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Martin kernels of general domains

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Abstract.

This note consists of our recent researches on Martin kernels of general domains. In particular, minimal Martin boundary points of a John domain, the boundary behavior of quotients of Martin kernels, and comparison estimates for the Green function and the Martin kernel are studied.

§1. Introduction

This note is a summary of our recent researches on the Martin boundary and the Martin kernel of general domains. To begin with, let us recall the notion of the Martin boundary and the Martin kernel. Let Ω be a Greenian domain in $\mathbb{R}^n$, where $n \geq 2$, possessing the Green function G_Ω for the Laplace operator. Let $x_0 \in \Omega$ be fixed, and let $\{y_j\}$ be a sequence in Ω with no limit point in Ω. If ω is an open subset of Ω such that the closure $\overline{\omega}$ is compact in Ω, then there exists j_0 such that $\{G_\Omega(\cdot, y_j)/G_\Omega(x_0, y_j)\}_{j=j_0}^{\infty}$ is a uniformly bounded sequence of positive harmonic functions in ω. Therefore there is a subsequence of $\{G_\Omega(\cdot, y_j)/G_\Omega(x_0, y_j)\}_j$ converging to a positive harmonic function in Ω. The collection of all such limit functions in Ω gives an ideal boundary of Ω, referred to as the *Martin boundary* of Ω and denoted by $\Delta(\Omega)$. For $\zeta \in \Delta(\Omega)$, we write $K_\Omega(\cdot, \zeta)$ for the positive harmonic function in Ω corresponding to ζ, and call K_Ω the *Martin kernel.* We say that a positive harmonic function h is *minimal* if every positive harmonic function less than or equal to h coincides with a constant multiple of h. The collection of all minimal elements in $\Delta(\Omega)$ is called the *minimal Martin boundary* of Ω, and is denoted by $\Delta_1(\Omega)$. The importance of the Martin boundary appears in the representation theorem for positive harmonic

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functions h in general domains: there exists a measure μ_h on $\Delta(\Omega)$ such that $\mu_h(\Delta(\Omega) \setminus \Delta_1(\Omega)) = 0$ and

$$h(x) = \int_{\Delta(\Omega)} K_\Omega(x,\zeta)d\mu_h(\zeta) \quad \text{for } x \in \Omega.$$

So, for general domains, it is valuable to investigate the Martin boundary and the behavior of the Martin kernel.

This note is organized as follows. In Section 2, we state the results, obtained in [2], about the number of minimal Martin boundary points of John domains. In Sections 3 and 4, we give the results, studied in [17] and [18], about the boundary behavior of the Martin kernels and comparison estimates for the Green function and the Martin kernel.

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§2. Minimal Martin boundary points of John domains

From the viewpoint of the representation theorem, the most interesting problem is to investigate that for what kind of domains the Martin boundary and the minimal Martin boundary are homeomorphic to the Euclidean boundary. For instance, see [19] for Lipschitz domains, [20] for NTA domains, and [1] for uniform domains. However, in general, the Martin boundary need not to be homeomorphic to the Euclidean boundary. There may be even infinitely many minimal Martin boundary points at a Euclidean boundary point (cf. [22, Example 3]). Here, a Martin boundary point at $y \in \partial\Omega$ (the Euclidean boundary of Ω) is a positive harmonic function in Ω which can be obtained as the limit of $\{G_\Omega(\cdot, y_j)/G_\Omega(x_0, y_j)\}_j$ for some sequence $\{y_j\}$ in Ω converging to y. It is also interesting to investigate that for what kind of domains the number of minimal Martin boundary points at every Euclidean boundary point is finite. For example, see [7] for Denjoy domains, [5, 6] and [13] for Lipschitz-Denjoy domains, [15] for sectorial domains, and [21] for quasi-sectorial domains. One of the main interests of these papers was to give a criterion for the number of minimal Martin boundary points at a fixed Euclidean boundary point. As a generalization of some parts of them, we study minimal Martin boundary points of John domains. A domain Ω is said to be a general John domain with John constant $c_J > 0$ and John center K_0, a compact subset of Ω, if each point x in Ω can be connected to some point in K_0 by a rectifiable curve γ in Ω such that

$$\operatorname{dist}(z, \partial\Omega) \geq c_J \ell(\gamma(x, z)) \quad \text{for all } z \in \gamma,$$

where $\operatorname{dist}(z, \partial\Omega)$ stands for the distance from z to $\partial\Omega$ and $\ell(\gamma(x, z))$ denotes the length of the subarc $\gamma(x, z)$ of γ from x to z. Note that every general John domain is bounded. We can obtain the following.

Theorem 2.1 ([2, Theorem 1.1]). *Let Ω be a general John domain with John constant c_J, and let $y \in \partial\Omega$. Then the following statements hold:*

(i) *The number of minimal Martin boundary points at y is bounded by a constant depending only on c_J and n.*

(ii) *If $c_J > \sqrt{3}/2$, then the number of minimal Martin boundary points at y is at most two.*

Remark 2.2. The bound $c_J > \sqrt{3}/2$ in Theorem 2.1(ii) is sharp. See [2, Remark 1.1].

For a class of general John domains represented as the union of open convex sets, we give a sufficient condition for the Martin boundary to be homeomorphic to the Euclidean boundary. For $0 < \theta < \pi$, we write $\Gamma_\theta(z, w) = \{x \in \mathbb{R}^n : \angle xzw < \theta\}$ for the open circular cone with vertex at z, axis $[z, w]$ and aperture θ. Let $A_0 \geq 1$ and $\rho_0 > 0$. We consider a bounded domain Ω with the following properties:

(I) Ω is the union of a family of open convex sets $\{C_\lambda\}_{\lambda \in \Lambda}$ such that

$$B(z_\lambda, \rho_0) \subset C_\lambda \subset B(z_\lambda, A_0\rho_0);$$

(II) for each $y \in \partial\Omega$, there are positive constants $\theta_1 \leq \sin^{-1}(1/A_0)$ and $\rho_1 \leq \rho_0 \cos\theta_1$ such that

$$\bigcup_{\substack{w\in\Omega \\ \Gamma_{\theta_1}(y,w)\cap B(y,2\rho_1)\subset\Omega}} \Gamma_{\theta_1}(y, w) \cap B(y, 2\rho_1) \quad \text{is connected and non-empty.}$$

Obviously, a bounded domain satisfying (I) is a general John domain with John center $\overline{\{z_\lambda\}_{\lambda\in\Lambda}}$ and John constant A_0^{-1}.

Theorem 2.3 ([2, Theorem 1.2]). *Let Ω be a bounded domain satisfying (I). If $y \in \partial\Omega$ satisfies (II), then there is a unique Martin boundary point at y and it is minimal. Furthermore, if every Euclidean boundary point satisfies (II), then the Martin boundary of Ω is homeomorphic to the Euclidean boundary.*

Remark 2.4. The bounds $\theta_1 \leq \sin^{-1}(1/A_0)$ and $\rho_1 \leq \rho_0 \cos\theta_1$ are sharp. See [2, Examples 8.1 and 8.2].

Theorem 2.3 is a generalization of Ancona's result [4]. He considered a bounded domain represented as the union of open balls with the same radius. His key lemma [4, Lemme 1] relies on the reflection with respect to a hyperplane, and is applied to a ball by the Kelvin transform. This approach is not applicable to our domains. Our approach is based on a new geometrical notion, the system of local reference points. We define the quasi-hyperbolic metric on Ω by

$$k_\Omega(x,y) = \inf_\gamma \int_\gamma \frac{ds(z)}{\mathrm{dist}(z,\partial\Omega)} \quad \text{for } x,y \in \Omega,$$

where the infimum is taken over all rectifiable curves γ in Ω connecting x to y and ds stands for the line element on γ. Let $N \in \mathbb{N}$ and $0 < \eta < 1$. We say that $y \in \partial\Omega$ has *a system of local reference points of order N with factor η* if there exist $R_y > 0$ and $A_y > 1$ with the following property: for each positive $R < R_y$ there are N points $y_1 = y_1(R), \cdots, y_N = y_N(R) \in \Omega \cap \partial B(y,R)$ such that $\mathrm{dist}(y_j, \partial\Omega) \geq A_y^{-1}R$ for $j = 1, \cdots, N$ and

$$\min_{j=1,\cdots,N}\{k_{\Omega\cap B(y,\eta^{-3}R)}(x,y_j)\} \leq A_y \log \frac{R}{\mathrm{dist}(x,\partial\Omega)} + A_y$$

for $x \in \Omega \cap B(y,\eta R)$. For example, if Ω is a (sectorial) domain in $\mathbb{R}^2$ whose boundary near $y \in \partial\Omega$ lies on m-distributed rays emanating from y, then y has a system of local reference points of order $N = m$. For a general John domain Ω with John constant c_J, we can show that

- each $y \in \partial\Omega$ has a system of local reference points of order N with $N \leq N(c_J, n) < \infty$. Moreover, if $c_J > \sqrt{3}/2$, then we can let $N \leq 2$ by choosing a suitable factor η.
- if Ω satisfies (I) and $y \in \partial\Omega$ satisfies (II), then y has a system of local reference points of order 1.

These observations played essential roles in the proofs of Theorems 2.1 and 2.3. Indeed, Theorems 2.1 and 2.3 can be reunderstood as follows.

Proposition 2.5 ([2, Proposition 2.3]). *Let Ω be a general John domain, and suppose that $y \in \partial\Omega$ has a system of local reference points of order N. Then the following statements hold:*

(i) *The number of minimal Martin boundary points at y is bounded by a constant depending only on N.*
(ii) *If $N \leq 2$, then there are at most N minimal Martin boundary points at y. Moreover, if $N = 1$, then there is a unique Martin boundary point at y and it is minimal.*

In Proposition 2.5(ii), the condition $N \le 2$ may be omitted, and we expect that the number of minimal Martin boundary points at y is at most N even for $N \ge 3$. We raise the following question.

Problem. *Let Ω be a general John domain and let $N \ge 3$. Suppose that $y \in \partial\Omega$ has a system of local reference points of order N. Is the number of minimal Martin boundary points at y at most N?*

§3. Boundary behavior of quotients of Martin kernels

In [9, 10], Burdzy obtained a result on the angular derivative problem of analytic functions in a Lipschitz domain. The important step was to study the boundary behavior of the Green function. We now write $\mathbf{0}$ for the origin of $\mathbb{R}^n$ to distinguish from $0 \in \mathbb{R}$, and denote $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $e = (\mathbf{0}', 1)$. Suppose that $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function such that $\phi(\mathbf{0}') = 0$, and put $\Omega_\phi = \{(x', x_n) : x_n > \phi(x')\}$. We set

(3.1)
$$I^+ = \int_{\{|x'|<1\}} \frac{\max\{\phi(x'), 0\}}{|x'|^n} dx', \quad I^- = \int_{\{|x'|<1\}} \frac{\max\{-\phi(x'), 0\}}{|x'|^n} dx'.$$

Theorem A. *Let I^+ and I^- be as in* (3.1). *Then the following statements hold:*

(i) *If $I^+ < \infty$ and $I^- = \infty$, then*
$$\lim_{t \to 0+} \frac{G_{\Omega_\phi}(te, e)}{t} = \infty.$$

(ii) *If $I^+ = \infty$ and $I^- < \infty$, then*
$$\lim_{t \to 0+} \frac{G_{\Omega_\phi}(te, e)}{t} = 0.$$

(iii) *If $I^+ < \infty$ and $I^- < \infty$, then the limit of $G_{\Omega_\phi}(te, e)/t$, as $t \to 0+$, exists and*
$$0 < \lim_{t \to 0+} \frac{G_{\Omega_\phi}(te, e)}{t} < \infty.$$

Burdzy's approach was based on probabilistic methods. Analytic proofs were given by Carroll [11, 12] and Gardiner [16]. As we see from their proofs, the convergence of the integrals I^+ and I^- are related to the minimal thinness of the differences $\Omega_\phi \setminus \mathbb{R}^n_+$ and $\mathbb{R}^n_+ \setminus \Omega_\phi$, where

$\mathbb{R}^n_+ = \{(x', x_n) : x_n > 0\}$. A subset E of Ω is said to be *minimally thin* at $\xi \in \Delta_1(\Omega)$ with respect to Ω if

$$ {}^{\Omega}\widehat{R}^{E}_{K_\Omega(\cdot,\xi)}(z) < K_\Omega(z,\xi) \quad \text{for some } z \in \Omega, $$

where ${}^{\Omega}\widehat{R}^{E}_{u}$ denotes the regularized reduced function of a positive superharmonic function u relative to E in Ω. We say that a function f, defined on a minimal fine neighborhood U of ξ, has *minimal fine limit* l at ξ with respect to Ω if there is a subset E of Ω, minimally thin at ξ with respect to Ω, such that $f(x) \to l$ as $x \to \xi$ along $U \setminus E$, and then we write

$$ \underset{\Omega}{\mathrm{mf}}\text{-}\lim_{x\to\xi} f(x) = l. $$

Theorem A was shown by using Naïm's characterization [23, Théorème 11] of the minimal thinness for a difference of domains in terms of the boundary behavior of the quotient of the Green functions.

We are now interested in the boundary behavior of Martin kernels. In this case, we can not apply the Naïm's characterization. Alternatively, we can characterize the minimal thinness for a difference of domains in terms of the boundary behavior of the quotient of the Martin kernels (see [17, Lemma 3.1]), and then obtain the following general result.

Theorem 3.1 ([17, Theorem 2.1]). *Suppose that Ω and D are Greenian domains such that $\Omega \cap D$ is a non-empty domain. Let $\xi \in \Delta_1(\Omega)$, where ξ is in the closure of $\Omega \cap D$ in the Martin compactification of Ω. Let $\zeta \in \Delta_1(D)$, where ζ is in the closure of $\Omega \cap D$ in the Martin compactification of D. If $\Omega \setminus D$ is minimally thin at ξ with respect to Ω, then $K_D(\cdot,\zeta)/K_\Omega(\cdot,\xi)$ has a finite minimal fine limit at ξ with respect to Ω. Furthermore, the following statements hold:*

(i) *If $D \setminus \Omega$ is not minimally thin at ζ with respect to D, then*

$$ \underset{\Omega}{\mathrm{mf}}\text{-}\lim_{x\to\xi} \frac{K_D(x,\zeta)}{K_\Omega(x,\xi)} = 0. $$

(ii) *If $D \setminus \Omega$ is minimally thin at ζ with respect to D, where ζ is the point such that*

$$ K_D(\cdot,\zeta) - {}^{D}\widehat{R}^{D\setminus\Omega}_{K_D(\cdot,\zeta)} = \alpha\big(K_\Omega(\cdot,\xi) - {}^{\Omega}\widehat{R}^{\Omega\setminus D}_{K_\Omega(\cdot,\xi)}\big) \quad \textit{on } \Omega\cap D \tag{3.2} $$

for some positive constant α, then

$$ 0 < \underset{\Omega}{\mathrm{mf}}\text{-}\lim_{x\to\xi} \frac{K_D(x,\zeta)}{K_\Omega(x,\xi)} < \infty. $$

(iii) *If $D \setminus \Omega$ is minimally thin at ζ with respect to D, where ζ is a point such that* (3.2) *is not satisfied, then*

$$\operatorname*{mf-lim}_{\Omega}\lim_{x\to\xi} \frac{K_D(x,\zeta)}{K_\Omega(x,\xi)} = 0.$$

As a consequence of Theorem 3.1, we can obtain a result corresponding to Theorem A. Note that Ω_ϕ has a unique Martin boundary point at the origin $\mathbf{0}$, so we write $K_{\Omega_\phi}(\cdot,\mathbf{0})$ for the Martin kernel at $\mathbf{0}$.

Corollary 3.2 ([17, Theorem 1.1]). *Let I^+ and I^- be as in* (3.1). *Then the following statements hold:*

(i) *If $I^+ < \infty$ and $I^- = \infty$, then*

$$\lim_{t\to 0+} t^{n-1} K_{\Omega_\phi}(te,\mathbf{0}) = 0.$$

(ii) *If $I^+ = \infty$ and $I^- < \infty$, then*

$$\lim_{t\to 0+} t^{n-1} K_{\Omega_\phi}(te,\mathbf{0}) = \infty.$$

(iii) *If $I^+ < \infty$ and $I^- < \infty$, then the limit of $t^{n-1}K_{\Omega_\phi}(te,\mathbf{0})$, as $t \to 0+$, exists and*

$$0 < \lim_{t\to 0+} t^{n-1} K_{\Omega_\phi}(te,\mathbf{0}) < \infty.$$

Remark 3.3. When $I^+ = \infty$ and $I^- = \infty$, the limit of $t^{n-1}K_{\Omega_\phi}(te,\mathbf{0})$ may take any values 0, positive and finite, or ∞ (see [17, Example 1.2]).

§4. Comparison estimates for the Green function and the Martin kernel

For two positive functions f_1 and f_2, the symbol $f_1 \approx f_2$ means that there exists a constant $A > 1$ such that $A^{-1}f_2 \le f_1 \le Af_2$. ¿From Theorem A and Corollary 3.2, we expect the following relationship between the Green function and the Martin kernel:

$$G_{\Omega_\phi}(te,e)K_{\Omega_\phi}(te,\mathbf{0}) \approx t^{2-n} \quad \text{for } 0 < t < 2^{-1},$$

or, more generally, if Ω is a Lipschitz domain and $\xi \in \partial\Omega$, then

$$G_\Omega(x,x_0)K_\Omega(x,\xi) \approx |x-\xi|^{2-n} \quad \text{for } x \in \Gamma_\alpha(\xi) \setminus B(x_0, 2^{-1}\operatorname{dist}(x_0,\partial\Omega)), \tag{4.1}$$

where $\Gamma_\alpha(\xi) = \{x \in \Omega : |x-\xi| < \alpha \operatorname{dist}(x,\partial\Omega)\}$ with $\alpha > 1$ large enough. If we restrict to the case of bounded Lipschitz domains Ω in $\mathbb{R}^n$ with

$n \geq 3$, then only the upper estimate in (4.1) can be obtained from the following 3G inequality:

$$\frac{G_\Omega(x,z)G_\Omega(x,y)}{G_\Omega(z,y)} \leq A(|x-y|^{2-n} + |x-z|^{2-n}) \quad \text{for } x,y,z \in \Omega,$$

which was first proved by Cranston, Fabes and Zhao [14] in the study of conditional gauge theory for the Schrödinger operator. See also Bogdan [8]. Recently, Aikawa and Lundh [3] extended this inequality to the case of bounded uniformly John domains in $\mathbb{R}^n$ with $n \geq 3$.

Now, let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 3$ and let $\{y_j\}$ be a sequence in Ω converging to $\xi \in \partial\Omega$. Then, substituting $z = x_0$ and $y = y_j$ into the 3G inequality and letting $j \to \infty$, we obtain the upper estimate: for $x \in \Omega \setminus B(x_0, 2^{-1}\operatorname{dist}(x_0, \partial\Omega))$,

$$G_\Omega(x,x_0)K_\Omega(x,\xi) \leq A(|x-\xi|^{2-n} + |x-x_0|^{2-n}) \leq A'|x-\xi|^{2-n}.$$

The lower estimate in (4.1) does not follow from the 3G inequality, but the boundary Harnack principle would enable us to obtain (4.1). We consider (4.1) in a uniform domain. A domain Ω is said to be *uniform* if there exists a constant $A_1 > 1$ such that each pair of points x and y in Ω can be connected by a rectifiable curve γ in Ω such that

$$\ell(\gamma) \leq A_1|x-y|,$$
$$\min\{\ell(\gamma(x,z)), \ell(\gamma(z,y))\} \leq A_1 \operatorname{dist}(z, \partial\Omega) \quad \text{for all } z \in \gamma.$$

It is known that if Ω is a uniform domain, then there is a unique (minimal) Martin boundary point at each Euclidean boundary point (cf. [1]). As above, we write $K_\Omega(\cdot, \xi)$ for the Martin kernel at $\xi \in \partial\Omega$. Our conclusions are different between $n \geq 3$ and $n = 2$, so we state them separately. See [18] for their proofs.

Theorem 4.1. *Let Ω be a uniform domain in $\mathbb{R}^n$, where $n \geq 3$, and let $\xi \in \partial\Omega$. Then*

(4.2)
$$G_\Omega(x,x_0)K_\Omega(x,\xi) \approx |x-\xi|^{2-n} \quad \textit{for } x \in \Gamma_\alpha(\xi) \cap B(\xi, 2^{-1}\operatorname{dist}(x_0, \partial\Omega)),$$

where the constant of comparison depends only on α and Ω.

When $n = 2$, the comparison estimate (4.2) does not hold in general as seen in the following example.

Example 4.2. Suppose that $n = 2$. Let $\Omega = B(\mathbf{0}, 1) \setminus \{\mathbf{0}\}$ and let $x_0 = (1/2, 0)$. Then Ω is a uniform domain, and we have for $x \in$

$B(\mathbf{0}, 1/4) \setminus \{\mathbf{0}\}$,

$$K_\Omega(x, \mathbf{0}) G_\Omega(x, x_0) = \frac{-\log|x|}{\log 2} \log\left(\frac{1}{2}\frac{|x-4x_0|}{|x-x_0|}\right) \approx \log\frac{1}{|x|}.$$

We say that $\xi \in \partial\Omega$ satisfies the exterior condition if there exists a positive constant κ such that for each $r > 0$ sufficiently small, there is $x_r \in B(\xi, r) \setminus \overline{\Omega}$ with $B(x_r, \kappa r) \subset \mathbb{R}^n \setminus \overline{\Omega}$.

Theorem 4.3. *Let Ω be a uniform domain in $\mathbb{R}^2$. Then the following statements hold:*

(i) *If $\xi \in \partial\Omega$ satisfies the exterior condition, then*

$$G_\Omega(x, x_0) K_\Omega(x, \xi) \approx 1 \quad \text{for } x \in \Gamma_\alpha(\xi) \cap B(\xi, 2^{-1}\operatorname{dist}(x_0, \partial\Omega)),$$

where the constant of comparison depends only on α and Ω.

(ii) *If $\xi \in \partial\Omega$ is an isolated point and Ω is bounded, then there exists $\delta > 0$ such that*

$$G_\Omega(x, x_0) K_\Omega(x, \xi) \approx \log\frac{1}{|x-\xi|} \quad \text{for } x \in B(\xi, \delta) \setminus \{\xi\},$$

where the constant of comparison is independent of x.

Finally, we note that if Ω is a Lipschitz domain, then every $\xi \in \partial\Omega$ satisfies the exterior condition and so Theorem 4.3(i) holds.

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