where DU is the generalized gradient of U (see [5; Theorem 2.4]). Then by [6; Theorem 4.16], U is locally bounded in G. Thus, $U \in W_{loc}^{1,p}(G)$ (see [3; Corollary 7.20]). Hence we see that $|\nu| \in (W_0^{1,p}(V))^*$ (cf. [6; p.142]). Thus, by (A.2), (A.3) and (B.3) we have

$$
\begin{aligned}
\alpha_1 \int_{B(x_0,r)} |\nabla u|^p \, dx &\le \int_{B(x_0,r)} \mathcal{A}(x,\nabla u)\cdot \nabla u \, dx \\
&= \int_{B(x_0,r)} (\mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla h))\cdot(\nabla u - \nabla h)\, dx \\
&\quad + \int_{B(x_0,r)} \mathcal{A}(x,\nabla h)\cdot(\nabla u - \nabla h)\, dx \\
&\quad + \int_{B(x_0,r)} \mathcal{A}(x,\nabla u)\cdot \nabla h \, dx \\
&\le \int_{B(x_0,R)} (\mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla h))\cdot(\nabla u - \nabla h)\, dx \\
&\quad + \alpha_2 \int_{B(x_0,R)} \left(|\nabla h|^{p-1}|\nabla u| + |\nabla u|^{p-1}|\nabla h|\right) dx \\
&\quad + \int_{B(x_0,R)} (\mathcal{B}(x,u) - \mathcal{B}(x,h))\,(u-h)\, dx \\
&= \int_{B(x_0,R)} (u-h)\, d\nu \\
&\quad + \alpha_2 \int_{B(x_0,r)} \left(|\nabla h|^{p-1}|\nabla u| + |\nabla u|^{p-1}|\nabla h|\right) dx,
\end{aligned} \tag{3.2}
$$

in the last inequality we have used that u is a solution of (E_ν), $|\nu| \in (W_0^{1,p}(V))^*$, h is $(\mathcal{A},\mathcal{B})$-harmonic and $u - h \in W_0^{1,p}(B(x,R))$. Set

$$
I_1 = \int_{B(x_0,R)} (u-h)\, d\nu
$$

and

$$
I_2 = \alpha_2 \int_{B(x_0,r)} \left(|\nabla h|^{p-1}|\nabla u| + |\nabla u|^{p-1}|\nabla h|\right) dx.
$$

Let $q = (N - p + \beta(p-1))/(\frac{N}{p} - 1)$ and $1/q + 1/q' = 1$. Since $u - h \in W_0^{1,p}(B(x,R))$, by Hölder's inequality, Adams' inequality and Young's inequality we have

$$
\begin{aligned}
&\int_{B(x_0,R)} |u-h|\,d|\nu| \\
&\le \left(\int_{B(x_0,R)} |u-h|^q\,d|\nu|\right)^{1/q} \left(\int_{B(x_0,R)} d|\nu|\right)^{1/q'} \\
&\le c\left(R^{N-p+\beta(p-1)}\right)^{1/q'} \left(\int_{B(x_0,R)} |u-h|^q\,d|\nu|\right)^{1/q} \\
&\le c\,R^{\frac{p-1}{p}(N-p+\beta p)} \left(\int_{B(x_0,R)} |\nabla(u-h)|^p\,dx\right)^{1/p} \\
&\le c\,R^{\frac{p-1}{p}(N-p+\beta p)} \\
&\quad \times \left\{\left(\int_{B(x_0,R)} |\nabla u|^p\,dx\right)^{1/p} + \left(\int_{B(x_0,R)} |\nabla h|^p\,dx\right)^{1/p}\right\} \\
&\le c\,R^{\frac{p-1}{p}(N-p+\beta p)} \\
&\quad \times \left\{\left(\int_{B(x_0,R)} |\nabla u|^p\,dx\right)^{1/p} + \left(\int_{B(x_0,R)} |u|^p\,dx\right)^{1/p} + R^{N/p}\right\} \\
&\le c\,R^{N-p+\beta p} + \frac{\alpha_1}{2}\,\varepsilon \int_{B(x_0,R)} |\nabla u|^p\,dx + c\int_{B(x_0,R)} |u|^p\,dx + c\,R^N,
\end{aligned}
$$

where we have used Lemma 2.1. Hence we have

$$
\begin{aligned}
I_1 &\le \int_{B(x_0,R)} |u-h|\,d|\nu| \\
&\le c\,R^{N-p+\beta p} + \frac{\alpha_1}{2}\,\varepsilon \int_{B(x_0,R)} |\nabla u|^p\,dx,
\end{aligned}
\tag{3.3}
$$

where we have used that $R \le 1$ and $N-p+\beta p \le N$ imply $R^N \le R^{N-p+\beta p}$. Here c depends on N, p, α_1, α_2, $\alpha_3(G)$, β, c_0, ε and L. Also, Young's inequality, Lemma 2.2 and (3.1) yield

$$\begin{aligned}
I_2 &\le \frac{\alpha_1}{2}\int_{B(x_0,r)} |\nabla u|^p\,dx + c\int_{B(x_0,r)} |\nabla h|^p\,dx \\
&\le \frac{\alpha_1}{2}\int_{B(x_0,r)} |\nabla u|^p\,dx + c\left(\frac{r}{R}\right)^{N-p+p\lambda}\int_{B(x_0,R)} |\nabla h|^p\,dx + c\,R^N \\
&\le \frac{\alpha_1}{2}\int_{B(x_0,r)} |\nabla u|^p\,dx \\
&\quad +c\left(\frac{r}{R}\right)^{N-p+p\lambda}\left(\int_{B(x_0,R)} |\nabla u|^p\,dx + \int_{B(x_0,R)} |u|^p\,dx\right) + c\,R^N \qquad (3.4)\\
&\le \frac{\alpha_1}{2}\int_{B(x_0,r)} |\nabla u|^p\,dx \\
&\quad +c\left(\frac{r}{R}\right)^{N-p+p\lambda}\int_{B(x_0,R)} |\nabla u|^p\,dx + c\,R^{N-p+\beta p},
\end{aligned}$$

where again we have used Lemma 2.1, (3.1) and $R^N \le R^{N-p+\beta p}$. It follows from (3.2), (3.3) and (3.4) that

$$\begin{aligned}
&\int_{B(x_0,r)} |\nabla u|^p\,dx \\
&\le c_1\left(\left(\frac{r}{R}\right)^{N-p+p\lambda} + \varepsilon\right)\int_{B(x_0,R)} |\nabla u|^p\,dx + c_2\,R^{N-p+p\beta}.
\end{aligned}$$

□

To achieve the aim in this section, we need the following two propositions in [2; III Lemma 2.1 and III Theorem 1.1].

Proposition 3.2. *Let A, γ_1 and γ_2 be positive constants such that $\gamma_2 < \gamma_1$. Then there exists a constant $\varepsilon_0 = \varepsilon_0(A,\gamma_1,\gamma_2) > 0$ with the following property: if $f(t)$ is a nonnegative nondecreasing function satisfying*

$$f(r) \le A\left\{\left(\frac{r}{R}\right)^{\gamma_1} + \varepsilon\right\} f(R) + B\,R^{\gamma_2}$$

for all $0 < r \le R \le R_0$ with $0 < \varepsilon \le \varepsilon_0$, $R_0 > 0$ and $B \ge 0$, then

$$f(r) \le c\left\{\left(\frac{r}{R}\right)^{\gamma_2} f(R) + B\,r^{\gamma_2}\right\}$$

for all $0 < r \le R \le R_0$ with a constant $c = c(A,\gamma_1,\gamma_2) > 0$.

Proposition 3.3. *Let $u \in W^{1,p}(B(x_0,R))$, $1 \le p \le N$. Suppose that for all $x \in B(x_0,R)$, all r, $0 < r \le \delta(x) = R - |x - x_0|$*

$$\int_{B(x,r)} |\nabla u|^p \, dx \le L^p \left(\frac{r}{\delta(x)}\right)^{N-p+p\beta}$$

holds with $0 < \beta \le 1$. Then, u is Hölder continuous in $B(x_0,\rho)$ with the exponent β for all $0 < \rho < R$.

Theorem 3.1. *Let G be a bounded open set and $u \in W^{1,p}_{loc}(G) \cap L^\infty_{loc}(G)$ is a solution of (E_ν) in G. Suppose that ν is a signed Radon measure on G such that there exist constants $M > 0$ and $0 < \beta < \lambda$, where $\lambda = \lambda(N,p,\alpha_1,\alpha_2,\alpha_3(G)) > 0$ is the number in Lemma 2.2 above, with*

$$|\nu|(B(x,r)) \le M \ r^{N-p+\beta(p-1)}$$

whenever $B(x,3r) \subset G$. Then u is locally Hölder continuous in G with the exponent β.

Proof. If $B(x_0,4R) \subset G$ with $R \le 1$, then Proposition 3.2 and Lemma 3.1 yield that

$$\int_{B(x,r)} |\nabla u|^p \, dx \le c \left\{ \int_{B(x_0,2R)} |\nabla u|^p \, dx + 1 \right\} \left(\frac{r}{R}\right)^{N-p+p\beta},$$

whenever $x \in B(x_0,R)$ and $0 < r \le R$, where $c > 0$ depends on N, p, α_1, α_2, $\alpha_3(G)$, M, β and $\sup_{B(x_0,2R)} |u|$. Hence, by Proposition 3.3, u is Hölder continuous in $B(x_0,\rho)$ with exponent β for $0 < \rho < R$. □

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