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Types of pasting arcs in two sheeted spheres

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Abstract.

Fix two disjoint nondegenerate continua A and B in the complex plane $\mathbb{C}$ with connected complements and choose a simple arc γ in the complex sphere $\widehat{\mathbb{C}}$ disjoint from $A \cup B$, which we call a pasting arc for A and B. Then form a covering Riemann surface $\widehat{\mathbb{C}}_\gamma$ over $\widehat{\mathbb{C}}$ by pasting two copies of $\widehat{\mathbb{C}} \setminus \gamma$ crosswise along the arc γ. Viewing A and B as embedded in the different two sheets $\widehat{\mathbb{C}} \setminus \gamma$ of $\widehat{\mathbb{C}}_\gamma$, consider the variational 2-capacity $\operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B)$ of the set A in $\widehat{\mathbb{C}}_\gamma$ with respect to the open subset $\widehat{\mathbb{C}}_\gamma \setminus B$ containing A. We are interested in the comparison of $\operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B)$ with $\operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B)$. We say that the pasting arc γ for A and B is subcritical, critical, or supercritical according as $\operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B)$ is less than, equal to, or greater than $\operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B)$, respectively. The purpose of this paper is to show the existence of subcritical arc γ for any arbitrarily given general pair of admissible A and B and then the existences of critical and also supercritical arcs γ under the additional condition imposed upon A and B that each of A and B is symmetric about a common straight line in $\widehat{\mathbb{C}}$, which is the case e.g. if A and B are disjoint closed discs.

§1. Introduction

Consider two disjoint compact subsets A and B in the complex plane $\mathbb{C}$ such that both of A and B are closures of analytic Jordan regions A^i and B^i in $\mathbb{C}$. Such compact subsets in $\mathbb{C}$ as A and B above will be referred to as being *admissible* in this paper. In actual fact, most of the results and especially those labeled as theorems are also valid for more general subsets A and B in $\mathbb{C}$ that are disjoint nonpolar (not necessarily

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connected) compact subsets with connected complements in $\mathbb{C}$. These results in the general setting as above can easily be deduced from the corresponding results in the present special setting proved in the sequel in this paper by the usual standard exhaustion method. However, just only for the sake of simplicity we assume that A and B are disjoint admissible compact subsets in the present paper. Take a simple arc γ in $\widehat{\mathbb{C}} \setminus (A \cup B)$. We denote by W_γ the Riemann surface obtained from $\widehat{\mathbb{C}} \setminus (A \cup \gamma)$ and $\widehat{\mathbb{C}} \setminus (B \cup \gamma)$ by pasting them crosswise along the arc γ, which is also denoted by the following impressive notation (cf. [4]):

$$W_\gamma = (\widehat{\mathbb{C}} \setminus (A \cup \gamma)) \ltimes\!\!\!\bigcup_\gamma (\widehat{\mathbb{C}} \setminus (B \cup \gamma)). \tag{1}$$

The arc γ in the above surface W_γ is referred to as the *pasting arc* for the set $A \cup B$. The surface W_γ may be viewed as a subsurface of the covering Riemann surface, the two sheeted sphere,

$$\widehat{\mathbb{C}}_\gamma := (\widehat{\mathbb{C}} \setminus \gamma) \ltimes\!\!\!\bigcup_\gamma (\widehat{\mathbb{C}} \setminus \gamma)$$

so that

$$W_\gamma = \widehat{\mathbb{C}}_\gamma \setminus (A \cup B), \tag{2}$$

where we understand that A (B, resp.) is situated e.g. in the upper (lower, resp.) sheet of $\widehat{\mathbb{C}}_\gamma$ although A and B are originally contained in the same $\mathbb{C}$. Here $\widehat{\mathbb{C}}_\gamma$ in general can be considered for any pasting arc γ for the empty set $\emptyset$, i.e. for any simple arc in $\widehat{\mathbb{C}}$ (i.e. $\widehat{\mathbb{C}} \setminus \emptyset$). Observe that W_γ and $\widehat{\mathbb{C}}_\gamma$ as Riemann surfaces are unchanged if the pasting arc γ is replaced by any pasting arc homotopic to γ in $\widehat{\mathbb{C}} \setminus (A \cup B)$ or in $\widehat{\mathbb{C}}$: $W_\gamma = W_{\gamma'}$ and $\widehat{\mathbb{C}}_\gamma = \widehat{\mathbb{C}}_{\gamma'}$ if γ and γ' are homotopic in $\widehat{\mathbb{C}} \setminus (A \cup B)$ and $\widehat{\mathbb{C}}$, respectively. Here in particular $\widehat{\mathbb{C}}_\gamma$ depends only upon the initial and the terminal points of γ and does not depends on the arc connecting these two points. In this sense we sometimes write $W_\gamma = W_{[\gamma]}$ and $\widehat{\mathbb{C}}_\gamma = \widehat{\mathbb{C}}_{[\gamma]}$, where $[\gamma]$ is the homotopy class of pasting arcs containing γ considered in $\widehat{\mathbb{C}} \setminus (A \cup B)$ or in $\widehat{\mathbb{C}}$.

Consider next the *capacity* $\operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B)$, or more precisely the variational 2-capacity (cf. e.g. [2]), of the compact subset A in $\widehat{\mathbb{C}}_\gamma$ with respect to the open subset $\widehat{\mathbb{C}}_\gamma \setminus B$ of $\widehat{\mathbb{C}}_\gamma$ containing A given by

$$\operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B) = \inf_\varphi D_{W_\gamma}(\varphi), \tag{3}$$

where φ in taking the infimum in (3) runs over the family of $\varphi \in C(\widehat{\mathbb{C}}_\gamma) \cap C^\infty(W_\gamma)$ with $\varphi|A = 1$ and $\varphi|B = 0$ and $D_{W_\gamma}(\varphi)$ indicates the Dirichlet integral of φ over W_γ defined by

$$D_{W_\gamma}(\varphi) = \int_{W_\gamma} d\varphi \wedge *d\varphi = \int_{W_\gamma} |\nabla\varphi(z)|^2 dxdy.$$

Here the second term in the above is the coordinate free expression of $D_{W_\gamma}(\varphi)$ and the third term is the expression of $D_{W_\gamma}(\varphi)$ in terms of local parameters $z = x + iy$ for W_γ and $\nabla\varphi(z)$ is the gradient vector $(\partial\varphi(z)/\partial x, \partial\varphi(z)/\partial y)$.

The variation (3) has the unique extremal function u_γ:

$$\text{(4)} \qquad \operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B) = D_{W_\gamma}(u_\gamma),$$

characterized by the conditions $u_\gamma \in C(\widehat{\mathbb{C}}_\gamma) \cap H(W_\gamma)$ with $u_\gamma|A = 1$ and $u_\gamma|B = 0$ (cf. e.g. [2]), where $H(X)$ denotes the class of harmonic functions defined on a Riemann surface X, so that the function $u_\gamma|W_\gamma$ is usually referred to as the *harmonic measure* of ∂A on W_γ. In addition to two characterizations (3) and (4) of the capacity we add one more interpretation. The surface W_γ is a doubly connected planar surface and hence W_γ is an annulus or conformally a ring region $\{z \in \mathbb{C} : 1 \leq |z| \leq e^M\}$ $(M > 0)$ and the conformal invariant M is called the *modulus* of W_γ and denoted by mod W_γ (cf. e.g. [6]). It is straightforward to deduce

$$\text{(5)} \qquad \operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B) = 2\pi/\operatorname{mod} W_\gamma.$$

The above remark concerning the dependence of the structure of W_γ and $\widehat{\mathbb{C}}_\gamma$ on the pasting arc γ also applies mutatis mutandis to the quantity $\operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B)$ so that $\operatorname{cap}(A, \widehat{\mathbb{C}}_{[\gamma]} \setminus B)$ is also meaningful.

We also consider the capacity $\operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B)$ of the set A in $\widehat{\mathbb{C}}$ contained in the open subset $\widehat{\mathbb{C}} \setminus B$. It is an important task to compare $\operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B)$ with $\operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B)$ and especially to clarify when the situation $\operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B) \leq \operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B)$ occurs from the viewpoints of various applications of capacities such as those to the classical and modern type problems (cf. e.g. [5], [8], [6], [4], [3], etc.). Actually there seems to have been an expectation among people who have been concerned with this question that this is almost always the case. The general *purpose* of this paper is to investigate the above problem and to claim that the reality is not that simple and tame as to support the above expectation. Since the occurence of the situation

$$\text{(6)} \qquad \operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B) = \operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B)$$

is very delicate in the sense that the relation is easily destroyed even if we change γ slightly but not preserving homotopy, we say that γ ($[\gamma]$, resp.) is a *critical* arc (homotopy class of arcs, resp.). The most desirable situation from the view point of our applications is the case in which the inequality

$$\text{(7)} \qquad \operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B) < \operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B)$$

holds and the pasting arc γ (the class $[\gamma]$, resp.) which makes (7) valid will be referred to as a *subcritical* arc (homotopy class, resp.). In the rest of (6) and (7) we have

$$\text{(8)} \qquad \operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B) > \operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B)$$

and in this case γ (the class $[\gamma]$) is called as a *supercritical* arc (homotopy class, resp.).

The purpose of this paper is to show that either one of the above three cases really occurs by suitably choosing a pasting arc γ of the surface $(\widehat{\mathbb{C}} \setminus (A \cup \gamma)) \ltimes\!\!\bigcup_\gamma (\widehat{\mathbb{C}} \setminus (B \cup \gamma))$ for an arbitrarily given pair of admissible compact sets A and B under an additional (technical) requirment that each of A and B is symmetric about a common straight line l. The simplest example of this case is the one when A and B are arbitrarily given disjoint closed discs. We believe that the above additional condition is only technical and not essential but at present it is merely a conjecture. More presice content of this paper is as follows. First, in entirely general case without any restriction we give two sufficient conditions for a given pasting arc to be subcritical. As a consequence of these results we see that there always exist a plenty of subcritical arcs for any general couple (A, B). Then we give an example of supercritical arc for any couple (A, B) satisfying the above symmetry requirment. In general, the existence of supercritical arcs always implies the existence of critical arcs as a consequence of the continuity of $\operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B)$ as the function of two end points of γ, and therefore we will be able to conclude the existence of a critical arc for any couple (A, B) satisfying the symmetry postulation.

§2. Subcritical arcs

Take an arbitrarily chosen pair (A, B) of two disjoint admissible compact subsets A and B in $\mathbb{C}$ as described in the introduction. In contrast with the notation $W_\gamma = \widehat{\mathbb{C}}_\gamma \setminus (A \cup B)$ we set

$$W := \widehat{\mathbb{C}} \setminus (A \cup B).$$

Recall that a pasting arc γ for W_γ is a simple arc in W and u_γ is the harmonic measure of ∂A with respect to the region W_γ. Similarly we denote by u the harmonic measure of ∂A with respect to W. In this section we will give two sufficient conditions for a given pasting arc γ to be subcritical.

Theorem 1. *For any point a in W and any real number $\rho > 0$ less than the distance between a and ∂W there exists an open disc $\Delta(a, r)$ of radius $0 < r < \rho$ centered at a such that every pasting arc γ lying in $\Delta(a, r)$ is subcritical.*

Proof. Since $u_\gamma = 1$ on ∂A and $u_\gamma = 0$ on ∂B and moreover ∂A and ∂B are analytic, we can find annular neighborhoods α and β of ∂A and ∂B such that every u_γ has a harmonic extention on $W'_\gamma := W_\gamma \cup \alpha \cup \beta$, as far as the pasting arc γ stays in an arbitrarily chosen but then fixed open disc $\Delta(a, r_0)$ $(0 < r_0 < \rho)$. Suppose erroneously that there is a pasting arc $\gamma(r)$ in $\Delta(a, r)$ for each $0 < r < r_0$ such that

$$\operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B) \le \operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma(r)} \setminus B) = D_{W_{\gamma(r)}}(u_{\gamma(r)}). \tag{9}$$

In view of $-1 \le u_\gamma \le 2$ on W'_γ, the family $\{u_\gamma : \gamma \subset \Delta(a, r), r_0 > r \downarrow 0\}$ forms a normal family (cf. e.g. [7], [1]) on each compact subset of

$$\Big(\Big((\widehat{\mathbb{C}} \setminus A) \cup \alpha\Big) \setminus \{a\}\Big) \bigcup \Big(\Big((\widehat{\mathbb{C}} \setminus B) \cup \beta\Big) \setminus \{a\}\Big).$$

Hence we can find a decreasing sequence $(r_n)_{n \ge 1}$ converging to zero with $r_1 < r_0$ such that $\big(u_{\gamma(r_n)}\big)_{n \ge 1}$ converges to a $v \in H(((\widehat{\mathbb{C}} \setminus A) \cup \alpha) \setminus \{a\})$ locally uniformly on $((\widehat{\mathbb{C}} \setminus A) \cup \alpha) \setminus \{a\}$ on which $-1 \le v \le 2$ along with each $u_{\gamma(r_n)}$. Hence, by the Riemann removability theorem v has a harmonic extension on $(\widehat{\mathbb{C}} \setminus A) \cup \alpha$ and $v|\partial A = 1$ implies that $v = 1$ identically on $(\widehat{\mathbb{C}} \setminus A) \cup \alpha$ and in particular on α. By the Green formula we see that

$$D_{W_{\gamma(r_n)}}(u_{\gamma(r_n)}) = \int_{\partial A} *du_{\gamma(r_n)} \to \int_{\partial A} *dv = 0.$$

By (9) we must conclude that $\operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B) = 0$, which is absurd. □

We turn to the other condition for an arc γ to be subcritical. We say that a pasting arc $\gamma \subset W$ *ranges homotopically between λ and μ* for an arbitrary given pair of real numbers λ and μ with $0 < \lambda \le \mu < 1$ if there is a pasting arc γ' in the homotopy class $[\gamma]$ containing γ in W such that

$$\gamma' \subset \{z \in W : \lambda \le u(z) \le \mu\}. \tag{10}$$

It may be covenient to write the above fact by $[\gamma] \subset \{z \in W : \lambda \leq u(z) \leq \mu\}$. We say that a pasting arc γ *stays homotopically at* $\lambda \in (0,1)$ if γ ranges homotopically between λ and itself so that there is a pasting arc $\gamma' \in [\gamma]$ such that γ' is contained in the level line $\{z \in W : u(z) = \lambda\}$ of u.

Theorem 2. *If a pasting arc γ ranges homotopically between λ and μ $(0 < \lambda \leq \mu < 1)$, then*

$$(1 - (\mu - \lambda))^2 \mathrm{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B) < \mathrm{cap}(A, \widehat{\mathbb{C}} \setminus B). \tag{11}$$

In particular, if γ stays homotopically at some λ in $(0,1)$, then γ is subcritical.

Proof. We may assume that $\gamma \subset \{z \in W : \lambda \leq u(z) \leq \mu\}$. We denote by X the connected part of W_γ lying over $\{z \in W : \lambda \leq u(z) \leq \mu\}$. There are two connected parts Y_1' and Y_2' in $\widehat{\mathbb{C}}_\gamma$ lying over $\{z \in W : u(z) \geq \mu\} \cup A$ such that $A \subset Y_1'$ and $A \cap Y_2' = \emptyset$. Then we set $Y_1 := Y_1' \setminus A$ and $Y_2 := Y_2'$. Similarly there are two connected parts Z_1' and Z_2' in $\widehat{\mathbb{C}}_\gamma$ lying over $\{z \in W : u(z) \leq \lambda\} \cup B$ such that $B \subset Z_1'$ and $B \cap Z_2' = \emptyset$. Then we set $Z_1 := Z_1' \setminus B$ and $Z_2 := Z_2'$. Then we have the decomposition

$$W_\lambda = X \bigcup (Y_1 \cup Y_2) \bigcup (Z_1 \cup Z_2)$$

of W_λ into 5 connected compact sets whose interiors are mutually disjoint. Consider the function v given by

$$v := \begin{cases} u - (\mu - \lambda) & \text{on } Y_1; \\ \lambda & \text{on } X \cup Y_2 \cup Z_2; \\ u & \text{on } Z_1. \end{cases}$$

Observe that v is a continuous and piecewise smooth function on $\overline{W_\gamma}$ such that v is not harmonic on W_γ and $v|\partial A = 1 - (\mu - \lambda)$ and $v|\partial B = 0$. Since the function $(1 - (\mu - \lambda))u_\gamma$ is the harmonization of v on W_γ preserving the boundary values on ∂W_γ, the Dirichlet principle assures that

$$D_{W_\gamma}((1 - (\mu - \lambda))u_\gamma) < D_{W_\gamma}(v). \tag{12}$$

We compute $D_{W_\gamma}(v)$ as follows:

$$\begin{aligned} D_{W_\gamma}(v) = D_{Y_1}(v) + D_{X \cup Y_2 \cup Z_2}(v) + D_{Z_1}(v) &= D_{Y_1}(u) + D_{X \cup Y_2 \cup Z_2}(\lambda) + D_{Z_1}(u) \\ &= D_{W \cap \{u > \mu\}}(u) + D_{W \cap \{u < \lambda\}}(u) \leq D_W(u). \end{aligned}$$

This with (12) yields $(1 - (\mu - \lambda))^2 D_{W_\gamma}(u_\gamma) < D_W(u)$, which is nothing but (11).

If $\lambda = \mu$, then (11) is reduced to the inequality defining for the arc γ to be subcritical. □

§3. Continuity of capacity

Although the capacity $\mathrm{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B) = D_{W_\gamma}(u_\gamma)$ depends not only upon the end points z and w of γ but also upon the homotopy class containing γ even if the end points z and w of γ are fixed in advance. Therefore the capacity is only a multivalued function of the branch points $\tilde{z}$ and $\tilde{w}$ of $\widehat{\mathbb{C}}_\gamma$ lying over z and w, which are the end points of γ, but it becomes a single valued function as far as we are concerned with their local behaviors. Thus fix two different points z_1 and w_1 in W and two discs U and V given by $U := \Delta(z_1, r_1)$ ($V := \Delta(w_1, r_1)$, resp.) centered at z_1 (w_1, resp.) with radius $r_1 > 0$ such that $\overline{U} \cup \overline{V} \subset W$ and $\overline{U} \cap \overline{V} = \emptyset$. For any $z \in U$ and $w \in V$, let $\gamma_1(z, w)$ be a pasting arc joining z and w. Then we can always find a pasting arc $\gamma(z, w) \in [\gamma_1(z, w)]$ such that $\gamma(z, w) \cap U$ ($\gamma(z, w) \cap V$, resp.) is a line segment joining $z_1 + r_1 \in \partial U$ ($w_1 + r_1 \in \partial V$, resp.) and $z \in U$ ($w \in V$, resp.) and $\gamma(z, w) \setminus U$ ($\gamma(z, w) \setminus V$, resp.) is the subarc of $\gamma(z, w)$ starting from w ($w_1 + r_1$, resp.) and ending at $z_1 + r_1$ (z, resp.). We assume that $[\gamma(z, w) \setminus (U \cup V)]$ is a fixed homotopy class. Then

$$c(z, w) := \mathrm{cap}(A, \widehat{\mathbb{C}}_{\gamma(z,w)} \setminus B) = D_{W_{\gamma(z,w)}}(u_{\gamma(z,w)}) \tag{13}$$

is a single valued function on the polydisc $U \times V$. We maintain:

Lemma 3. *The function $c(z, w)$ in* (13) *is continuous on the polydisc $U \times V$ so that capacities are continuous functions of branch points.*

Proof. The proof is similar to that of Theorem 1 but we repeat it here since the settings or the arrangements of the stage is superficially quite different from that for Theorem 1.

Choose and then fix an arbitrary point $\sigma_0 := (z_2, w_2) \in U \times V$. We only have to show that $c(\sigma_n) \to c(\sigma_0)$ for any sequence $(\sigma_n)_{n \geq 1}$ in $U \times V$ convergent to σ_0. For the purpose we choose two annular neighborhoods α and β of ∂A and ∂B, respectively, such that $(\alpha \cup \beta) \cap (U \cup V) = \emptyset$ and every $u_{\gamma(\sigma)} \in H(W_{\gamma(\sigma)})$ can be continued to $W'_{\gamma(\sigma)} := W_{\gamma(\sigma)} \cup \alpha \cup \beta$ so as to being $u_{\gamma(\sigma)} \in H(W'_{\gamma(\sigma)})$ and $-1 \leq u_{\gamma(\sigma)} \leq 2$ on $W'_{\gamma(\sigma)}$ for every $\sigma = (z, w) \in U \times V$. The possibility of such a choice of (α, β) comes from the fact that the reflection principle is applicable as a result of $u_{\gamma(\sigma)}|\partial A = 1$ and $u_{\gamma(\sigma)}|\partial B = 0$ and the analyticity of relative boundaries of A and B. Take an arbitrarily chosen and then fixed decreasing sequence $(r_m)_{m \geq 2}$ converging to zero with

$0 < r_m < r_1 - |z_2 - z_1|$ $(m \geq 2)$. Let $K_m := \pi^{-1}(\overline{\Delta(z_2, r_m)} \cup \overline{\Delta(w_2, r_m)})$, where the covering surface $(\widehat{\mathbb{C}}_{\gamma(\sigma_0)}, \widehat{\mathbb{C}}, \pi)$ over $\widehat{\mathbb{C}}$ with its projection π is considered here. Then $W'_{\gamma(\sigma)} \setminus K_m = W'_{\gamma(\sigma')} \setminus K_m$ for every (σ, σ') in $L_m := \pi^{-1}(\Delta(z_2, r_m)) \times \pi^{-1}(\Delta(w_2, r_m))$ for any arbitrarily fixed $m \geq 2$. We denote by W'_m (W_m, resp.) the surface $W'_{\gamma(\sigma)} \setminus K_m$ ($W_{\gamma(\sigma)} \setminus K_m$, resp.), which does not depend on the choice of $\sigma \in L_m$. Then $\{u_{\gamma(\sigma_n)} : \sigma_n \in L_m, \sigma_n \to \sigma_0\}$ forms a normal family on W'_m. Hence we can find a subsequence $(\sigma_{n'})$ of any given subsequence of (σ_n) such that $u_{\gamma(\sigma_{n'})}$ converges to a $v \in H(W'_{\gamma(\sigma_0)} \setminus \{\tilde{z_2}, \tilde{w_2}\})$ locally uniformly on $W'_{\gamma(\sigma_0)} \setminus \{\tilde{z_2}, \tilde{w_2}\}$, where $\tilde{z_2}$ and $\tilde{w_2}$ are the branch points of $W'_{\gamma(\sigma_2)}$ over z_2 and w_2. Clearly $-1 \leq v \leq 2$, $v|\partial A = 1$, and, $v|\partial B = 0$ on $W'_{\gamma(\sigma_0)} \setminus \{\tilde{z_2}, \tilde{w_2}\}$ along with each $u_{\gamma(\sigma_{n'})}$ on $W_{n'}$. Thus $v \in H(W'_{\gamma(\sigma_0)})$ so that $v = u_{\gamma(\sigma_0)}$ on $W_{\gamma(\sigma_0)}$. Hence the original sequence $u_{\gamma(\sigma_n)}$ converges to $u_{\gamma(\sigma_0)}$ locally uniformly on $W'_{\gamma(\sigma_0)} \setminus \{\tilde{z_2}, \tilde{w_2}\}$ and, in particular, not only $u_{\gamma(\sigma_n)} = 1$ converges to $u_{\gamma(\sigma_0)} = 1$ but also $*du_{\gamma(\sigma_n)}$ converges to $*du_{\gamma(\sigma_0)}$ uniformly on ∂A. Therefore, by the Green formula, we see that

$$c(\sigma_n) = D_{W'_{\gamma(\sigma_n)}}(u_{\gamma(\sigma_n)}) = \int_{\partial A} *du_{\gamma(\sigma_n)}$$

$$\to \int_{\partial A} *du_{\gamma(\sigma_0)} = D_{\gamma(\sigma_0)}(u_{\gamma(\sigma_0)}) = c(\sigma_0) \quad (n \to \infty),$$

which is to have been shown. □

Take an arbitrary pasting arc γ in W starting from a point z_0 and ending at a point z_1. We denote by γ_z the subarc of γ starting from z_0 and ending at some point $z \in \gamma$.

Lemma 4. *The range set* $\{\text{cap}(A, \widehat{\mathbb{C}}_{\gamma_z} \setminus B) : z \in \gamma\}$ *contains the closed interval* $[0, \text{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B)]$ *and*

$$\lim_{z \in \gamma, z \to z_0} \text{cap}(A, \widehat{\mathbb{C}}_{\gamma_z} \setminus B) = 0. \tag{14}$$

Proof. By Lemma 3, the function $z \mapsto \text{cap}(A, \widehat{\mathbb{C}}_{\gamma_z} \setminus B)$ is single valued and continuous on the set γ and a fortiori the intermediate value theorem assures the validity of the first half of the aove assertion.

To prove (14) we again use the normal family argument. We can view that $\{u_{\gamma_z} : z \in \gamma, z \to z_0\}$ forms a normal family on each compact subset of $\widehat{\mathbb{C}} \setminus A^i \cup \{z_0\}$, where $A^i = A \setminus \partial A$. Then we see that u_{γ_z} converges to a $v \in H(\widehat{\mathbb{C}} \setminus A \cup \{z_0\}) \cap C(\widehat{\mathbb{C}} \setminus A^i \cup \{z_0\})$ with $v|\partial A = 1$. Since $0 \leq v \leq 1$ on $\widehat{\mathbb{C}} \setminus A^i \cup \{z_0\}$, the Riemann removability theorem implies that $v \in H(\widehat{\mathbb{C}} \setminus A)$ so that $v|\partial A = 1$ yields that $v \equiv 1$ on $\widehat{\mathbb{C}} \setminus A^i$

and in particular $*dv = 0$ on ∂A. Hence $*du_{\gamma_z}$ converges to $*dv = 0$ uniformly on ∂A. Thus

$$\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma_z} \setminus B) = \int_{\partial A} *du_{\gamma_z} \to \int_{\partial A} *dv = 0 \quad (z \to z_0),$$

which proves (14). □

As a supplement to Theorems 1 and 2 which assure for an arc to be subcritical, we can state the following direct consequence of Lemma 4:

Theorem 5. *Any pasting arc γ in W contains a subarc γ_z $(z \in \gamma)$ which is subcritical; more precisely, there exists a point $z_1 \in \gamma$ such that γ_z is subcritical for every $z \in \gamma_{z_1}$.*

Proof. In view of (14) and also the first half of Lemma 4, we can find a $z_1 \in \gamma$ enough close to the initial point z_0 of γ such that

$$\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma_z} \setminus B) < \operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B)$$

for every $z \in \gamma_{z_1}$. □

Theorem 6. *If a pasting arc γ starting from z_0 and ending at z_1 is supercritical, then there is a subarc γ_z $(z \in \gamma)$ which is critical.*

Proof. Since $0 < \operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B) < \operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B)$, Lemma 4 assures that the quantity $\operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B)$ is contained in the range set $\{\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma_z} \setminus B) : z \in \gamma\}$ so that there is a point $z \in \gamma$ with $\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma_z} \setminus B) = \operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B)$, which shows that γ_z is critical. □

We define a distance $d(\gamma, \gamma')$ between two pasting arcs γ and γ' in $W := \widehat{\mathbb{C}} \setminus (A \cup B)$. Let σ be an arc in $W = \widehat{\mathbb{C}} \setminus (A \cup B)$ connecting one of end points of γ as its initial point with that of γ' as its terminal point and τ be another arc in W connecting the other end point of γ as its initial point with that of γ' as its terminal point such that $-\sigma + \gamma + \tau - \gamma'$ is a closed curve and $-\sigma + \gamma + \tau$ is a pasting arc homotopic to γ'. We denote by $\mathcal{F}$ the totality of pairs (σ, τ) of arcs σ and τ in W with the property described above and by $|\gamma''|$ the spherical length of an arc γ'' in W. Then the *distance* $d(\gamma, \gamma')$ of γ and γ' is given by

$$d(\gamma, \gamma') := \inf_{(\sigma, \tau) \in \mathcal{F}} (|\sigma| + |\tau|). \tag{15}$$

As the last consequence of Lemma 3 in this paper we state the following invariance of sub and supercriticality of pasting arcs γ under the small perturbation of γ in the sense of (15):

Theorem 7. *For any pasting arc γ in $\widehat{\mathbb{C}} \setminus (A \cup B)$, there exists a positive number ε such that any pasting arc γ' in $\widehat{\mathbb{C}} \setminus (A \cup B)$ with $d(\gamma, \gamma') < \varepsilon$ is subcritical (supercritical, resp.) if γ is subcritical (supercritical, resp.).*

Proof. Since there exist arcs σ and τ described above in the definition of the distance $d(\gamma, \gamma')$ such that $-\sigma + \gamma + \tau$ is homotopic to γ' and end points of $-\sigma + \gamma + \tau$ converge to the corresponding end points of γ as $|\sigma|$ and $|\tau|$ converge to zero as a consequence of the assumption $d(\gamma, \gamma') \to 0$, Lemma 3 assures that

$$\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma'} \setminus B) = \operatorname{cap}(A, \widehat{\mathbb{C}}_{-\sigma+\gamma+\tau} \setminus B) \to \operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma} \setminus B)$$

as $|\sigma| + |\tau| \to 0$ so that

$$\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma'} \setminus B) \to \operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma} \setminus B) \quad (\text{as } d(\gamma, \gamma') \to 0).$$

Hence $\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma'} \setminus B)$ is strictly greater (less, resp.) than $\operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B)$ for every γ' with sufficiently small $d(\gamma, \gamma')$ if and only if $\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma} \setminus B)$ is strictly greater (less, resp.) than $\operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B)$. Therefore γ' is supercritical (subcritical, resp.) for every γ' with sufficiently small $d(\gamma, \gamma')$ if and only if γ is supercritical (subcritical, resp.). □

§4. Supercritical arcs

Take a pair (A, B) of two disjoint admissible compact subsets A and B as described in Section 1 and a pasting arc γ in $W := \widehat{\mathbb{C}} \setminus (A \cup B)$. Then, as we saw in Section 2 and also in Section 3, the capacity $\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma} \setminus B)$ covers some small interval $(0, \varepsilon)$ $(\varepsilon > 0)$ by choosing γ enough short. On the other hand we ask how large the capacity $\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma} \setminus B)$ can be by a variety of choices of γ. In reality the capacity $\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma} \setminus B)$ cannot be too large no matter how we choose γ. In general we have the following relation:

$$0 < \operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma} \setminus B) < 2\operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B) \tag{16}$$

for every pasting arc γ in W. The proof goes as follows. Let $(\widehat{\mathbb{C}}_{\gamma}, \widehat{\mathbb{C}}, \pi)$ be the natural two sheeted sphere (i.e. the covering surface of $\widehat{\mathbb{C}}$) with altogether two branch points over respective end points of γ and with its projection π. Recall that u_{γ} (u, resp.) is the harmonic measure of ∂A on W_{γ} (W, resp.) and observe that $u \circ \pi$ is a nonharmonic competing function in (3) so that the Dirichlet principle shows that

$$D_{W_{\gamma}}(u_{\gamma}) < D_{W_{\gamma}}(u \circ \pi) = 2D_W(u),$$

which proves the above inequality (16).

The purpose of this section is to show the following central and main result of this paper: the existence of a supercritical arc γ in W characterized by the inequality $\mathrm{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B) > \mathrm{cap}(A, \widehat{\mathbb{C}} \setminus B)$. We believe that this result is always true for every admissible pair (A, B) but at present we need to have the following additional condition on (A, B) to prove the above result. We say that A and B are *symmetric* about a common straight line l if there is a straight line l in $\mathbb{C}$ such that the reflection T about l (i.e. the indirect conformal mapping T of $\mathbb{C}$ onto itself with the property that z and $T(z)$ are symmetric about l for every $z \in \mathbb{C}$) maps A onto itself and at the same time B onto itself so that, of course, T maps $\mathbb{C} \setminus (A \cup B)$ onto itself. A typical example is the case where A and B are disjoint closed discs; the line l in this situation is the one passing through centers of A and B.

Theorem 8. *Suppose that A and B are symmetric about a common straight line l. Then there exists a supercritical arc γ in $\widehat{\mathbb{C}} \setminus (A \cup B)$ and also a critical arc in $\widehat{\mathbb{C}} \setminus (A \cup B)$ which is a subarc of γ.*

Proof. The last assertion on the existence of critical arc follows at once from the above theorem 6. Thus we only have to concentrate ourselves to the proof of the existence of a supercritical arc γ.

By translating and rotating $\widehat{\mathbb{C}}$ if necessary we can assume that l is the real line $\{z \in \mathbb{C} : \Im z = 0\}$. Pick an arbitrary point $a \in l \setminus (A \cup B)$ and an analytic Jordan curve σ starting and ending at a and surrounding only B and hence separating B from A such that the subarc $\langle aa' \rangle$ ($\langle a'a \rangle$, resp.) of σ starting from a (a', resp.) and ending at a' (a, resp.) is situated in the upper (lower, resp.) half plane, where $l \cap \sigma = \{a, a'\}$. We also take a line segment $-\tau$ contained in l starting from a and terminating at a point b so that τ starts from b and ending at a which lies outside σ with $\tau \subset l \setminus (A \cup B')$, where B' is the region bounded by σ so that $\overline{B} \subset B'$. For $t > 0$ let $c(t)$ be a point on $\langle a'a \rangle \subset \sigma$ and σ'_t be the subarc of $\langle a'a \rangle \subset \sigma$ starting from $c(t)$ and ending at a (i.e. $\sigma'_t = \langle c(t)a \rangle$) with $|\sigma'_t| = t$, where $|\sigma'_t|$ is the length of the arc σ'_t. Finally let

$$\gamma_t := \tau + \sigma_t \quad (t > 0),$$

where $\sigma_t := \overline{\sigma \setminus \sigma'_t}$ (i.e. $\sigma_t = (\sigma \setminus \sigma'_t) \cup \{c(t), a\}$) is the subarc of σ starting from a and ending at $c(t)$. Next consider the surface $W_t := W_{\gamma_t}$ $(t > 0)$ given by

$$W_t := (\widehat{\mathbb{C}} \setminus (A \cup \gamma_t)) \ltimes\!\!\!\bigcup_{\gamma_t} \left(\widehat{\mathbb{C}} \setminus (B \cup \gamma_t)\right).$$

We denote by δ_t the segment lying over σ'_t, i.e. the union of two copies of σ'_t with two copies of a in each of the above copies being identified.

We also set

$$W_t' := W_t \setminus \delta_t \quad (t > 0),$$

which is a subsurface of any W_s $(0 < s \leq t)$. Consider one more surface

$$W_0 := \left(\widehat{\mathbb{C}} \setminus (A \cup B \cup \tau)\right) \underset{\tau}{\bowtie} \left(\widehat{\mathbb{C}} \setminus \tau\right).$$

Then we see that

$$W_0 \setminus \delta_t = W_t \setminus \delta_t \ (=: W_t')$$

for every $t > 0$.

Simply we write $u_t := u_{\gamma_t}$, the harmonic measure of ∂A on W_t. Consider the function w_t on $\overline{W_t \setminus \delta_t}$ harmonic on $W_t \setminus \delta_t$ and continuous on $\overline{W_t \setminus \delta_t}$ with boundary values $w_t|\partial A = w_t|\partial B = 0$ and $w_t|\delta_t = 1$. By the standard normal family argument we see that $w_t \downarrow 0$ locally uniformly on W_0. Clearly

$$|u_t - u_s| < w_t$$

on W_t' for every $0 < s < t$. This shows that $(u_t)_{t>0}$ converges to a continuous function v on $\overline{W}_0$ harmonic on W_0 with $v|\partial A = 1$ and $v|\partial B = 0$ locally uniformly on $W_0 \setminus \delta_t$ for every $t > 0$, where $\overline{W}_0$ is understood here as the Carathéodory compactification of W_0: $\overline{W}_0 = W_0 \cup \partial A \cup \partial B$. Hence we conclude that v is the harmonic measure of ∂A on W_0. Since $*du_t$ converges uniformly on ∂A to $*dv$ and $\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma_t} \setminus B) = \int_{\partial A} *du_t$, we see that

$$\lim_{t \downarrow 0} \operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma_t} \setminus B) = \int_{\partial A} *dv. \tag{17}$$

Let $\check{v}$ be the symmetric transformation of v on $\widehat{\mathbb{C}} \setminus (A \cup B \cup \tau)$ and also on $\widehat{\mathbb{C}} \setminus \tau$ about τ, where values on the upper edge τ^+ of τ are sent to those on the lower edge τ^- of τ and vice versa on each sheet so that $\check{v} = v \circ T$. It is not difficult to see that $\check{v}$ is also harmonic on W_0 along with v on W_0. Hence now we come to the crucial conclusion in our proof: the uniqueness of the harmonic measure of ∂A on W_0 assures that $v = \check{v}$ on W_0. The additional symmetry assumption is only made use of here to let this conclusion be valid. As a consequence of the above identity $v = \check{v}$ on W_0 , we deduce, in particular, that $v|\tau^- = v|\tau^+$, which shows that $v|(W \setminus \tau)$ can be continued to $v \in C(W)$ and $v|(\widehat{\mathbb{C}} \setminus \tau)$ to $v \in C(\widehat{\mathbb{C}})$, where $W := \widehat{\mathbb{C}} \setminus (A \cup B)$. Hence the Dirichlet principle can be applied on W to deduce

$$D_W(v) \geq D_W(u) = \operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B),$$

where we recall that u is the harmonic measure of ∂A on W. Then

$$\int_{\partial A} *dv = D_{W_0}(v) = D_{\widehat{\mathbb{C}}\setminus(A\cup B\cup\tau)}(v) + D_{\widehat{\mathbb{C}}\setminus\tau}(v) = D_W(v) + D_{\widehat{\mathbb{C}}}(v).$$

Thus we can conclude that

$$\int_{\partial A} *dv \geq \mathrm{cap}(A, \widehat{\mathbb{C}} \setminus B) + D_{\widehat{\mathbb{C}}}(v).$$

This with (17) implies that

$$\lim_{t\downarrow 0} \mathrm{cap}(A, \widehat{\mathbb{C}}_{\gamma_t} \setminus B) > \mathrm{cap}(A, \widehat{\mathbb{C}} \setminus B)$$

since $D_{\widehat{\mathbb{C}}}(v) > 0$. This shows that, if $t > 0$ is sufficiently small, then

$$\mathrm{cap}(A, \widehat{\mathbb{C}}_{\gamma_t} \setminus B) > \mathrm{cap}(A, \widehat{\mathbb{C}} \setminus B),$$

i.e. γ_t is supercritical. □

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