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Representations of nonnegative solutions for parabolic equations

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§1. Introduction

This paper is an announcement of results on integral representations of nonnegative solutions to parabolic equations, and gives a representation theorem which is general and applicable to many concrete examples for establishing explicit integral representations.

We consider nonnegative solutions of a parabolic equation

$$(\partial_t + L)u = 0 \quad \text{in} \quad D \times (0, T), \tag{1.1}$$

where T is a positive number, D is a non-compact domain of a Riemannian manifold M, $\partial_t = \partial/\partial t$, and L is a second order elliptic operator on D. We study the problem:

Determine all nonnegative solutions of the parabolic equation (1.1). This problem is closely related to the Widder type uniqueness theorem for a parabolic equation, which asserts that any nonnegative solution is determined uniquely by its initial value. (For Widder type uniqueness theorems, see [1], [5], [10], [13] and references therein.) We say that **[UP]**(i.e., uniqueness for the positive Cauchy problem) holds for (1.1) when any nonnegative solution of (1.1) with zero initial value is identically zero. When [UP] holds for (1.1) the answer to our problem is extremely simple: for any nonnegative solution of (1.1)there exists a

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unique Borel measure μ on D such that

$$u(x,t) = \int_D p(x,y,t)d\mu(y), \quad x \in D,\ 0 < t < T,$$

where p is the minimal fundamental solution for (1.1) (cf. [2], [1]). While [UP] does not hold, however, only few explicit integral representations of nonnegative solutions to parabolic equations are given (cf. [8], [4], [14]). On the other hand, for elliptic equations, there has been a significant progress in determining explicitly Martin boundaries in many important cases (cf. [12] and references therein). Recall that any nonnegative solution of an elliptic equation is represented by an integral of Martin kernels with respect to a Borel measure on the Martin boundary.

The aim of this paper is to give explicit integral representations of nonnegative solutions to parabolic equations for which [UP] does not hold. We give a general and sharp condition under which any nonnegative solution of (1.1) with zero initial value is represented by an integral on the product of the Martin boundary of D for an elliptic operator associated with L and the time interval $[0,T)$.

§2. Main results

Let M be a connected separable n-dimensional smooth manifold with Riemannian metric of class C^0. Denote by ν the Riemannian measure on M. T_xM and TM denote the tangent space to M at $x \in M$ and the tangent bundle, respectively. We denote by $\mathrm{End}(T_xM)$ and $\mathrm{End}(TM)$ the set of endmorphisms in T_xM and the corresponding bundle, respectively. The inner product on TM is denoted by $\langle X, Y\rangle$, where $X, Y \in TM$; and $|X| = \langle X, X\rangle^{1/2}$. The divergence and gradient with respect to the metric on M are denoted by div and ∇, respectively. Let D be a non-compact domain of M. Let L be an elliptic differential operator on D of the form

$$Lu = -m^{-1}\mathrm{div}(mA\nabla u) + Vu, \tag{2.1}$$

where m is a positive measurable function on D such that m and m^{-1} are bounded on any compact subset of D, A is a symmetric measurable section on D of $\mathrm{End}(TM)$, and V is a real-valued measurable function on D such that

$$V \in L^p_{\mathrm{loc}}(D, md\nu), \qquad \text{for some } p > \max(\frac{n}{2}, 1).$$

Here $L^p_{\mathrm{loc}}(D, md\nu)$ is the set of real-valued functions on D locally p-th integrable with respect to $md\nu$. We assume that L is locally uniformly elliptic on D, i.e., for any compact set K in D there exists a positive constant λ such that

$$\lambda|\xi|^2 \leq \langle A\xi, \xi\rangle \leq \lambda^{-1}|\xi|^2, \quad x \in K,\ (x,\xi) \in TM.$$

We assume that the quadratic form Q on $C_0^\infty(D)$ defined by

$$Q[u] = \int_D (\langle A\nabla u, \nabla u\rangle + V|u|^2) m d\nu$$

is bounded from below, and put

$$\lambda_0 = \inf\{Q[u]; u \in C_0^\infty(D),\ \int_D |u|^2 m d\nu = 1\}.$$

Denote by L_D the selfadjoint operator in $L^2(D; md\nu)$ associated with the closure of Q. We assume that λ_0 is an eigenvalue of L_D. Let ϕ_0 be the normalized positive eigenfunction for λ_0. Let $p(x,y,t)$ be the minimal fundamental solution for (1.1), which is equal to the integral kernel of the semigroup e^{-tL_D} on $L^2(D, md\nu)$.

Our main assumptions are [IU] (i.e., intrinsic ultracontractivity) and [SSP] (i.e., semismall perturbation) as follows.

[IU] For any $t > 0$, there exists $C_t > 0$ such that

$$p(x,y,t) \leq C_t\ \phi_0(x)\phi_0(y), \quad x, y \in D.$$

This condition implies that L_D admits a complete orthonormal base of eigenfunctions $\{\phi_j\}_{j=0}^\infty$ with eigenvalues $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ repeated according to multiplicity. Furthermore,

$$p(x,y,t) = \sum_{j=0}^{\infty} e^{-\lambda_j t}\phi_j(x)\phi_j(y) \tag{2.2}$$

(cf. [3], [12] and references therein). Recall that if [IU] holds, then [UP] does not hold for (1.1) and the equation admits a positive solution with zero initial value (cf. [9]); and for a class of parabolic equations, [IU] is equivalent to the existence of such a solution (cf. [10]).

[SSP] For some $a < \lambda_0$, 1 is a semismall perturbation of $L - a$ on D, i.e., for any $\varepsilon > 0$ there exists a compact subset K of D such that for any $y \in D \setminus K$

$$\int_{D\setminus K} G(x^0, z)G(z,y)m(z)d\nu(z) \leq \varepsilon G(x^0, y),$$

where G is the Green function of $L - a$ on D, and x^0 is a reference point fixed in D.

This condition implies that for any $j = 1, 2, \cdots$ the function ϕ_j/ϕ_0 has a continuous extension $[\phi_j/\phi_0]$ up to the Martin boundary $\partial_M D$ of D for $L - a$. (For semismall perturbations, see [11], [16], [12].) The union $D \cup \partial_M D$ is a compact metric space called the Martin compactification of D for $L - a$. We denote by $\partial_m D$ the minimal Martin boundary of D for $L - a$. This is a Borel subset of $\partial_M D$. Here, we note that $\partial_M D$ and $\partial_m D$ are independant of a in the following sense: if [SSP] holds, then for any $b < \lambda_0$ there is a homeomorphism Φ from the Martin compactification of D for $L - a$ onto that for $L - b$ such that $\Phi|_D = identity$ and Φ maps the Martin boundary and minimal Martin boundary of D for $L - a$ onto those for $L - b$, respectively (cf. Theorem 1.4 of [11]).

Now, we are ready to state our main theorem.

Theorem 2.1. *Assume* [IU] *and* [SSP]. *Then, for any nonnegative solution u of* (1.1) *there exists a unique pair of Borel measures μ on D and λ on $\partial_M D \times [0, T)$ such that λ is supported by the set $\partial_m D \times [0, T)$,*

$$\begin{aligned} u(x,t) &= \int_D p(x,y,t)\,d\mu(y) \\ &\quad + \int_{\partial_M D \times [0,t)} q(x,\xi,t-s)\,d\lambda(\xi,s), \end{aligned} \tag{2.3}$$

for any $x \in D$, $0 < t < T$. Here $q(x, \xi, \tau)$ is a continuous function on $D \times \partial_M D \times (-\infty, \infty)$ defined by

$$\begin{aligned} q(x,\xi,\tau) &= \sum_{j=0}^{\infty} e^{-\lambda_j \tau} \phi_j(x) [\phi_j/\phi_0](\xi), \quad \tau > 0, \\ q(x,\xi,\tau) &= 0, \qquad \tau \le 0, \end{aligned} \tag{2.4}$$

where the series in (2.4) *converges uniformly on $K \times \partial_M D \times (\delta, \infty)$ for any compact subset K of D and $\delta > 0$. Furthermore,*

$$q > 0 \quad on \quad D \times \partial_M D \times (0, \infty), \tag{2.5}$$

$$(\partial_t + L) q(\cdot, \xi, \cdot) = 0 \quad on \quad D \times (-\infty, \infty). \tag{2.6}$$

Conversely, for any Borel measures μ on D and λ on $\partial_M D \times [0, T)$ such that λ is supported by $\partial_m D \times [0, T)$ and

$$\int_D p(x^0, y, t)\,d\mu(y) < \infty, \quad 0 < t < T, \tag{2.7}$$

$$(2.8) \qquad \int_{\partial_M D \times [0,t)} q(x^0, \xi, t-s) d\lambda(\xi, s) < \infty, \quad 0 < t < T,$$

where x^0 *is a point fixed in* D*, the right hand side of* (2.3) *is a nonnegative solution of* (1.1).

The proof of this theorem is based upon the abstract parabolic Martin representation theorem and Choquet's theorem (cf. [7], [6], [15]), and its key step is to identify the parabolic Martin boundary.

§3. Examples

In this section we give concrete examples as applications of Theorem 2.1.

Example 3.1. *Let* $\alpha \in \mathbf{R}$ *and*

$$L = -\Delta + (1 + |x|^2)^{\alpha/2} \quad on \quad D = \mathbf{R}^n.$$

Then [UP] *holds for* (1.1) *if and only if* $\alpha \leq 2$*; while* [IU] *(or* [SSP] *with* $a = -1$*) is satisfied if and only if* $\alpha > 2$ *(cf.* [10], [12]*).*

(i) *Suppose that* $\alpha \leq 2$. *Then for any nonnegative solution* u *of* (1.1) *there exists a unique Borel measure* μ *on* D *such that*

$$(3.1) \qquad u(x,t) = \int_D p(x,y,t) d\mu(y), \quad x \in D, \ 0 < t < T.$$

Conversely, for any Borel measure μ *on* D *satisfying* (2.7), *the right hand side of* (3.1) *is a nonnegative solution of* (1.1).

(ii) *Suppose that* $\alpha > 2$. *Then the conclusions of Theorem* 2.1 *hold with*

$$(3.2) \qquad \partial_M D = \partial_m D = \infty S^{n-1},$$

where ∞S^{n-1} *is the sphere at infinity of* $\mathbf{R}^n$*, and the Martin compactification* D^* *of* $D = \mathbf{R}^n$ *with respect to* L *is obtained by attaching a sphere* S^{n-1} *at infinity:* $D^* = \mathbf{R}^n \sqcup \infty S^{n-1}$.

Note that the Martin boundary $\partial_M D$ *in the case* $-2 < \alpha \leq 2$ *is also equal to that for* $\alpha > 2$. *Nevertheless, when* [UP] *holds, the elliptic Martin boundary disappears in the parabolic representation theorem; while it enters when* [UP] *does not hold.*

Example 3.2. *Let* $L = -\Delta$ *on a bounded John domain* $D \subset \mathbf{R}^n$*, i.e.* D *is a bounded domain, and there exist a point* $z^0 \in D$ *and a positive* *constant* c_J *such that each* $z \in D$ *can be joined to* z^0 *by a rectifiable curve* $\gamma(t), 0 \leq t \leq 1$, *with* $\gamma(0) = z,\ \gamma(1) = z^0,\ \gamma \subset D$, *and*

$$dist(\gamma(t), \partial D) \geq c_J \ell(\gamma[0,t]), \quad 0 \leq t \leq 1,$$

where $\ell(\gamma[0,t])$ *is the length of the curve* $\gamma(s), 0 \leq s \leq t$. *Then the conditions* [IU] *and* [SSP] *with* $a = 0$ *are satisfied (cf. Example* 10.4 *of* [12]*). Thus the conclusions of Theorem* 2.1 *hold.*

Note that the Martin boundary $\partial_M D$ *of* D *with respect to* $L = -\Delta$ *may be different from the topological boundary* ∂D *in* $\mathbf{R}^n$, *although they are equal if* ∂D *is not bad (for example, when* D *is a Lipschitz domain).*

Note added in proof. It has turned out that the condition [IU] implies the condition [SSP] (see Theorem 1.1 of the paper: M. Murata and M. Tomisaki, Integral representations of nonnegative solutions for parabolic equations and elliptic Martin boundaries, Preprint, April 2006). Thus the assumption [SSP] of Theorem 2.1 in this paper is redundant.

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