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A decomposition of the Schwartz class by a derivative space and its complementary space

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Abstract.

Let $\mathcal{D}(R^n)$ be the class of all C^∞−functions on R^n with compact support. For a multi-index α we denote $\mathcal{D}^\alpha(R^n) = \{D^\alpha\varphi : \varphi \in \mathcal{D}(R^n)\}$. We give a direct sum decomposition of $\mathcal{D}(R^n)$ by $\mathcal{D}^\alpha(R^n)$ and its complementary space.

§1. Introduction

Let R^n be the n-dimensional Euclidean space. The points of R^n are ordered n-tuples $x = (x_1, \cdots, x_n)$, which each x_i is a real number. If $\alpha = (\alpha_1, \cdots, \alpha_n)$ is an n-tuple of nonnegative integers, then α is called a multi-index, and we let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. The partial derivative operators are denoted by $D_j = \partial/\partial x_j$ for $1 \le j \le n$, and the higher order derivatives by

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Following L.Schwartz [4] the notation $\mathcal{D}(R^n)$ (the Schwartz class) stands for the class of all infinitely differentiable functions on R^n with compact support.

L.Schwartz uses the following fact about $\mathcal{D}(R^n)$ in the discussion of primitives of distributions [4: sections 4 and 5 in Chap.II].

Fact. Let $\theta_0(t) \in \mathcal{D}(R^1)$ be a function which satisfies

$$\int_{-\infty}^{\infty} \theta_0(t)dt = 1.$$

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Then $\varphi \in \mathcal{D}(R^n)$ can be decomposed uniquely as follows:

$$\varphi(x) = \chi(x) + \lambda(x_2, \cdots, x_n)\theta_0(x_1) \tag{1.1}$$

where $\chi \in \mathcal{D}^1(R^n) = \{D_1\varphi : \varphi \in \mathcal{D}(R^n)\}$ and $\lambda \in \mathcal{D}(R^{n-1})$. Namely, if we put

$$\mathcal{U}^1(R^n) = \{\lambda(x_2, \cdots, x_n)\theta_0(x_1) : \lambda \in \mathcal{D}(R^{n-1})\},$$

then

$$\mathcal{D}(R^n) = \mathcal{D}^1(R^n) \oplus \mathcal{U}^1(R^n)$$

where the symbol $\oplus$ means a direct sum.

Moreover some authors deal with orthogonal decompositions of the Lebesgue spaces and the Sobolev spaces related to certain kinds of differential operators ([1], [2], [3]).

In this article we are concerned with a direct sum decomposition of the Schwartz class $\mathcal{D}(R^n)$ related to the higher order differential operator D^α. Namely we treat the following problem.

Problem. Let α be a nonzero multi-index and $\mathcal{D}^\alpha(R^n) = \{D^\alpha\varphi : \varphi \in \mathcal{D}(R^n)\}$. Give a direct sum decomposition of $\mathcal{D}(R^n)$ by means of $\mathcal{D}^\alpha(R^n)$ and its complementary space.

The next section is devoted to study of the problem.

§2. A decomposition of the Schwartz class

As we saw in section 1, $\mathcal{D}(R^n) = \mathcal{D}^1(R^n) \oplus \mathcal{U}^1(R^n)$. In order to give a direct sum decomposition of $\mathcal{D}(R^n)$ by means of $\mathcal{D}^\alpha(R^n)$, we must construct the complementary space of $\mathcal{D}^\alpha(R^n)$. We need two preparations. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ we set

$$M_\alpha = \{j \in \{1, \cdots, n\} : \alpha_j \neq 0\}$$

and for $j = 1, \cdots, n$ let

$$R^{n,j} = \{(x_1, \cdots, x_n) \in R^n : x_j = 0\}.$$

First we note that

Lemma 2.1. *(cf. [4: Section 5 in Chap. II]) Let $f \in \mathcal{D}(R^n)$ and α be a nonzero multi-index. Then the following two conditions are equivalent:*

(I) *There exists $u \in \mathcal{D}(R^n)$ such that $D^\alpha u = f$.*

(II)

$$\int_{-\infty}^{\infty} f(x_1, \cdots, x_j, \cdots, x_n)x_j^\ell dx_j = 0$$

for $j \in M_\alpha, \ell = 0, \cdots, \alpha_j - 1$ and any $(x_1, \cdots, 0, \cdots, x_n) \in R^{n,j}$.

Secondly, we need the following fact: For a nonnegative integer m, there exist functions $\{\theta_j\}_{j=0,1,\cdots,m} \subset \mathcal{D}(R^1)$ which satisfy the condition

$$\int_{-\infty}^{\infty} \theta_j(t)t^i dt = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases} \qquad (j, i = 0, 1, \cdots, m). \tag{2.1}$$

The first step is

Lemma 2.2. *For a nonnegative integer m there exists a function $\theta(t) \in \mathcal{D}(R^1)$ such that*

$$\int_{-\infty}^{\infty} \theta(t)t^i dt = \begin{cases} 1, & i = 0 \\ 0, & i = 1, \cdots, m. \end{cases} \tag{2.2}$$

Proof. We take a function $\eta(t) \in \mathcal{D}(R^1)$ such that

$$\int_{-\infty}^{\infty} \eta(t)dt = 1. \tag{2.3}$$

We put

$$\theta(t) = \eta(t) + \sum_{j=1}^{m} c_j \eta^{(j)}(t)$$

where $\eta^{(j)}$ is the derivative of order j of η and c_j $(j = 1, \cdots, m)$ are constants. We show that we can choose c_j $(j = 1, \cdots, m)$ such that $\theta(t)$ satisfies (2.2). Since $\int_{-\infty}^{\infty} \eta^{(j)}(t)dt = 0$ $(j = 1, \cdots, m)$, by (2.3) we have

$$\int_{-\infty}^{\infty} \theta(t)dt = 1.$$

Hence we must choose the constants c_j $(j = 1, \cdots, m)$ which satisfy

$$\sum_{j=1}^{m} c_j \int_{-\infty}^{\infty} \eta^{(j)}(t)t^i dt = -\int_{-\infty}^{\infty} \eta(t)t^i dt, \qquad i = 1, 2, \cdots, m. \tag{2.4}$$

We show that the linear equation (2.4) with respect to c_j $(j = 1, 2, \cdots, m)$ has a solution. We consider the coefficient matrix

$$A = \left(\int_{-\infty}^{\infty} \eta^{(j)}(t)t^i dt \right)_{i,j=1,\cdots,m}.$$

We see that for $j > i$

$$\int_{-\infty}^{\infty} \eta^{(j)} t^i dt = (-1)^i i! \int_{-\infty}^{\infty} \eta^{(j-i)}(t)dt = 0$$

and
$$\int_{-\infty}^{\infty} \eta^{(j)}(t)t^j dt = (-1)^j j! \int_{-\infty}^{\infty} \eta(t)dt = (-1)^j j!.$$

Therefore the matrix A is a triangular matrix and the diagonal elements are $(-1)^j j!$ $(j = 1, \cdots, m)$. Hence the determinant of A is not zero. Consequently the linear equation (2.4) has a solution. Thus we obtain the lemma. □

Using Lemma 2.2 we prove

Lemma 2.3. *For a nonnegative integer m, there exist functions $\theta_j(t) \in \mathcal{D}(R^1)$ $(j = 0, 1, \cdots, m)$ which satisfy* (2.1).

Proof. By Lemma 2.2 there exists a function $\theta(t) \in \mathcal{D}(R^1)$ which satisfies (2.2). We put

$$\theta_j(t) = \frac{(-1)^j}{j!}\theta^{(j)}(t), \qquad j = 0, 1, \cdots, m.$$

Then for $j > i$ by integration by parts we have

$$\int_{-\infty}^{\infty} \theta_j(t)t^i dt = \frac{(-1)^j}{j!}\int_{-\infty}^{\infty} \theta^{(j)}(t)t^i dt = \frac{(-1)^{j+i} i!}{j!}\int_{-\infty}^{\infty} \theta^{(j-i)}(t)dt = 0.$$

For $j < i$ $(\leq m)$ it follows from integration by parts and (2.2) that

$$\int_{-\infty}^{\infty} \theta_j(t)t^i dt = (-1)^{2j} \binom{i}{j} \int_{-\infty}^{\infty} \theta(t)t^{i-j} dt = 0$$

where $\binom{i}{j} = \frac{i!}{j!(i-j)!}$. Moreover, for $j = i$ by integration by parts and (2.2) we see that

$$\int_{-\infty}^{\infty} \theta_j(t)t^j dt = (-1)^{2j}\frac{j!}{j!}\int_{-\infty}^{\infty} \theta(t)dt = 1.$$

Thus the functions θ_j $(j = 0, 1, \cdots, m)$ satisfy (2.1). The lemma was proved. □

From now on let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a nonzero multi-index with $M_\alpha = \{j_1, \cdots, j_k\}$ and $m = \max_{i=1,\cdots,n}(\alpha_i - 1)$. For the nonnegative integer m we take the functions $\{\theta_j(t)\}_{j=0,1,\cdots,m} \subset \mathcal{D}(R^1)$ in Lemma 2.3.

In the decomposition (1.1) of $\varphi \in \mathcal{D}(R^n)$, $\lambda(x_2, \cdots, x_n)$ is given by

$$\lambda(x_2, \cdots, x_n) = \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_n)dx_1.$$

Indeed, if we put

$$\chi(x) = \varphi(x) - \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_n) dx_1 \theta_0(x_1),$$

then

$$\int_{-\infty}^{\infty} \chi(x_1, \cdots, x_n) dx_1$$
$$= \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_n) dx_1 - \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_n) dx_1 \int_{-\infty}^{\infty} \theta_0(x_1) dx_1 = 0$$

because of $\int_{-\infty}^{\infty} \theta_0(x_1) dx_1 = 1$. Hence by Lemma 2.1 we see that $\chi \in \mathcal{D}^1(R^n)$.

Taking the above situation into account, we introduce linear operators T^α and U^α on $\mathcal{D}(R^n)$. First, for $\varphi \in \mathcal{D}(R^n)$ and $1 \le i \le n, 0 \le \ell \le m$, we put

$$S_{i,\ell}\varphi(x) = \sum_{j=0}^{\ell} \left(\int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_i, \cdots, x_n) x_i^j dx_i \right) \theta_j(x_i).$$

We note that $S_{i,\ell}\varphi \in \mathcal{D}(R^n)$. For $1 \le p \le k$, let $M_{\alpha,p}$ denote the collection of subsets of M_α which have p elements. For $\{i_1, \cdots, i_p\} \in M_{\alpha,p}$ we set

$$S^{\alpha(i_1,\cdots,i_p)} = S_{i_1,\alpha_{i_1}-1} \cdots S_{i_p,\alpha_{i_p}-1}.$$

For $\varphi \in \mathcal{D}(R^n)$, by Fubini's theorem $S^{\alpha(i_1,\cdots,i_p)}\varphi(x)$ is given by

$$S^{\alpha(i_1,\cdots,i_p)}\varphi(x) = \sum_{s_1=0}^{\alpha_{i_1}-1} \cdots \sum_{s_p=0}^{\alpha_{i_p}-1} \tag{2.5}$$

$$\left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_n) x_{i_1}^{s_1} \cdots x_{i_p}^{s_p} dx_{i_1} \cdots dx_{i_p} \right) \theta_{s_1}(x_{i_1}) \cdots \theta_{s_p}(x_{i_p}).$$

In particular, the order in the definition of $S^{\alpha(i_1,\cdots,i_p)}$ is irrelevant.

Next, for $1 \le i \le n$ and $0 \le \ell \le m$ we set

$$T_{i,\ell} = I - S_{i,\ell}.$$

Further we define

$$T^\alpha = T_{j_1,\alpha_{j_1}-1} \cdots T_{j_k,\alpha_{j_k}-1}.$$

It follows from the definition that

$$(2.6)\qquad T^\alpha = (I - S_{j_1,\alpha_{j_1}-1})\cdots(I - S_{j_k,\alpha_{j_k}-1})$$

$$= I - \sum_{p=1}^{k}(-1)^{p+1}\sum_{\{i_1,\cdots,i_p\}\in M_{\alpha,p}} S^{\alpha(i_1,\cdots,i_p)}.$$

Finally we define

$$(2.7)\qquad U^\alpha = I - T^\alpha = \sum_{p=1}^{k}(-1)^{p+1}\sum_{\{i_1,\cdots,i_p\}\in M_{\alpha,p}} S^{\alpha(i_1,\cdots,i_p)}.$$

For a subset $\{k_1,\cdots,k_s\}\subset\{1,\cdots,n\}$, the notation $f(\{x_{k_1},\cdots,x_{k_s}\}^c)$ stands for a function of the remaining variables of $\{x_{k_1},\cdots,x_{k_s}\}$. For example, $f(\{x_1\}^c) = f(x_2,\cdots,x_n)$.

Referring to (2.5) and (2.7) we define tensor product functions of order α (associated with $\{\theta_j\}_{j=0,1,\cdots,m}$) as follows. If a function $f\in\mathcal{D}(R^n)$ which has the following form

$$(2.8)\qquad f(x) = \sum_{p=1}^{k}(-1)^{p+1}\sum_{\{i_1,\cdots,i_p\}\in M_{\alpha,p}}\sum_{s_1=0}^{\alpha_{i_1}-1}\cdots\sum_{s_p=0}^{\alpha_{i_p}-1}$$

$$\lambda_{i_1,\cdots,i_p;s_1,\cdots,s_p}(\{x_{i_1},\cdots,x_{i_p}\}^c)\theta_{s_1}(x_{i_1})\cdots\theta_{s_p}(x_{i_p})$$

satisfies the conditions

$$(2.9)\qquad \text{(i)}\quad \lambda_{i_1,\cdots,i_p;s_1,\cdots,s_p}\in\mathcal{D}(R^{n-p}),$$

(2.10) (ii) for $2\le p\le k, \{i_1,\cdots,i_p\}\in M_{\alpha,p}$ and

$0\le s_1\le\alpha_{i_1}-1,\cdots,0\le s_p\le\alpha_{i_p}-1$,

$$\lambda_{i_1,\cdots,i_p;s_1,\cdots,s_p}(\{x_{i_1},\cdots,x_{i_p}\}^c)$$
$$=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\lambda_{i_\ell;s_\ell}(\{x_{i_\ell}\}^c)x_{i_1}^{s_1}\cdots\widehat{x_{i_\ell}^{s_\ell}}\cdots x_{i_p}^{s_p}dx_{i_1}\cdots\widehat{dx_{i_\ell}}\cdots dx_{i_p}$$

$(\ell = 1,2,\cdots,p)$,

then we call f a tensor product function of order α, where the symbol $\widehat{\ }$ indicates that the variable underneath is deleted. We denote by $\mathcal{U}^\alpha(R^n)$ the set of all tensor product functions of order α.

A fundamental property of tensor product functions of order α is the following.

Lemma 2.4. *Let f be a tensor product function of order α with the form* (2.8). *Then*

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x_1,\cdots,x_n)x_{k_1}^{t_1}\cdots x_{k_q}^{t_q}dx_{k_1}\cdots dx_{k_q} = \lambda_{k_1,\cdots,k_q;t_1,\cdots,t_q}(\{x_{k_1},\cdots,x_{k_q}\}^c)$$

for $1\le q\le k, \{k_1,\cdots,k_q\}\in M_{\alpha,q}$ *and* $0\le t_1\le \alpha_{k_1}-1,\cdots,0\le t_q\le \alpha_{k_q}-1$.

Proof. First we prove

$$\int_{-\infty}^{\infty} f(x_1,\cdots,x_j,\cdots,x_n)x_j^t dx_j = \lambda_{j;t}(\{x_j\}^c) \tag{2.11}$$

for $j=j_1,\cdots,j_k$ and $t=0,1,\cdots,\alpha_j-1$. For $\{i_1,\cdots,i_p\}\in M_{\alpha,p}$ we put

$$I^{i_1,\cdots,i_p}(\{x_j\}^c) = \sum_{s_1=0}^{\alpha_{i_1}-1}\cdots\sum_{s_p=0}^{\alpha_{i_p}-1}\int_{-\infty}^{\infty}\lambda_{i_1,\cdots,i_p;s_1,\cdots,s_p}(\{x_{i_1},\cdots,x_{i_p}\}^c) \times\theta_{s_1}(x_{i_1})\cdots\theta_{s_p}(x_{i_p})x_j^t dx_j.$$

Then we have

$$\begin{aligned}\int_{-\infty}^{\infty} & f(x_1,\cdots,x_j,\cdots,x_n)x_j^t dx_j \\ &= \sum_{p=1}^{k}(-1)^{p+1}\sum_{\{i_1,\cdots,i_p\}\in M_{\alpha,p}} I^{i_1,\cdots,i_p}(\{x_j\}^c) \\ &= \sum_{p=1}^{k-1}(-1)^{p+1}(I_p(\{x_j\}^c)+J_p(\{x_j\}^c)) + (-1)^{k+1}I_k(\{x_j\}^c)\end{aligned}$$

where

$$I_p(\{x_j\}^c) = \sum_{\{i_1,\cdots,i_p\}\in M_{\alpha,p}, j\in\{i_1,\cdots,i_p\}} I^{i_1,\cdots,i_p}(\{x_j\}^c) \qquad (p=1,\cdots,k)$$

and

$$J_p(\{x_j\}^c) = \sum_{\{i_1,\cdots,i_p\}\in M_{\alpha,p}, j\notin\{i_1,\cdots,i_p\}} I^{i_1,\cdots,i_p}(\{x_j\}^c) \qquad (p=1,\cdots,k-1).$$

We show that for $p=1,2,\cdots,k-1$

$$J_p(\{x_j\}^c) = I_{p+1}(\{x_j\}^c). \tag{2.12}$$

First we consider $J_p(\{x_j\}^c)$. For $j \notin \{i_1, \cdots, i_p\}$, let $i_{\ell-1} < j < i_\ell$ ($\ell = 1, 2, \cdots, p+1$) with $i_0 = 0$ and $i_{p+1} = n+1$. Then by (2.10)

$$\int_{-\infty}^{\infty} \lambda_{i_1,\cdots,i_p;s_1,\cdots,s_p}(\{x_{i_1}, \cdots, x_{i_p}\}^c) x_j^t dx_j$$
$$= \lambda_{i_1,\cdots,i_{\ell-1},j,i_\ell,\cdots,i_p;s_1,\cdots,s_{\ell-1},t,s_\ell,\cdots,s_p}(\{x_{i_1}, \cdots, x_{i_{\ell-1}}, x_j, x_{i_\ell}, \cdots, x_{i_p}\}^c)$$

where

$$\lambda_{i_1,\cdots,i_{\ell-1},j,i_\ell,\cdots,i_p;s_1,\cdots,s_{\ell-1},t,s_\ell,\cdots,s_p} = \begin{cases} \lambda_{j,i_1,\cdots,i_p;t,s_1,\cdots,s_p} & \text{if } \ell = 1, \\ \lambda_{i_1,\cdots,i_p,j;s_1,\cdots,s_p,t} & \text{if } \ell = p+1. \end{cases}$$

Hence

$$(2.13) \qquad J_p(\{x_j\}^c) = \sum_{\{i_1,\cdots,i_p\} \in M_{\alpha,p}, j \notin \{i_1,\cdots,i_p\}} \sum_{s_1=0}^{\alpha_{i_1}-1} \cdots \sum_{s_p=0}^{\alpha_{i_p}-1}$$
$$\lambda_{i_1,\cdots,i_{\ell-1},j,i_\ell,\cdots,i_p;s_1,\cdots,s_{\ell-1},t,s_\ell,\cdots,s_p}(\{x_{i_1}, \cdots, x_{i_{\ell-1}}, x_j, x_{i_\ell}, \cdots, x_{i_p}\}^c)$$
$$\times \theta_{s_1}(x_{i_1}) \cdots \theta_{s_p}(x_{i_p})$$

with $i_{\ell-1} < j < i_\ell$. Next we consider $I_{p+1}(\{x_j\}^c)$. For $\{i_1, \cdots, i_{p+1}\} \in M_{\alpha,p+1}$ with $\{i_1, \cdots, i_{p+1}\} \ni j$, let $j = i_\ell$. Since

$$\int_{-\infty}^{\infty} \theta_{s_\ell}(x_{i_\ell}) x_j^t dx_j = \int_{-\infty}^{\infty} \theta_{s_\ell}(x_j) x_j^t dx_j = \begin{cases} 1, & s_\ell = t, \\ 0, & s_\ell \neq t \end{cases}$$

by (2.1), we have

$$I^{i_1,\cdots,i_{p+1}}(\{x_j\}^c)$$
$$= \sum_{s_1=0}^{\alpha_{i_1}-1} \cdots \sum_{s_{p+1}=0}^{\alpha_{i_{p+1}}-1} \int_{-\infty}^{\infty} \lambda_{i_1,\cdots,i_{p+1};s_1,\cdots,s_{p+1}}(\{x_{i_1}, \cdots, x_{i_{p+1}}\}^c)$$
$$\times \theta_{s_1}(x_{i_1}) \cdots \theta_{s_{p+1}}(x_{i_{p+1}}) x_j^t dx_j$$
$$= \sum_{s_1=0}^{\alpha_{i_1}-1} \cdots \sum_{s_{p+1}=0}^{\alpha_{i_{p+1}}-1} \lambda_{i_1,\cdots,i_{p+1};s_1,\cdots,s_{p+1}}(\{x_{i_1}, \cdots, x_{i_{p+1}}\}^c)$$
$$\times \theta_{s_1}(x_{i_1}) \cdots \widehat{\theta_{s_\ell}(x_{i_\ell})} \cdots \theta_{s_{p+1}}(x_{i_{p+1}}) \int_{-\infty}^{\infty} \theta_{s_\ell}(x_j) x_j^t dx_j$$
$$= \sum_{s_1=0}^{\alpha_{i_1}-1} \cdots \sum_{s_{\ell-1}=0}^{\alpha_{i_{\ell-1}}-1} \sum_{s_{\ell+1}=0}^{\alpha_{i_{\ell+1}}-1} \cdots \sum_{s_{p+1}=0}^{\alpha_{i_{p+1}}-1}$$
$$\lambda_{i_1,\cdots,i_\ell,\cdots,i_{p+1};s_1,\cdots,s_{\ell-1},t,s_{\ell+1},\cdots,s_{p+1}}(\{x_{i_1}, \cdots, x_{i_\ell}, \cdots, x_{i_{p+1}}\}^c)$$

$$\times\theta_{s_1}(x_{i_1})\cdots\theta_{s_{\ell-1}}(x_{i_{\ell-1}})\theta_{s_{\ell+1}}(x_{i_{\ell+1}})\cdots\theta_{s_{p+1}}(x_{i_{p+1}}).$$

By putting $u_1 = s_1, \cdots, u_{\ell-1} = s_{\ell-1}, u_\ell = s_{\ell+1}, \cdots, u_p = s_{p+1}$, we have

$$I^{i_1,\cdots,i_{p+1}}(\{x_j\}^c) = \sum_{u_1=0}^{\alpha_{i_1}-1}\cdots\sum_{u_{\ell-1}=0}^{\alpha_{i_{\ell-1}}-1}\sum_{u_\ell=0}^{\alpha_{i_{\ell+1}}-1}\cdots\sum_{u_p=0}^{\alpha_{i_{p+1}}-1}$$

$$\lambda_{i_1,\cdots,i_\ell,\cdots,i_{p+1};u_1,\cdots,u_{\ell-1},t,u_\ell,\cdots,u_p}(\{x_{i_1},\cdots,x_{i_\ell},\cdots,x_{i_{p+1}}\}^c)$$
$$\times\theta_{u_1}(x_{i_1})\cdots\theta_{u_{\ell-1}}(x_{i_{\ell-1}})\theta_{u_\ell}(x_{i_{\ell+1}})\cdots\theta_{u_p}(x_{i_{p+1}})$$

with $i_\ell = j$. Moreover, for $\{i_1,\cdots,i_{p+1}\}\in M_{\alpha,p+1}$ with $\{i_1,\cdots,i_{p+1}\}\ni j = i_\ell$ we put

$$m_1 = i_1, \cdots, m_{\ell-1} = i_{\ell-1}, m_\ell = i_{\ell+1}, \cdots, m_p = i_{p+1}.$$

We denote $M_{\alpha,p,j} = \{\{i_1,\cdots,i_p\}\in M_{\alpha,p} : j\in\{i_1,\cdots,i_p\}\}$ and $M^c_{\alpha,p,j} = \{\{i_1,\cdots,i_p\}\in M_{\alpha,p} : j\notin\{i_1,\cdots,i_p\}\}$. We note that the above correspondence $\{i_1,\cdots,i_{p+1}\}\to\{m_1,\cdots,m_p\}$ is a one-to-one and onto mapping from $M_{\alpha,p+1,j}$ to $M^c_{\alpha,p,j}$. Hence

$$(2.14)\qquad I_{p+1}(\{x_j\}^c)$$

$$= \sum_{\{i_1,\cdots,i_{p+1}\}\in M_{\alpha,p+1},j\in\{i_1,\cdots,i_{p+1}\}}\ \sum_{u_1=0}^{\alpha_{i_1}-1}\cdots\sum_{u_{\ell-1}=0}^{\alpha_{i_{\ell-1}}-1}\sum_{u_\ell=0}^{\alpha_{i_{\ell+1}}-1}\cdots\sum_{u_p=0}^{\alpha_{i_{p+1}}-1}$$
$$\lambda_{i_1,\cdots,i_\ell,\cdots,i_{p+1};u_1,\cdots,u_{\ell-1},t,u_\ell,\cdots,u_p}(\{x_{i_1},\cdots,x_{i_\ell},\cdots,x_{i_{p+1}}\}^c)$$
$$\times\theta_{u_1}(x_{i_1})\cdots\theta_{u_{\ell-1}}(x_{i_{\ell-1}})\theta_{u_\ell}(x_{i_{\ell+1}})\cdots\theta_{u_p}(x_{i_{p+1}})$$

$$= \sum_{\{m_1,\cdots,m_p\}\in M_{\alpha,p},j\notin\{m_1,\cdots,m_p\}}\ \sum_{u_1=0}^{\alpha_{m_1}-1}\cdots\sum_{u_p=0}^{\alpha_{m_p}-1}$$
$$\lambda_{m_1,\cdots,m_{\ell-1},j,m_\ell,\cdots,m_p;u_1,\cdots,u_{\ell-1},t,u_\ell,\cdots,u_p}(\{x_{m_1},\cdots,x_{m_{\ell-1}},x_j,$$
$$x_{m_\ell},\cdots,x_{m_p}\}^c)\theta_{u_1}(x_{m_1})\cdots\theta_{u_p}(x_{m_p})$$

with $i_\ell = j$ and $m_{\ell-1} < j < m_\ell$. Comparing (2.13) and (2.14) we obtain (2.12). Therefore, by (2.1)

$$\int_{-\infty}^{\infty} f(x_1,\cdots,x_j,\cdots,x_n)x_j^t dt = I_1(\{x_j\}^c)$$
$$= \sum_{s=0}^{\alpha_j-1}\int_{-\infty}^{\infty}\lambda_{j;s}(\{x_j\}^c)\theta_s(x_j)x_j^t dx_j = \lambda_{j;t}(\{x_j\}^c).$$

Thus we obtain (2.11). Next, for $2 \leq q \leq k, \{k_1, \cdots, k_q\} \in M_{\alpha,q}$ and $0 \leq t_1 \leq \alpha_{k_1} - 1, \cdots, 0 \leq t_q \leq \alpha_{k_q} - 1$, by (2.10) and (2.11) we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \cdots, x_n) x_{k_1}^{t_1} \cdots x_{k_q}^{t_q} dx_{k_1} \cdots dx_{k_q} \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x_1, \cdots, x_n) x_{k_1}^{t_1} dx_{k_1} \right) x_{k_2}^{t_2} \cdots x_{k_q}^{t_q} dx_{k_2} \cdots dx_{k_q} \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \lambda_{k_1;t_1}(\{x_{k_1}\}^c) x_{k_2}^{t_2} \cdots x_{k_q}^{t_q} dx_{k_2} \cdots dx_{k_q} \\
&= \lambda_{k_1,k_2,\cdots,k_q;t_1,t_2,\cdots,t_q}(\{x_{k_1}, x_{k_2}, \cdots, x_{k_q}\}^c).
\end{aligned}$$

We have completed the proof of the lemma. □

We state properties of the operators T^α and U^α which are important for a direct sum decomposition of $\mathcal{D}(R^n)$. We denote by e_j the multi-index which has 1 in the jth spot and 0 everywhere else ($j = 1, 2, \cdots, n$).

Lemma 2.5. *Let $\gamma = (\gamma_1, \cdots, \gamma_n)$ be a multi-index with $\gamma_j = 0$ and $0 \leq \ell \leq m$. Then for $\varphi \in \mathcal{D}^\gamma(R^n)$, $T_{j,\ell}\varphi \in \mathcal{D}^{\gamma+(\ell+1)e_j}(R^n)$.*

Proof. We note that $j \notin M_\gamma$ by the condition $\gamma_j = 0$. Let $\varphi \in \mathcal{D}^\gamma(R^n)$. In order to show that $T_{j,\ell}\varphi \in \mathcal{D}^{\gamma+(\ell+1)e_j}(R^n)$, by Lemma 2.1 it suffices to prove

$$\int_{-\infty}^{\infty} T_{j,\ell}\varphi(x_1, \cdots, x_i, \cdots, x_n) x_i^s dx_i = 0$$

for $s = 0, 1, \cdots, \gamma_i - 1$ if $i \in M_\gamma$ and $s = 0, 1, \cdots, \ell$ if $i = j$. First, let $i \in M_\gamma$. Then by the condition $\varphi \in \mathcal{D}^\gamma(R^n)$ and Lemma 2.1 we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} T_{j,\ell}\varphi(x_1, \cdots, x_i, \cdots, x_n) x_i^s dx_i \\
&= \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_i, \cdots, x_n) x_i^s dx_i \\
&\quad - \int_{-\infty}^{\infty} \left(\sum_{t=0}^{\ell} \left(\int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n) x_j^t dx_j \right) \theta_t(x_j) \right) x_i^s dx_i \\
&= \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_i, \cdots, x_n) x_i^s dx_i \\
&\quad - \sum_{t=0}^{\ell} \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n) x_i^s dx_i \right) x_j^t dx_j \right) \theta_t(x_j) \\
&= 0
\end{aligned}$$

for $s = 0, 1, \cdots, \gamma_i - 1$. Next, let $i = j$. Then for $s = 0, 1, \cdots, \ell$ we see that

$$\begin{aligned}
&\int_{-\infty}^{\infty} T_{j,\ell}\varphi(x_1, \cdots, x_j, \cdots, x_n) x_j^s dx_j \\
&\quad = \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_j, \cdots, x_n) x_j^s dx_j \\
&\quad\quad - \int_{-\infty}^{\infty} \left(\sum_{t=0}^{\ell} \left(\int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_j, \cdots, x_n) x_j^t dx_j \right) \theta_t(x_j) \right) x_j^s dx_j \\
&\quad = \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_j, \cdots, x_n) x_j^s dx_j \\
&\quad\quad - \sum_{t=0}^{\ell} \left(\int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_j, \cdots, x_n) x_j^t dx_j \right) \int_{-\infty}^{\infty} \theta_t(x_j) x_j^s dx_j \\
&\quad = \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_j, \cdots, x_n) x_j^s dx_j - \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_j, \cdots, x_n) x_j^s dx_j \\
&\quad = 0
\end{aligned}$$

because the functions $\{\theta_j\}_{j=0,1,\cdots,m}$ satisfy (2.1). Thus we obtain the lemma. □

Lemma 2.6. (i) *If $\varphi \in \mathcal{D}(R^n)$, then $T^\alpha \varphi \in \mathcal{D}^\alpha(R^n)$.*
(ii) *If $\varphi \in \mathcal{D}(R^n)$, then $U^\alpha \varphi \in \mathcal{U}^\alpha(R^n)$.*

Proof. (i) By using Lemma 2.5 repeatedly we obtain (i).
(ii) Let $\varphi \in \mathcal{D}(R^n)$. By (2.5) and (2.7) $U^\alpha \varphi$ has the following form:

$$U^\alpha \varphi(x) = \sum_{p=1}^{k} (-1)^{p+1} \sum_{\{i_1, \cdots, i_p\} \in M_{\alpha,p}} \sum_{s_1=0}^{\alpha_{i_1}-1} \cdots \sum_{s_p=0}^{\alpha_{i_p}-1}$$

$$\left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_n) x_{i_1}^{s_1} \cdots x_{i_p}^{s_p} dx_{i_1} \cdots dx_{i_p} \right) \theta_{s_1}(x_{i_1}) \cdots \theta_{s_p}(x_{i_p}).$$

For $\{i_1, \cdots, i_p\} \in M_{\alpha,p}$ and $0 \le s_1 \le \alpha_{i_1} - 1, \cdots, 0 \le s_p \le \alpha_{i_p} - 1$ we set

$$\begin{aligned}
&\lambda_{i_1, \cdots, i_p; s_1, \cdots, s_p}(\{x_{i_1}, \cdots, x_{i_p}\}^c) \\
&\quad = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_n) x_{i_1}^{s_1} \cdots x_{i_p}^{s_p} dx_{i_1} \cdots dx_{i_p}.
\end{aligned}$$

It is clear that $\lambda_{i_1,\cdots,i_p;s_1,\cdots,s_p} \in \mathcal{D}(R^{n-p})$. Moreover, for $2 \le p \le k$ and $1 \le \ell \le p$ we have

$$\begin{aligned}
&\lambda_{i_1,\cdots,i_p;s_1,\cdots,s_p}(\{x_{i_1},\cdots,x_{i_p}\}^c) \\
&= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\varphi(x_1,\cdots,x_{i_\ell},\cdots,x_n)x_{i_\ell}^{s_\ell}dx_{i_\ell}\right) \\
&\qquad\qquad \times x_{i_1}^{s_1}\cdots\widehat{x_{i_\ell}^{s_\ell}}\cdots x_{i_p}^{s_p}dx_{i_1}\cdots\widehat{dx_{i_\ell}}\cdots dx_{i_p} \\
&= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\lambda_{i_\ell;s_\ell}(\{x_{i_\ell}\}^c)x_{i_1}^{s_1}\cdots\widehat{x_{i_\ell}^{s_\ell}}\cdots x_{i_p}^{s_p}dx_{i_1}\cdots\widehat{dx_{i_\ell}}\cdots dx_{i_p}.
\end{aligned}$$

Thus $U^\alpha\varphi$ satisfies (2.9) and (2.10), and hence $U^\alpha\varphi$ is a tensor product function of order α. The lemma was proved. □

Lemma 2.7. (i) *If $\varphi \in \mathcal{D}^\alpha(R^n)$, then $T^\alpha\varphi = \varphi$.*
(ii) *If $\varphi \in \mathcal{U}^\alpha(R^n)$, then $U^\alpha\varphi = \varphi$.*

Proof. (i) Let $\varphi \in \mathcal{D}^\alpha(R^n)$. For $p = 1, 2, \cdots, k$ and $\{i_1,\cdots,i_p\} \in M_{\alpha,p}$, since $S^{\alpha(i_1,\cdots,i_p)}\varphi$ is given by (2.5), we see that $S^{\alpha(i_1,\cdots,i_p)}\varphi = 0$ by the condition $\varphi \in \mathcal{D}^\alpha(R^n)$, Lemma 2.1 and Fubini's theorem. Hence (2.6) implies that $T^\alpha\varphi = \varphi$.

(ii) Let $\varphi \in \mathcal{U}^\alpha(R^n)$ and φ have the form (2.8). Then by (2.5), (2.7) and Lemma 2.4 we have

$$\begin{aligned}
U^\alpha\varphi(x) &= \sum_{p=1}^{k}(-1)^{p+1}\sum_{\{i_1,\cdots,i_p\}\in M_{\alpha,p}}\sum_{s_1=0}^{\alpha_{i_1}-1}\cdots\sum_{s_p=0}^{\alpha_{i_p}-1} \\
&\quad \left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\varphi(x_1,\cdots,x_n)x_{i_1}^{s_1}\cdots x_{i_p}^{s_p}dx_{i_1}\cdots dx_{i_p}\right)\theta_{s_1}(x_{i_1})\cdots\theta_{s_p}(x_{i_p}) \\
&= \sum_{p=1}^{k}(-1)^{p+1}\sum_{\{i_1,\cdots,i_p\}\in M_{\alpha,p}}\sum_{s_1=0}^{\alpha_{i_1}-1}\cdots\sum_{s_p=0}^{\alpha_{i_p}-1} \\
&\qquad \lambda_{i_1,\cdots,i_p;s_1,\cdots,s_p}(\{x_{i_1},\cdots,x_{i_p}\}^c)\theta_{s_1}(x_{i_1})\cdots\theta_{s_p}(x_{i_p}) \\
&= \varphi(x).
\end{aligned}$$

Hence we obtain (ii). □

Now we establish our main result.

Theorem 2.1. $\mathcal{D}(R^n) = \mathcal{D}^\alpha(R^n) \oplus \mathcal{U}^\alpha(R^n)$.

Proof. Since $\varphi = T^\alpha\varphi + U^\alpha\varphi$ for $\varphi \in \mathcal{D}(R^n)$, Lemma 2.6 gives $\mathcal{D}(R^n) = \mathcal{D}^\alpha(R^n) + \mathcal{U}^\alpha(R^n)$. Moreover, let $\varphi \in \mathcal{D}^\alpha(R^n) \cap \mathcal{U}^\alpha(R^n)$. Then Lemma 2.7 implies

$$\varphi = T^\alpha\varphi + U^\alpha\varphi = \varphi + \varphi = 2\varphi.$$

Hence $\varphi = 0$. Therefore $\mathcal{D}^\alpha(R^n) \cap \mathcal{U}^\alpha(R^n) = \{0\}$. Thus we obtain the theorem. □

Remark 2.1. We note that Lemmas 2.6 (i) and 2.7 (i) (resp. Lemmas 2.6 (ii) and 2.7 (ii)) imply $T^\alpha(\mathcal{D}(R^n)) = \mathcal{D}^\alpha(R^n)$ (resp. $U^\alpha(\mathcal{D}(R^n)) = \mathcal{U}^\alpha(R^n)$). Moreover, $(T^\alpha)^{-1}(0) = \mathcal{U}^\alpha(R^n)$ and $(U^\alpha)^{-1}(0) = \mathcal{D}^\alpha(R^n)$. We give the proof of $(U^\alpha)^{-1}(0) = \mathcal{D}^\alpha(R^n)$. Let $\varphi \in \mathcal{D}^\alpha(R^n)$. Then $U^\alpha\varphi = \varphi - T^\alpha\varphi = \varphi - \varphi = 0$ by Lemma 2.7 (i). Hence $\mathcal{D}^\alpha(R^n) \subset (U^\alpha)^{-1}(0)$. Conversely let $\varphi \in (U^\alpha)^{-1}(0)$. Then $0 = U^\alpha\varphi = \varphi - T^\alpha\varphi$. Hence $\varphi = T^\alpha\varphi \in \mathcal{D}^\alpha(R^n)$ by Lemma 2.6 (i). Therefore $(U^\alpha)^{-1}(0) \subset \mathcal{D}^\alpha(R^n)$. The proof of $(T^\alpha)^{-1}(0) = \mathcal{U}^\alpha(R^n)$ is the same.

References

[1] H. Begehr, Orthogonal decompositions of the function space $L_2(\overline{D};\mathbb{C})$, J. Reine Angew. Math., **549** (2002), 191–219, MR1916655 (2003f:46037).

[2] H. Begehr and Y. Dubinskiĭ, Some orthogonal decompositions of Sobolev spaces and applications, Colloq. Math., **89** (2001), 199–212, MR1854703 (2002h:46050).

[3] Y. A. Dubinskiĭ, On an orthogonal decomposition of the spaces $\overset{\circ}{W}{}_2^1$ and W_2^{-1} and its application to the Stokes problem, Dokl. Akad. Nauk, **374** (2000), 13–16, MR1799506 (2001m:46071).

[4] L. Schwartz, Théorie des distributions, Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X. Nouvelle édition, entiérement corrigée, refondue et augmentée, Hermann, Paris, 1966, MR0209834 (35 #730).

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